Coherent Functors *

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Let $\mathscr{C}$ be an abelian category and F a (covariant) functor from $\mathscr{C}$ to abelian groups. We say that F is a coherent functor if there exists an exact sequence $(X, \_) \to (Y, \_) \to F \to 0$ where (X, A) denotes the maps from X to A. The main purpose of this paper is to initiate a study of the full subcategory $\check{\mathscr{C}}$ of coherent functors and give some applications to the theory of complexes in abelian categories as well as to some more specialized questions concerning modules over rings.

The first two sections of the paper are devoted to questions of notation and some of the more elementary questions concerning the category $\check{\mathscr{C}}$. For instance, it is shown that if $0 \to F_1 \to F_2 \to F_3 \to F_4 \to 0$ is an exact sequence of functors with F_2 and F_3 coherent, then F_1 and F_4 are also coherent. It is also shown that $\check{\mathscr{C}}$ is closed under extensions. Thus if X is a complex in C, then the cohomology functors $H^i(X, \_)$ are in $\check{\mathscr{C}}$. Since $\check{\mathscr{C}}$ has enough projectives, it makes sense to talk about the global dimension of $\check{\mathscr{C}}$. It is shown that the gl. dim $\check{\mathscr{C}} = 0$ or 2 and that gl. dim $\check{\mathscr{C}} = 0$ if and only if the gl. dim $\mathscr{C} = 0$.

We of course have the usual right exact functor $u : \mathscr{C} \to (\check{\mathscr{C}})^0$ given by $u(A) = (A, \_)$. If $\mathscr{D}$ is an abelian category, then we have the induced functor $u. : \mathscr{H}om(\mathscr{C}, \mathscr{D}) \to \mathscr{H}om((\check{\mathscr{C}})^0, \mathscr{D})$. Section two ends by showing that the functor $u.$ always has an adjoint $u^{\cdot} : \mathscr{H}om((\check{\mathscr{C}})^0, \mathscr{D}) \to$ $\to \mathscr{H}om(\mathscr{C}, \mathscr{D})$ having the following properties: a) if $F : \mathscr{C} \to \mathscr{D}$, then $(u^{\cdot} F) u = F$ and $u^{\cdot} F$ is left exact and $u^{\cdot} F$ is exact if F is right exact; b) if $\mathscr{C}$ has enough projectives and $F : \mathscr{C} \to \mathscr{D}$ is right exact, then $(L^i F)(A) = u^{\cdot} F(\operatorname{Ext}^i(A, \_))$, where $u^{\cdot} F$ is exact.

As seen above, the identity functor $I : \mathscr{C} \to \mathscr{C}$ can be factored through $(\check{\mathscr{C}})^0$ as $I = (u^{\cdot} I) u$ where $u^{\cdot} I : (\check{\mathscr{C}})^0 \to \mathscr{C}$ is exact. It is this functor $u^{\cdot} I$ which is studied in section three. Denoting by $\check{\mathscr{C}}_0$, the full subcategory of $\check{\mathscr{C}}$ such that $u^{\cdot} I$ sends the objects in $(\check{\mathscr{C}}_0)^0$ to zero, we have that $\check{\mathscr{C}}_0$ is a dense subcategory of $\mathscr{C}$ and that $\mathscr{C}$ is equivalent to $(\check{\mathscr{C}}/\check{\mathscr{C}}_0)^0$. Since

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the objects of $\check{\mathscr{C}}_0$ can be described as those objects F in $\check{\mathscr{C}}$ such that $(F, G) = 0$ for all projective objects G in $\check{\mathscr{C}}$, it follows that up to equivalence of categories, the category $\check{\mathscr{C}}$ determines the category $\mathscr{C}$. An exact sequence of fundamental importance in this section and the rest of the paper is introduced in the course of the proofs of the above statements. Namely for F in $\check{\mathscr{C}}$, it is shown that the $R^0F \approx (u^{\cdot} I(F), \_)$ and that there exists an exact sequence $0 \to F_0 \to F \to R^0F \to F_1 \to 0$ with F_0 and F_1 in $\check{\mathscr{C}}_0$.

