Lecture 9

II.3 Faddeev equations for the 3-body bound state

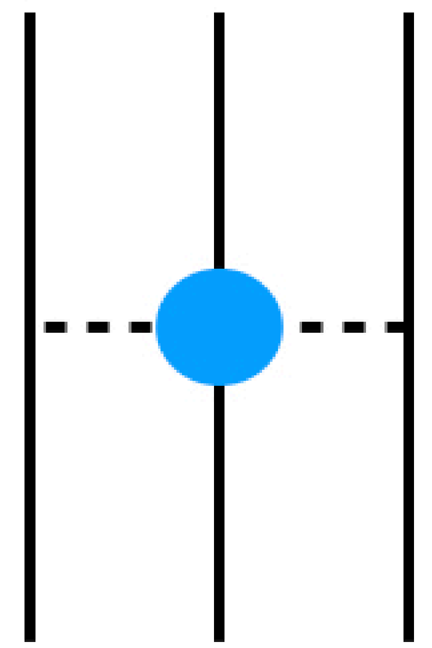
The aim is the implementation of a **three-body bound** state problem using so-called **Faddeev equations**

Schrödinger equation:

neglect three-body force

$$\hat{H}\,|\Psi\rangle = (\hat{H}_0 + \hat{V}_{12} + \hat{V}_{23} + \hat{V}_{31} + \cancel{\hat{V}_{123}})\,|\Psi\rangle = E\,|\Psi\rangle$$

Three **pair interactions** V_{ij} contribute.



$\hat{H}_0$ is the **internal kinetic energy** (without CM motion)

$\hat{V}_{123}$ is a possible three-body interaction (could be studied as a project later on, will be omitted here)

We will here also assume that the three particles are **identical bosons**.

Therefore, we have to obey the Pauli principle but we will also be able to relate the subsystems to each other!

Definition of permutation operators

Let us first look at the interactions and the relation of these using permutation operators P_{ij}

($\hat{}$ for operators is omitted now)

$$H = \underbrace{T_1 + T_2 + T_3}_{H_0} + \underbrace{V_{12} + V_{23} + V_{31}}_{V}$$

The interactions can be related by permutation operators in the following way:

$$V_{23} = P_{12} P_{23} \, V_{12} \, P_{23} P_{12}$$

$$V_{31} = P_{13} P_{23} \, V_{12} \, P_{23} P_{13}$$

Consider single particle coordinates (momenta) to derive these relations:

$$\langle \vec{a}' \, \vec{b}' \, \vec{c}' \, | \, V_{12} \, | \, \vec{a} \; \vec{b} \; \vec{c} \rangle = \delta^{(3)}(\vec{c}' - \vec{c}) \; V(\vec{a}' \, \vec{b}' \, , \, \vec{a} \; \vec{b})$$

particle 3 is not affected

V is the same for each pair

Therefore, for pair (23):

$$\langle \vec{a}' \, \vec{b}' \, \vec{c}' \, | \, V_{23} \, | \, \vec{a} \; \vec{b} \; \vec{c} \rangle = \delta^{(3)}(\vec{a}' - \vec{a}) \; V(\vec{b}' \, \vec{c}' \, , \, \vec{b} \; \vec{c})$$

permutations are hermitian

$$\langle \vec{a}' \, \vec{b}' \, \vec{c}' \, | \, P_{12} P_{23} \, V_{12} \, P_{23} P_{12} \, | \, \vec{a} \; \vec{b} \; \vec{c} \rangle = \langle \vec{b}' \, \vec{a}' \, \vec{c}' \, | \, P_{23} \, V_{12} \, P_{23} \, | \, \vec{b} \; \vec{a} \; \vec{c} \rangle = \langle \vec{b}' \, \vec{c}' \, \vec{a}' \, | \; V_{12} \; | \, \vec{b} \; \vec{c} \; \vec{a} \rangle$$

$$= \delta^{(3)}(\vec{a}' - \vec{a}) \; V(\vec{b}' \, \vec{c}' \, , \, \vec{b} \; \vec{c}) = \langle \vec{a}' \, \vec{b}' \, \vec{c}' \, | \, V_{23} \, | \, \vec{a} \; \vec{b} \; \vec{c} \rangle$$

Rewriting the Schrödinger equation

In the following, we again assume that $E < 0$ (even more $E < E_{2b}$)

