Exercise 06: Form factor of a two-boson bound state

Computational Physics WS20/21

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1 Introduction

In the lecture, we found the wave functions of some two-boson bound state. In this exercise we would like to further study the properties of the deuteron bound state with the experimentally known binding energy $E = -2{,}225\,\mathrm{MeV}$. The correct parameters of the interaction were found in the lecture and are summed up in the exercise sheet. In this exercise we want to study the interaction of an external probe (a photon) with our two-boson system. The goal is to calculate the form factor of the deuteron.

2 Theory

In the case, where the absorption of the photon leads only to a momentum transfer and not an energy transfer, the charge density operator is given by the momentum conserving δ-function

$$\left\langle \vec{k_1\prime} \middle| \rho(\vec{q}) \middle| \vec{k_1} \right\rangle = \delta(\vec{k_1\prime} - \vec{q} - \vec{k_1}). \tag{1}$$

Here we assume, the photon with momentum $\vec{q}$ only couples to the proton with momentum $\vec{k_1}$ (after interaction: $\vec{k_1\prime}$), not to the neutron ($\vec{k_2}$). The charge form factor $F(\vec{q}^2)$ is then given by

$$\left\langle \psi\vec{P\prime} \middle| \rho(\vec{q}) \middle| \psi\vec{P} \right\rangle = F(\vec{q}^2)\delta\delta\left(\vec{P\prime} - \vec{q} - \vec{P}\right) \tag{2}$$

In the following, we will also assume two particles with equal mass $m_1 = m_2 = m_N = 938{,}92\,\mathrm{MeV}$.

1. The vectors $\vec{p}$ and $\vec{P}$ are defined as

$$\vec{p} = \frac{1}{m_1 + m_2}\left(m_2\vec{k_1} - m_1\vec{k_2}\right) = \frac{1}{2}\left(\vec{k_1} - \vec{k_2}\right), \quad \vec{P} = \vec{k_1} + \vec{k_2} \tag{3}$$

Then, it is $\vec{p'} = 1/2(\vec{k'_1} - \vec{k_2})$ and $\vec{P'} = \vec{k'_1} + \vec{k_2}$. We can see, that the charge density operator can also be expressed as

$$\left\langle \vec{p'}, \vec{P'} \middle| \rho(\vec{q}) \middle| \vec{p}, \vec{P} \right\rangle = \delta\left(\vec{p'} - \vec{p} - \frac{1}{2}\vec{q}\right) \delta\left(\vec{P'} - \vec{q} - \vec{P}\right) \tag{4}$$

With that, it is

$$\left\langle \psi \vec{P'} \middle| \rho(\vec{q}) \middle| \psi \vec{P} \right\rangle = \int \mathrm{d}^3 p' \int \mathrm{d}^3 p \, \psi^*(\vec{p'}) \delta\left(\vec{p'} - \vec{p} - \frac{1}{2}\vec{q}\right) \delta\left(\vec{P'} - \vec{q} - \vec{P}\right) \psi(\vec{p}) \tag{5}$$

$$= \int \mathrm{d}^3 p' \, \psi^*(\vec{p'}) \psi\left(\vec{p'} - \frac{1}{2}\vec{q}\right) \delta\left(\vec{P'} - \vec{q} - \vec{P}\right) \tag{6}$$

and from comparison, we find

$$F(\vec{q}^2) = 2\pi \int \mathrm{d}^3 p' \, \psi^*(\vec{p'}) \psi\left(\vec{p'} - \frac{1}{2}\vec{q}\right) \tag{7}$$

2. The wave functions can be written in terms of partial waves $\sum_{l,l_z} \psi_{ll_z}(p) Y_{ll_z}(\hat{p})$, with the spherical harmonics $Y_{ll_z}(\hat{p})$. We assume, that $\vec{q} = q\hat{e_z}$ and that only the partial wave ll_z contributes. In this case l_z is conserved and the integration over the longitudinal angle ϕ simplifies to the factor 2π. We get

$$F(\vec{q}^2) = 2\pi \int \mathrm{d}p' \, p'^2 \int_{-1}^{1} \mathrm{d}x \, Y^*_{ll_z}(\hat{p'}) Y_{ll_z}(\widehat{\vec{p'} - \frac{1}{2}\vec{q}}) \psi^*_{ll_z}(p') \psi_{ll_z}(\left|\vec{p'} - \frac{1}{2}\vec{q}\right|) \tag{8}$$

Here, it is $\vec{p'} = \left(p'^2 \sqrt{1 - x^2}, 0, p'x\right)$.