Section four is devoted to a discussion of half exact coherent functors. Under certain very mild conditions on $\mathscr{C}$ it is shown that if $\mathscr{C}$ has enough projectives, then a functor F is isomorphic to $\mathrm{Ext}^1(C, \_)$ for some C in $\mathscr{C}$ if and only if it is coherent, $(F, G) = 0$ for all representable functors G and F is half exact. From this it follows that if F is coherent and half exact, then we have for some A and B in $\mathscr{C}$ an exact sequence

$$0 \to \mathrm{Ext}^1(B, \_) \to F \to (A, \_) \to \mathrm{Ext}^2(B, \_).$$

Introducing a suitable equivalence relation in such exact sequences for a fixed A and B, it is shown that the elements of $\mathrm{Ext}^2(B, A)$ are in natural one to one correspondence with the equivalence classes of such exact sequences with half exact F. In particular, if the $\mathrm{gl.dim}\,\mathscr{C} \leqq 1$, we have that the half exact coherent functors are all of the form $\mathrm{Ext}^1(B, \_) + (A, \_)$ (direct sum).

Section five is devoted to applying some of these results to complexes in an abelian category. Among other things it is shown that if X is a complex and $F : \mathscr{C} \to \mathscr{Ab}$ is right exact, then

$$H_i(F(X)) \approx (H^i(X, \_), F)$$

for all i.

In section six and seven we apply the theory of coherent functors to rings. It is shown that $\otimes M$ is coherent if and only if M is a finitely presented R-module. And a new proof is given for our earlier result that if $0 \to A \to B \to C \to 0$ is an exact sequence of R-modules, with C finitely presented, which remains exact when tensored with arbitrary modules, then the exact sequence splits. We also show that if R is noetherian and F is a coherent functor, then $F \approx \mathrm{Tor}_1(\_, A)$ for some R-module A if and only if F is half exact, commutes with direct limits over directed sets, and $F(R) = 0$.

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1. Preliminaries

Let $\mathfrak{U}$ be a universe in the sense of GROTHENDIECK (see [2] for details). We shall say that a category $\mathscr{C}$ is a $\mathfrak{U}$-category if the objects of $\mathscr{C}$ and the maps of $\mathscr{C}$ are both sets which are elements of $\mathfrak{U}$. If $\mathscr{C}$ and $\mathscr{D}$ are $\mathfrak{U}$-categories then we shall denote by $\mathscr{H}om(\mathscr{C}, \mathscr{D})$ the category of covariant functors from $\mathscr{C}$ to $\mathscr{D}$ which is easily seen to be a $\mathfrak{U}$-category.

A particular interesting category associated with the universe $\mathfrak{U}$ is the category of $\mathfrak{U}$-sets $\mathscr{S}$. The objects of $\mathscr{S}$ are the elements of $\mathfrak{U}$ and for A and B in $\mathscr{S}$ the set $\mathscr{S}(A, B)$ of maps from A to B is defined to be ordinary set maps. Although $\mathscr{S}(A, B)$ is in $\mathfrak{U}$ for all A and B in $\mathscr{S}$, it can easily be seen that $\mathscr{S}$ is not a $\mathfrak{U}$-category. However if we let $\mathfrak{U}'$ be a universe containing $\mathfrak{U}$ as an element, then $\mathscr{S}$ is a $\mathfrak{U}'$-category. This suggests the following language which is sufficient for the purposes of this paper.

Let $\mathfrak{U}$ bc a fixed universe and $\mathfrak{U}'$ the smallest universe containing $\mathfrak{U}$ as an element. By a category we shall mean a $\mathfrak{U}'$-category. Thus as observed above, if $\mathscr{C}$ and $\mathscr{D}$ are categories, then $\mathscr{H}om(\mathscr{C}, \mathscr{D})$, the category of covariant functors from $\mathscr{C}$ to $\mathscr{D}$ is, also a category. Also $\mathscr{S}$ as defined above is a category. Similarly $\mathscr{A}b$, the category of abelian groups whose underlying sets are elements of $\mathfrak{U}$ is a category in this sense.