Start with the Schrödinger equation and the bound state wave function Ψ

symmetry (Pauli principle)

$$(E - H_0)\, |\, \Psi \,\rangle = V_{12}\, |\, \Psi \,\rangle + P_{12}P_{23}\, V_{12}\, P_{23}P_{12}\, |\, \Psi \,\rangle + P_{13}P_{23}\, V_{12}\, P_{23}P_{13}\, |\, \Psi \,\rangle$$

$$= \Big(\mathbb{1} + \underbrace{P_{12}P_{23} + P_{13}P_{23}}_{\equiv P} \Big)\; V_{12}\, |\, \Psi \,\rangle$$

abbreviation P

This allows to define a Faddeev component and relate it to the wave function:

invertible since E negative

symmetric operator! Why?

$$|\, \Psi \,\rangle = \frac{1}{E - H_0}\, (\mathbb{1} + P)\; V_{12}\, |\, \Psi \,\rangle = (\mathbb{1} + P)\; \underbrace{\frac{1}{E - H_0}\; V_{12}\, |\, \Psi \,\rangle}_{\equiv |\, \psi_{12} \,\rangle} = (\mathbb{1} + P)\, |\, \psi_{12} \,\rangle$$

Faddeev component

For the Faddeev component, one finds

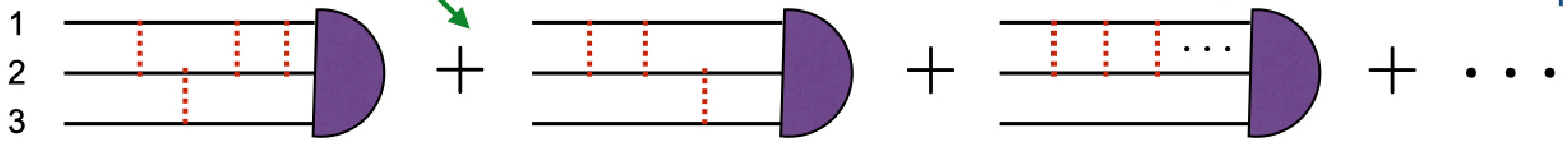
$$|\, \psi_{12} \,\rangle = \frac{1}{E - H_0}\; V_{12}\, |\, \Psi \,\rangle = \frac{1}{E - H_0}\; V_{12}\; (\mathbb{1} + P)\, |\, \psi_{12} \,\rangle$$

many contributions by reinserting eq.

first interaction is in (12)

one part only consists of interactions in (12)

particle 3 is not affected here



Treatment of the not-connected term

The last piece is troublesome: when in part of the operator one of the particles is not affect (unit operator in this coordinate)

- *eigenvalue spectrum will have more than the accumulation point at zero*
- *numerical representation requires delta-function*
- *cannot be approximated in a finite dimensional space*

Therefore, we work further to deal with it analytically:

$$|\psi_{12}\rangle - \frac{1}{E-H_0} V_{12} |\psi_{12}\rangle = \frac{1}{E-H_0} V_{12}\ P\,|\psi_{12}\rangle \longrightarrow \underbrace{\left(\mathbb{1} - \frac{1}{E-H_0} V_{12}\right)}_{\text{Are we able to invert this?}} |\psi_{12}\rangle = \frac{1}{E-H_0} V_{12}\ P\,|\psi_{12}\rangle$$

yes, use LS equation (see scattering)

$$\left(\mathbb{1} + \frac{1}{E-H_0} t_{12}\right)\left(\mathbb{1} - \frac{1}{E-H_0} V_{12}\right) = \mathbb{1} - \frac{1}{E-H_0} V_{12} + \frac{1}{E-H_0} t_{12} - \frac{1}{E-H_0} t_{12} \frac{1}{E-H_0} V_{12}$$

$$= \mathbb{1} - \frac{1}{E-H_0} \underbrace{\left(t_{12} - V_{12} - t_{12} \frac{1}{E-H_0} V_{12}\right)}_{\text{0 because of LS eq.}} = \mathbb{1}$$

Finally, we get the **Faddeev equations**

$$|\psi_{12}\rangle = \left(\mathbb{1} + \frac{1}{E-H_0} t_{12}\right) \frac{1}{E-H_0} V_{12}\ P\,|\psi_{12}\rangle = \frac{1}{E-H_0}\left(V_{12} + t_{12} \frac{1}{E-H_0} V_{12}\right) P\,|\psi_{12}\rangle = \frac{1}{E-H_0} t_{12}\ P\,|\psi_{12}\rangle$$