3 Implementation and Testing

3. The form factor $F(\vec{q}^2)$ is implemented in the file `onebosonexchange.py`. The input angles (ϕ, θ) for the spherical harmonics can be calculated from the cartesian coordinates (x, y, z)

$$\phi = \begin{cases} \arccos \frac{x}{\sqrt{x^2+y^2}} & y \geq 0 \\ 2\pi - \arccos \frac{x}{\sqrt{x^2+y^2}} & y < 0 \end{cases} \tag{9}$$

$$\theta = \arccos \frac{z}{\sqrt{x^2 + y^2 + z^2}}. \tag{10}$$

We get $\phi = \phi' = 0$ and

$$\theta = \arccos x, \quad \theta' = \arccos \frac{p'x - \frac{1}{2}q}{\sqrt{p'^2(1 - x^2) + (p'x - \frac{1}{2}q)^2}} \tag{11}$$

We use the functions `scipy.interpolate.splrep` and `scipy.interpolate.splev` to get the grid points for $\psi_{ll_z}(\left|\vec{p'} - \frac{1}{2}\vec{q}\right|)$.

4. We test our implementation for $\Lambda = 1200\,\mathrm{MeV}$ and momentum transfers q up to $10\,\mathrm{fm}^{-1}$ to check the numerical accuracy of our result. By changing the number of grid points for the angular momentum, we found that 20 grid points for the angular integration are sufficients so that the numerical error is smaller than 10^{-6}.

5. The normalization of the form factor is

$$F(0) = \int \mathrm{d}^3p'\, \psi^*(\vec{p'})\psi\left(\vec{p'} - \frac{1}{2}\vec{0}\right) \tag{12}$$
$$= \int \mathrm{d}^3p'\, \psi^*(\vec{p'})\psi\left(\vec{p'}\right) \tag{13}$$
$$= \int \mathrm{d}^3p' \left|\psi(\vec{p'})\right|^2 \tag{14}$$
$$= 1, \tag{15}$$

as the wave functions are normalized. Computing the first derivative of $F(\vec{q}^2)$, we find

$$\frac{\partial F(\vec{q}^2)}{\partial \vec{q}^2}\Big\|_{q^2=0} = \frac{\partial}{\partial \vec{q}^2}\int \mathrm{d}^3p'\, \psi^*(\vec{p'})\psi\left(\vec{p'} - \frac{1}{2}\vec{0}\right)\Big\|_{q^2=0} \tag{16}$$
$$= \frac{\partial}{\partial \vec{q}^2}\int \mathrm{d}^3p' \int \frac{\mathrm{d}^3r'}{(2\pi)^{3/2}}\frac{\mathrm{d}^3r}{(2\pi)^{3/2}}\left(\psi(\vec{r'})e^{i\vec{p'}\vec{r'}}\right)^*\psi(\vec{r})e^{i\vec{r}(\vec{p'}-\frac{1}{2}\vec{q})}\Big\|_{q^2=0} \tag{17}$$
$$= \frac{\partial}{\partial \vec{q}^2}\int \mathrm{d}^3r' \int \mathrm{d}^3r\, \delta(\vec{r} - \vec{r'})\psi(\vec{r'})^*\psi(\vec{r})e^{-1/2i\vec{r}\vec{q}}\Big\|_{q^2=0} \tag{18}$$
$$= \frac{\partial}{\partial \vec{q}^2}\int \mathrm{d}^3r\, \psi(\vec{r})^*\psi(\vec{r})e^{-1/2i\vec{r}\vec{q}}\Big\|_{q^2=0} \tag{19}$$
$$\approx \frac{\partial}{\partial \vec{q}^2}\int \mathrm{d}^3r\, |\psi(\vec{r})|^2(1 - \frac{1}{2}\vec{q}^2\vec{r}^2 + \ldots)\Big\|_{q^2=0} \tag{20}$$
$$= 1 - \frac{1}{6}\left\langle r^2\right\rangle \tag{21}$$

We want to see if our code fulfills these properties. For $F(0)$ we get indeed a value of 1,000 000 082 832 141 8, which is close enough to 1. The expectation value of the radius squared can be calculated with the functions `fourier` and `rms` in the TwoBody class. We get here a value of $\langle r^2\rangle = (0{,}999\,784 \pm 4{,}141\,728)\,\mathrm{fm}^2$. In order to find the derivative of $F(\vec{q^2})$ at $q = 0$ we fit a polynomial

$$P(x) = c + bx - \frac{1}{6}ax^2 \tag{22}$$

to the values of the form factor for some $q \in (0, 3)$fm (see Fig. (3.1)). We get a parameter $a = (4{,}974 \pm 0{,}357)\,\mathrm{fm}^2$, which is not agreement with the calculated value. Also as one can see in Fig. (3.1), the fit is not very good. Unfortunately, we did not have time to investigate the problem further.