Now let $\mathscr{I}$ and $\mathscr{C}$ be categories. Then we always have the covariant functor $i: \mathscr{C} \to \mathscr{H}om(\mathscr{I}, \mathscr{C})$ given by $i(C)(X) = C$ for all X in $\mathscr{I}$ and C in $\mathscr{C}$ and for $f: C \to D$ we have $i(f): i(C)(X) \to i(D)(X)$ is the map f, for each X in $\mathscr{I}$. We shall say that $\mathscr{C}$ has $\mathscr{I}$-direct limits if the functor i has an adjoint, i.e. there exists a functor $\varinjlim: \mathscr{H}om(\mathscr{I}, \mathscr{C}) \to \mathscr{C}$ such that $\mathscr{C}(\varinjlim \alpha, C) \approx \mathscr{H}om(\mathscr{I}, \mathscr{C})(\alpha, i(C))$ for all α in $\mathscr{H}om(\mathscr{I}, \mathscr{C})$ and C in $\mathscr{C}$, the isomorphism being functorial in α and C. We shall say that $\mathscr{C}$ has direct limits if $\mathscr{C}$ has $\mathscr{I}$-direct limits for all $\mathfrak{U}$-categories $\mathscr{I}$ which are directed sets. If $\mathscr{C}$ is an abelian category, then $\mathscr{H}om(\mathscr{I}, \mathscr{C})$ is an abelian category. If $\mathscr{I}$-direct limits exist, then the functor $\varinjlim$ is right exact since it is an adjoint. We shall say that an abelian category $\mathscr{C}$ has exact direct limits if for each $\mathfrak{U}$-category $\mathscr{I}$ which is a directed set, $\mathscr{I}$-direct limits exist and the functor $\varinjlim: \mathscr{H}om(\mathscr{I}, \mathscr{C}) \to \mathscr{C}$ is exact. The category $\mathscr{A}b$ is an example of an abelian category with exact direct limits. Also if $\mathscr{C}$ is an abelian category with exact direct limits and $\mathscr{I}$ is an arbitrary category, then $\mathscr{H}om(\mathscr{I}, \mathscr{C})$ is an abelian category with exact direct limits.

Let $\mathscr{C}$ be a category. When we speak of a family of objects or maps in $\mathscr{C}$ we shall always mean a family which can be indexed by a set in $\mathfrak{U}$. If $(A_i)_{i\in I}$ is a family of objects in $\mathscr{C}$ and A is an object in $\mathscr{C}$, then a map of $(A_i)_{i\in I}$ to A is simply a family of maps $f_i : A_i \to A$. We shall say that $f_i : A_i \to A$ is an epimorphism if the induced maps $\mathscr{C}(A, X) \to \to \prod \mathscr{C}(A_i, X)$ are monomorphisms for all X in $\mathscr{C}$. An object A in $\mathscr{C}$ will be said to be finitely generated if given an epimorphism $f_i : A_i \to A$ with $i \in I$, there exists a finite subset J of I such that $f_j : A_j \to A$ (with $j \in J$) is an epimorphism.

Suppose now that $\mathscr{C}$ is an abelian category with exact direct limits. Let $(A_i)_{i\in I}$ be a family of objects in $\mathscr{C}$ and A an object in $\mathscr{C}$. Then a map from $(A_i)_{i\in I}$ to A is nothing more than a map of $\sum\limits_{i\in I} A_i$ to A, where $\sum\limits_{i\in I} A_i$ denotes the direct sum of the family $(A_i)_{i\in I}$. Also the condition that the map $(A_i)_{i\in I} \to A$ is an epimorphism is equivalent to the statement that the associated map $\sum\limits_{i\in I} A_i \to A$ is an epimorphism. Thus an object A in $\mathscr{C}$ is finitely generated if and only if given any family $(A_i)_{i\in I}$ in $\mathscr{C}$ and an epimorphism $f : \sum A_i \to A$, there exists a finite subset J of I such that f restricted to $\sum\limits_{j\in J} A_j$ is an epimorphism. The rest of this section is devoted to indicating how the usual terminology and fundamental properties of finitely generated objects in the category of modules over a ring can be extended to an abelian category $\mathscr{C}$ with exact direct limits.