together with the **Lippmann-Schwinger equation**

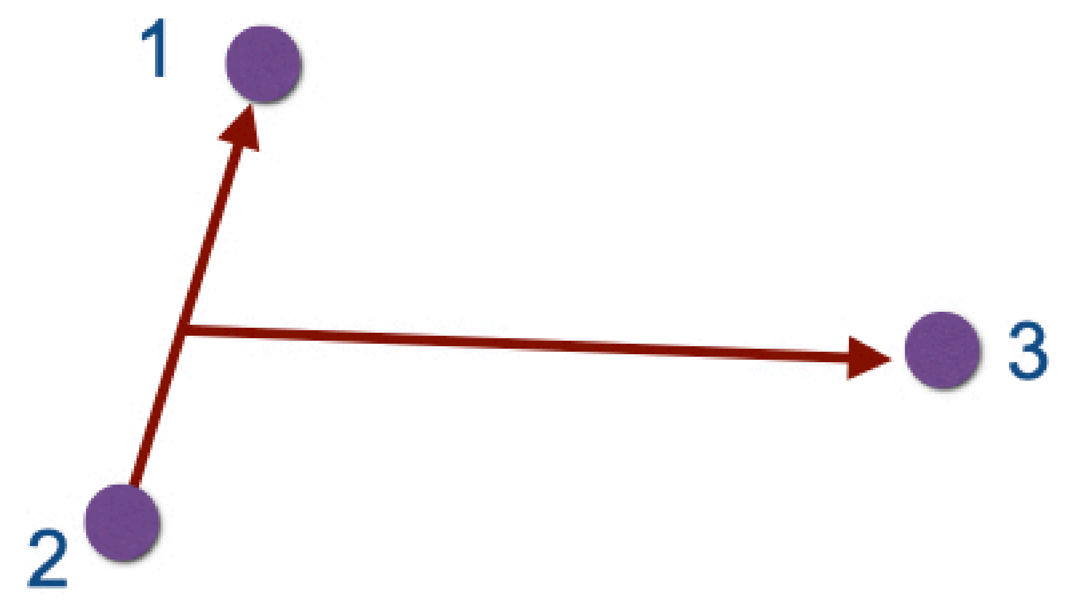
$$t_{12} = V_{12} + V_{12} \frac{1}{E-H_0} t_{12}$$

E is the binding energy and not directly related to the two-body state

Jacobi coordinates and permutation operators

This is the essential new aspect of more than two-body systems
(for non-identical particles a similar problem arises with coordinate transformations)

Single particle momenta: $\vec{k}_1 \;\; \vec{k}_2 \;\; \vec{k}_3$



Jacobi momenta:

$$\vec{p}_{12} = \frac{1}{2}\left(\vec{k}_1 - \vec{k}_2\right) \qquad \vec{p}_3 = \frac{2}{3}\,\vec{k}_3 - \frac{1}{3}\left(\vec{k}_1 + \vec{k}_2\right) \qquad \vec{P} = \vec{k}_1 + \vec{k}_2 + \vec{k}_3$$

but other choices are equally well possible

$$\vec{p}_{23} = \frac{1}{2}\left(\vec{k}_2 - \vec{k}_3\right) \;\; \vec{p}_1 = \frac{2}{3}\,\vec{k}_1 - \frac{1}{3}\left(\vec{k}_2 + \vec{k}_3\right) \quad \text{and} \quad \vec{p}_{31} = \frac{1}{2}\left(\vec{k}_3 - \vec{k}_1\right) \;\; \vec{p}_2 = \frac{2}{3}\,\vec{k}_2 - \frac{1}{3}\left(\vec{k}_3 + \vec{k}_1\right)$$

with some algebra, one can related the different sets with each other

$$\vec{p}_{23} = -\frac{1}{2}\,\vec{p}_{12} - \frac{3}{4}\,\vec{p}_3 \qquad \vec{p}_1 = \vec{p}_{12} - \frac{1}{2}\,\vec{p}_3 \quad \text{and} \quad \vec{p}_{31} = -\frac{1}{2}\,\vec{p}_{12} + \frac{3}{4}\,\vec{p}_3 \qquad \vec{p}_2 = -\vec{p}_{12} - \frac{1}{2}\,\vec{p}_3$$

Based on these relations, momentum eigenstates of the different Jacobi coordinates can be related to each other

$$_{(12)3}\langle \vec{p}\,'_{12}\,\vec{p}\,'_3 \,|\, \vec{p}_{23}\,\vec{p}_1 \rangle_{(23)1} = \delta^{(3)}(\vec{p}_{23} + \frac{1}{2}\,\vec{p}\,'_{12} + \frac{3}{4}\,\vec{p}\,'_3)\,\delta^{(3)}(\vec{p}_1 - \vec{p}\,'_{12} + \frac{1}{2}\,\vec{p}\,'_3)$$

