Solving the heat equation using the finite difference method

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1 Problem statement

The goal is to approximate the solution to the heat equation through numerical integration over a grid, given an initial condition of a sine wave at time t = 0. Starting from the homogeneous heat equation:

$$\frac{du}{dt} = \frac{d^2u}{dx^2}$$

We want to derive an approximation for this continuous problem using discrete functions. This can be achieved through use of the centered finite difference approximations of the first and second derivative. These are written as follows.

First derivative:

$$f'(x) = \frac{f(x + \Delta x) - f(x - \Delta x)}{2\Delta x}$$

Second derivative:

$$f''(x) = \frac{f(x + \Delta x) - 2f(x) + f(x - \Delta x)}{\Delta x^2}$$

This problem can be solved analytically through use of separation of variables. The analytic solution of this problem is stated as follows.

$$u(x,t) = e^{-\pi^2 t} sin(\pi x)$$

The analytical solution is used in the program as a comparison for the solution the finite difference method arrives at.

2 Derivation

Substituting for the first and second derivative in the heat equation:

$$\frac{u(x, t + \Delta t) - u(x, t - \Delta t)}{2\Delta t} = \frac{u(x + \Delta x, t) - 2u(x, t) + u(x - \Delta x, t)}{\Delta x^2}$$

One can choose Δ t such that each step is a half integer:

$$\Delta t = \frac{1}{2}\Delta t$$

This removes the factor of 2 in the denominator of the first term. From this, we can find an update function for integration of the differential equation. We are seeking to update $u(x, t + \Delta t)$; thus, we rearrange our equation as follows:

$$u(x, t + \Delta t) = u(x, t - \Delta t) + \frac{\Delta t}{\Delta x^2}(u(x + \Delta x, t) - 2u(x, t) + u(x - \Delta x, t))$$

Iterating over this acts as a modified version of the finite difference method.

2.1 Stability condition

It should be noted that there is for difference equations a stability requirement that can be stated as:

$$\Delta t < \frac{\Delta x^2}{2}$$

This instability is demonstrated later in the analysis.

3 Program Notes

I have defined the number of steps forward in time as the variable N and the number of steps in space as the variable I. The variables dt and dx are defined as the change in time and space respectively per discrete step. As such, these are defined as the difference between their respective endpoints divided by the number of steps in that dimension.

The array xf of dimension (N,I) is initially populated with zeroes. An initial condition is set for the problem, adding to the first row of the array (representing time t = 0) a sine wave along the position axis.

The numerical solution is achieved by iterating through the update function derived in section 2 over the N time steps. The new values are entered to array xf for calculation of the next set. The final numerical solution is entered into array xf at the completion of the loop.

The function u(x,t) defines the analytical solution to the homogeneous heat equation.

4 Results

The following figures show the results for the integration varying the number of steps in t and x. For parameters that satisfy the stability condition, the program is shown to approximate the diffusion of the initial sine wave over time.