Let $(A_i)_{i\in I}$ be a family of subobjects of an object A in $\mathscr{C}$. Then the $\operatorname{Im}(\sum A_i \to A)$ will be called the subobject of A generated by $(A_i)_{i\in I}$. We shall say that the family $(A_i)_{i\in I}$ generates A if the map $\sum A_i \to A$ is an epimorphism.

Thus A is finitely generated if and only if given any family of subobjects of A which generate A there is a finite subfamily which generates A.

Another useful formulation of when an object A is finitely generated is the following. We shall say that a family $(A_i)_{i\in I}$ of subobjects of A is directed if given i_0, i_1 in I there is an i_2 in I such that $A_{i_0} \subset A_{i_2}$ and $A_{i_1} \subset A_{i_2}$. Thus an object A is finitely generated if and only if given any directed family $(A_i)_{i\in I}$ of subobjects of A which generates A there is an $i \in I$ such that $A = A_i$. It should be observed that if $(A_i)_{i\in I}$ is a directed family of subobjects of A, then we can introduce the partial ordering $i_1 \leqq i_2$ if $A_{i_1} \subset A_{i_2}$ which makes I into a directed $\mathfrak{U}$-category $\mathscr{I}$. Then the subobject of A generated by $(A_i)_{i\in I}$ is $\varinjlim\limits_{i\in I} A_i$, where by $\varinjlim\limits_{i\in I} A_i$ we mean $\varinjlim F$ where F is the functor $\mathscr{I} \to \mathscr{C}$ which associates to i the object A_i.

These descriptions of finitely generated objects together with the following technical lemma, give most of the properties of finitely generated objects.

Lemma 1.1. *Let $\mathscr{C}$ be an abelian category with exact direct limits. Suppose $0 \to A' \xrightarrow{g} A \xrightarrow{f} A'' \to 0$ is an exact sequence in $\mathscr{C}$ and $(A_i)_{i \in I}$ is a directed set of subobjects of A. Then the families $(g^{-1}(A_i))_{i \in I}$ and $(f(A_i))_{i \in I}$ are directed sets of subobjects of A' and A'' and $(g^{-1}(A_i))_{i \in I}$ and $(f(A_i))_{i \in I}$ generate A' and A'' respectively if and only if $(A_i)_{i \in I}$ generates A.*

Proof. The fact that they are directed sets of subobjects is trivial to show. From the exact sequences

$$0 \to A'/g^{-1}(A_i) \to A/A_i \to A''/f(A_i) \to 0$$

we obtain passing to the limit, the exact sequence

$$0 \to A'/\varinjlim g^{-1}(A_i) \to A/\varinjlim A_i \to A''/\varinjlim f(A_i) \to 0.$$

From this we see immediately that $A = \varinjlim A_i$ if and only if

$$A' = \varinjlim g^{-1}(A_i) \quad \text{and} \quad A'' = \varinjlim f(A_i),$$

which gives us our desired result.

As an immediate consequence we obtain

Proposition 1.2. *Let $\mathscr{C}$ be an abelian category with exact direct limits. Let $0 \to A' \to A \to A'' \to 0$ be an exact sequence in $\mathscr{C}$.*

a) If A is finitely generated, then A'' is finitely generated.

b) If A' and A'' are finitely generated, then A is finitely generated.

Corollary 1.3. *Let $\mathscr{C}$ be as above. Then any object which is generated by a finite number of finitely generated objects, is finitely generated.*

Proposition 1.4. *Let $\mathscr{C}$ be an abelian category with exact direct limits. Suppose $\mathscr{I}$ is a $\mathfrak{U}$-category which is a directed set, $F : \mathscr{I} \to \mathscr{C}$ a functor and $C = \varinjlim F$.*

If A is a finitely generated object in $\mathscr{C}$, then the map $\varinjlim \mathscr{C}(A, F(i)) \to \mathscr{C}(A, C)$ is a monomorphism and is an isomorphism if each of the maps $F(i) \to C$ is a monomorphism.