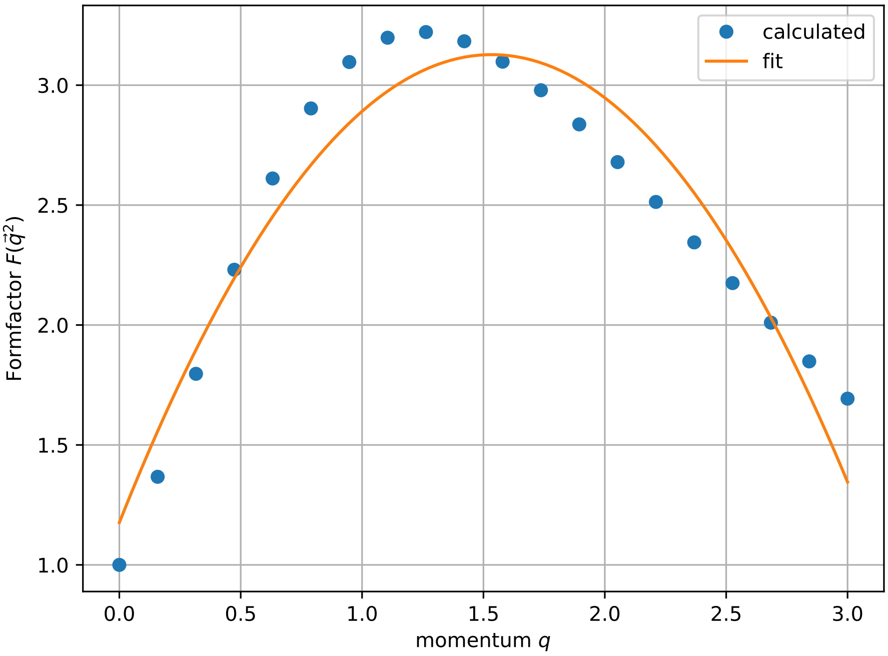

Figure 3.1: Formfactor $F(\vec{q}^2)$ as implemented in `onebosonexchange.py` and fit to values.

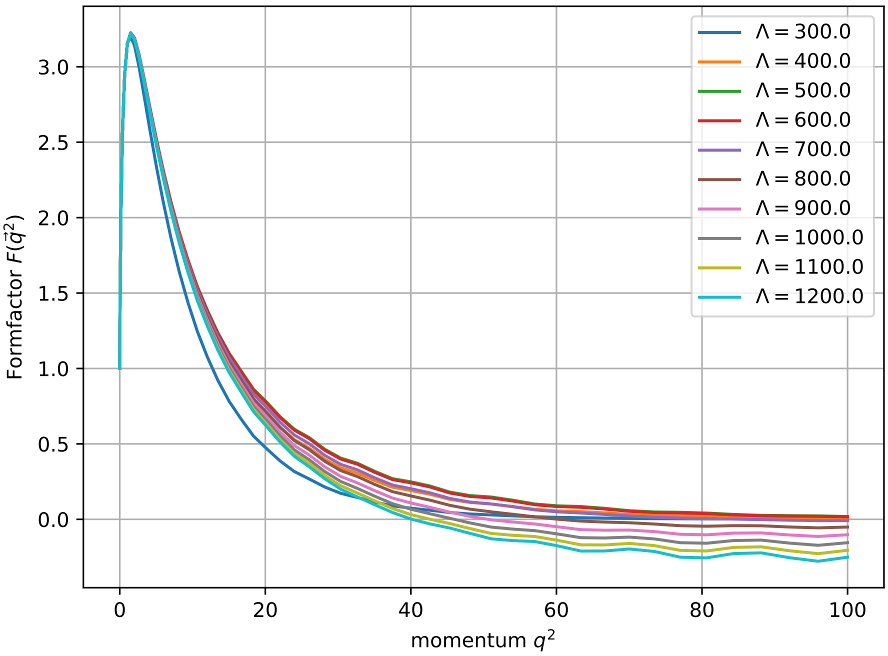

Figure 3.2: Formfactor $F(\vec{q}^2)$ as implemented in `onebosonexchange.py` for different cutoffs Λ.

6. Finally the form factors $F(\vec{q}^2)$ are plotted for a few different cutoffs Λ in Fig. (3.2).

   We can see that for small values of $\vec{q}^2$ there is almost no difference in the form factors. They differ, however, for higher $\vec{q}^2$. The larger the cutoff Λ, the smaller the form factor for high $\vec{q}^2$.