Indices indicate the kind of Jacobi coordinate!
often I skip (12)3 to be shorter

Jacobi coordinates and permutation operators

Permutation operators are essentially coordinate transformations!

$$
{}_{(12)3}\langle \vec{p}^{\,\prime}_{12}\, \vec{p}^{\,\prime}_{3}\, \vec{P}^{\,\prime} \,|\, P_{12}P_{23} \,|\, \vec{p}_{12}\, \vec{p}_{3}\ \vec{P}\, \rangle_{(12)3}
$$

$$
= \int d^3k_1\, d^3k_2\, d^3k_3\ {}_{(12)3}\langle \vec{p}^{\,\prime}_{12}\, \vec{p}^{\,\prime}_{3}\, \vec{P}^{\,\prime} \,|\, \vec{k}_1\, \vec{k}_2\, \vec{k}_3 \,\rangle \langle \vec{k}_1\, \vec{k}_2\, \vec{k}_3 \,|\, P_{12}P_{23} \,|\, \vec{p}_{12}\, \vec{p}_{3}\, \vec{P}\, \rangle_{(12)3}
$$

$$
= \int d^3k_1\, d^3k_2\, d^3k_3\ {}_{(12)3}\langle \vec{p}^{\,\prime}_{12}\, \vec{p}^{\,\prime}_{3}\, \vec{P}^{\,\prime} \,|\, \vec{k}_1\, \vec{k}_2\, \vec{k}_3 \,\rangle\, \underbrace{\langle \vec{k}_2\, \vec{k}_3\, \vec{k}_1 \,|\, \vec{p}_{12}\, \vec{p}_{3}\, \vec{P}\, \rangle_{(12)3}}_{=\langle \vec{k}_1\, \vec{k}_2\, \vec{k}_3 \,|\, \vec{p}_{12}\, \vec{p}_{3}\, \vec{P}\, \rangle_{(23)1}} \quad = {}_{(12)3}\, \langle \vec{p}^{\,\prime}_{12}\, \vec{p}^{\,\prime}_{3}\, \vec{P}^{\,\prime} \,|\, \vec{p}_{12}\, \vec{p}_{3}\ \vec{P}\, \rangle_{(23)1}
$$

by the change of single particle momenta, effectively, p_{23} is set to p_{12} and p_1 to p_3

Permutation operators can be expressed in terms of such coordinate transformations (see previous slide!)

Partial wave representation of the permutation operator

Because of angular momentum conservation, it is again useful to expand in terms of partial waves.

$$\langle \vec{p}_{12} \,|\, p'_{12}\, l'_{12}\, m'_{12} \rangle = \frac{\delta(p_{12}-p'_{12})}{p_{12}\, p'_{12}}\, Y_{l'_{12} m'_{12}}(\hat{p}_{12}) \qquad \langle \vec{p}_{3} \,|\, p'_{3}\, l'_{3}\, m'_{3} \rangle = \frac{\delta(p_{3}-p'_{3})}{p_{3}\, p'_{3}}\, Y_{l'_{3} m'_{3}}(\hat{p}_{3})$$

Conserved is the **total angular momentum** $\hat{\vec{L}} = \hat{\vec{l}}_{12} + \hat{\vec{l}}_3$!

$$|\, p_{12} p_3\, (l_{12}\, l_3)\, L\, M_L \rangle = \sum_{m_{12} m_3} (l_{12}\, l_3\, L\,,\, m_{12}\, m_3\, M_L)\, |\, p_{12}\, l_{12}\, m_{12} \rangle\, |\, p_3\, l_3\, m_3 \rangle$$

Clebsch-Gordan coefficient

Completeness relation reads

$$\mathbb{1} = \sum_{l_{12}\, l_3\, L\, M_L} \int dp_{12} p_{12}^2\, dp_3 p_3^2\, |\, p_{12} p_3\, (l_{12}\, l_3)\, L\, M_L \rangle \langle p_{12} p_3\, \underbrace{(l_{12}\, l_3)\, L}_{\equiv \alpha}\, M_L\, |$$