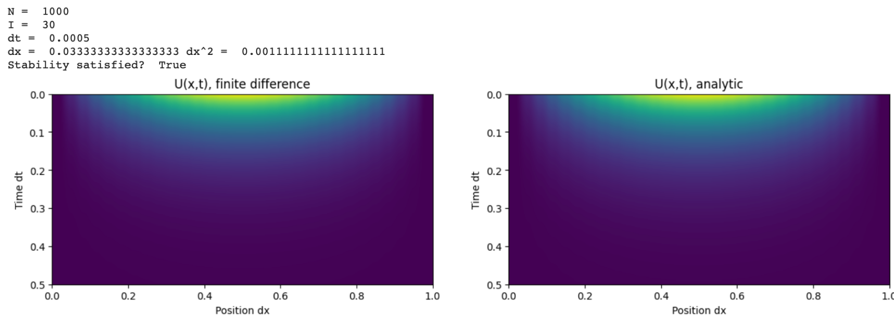

Figure 1: Fig. 1: Stable, N = 1000, I = 30

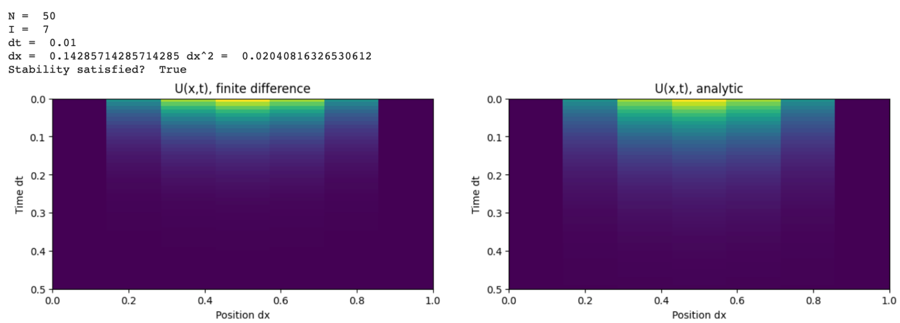

Figure 2: Fig. 2: Stable, N = 50, I = 7

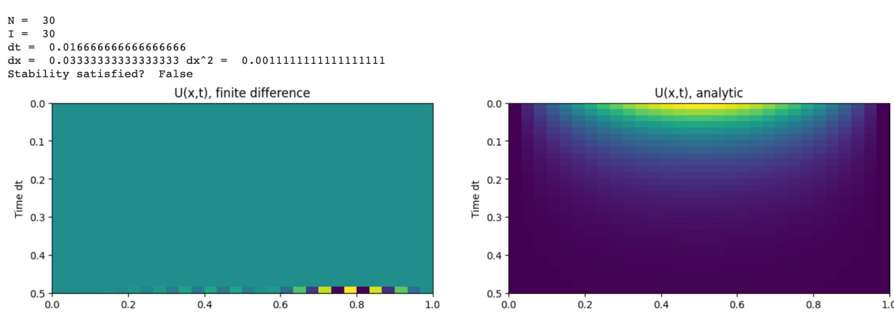

Figure 3: Fig. 3: Unstable, N = 30, I = 30

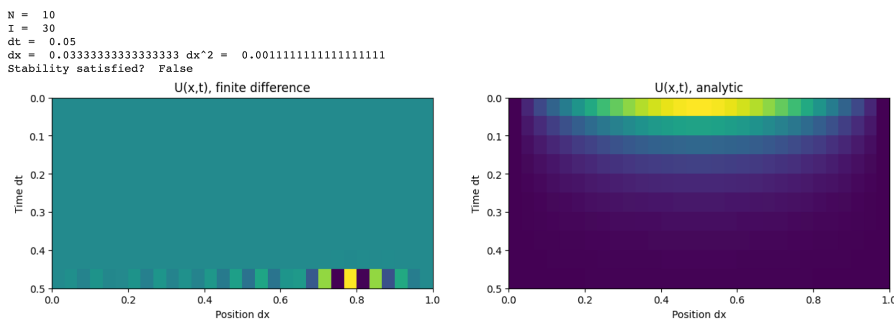

Figure 4: Fig. 4: Unstable, N = 10, I = 30

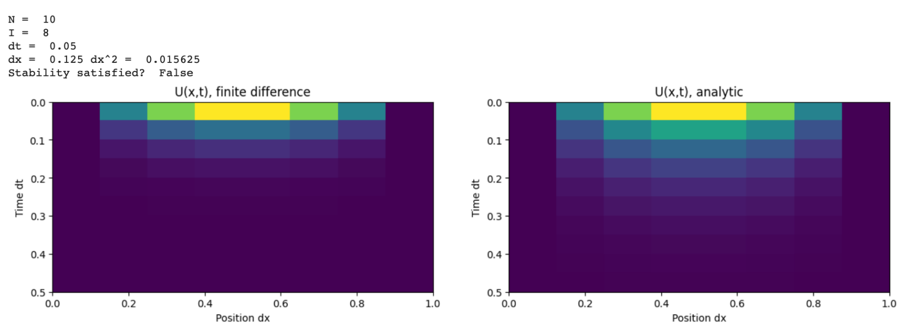

Figure 5: Fig. 5: Unstable, N = 10, I = 8

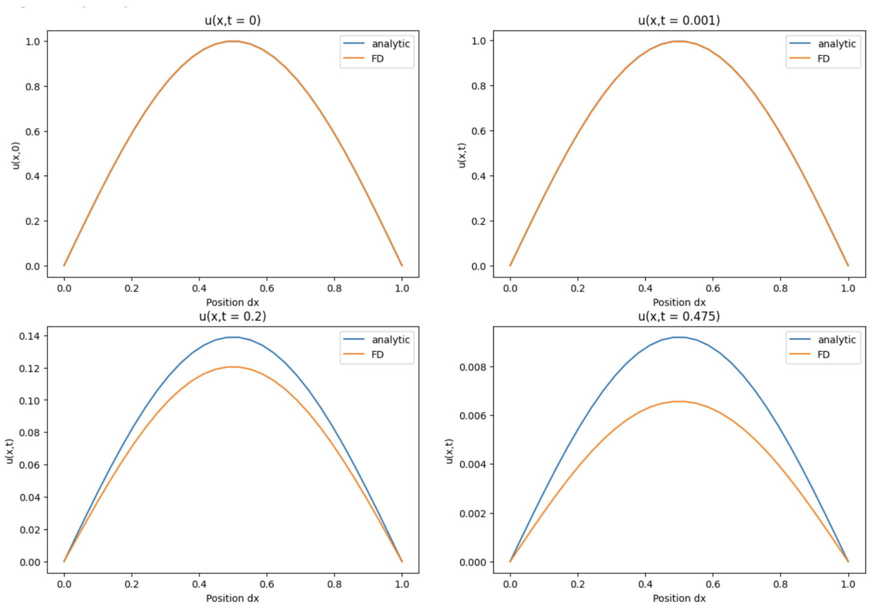

Figure 6: Fig. 6: Snapshots of difference between approximation and function at various time steps.

5 Summary

These plots demonstrate that the stability condition must be upheld, or the solutions will become unstable, which on smaller scales resulted in a solution that decayed more rapidly and on larger scales failed to produce a solution at all. The finite difference method is shown to form a fair approximation of the heat equation solution when the stability condition is followed.

Figure 5 shows the increase over time of the difference between the approximation and true solution to the heat equation. This difference grows larger over time as a result of repeated truncations of numbers to perform the computation.