Exercises for Computational Physics (physics760)
WS 2020 / 21

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Exercise 6 (20 pts. total)

Homework (due Dec. 16th at 18:00)

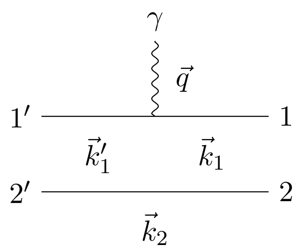
Please note the due date of this homework. Submission of homework *requires* submitting your solutions (e.g. answers to questions, graphs, results in tabular form) in the form of a brief report (please no 100 page submissions!) **AND** a copy of your code that you used to do the simulations.

7 Form factor of a two-boson bound state

In this exercise, we would like to further study the properties of bound states of two bosons. The example will be based on the calculations shown during the lecture. We assume that the bound state problem was solved for the OBE using several cutoffs. Thereby, the interactions have been tuned so that our two-boson system corresponds to the deuteron bound state in nuclear physics and reproduce the experimentally known binding energy of $E = -2.225$ MeV. Based on the lecture, this condition is fulfilled for the following parameter sets (using $A = -0.1544435$ and $m_B = 138.0$ MeV for the long range part of the interaction) .

Λ[MeV]	C_0[fm^2]
300.000	-9.827953e-02
400.000	-2.820315e-02
500.000	-4.221894e-04
600.000	1.285743e-02
700.000	2.016719e-02
800.000	2.470795e-02
900.000	2.786520e-02
1000.000	3.030801e-02
1100.000	3.239034e-02
1200.000	3.431611e-02

We are now interested in the interaction of an external probe with our two-boson system. The interaction with the external probe (e.g. a photon) can often be treated perturbatively. The most important contribution is then given by the following diagram



which represents the absorption of the "photon". We will work in the Breit frame, the energy transfer is zero in this case, and we only need to consider three-momentum transfer.
In this simple case, where the particles are scalars, the charge density operator is just given by the momentum conserving δ-function

$$\langle \vec{k}_1' | \rho(\vec{q}) | \vec{k}_1 \rangle = \delta(\vec{k}_1' - \vec{q} - \vec{k}_1) \ .$$

We assume that the "photon" only couples to the first particle (e.g. the first particle is a proton and the second a neutron). In this case, the (charge) form factor $F(\vec{q}^{\,2})$ of the two-body system is given by

$$\langle \psi \vec{P}' | \rho(\vec{q}) | \psi \vec{P} \rangle = F(\vec{q}^{\,2}) \delta(\vec{P}' - \vec{q} - \vec{P}) \ .$$

Here ψ is the two-body bound state and $\vec{P}$ and $\vec{P}'$ are the initial and final center of mass momentum. In the following, we will assume two particles with equal mass $m_N = 938.92$ MeV.

1. Confirm that the form factor can be obtained based on the internal wave functions as

$$F(\vec{q}^{\,2}) = \int d^3p' \ \psi^*(\vec{p}\,') \psi(\vec{p}\,' - \frac{1}{2}\vec{q}) \ .$$

2p

2. Express this relation in terms of partial wave amplitudes. In order to simplify the expression, you can assume that $\vec{q} = q\hat{e}_z$ and that only the partial wave ll_z contributes to the bound state wave function. In this case, l_z is conserved and the solid angle integration can be simplified to the integration over $x = \cos(\theta)$. Verify the relation

$$F(\vec{q}^{\,2}) = 2\pi \int dp' \, p'^2 \int_{-1}^{1} dx \ Y^*_{ll_z}(\hat{p}') \ Y_{ll_z}(\widehat{\vec{p}\,' - \frac{1}{2}\vec{q}}) \psi^*_{ll_z}(p') \psi_{ll_z}(\left| \vec{p}\,' - \frac{1}{2}\vec{q} \right|) \ .$$

In this relation, the momentum is given by $\vec{p}\,' = (p'\sqrt{1-x^2}), 0, p'x)$. **3p**

3. Implement the form factor based on the wave functions obtained in the lecture that are defined on a finite grid (you may extend the notebook provided and reuse the TwoBody class). *Hint: the wave function $\psi_{ll_z}(|\vec{p}\,' - 1/2\,\vec{q}\,|)$ at momenta different from the momentum grid can be obtained using (cubic) splines. Argument of the spherical harmonic $\vec{p}\,' - 1/2\,\vec{q}$ is easiest obtained in terms of its x and z component. How do you get the angles from these components?* **7p**

4. Use the wave function for $\Lambda = 1200$ MeV and selected momentum transfers $|\,\vec{q}\,|$ in the range up to 10 fm^{-1} to check the numerical accuracy of your result, especially with respect to the number of grid points used for angular integration. **2p**

5. Show that
$$F(0) = 1 \qquad \text{and} \qquad \left.\frac{\partial F(\vec{q}^{\,2})}{\partial \vec{q}^{\,2}}\right|_{q^2=0} = -\frac{1}{6}\langle r^2\rangle$$
where $\langle r^2\rangle$ is the expectation value of the square of the position of the first particle with respect to the center of mass. Confirm that your code reproduces the normalization of the form factor and the rms radius known from the Fourier transformation. *Hint: the derivative can also be obtained by fitting F at low momentum transfers to q polynomial in $\vec{q}^{\,2}$. The python package "lmfit" provides the necessary tools.* **3p**

6. Plot form factor for several Λ in the range $\vec{q} = 0-10$ fm^{-1} and **briefly** discuss differences and similarities at low and high momenta. **3p**