We need ${}_{(12)3}\langle p'_{12} p'_3\, \alpha'\, M_L \,|\, p_{12} p_3\, \alpha\, M_L \rangle_{(23)1} \longleftarrow$ **!**

and use

$$\frac{1}{2L+1} \sum_{M_L} {}_{(12)3}\langle p'_{12} p'_3\, \alpha'\, M_L \,|\, p_{12} p_3\, \alpha\, M_L \rangle_{(23)1} \quad \text{for } L = L'$$

$$= \frac{1}{2L+1} \sum_{M_L} \int d^3\tilde{p}'_{12} d^3\tilde{p}'_3\, d^3\tilde{p}_{12} d^3\tilde{p}_3\; {}_{(12)3}\langle p'_{12} p'_3\, \alpha'\, M_L \,|\, \vec{\tilde{p}}'_{12}\, \vec{\tilde{p}}'_3 \rangle_{(12)3}$$

$${}_{(12)3}\langle \vec{\tilde{p}}'_{12}\, \vec{\tilde{p}}'_3 \,|\, \vec{\tilde{p}}_{12}\, \vec{\tilde{p}}_3 \rangle_{(23)1} \quad {}_{(23)1}\langle \vec{\tilde{p}}_{12}\, \vec{\tilde{p}}_3 \,|\, p_{12} p_3\, \alpha\, M_L \rangle_{(23)1}$$

M independence!

averaging remove preference for quantization axis

Partial wave representation of the permutation operator

$$\frac{1}{2L+1}\sum_{M_L} {}_{(12)3}\langle\, p'_{12}p'_3\,\alpha'\,M_L\,|\,p_{12}p_3\,\alpha\,M_L\,\rangle_{(23)1}$$

$$= \frac{1}{2L+1}\sum_{M_L}\int d^3\tilde{p}'_{12}d^3\tilde{p}'_3\,d^3\tilde{p}_{12}d^3\tilde{p}_3\,\frac{\delta(\,p'_{12}-\tilde{p}'_{12}\,)}{p'_{12}\,\tilde{p}'_{12}}\,\frac{\delta(\,p'_3-\tilde{p}'_3\,)}{p'_3\,\tilde{p}'_3}\,\frac{\delta(\,p_{12}-\tilde{p}_{12}\,)}{p_{12}\,\tilde{p}_{12}}\,\frac{\delta(\,p_3-\tilde{p}_3\,)}{p_3\,\tilde{p}_3}$$

$$\delta^{(3)}(\,\vec{\tilde{p}}_{12}+\frac{1}{2}\,\vec{\tilde{p}}'_{12}+\frac{3}{4}\,\vec{\tilde{p}}'_3\,)\,\delta^{(3)}(\,\vec{\tilde{p}}_3-\vec{\tilde{p}}'_{12}+\frac{1}{2}\,\vec{\tilde{p}}'_3\,)\quad \mathcal{Y}^{*\,L'\,M'_L}_{l'_{12}l'_3}(\,\hat{\tilde{p}}'_{12}\,\hat{\tilde{p}}'_3\,)\mathcal{Y}^{L\,M_L}_{l_{12}l_3}(\,\hat{\tilde{p}}_{12}\,\hat{\tilde{p}}_3\,)$$

$$= \frac{1}{2L+1}\sum_{M_L}\int d\hat{\tilde{p}}_3 d\hat{\tilde{p}}'_3\,\frac{\delta\left(p_{12}-\left|-\vec{\tilde{p}}'_3-\frac{1}{2}\,\vec{\tilde{p}}_3\right|\right)}{p_{12}^2}\,\frac{\delta\left(p'_{12}-\left|\vec{\tilde{p}}_3+\frac{1}{2}\,\vec{\tilde{p}}'_3\right|\right)}{p'^2_{12}}$$

$$\mathcal{Y}^{*\,L'\,M'_L}_{l'_{12}l'_3}(\,\widehat{\vec{\tilde{p}}_3+\frac{1}{2}\,\vec{\tilde{p}}'_3}\,\hat{\tilde{p}}'_3\,)\;\mathcal{Y}^{L\,M_L}_{l_{12}l_3}(\,\widehat{-\vec{\tilde{p}}'_3-\frac{1}{2}\,\vec{\tilde{p}}_3}\,\hat{\tilde{p}}_3)$$

$$= \int_{-1}^{1}dx\;\frac{\delta\,(\,p_{12}-\pi_{12}(p'_3p_3\,x))}{p_{12}^2}\,\frac{\delta\,(\,p'_{12}-\pi'_{12}(p'_3p_3\,x))}{p'^2_{12}}\quad\frac{1}{2L+1}\sum_{M_L}\mathcal{Y}^{*\,L'\,M'_L}_{l'_{12}l'_3}(\,\widehat{\vec{\tilde{p}}_3+\frac{1}{2}\,\vec{\tilde{p}}'_3}\,\hat{\tilde{p}}'_3\,)\;\mathcal{Y}^{L\,M_L}_{l_{12}l_3}(\,\widehat{-\vec{\tilde{p}}'_3-\frac{1}{2}\,\vec{\tilde{p}}_3}\,\hat{\tilde{p}}_3)$$

$$= \int_{-1}^{1}dx\;\frac{\delta\,(\,p_{12}-\pi_{12}(p'_3p_3\,x))}{p_{12}^2}\,\frac{\delta\,(\,p'_{12}-\pi'_{12}(p'_3p_3\,x))}{p'^2_{12}}\;G_{\alpha'\alpha}(p'_3p_3\,x)$$

Using $\hat{\tilde{p}}'_3=\hat{e}_z$ and $\hat{\tilde{p}}_3$ in x-z-plane

$$\int d\hat{\tilde{p}}_3 d\hat{\tilde{p}}'_3 \longrightarrow 4\pi\,2\pi\int_{-1}^{1}dx$$

$$\vec{\tilde{p}}'_3=\begin{pmatrix}0\\0\\p'_3\end{pmatrix}\qquad \vec{\tilde{p}}_3=\begin{pmatrix}p_3\sqrt{1-x^2}\\0\\p_3x\end{pmatrix}$$

and coupled spherical harmonics

$$\mathcal{Y}^{L\,M_L}_{l_{12}l_3}(\hat{p}_{12}\hat{p}_3)=\sum_{m_{12}m_3}(l_{12}\,l_3\,L\,,\,m_{12}\,m_3\,M_L)\;Y_{l_{12}m_{12}}(\hat{p}_{12})\,Y_{l_3m_3}(\hat{p}_3)$$

Partial waves allowed and permutation operators

For identical bosons: $P_{12}\;|\,\vec{p}_{12}\,\vec{p}_3\,\rangle = |\,-\vec{p}_{12}\,\vec{p}_3\,\rangle$

In partial wave presentation: $P_{12}\;|\,p_{12}p_3\,(\,l_{12}\,l_3\,)\,L\,M_L\,\rangle = \sum_{m_{12}m_3} \int d\hat{p}_{12}\,d\hat{p}_3 (\,l_{12}\,l_3\,L\,,\,m_{12}\,m_3\,M_L\,)\;\mathrm{Y}_{l_{12}m_{12}}(\hat{p}_{12})\,\mathrm{Y}_{l_3m_3}(\hat{p}_3)\,P_{12}|\,\vec{p}_{12}\,\vec{p}_3\,\rangle$

$$= \sum_{m_{12}m_3} \int d\hat{p}_{12}\,d\hat{p}_3 (\,l_{12}\,l_3\,L\,,\,m_{12}\,m_3\,M_L\,)\;(-)^{l_{12}}\mathrm{Y}_{l_{12}m_{12}}(-\hat{p}_{12})\,\mathrm{Y}_{l_3m_3}(\hat{p}_3)\,|\,-\vec{p}_{12}\,\vec{p}_3\,\rangle$$

$$= (-)^{l_{12}}\;|\,p_{12}p_3\,(\,l_{12}\,l_3\,)\,L\,M_L\,\rangle$$

Only even partial waves are allowed! Further

$$\langle\,p'_{12}\,p'_3\,\alpha'\,|\,P_{12}P_{23} + P_{13}P_{23}\,|\,p_{12}\,p_3\,\alpha\,\rangle = \langle\,p'_{12}\,p'_3\,\alpha'\,|\,P_{12}P_{23} + P_{12}P_{13}P_{23}P_{12}\,|\,p_{12}\,p_3\,\alpha\,\rangle$$

$$= \langle\,p'_{12}\,p'_3\,\alpha'\,|\,P_{12}P_{23} + P_{12}P_{23}\,\underbrace{P_{12}P_{12}}_{=\mathbb{1}}\,|\,p_{12}\,p_3\,\alpha\,\rangle = 2\,\langle\,p'_{12}\,p'_3\,\alpha'\,|\,P_{12}P_{23}\,|\,p_{12}\,p_3\,\alpha\,\rangle$$

The combination of permutation operators can in practice be replaces by just one.

Note that the permutation operator conserves L but couples partial wave channels with different l_{12} and l_3 !

Partial wave representation of the Faddeev equation

Faddeev component will be independent of M_L $\psi^{\alpha}_{12}(p_{12}p_3) = \langle\, p_{12}\, p_3\, \alpha \,|\, \psi_{12}\,\rangle$ where $\alpha = (\, l_{12}\, l_3\,)\, L$

Two-body interaction:

$$\langle\, p'_{12}\, p'_3\, \alpha' \,|\, V_{12} \,|\, p_{12}\, p_3\, \alpha\,\rangle = \delta_{\alpha\alpha'}\, \frac{\delta(p_3 - p'_3)}{p_3\, p'_3}\, V_{l_{12}}(p'_{12}, p_{12})$$

Kinetic energy:

$$H_0 = \frac{k_1^2}{2m} + \frac{k_2^2}{2m} + \frac{k_3^2}{2m} = \frac{p_{12}^2}{2\mu_{12}} + \frac{p_3^2}{2\mu_3} + \frac{P^2}{2M} = \frac{p_{12}^2}{m} + \frac{3p_3^2}{4m} + \cancel{\frac{P^2}{6m}}$$

CM momentum separates from rest

Resolvent operator (no singularity here):

$$\langle\, p'_{12}\, p'_3\, \alpha' \,|\, G_0(E) \,|\, p_{12}\, p_3\, \alpha\,\rangle = \delta_{\alpha\alpha'}\, \frac{\delta(p_{12} - p'_{12})}{p_{12}\, p'_{12}}\, \frac{\delta(p_3 - p'_3)}{p_3\, p'_3}\, \frac{1}{E - \frac{p_{12}^2}{m} - \frac{3p_3^2}{4m}}$$

t-matrix embedded in three-body space (off-shell energies)

$$\langle\, p'_{12}\, p'_3\, \alpha' \,|\, t_{12} \,|\, p_{12}\, p_3\, \alpha\,\rangle = \delta_{\alpha\alpha'}\, \frac{\delta(p_3 - p'_3)}{p_3\, p'_3}\, t_{l_{12}}(p'_{12}, p_{12}; p_3)$$

The Lippmann-Schwinger equation can then be written

$$t_{12} = V_{12} + V_{12}\, \frac{1}{E - H_0}\, t_{12} \quad \longrightarrow \quad t_{l_{12}}(p'_{12}, p_{12}; p_3) = V_{l_{12}}(p'_{12}, p_{12}) + \int dp''_{12} {p''_{12}}^2\; V_{l_{12}}(p'_{12}, p''_{12})\, \frac{1}{\tilde{E} - \frac{{p''_{12}}^2}{m}}\, t_{l_{12}}(p''_{12}, p_{12}; p_3)$$

where the shifted energy is shifted by the spectator energy: $\tilde{E} = E - \dfrac{3{p_3}^2}{4m}$

Discretized representation of the Faddeev equation

All parts together result in

$$\psi_{12}^{\alpha'}(p'_{12}p'_3) = \frac{1}{E - \frac{p'^2_{12}}{m} - \frac{3p'^2_3}{4m}} \int_{-1}^{1} dx \int_0^\infty dp_3\, p_3^2\; t_{\alpha'}(p'_{12}\pi'_{12}(p'_3 p_3 x); p'_3) G_{\alpha'\alpha}(p'_3 p_3 x)\psi_{12}^{\alpha}(\pi_{12}(p'_3 p_3 x)\, p_3)$$

which is again an eigenvalue equation where E needs to be varied until the eigen value 1 is in the spectrum

A few notes on the discretization:

momenta need to be discretized using N_p and N_q momenta

Faddeev components can be label by one common index

$$\psi_{12}^{\alpha}(p_{12}p_3) \longrightarrow \psi(p_i q_j \alpha) \equiv \psi(i + j\, N_p + \alpha\, N_p\, N_q)$$

for the shifted momenta an interpolation is necessary (see next slides)

use representation as sum of function values

$$f(\pi) = \sum_{i=0}^{N_p-1} S_i(\pi)\, f(p_i)$$

as usual we have grid points and corresponding integration weights: p_i, ω_i^p , q_j, ω_j^q and x_k, ω_k^x

This leads to the matrix equation (depending on 3-index i, k, α ..

$$\psi(ij\alpha) = \sum_{i'j'\alpha'} K(ij\alpha, i'j'\alpha')\; \psi(i'j'\alpha')$$

Discretized representation of the Faddeev equation

The matrix is found as

$$K(ij\alpha, i'j'\alpha') = \frac{1}{E - \frac{p_i^2}{m} - \frac{3q_j^2}{4m}} \sum_k \omega_k^x \omega_{j'}^q q_{j'}^2 \sum_m S_m(\pi'_{12}(q_j q_{j'} x_k)) t_{\alpha'}(p_i p_m; q_j) G_{\alpha\alpha'}(q_j q_{j'} x_k) S_{i'}(\pi_{12}(q_j q_{j'} x_k))$$

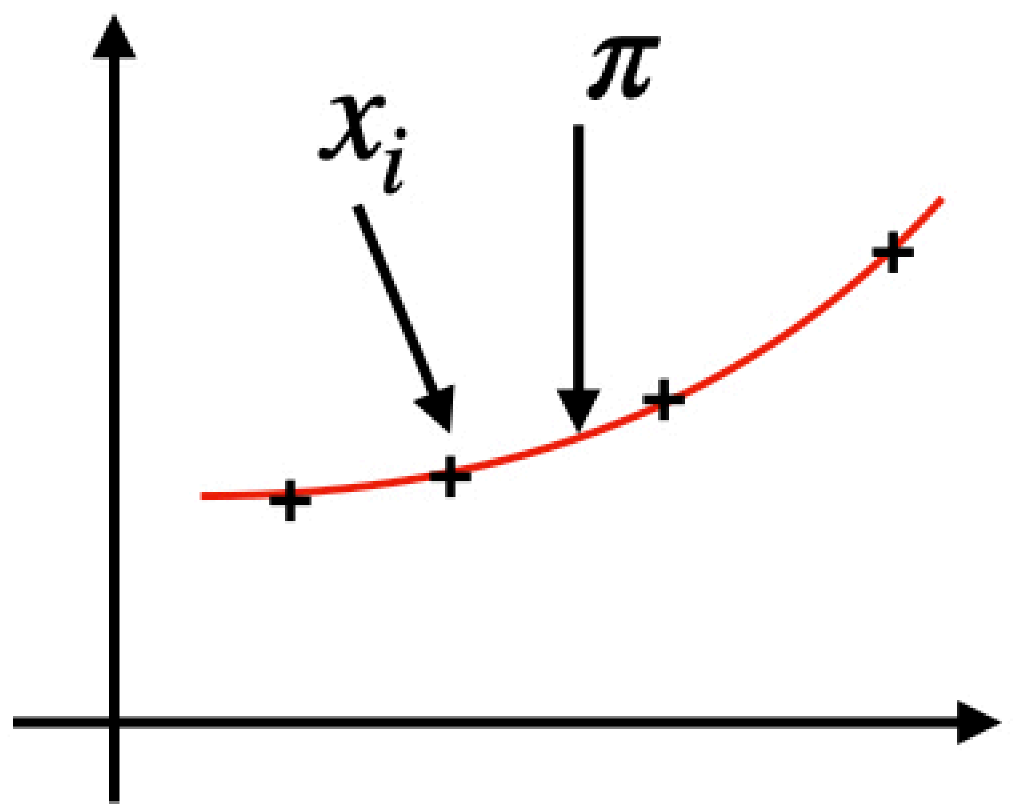
Note that there are no integration weights for p required (just interpolations).
Also note that the matrix equation introduces an additional sum over $i'j'\alpha'$

Interpolation in terms of spline functions $S_n(\pi)$

There are several methods for interpolation. Maybe the simplest, but still very useful, scheme is based on four grid points next to the interpolated value.

1) look for nearest grid point x_i below π. This defines $x_{i-1}, x_i, x_{i+1}, x_{i+2}$

2) find cubic polynomial that reproduces the function and its numerical derivative at x_i and x_{i+1}



$\longrightarrow$ "cubic hermitian splines"

Nice feature: the coefficients of the polynomial are linear in the function values.
Therefore, the evaluation of the polynomials at π can be written as wanted

$$f(\pi) = \sum_{i=0}^{N_p-1} S_i(\pi)\, f(p_i)$$

Here $S_j(\pi) = 0$ for $j \neq i-1, i, i+1, i+2$.

Close to zero a linear inter/extrapolation is useful.
At high momenta, extrapolation by zero is fine.

Example notebook

The provided notebook implements the solution.

- need to discuss in open hour
- note the implementation of the interpolation and accompanying notes
- run times could be optimized, but converged results also in partial wave channels can be obtained
- binding energies are not very cutoff dependent
- the solution of the Lippmann-Schwinger equation is implemented very similarly to the one for scattering (just simpler because of the missing singularities!)
- At low momenta, partial wave decomposed amplitudes have a definite behavior as $\psi_l(p) \propto p^l$
  At high orbital angular momenta, this is not well approximated by a cubic polynomial.
  Therefore, the notebook uses (requires good threshold behavior of f!)

$$f_l(\pi) = \sum_n S_n(\pi)\ \pi^l\ \frac{f_l(p_n)}{p_n^l}$$