If A is a projective object in $\mathscr{C}$, then the following statements are equivalent:

a) A is finitely generated;

b) $\varinjlim \mathscr{C}(A, F(i)) = \varinjlim \mathscr{C}(A, C)$;

c) the functor $\mathscr{C}(A, \_)$ commutes with direct sums.

Proof. Suppose A is finitely generated. In order to show that the map $\varinjlim \mathscr{C}(A, F(i)) \to \mathscr{C}(A, C)$ is a monomorphism, it suffices to show that given any $f : A \to F(i)$ such that the composite $A \xrightarrow{f} F(i) \to C$ is zero, there is a $j \geqq i$ such that the composite $A \xrightarrow{f} F(i) \to F(j)$ is zero. Let J consist of all $j \in I$ such that $j \geqq i$. Then $\varinjlim F|J = C$. Thus

$$0 = \operatorname{Im}(A \to F(i) \to C) = \varinjlim_{J} \operatorname{Im}(A \to F(i) \to F(j))$$

$$= \varinjlim_{J} A/A_j \text{ where } A_j = \operatorname{Ker}(A \to F(i) \to F(j)).$$

Thus $(A_j)_{j \in J}$ is a directed set of subobjects of A which generate A. Therefore $A = A_{j_0}$ for some $j_0 \in J$, since A is finitely generated. Thus the map $A \to F(i) \to F(j_0)$ is the zero map, which gives us our desired result.

Suppose now that each $F(i) \to C$ is a monomorphism. Let $C_i = \operatorname{Im}(F(i) \to C)$. Then $(C_i)_{i \in I}$ is a directed set of subobjects of C which generate C. Let $f : A \to C$ and $A_i = f^{-1}(C_i)$. Then it is easily seen that $(A_i)_{i \in I}$ is a directed family of subobjects of A which generate A. Since A is finitely generated we know that $A = A_{i_0}$ for some $i_0 \in I$, i.e. $\operatorname{Im} f \subset A_{i_0}$. Therefore it follows that f is in the $\operatorname{Im}(\mathscr{C}(A, F(i_0)) \to \mathscr{C}(A, C))$. Thus we obtain that $\varinjlim \mathscr{C}(A, F(i)) \to \mathscr{C}(A, C)$ is an isomorphism, which finishes the proof of the first part of the proposition.

Suppose A is a projective object in $\mathscr{C}$. a) $\Rightarrow$ b). By the first part of the proposition we know that $\varinjlim \mathscr{C}(A, F(i)) \to \mathscr{C}(A, C)$ is a monomorphism. Let $C_i = \operatorname{Im}(F(i) \to C)$. Then $(C_i)_{i \in I}$ is a directed family of subobjects of C which generate C. Thus given $f : A \to C$ we know by the first part of the proposition that $\operatorname{Im} f \subset C_{i_0}$ for some $i_0 \in I$. Since A is projective we know there is a map $A \to F(i_0)$ such that $A \to F(i_0) \to C_{i_0} \to C$ is the map f. Therefore f is in the $\operatorname{Im}(\mathscr{C}(A, F(i_0)) \to \mathscr{C}(A, C)$ which shows that $\varinjlim \mathscr{C}(A, F(i)) = \mathscr{C}(A, C)$.

b) $\Rightarrow$ c). Trivial.

c) $\Rightarrow$ a). Suppose $(A_i)_{i \in I}$ is a family of subobjects of A which generate A. Then the map $\sum A_i \to A$ is an epimorphism. Since A is projective, we know there is a map $g : A \to \sum A_i$ such that the composite $A \xrightarrow{g} \sum A_i \to A$ is the identity. But $\mathscr{C}(A, \_)$ commutes with direct sums. Thus there is a finite subset J of I such that $\operatorname{Im} g \subset \sum_{i \in J} A_i$. From this it follows that $(A_i)_{i \in J}$ generate A and thus that A is finitely generated.

Let $\mathscr{C}$ be an arbitrary category. Then the set whose objects are of the form $\mathscr{C}(A, B)$ for all A and B in $\mathscr{C}$ is easily seen to be in $\mathfrak{U}'$. Similarly the union of all sets of the form set maps from $\mathscr{C}(A, B)$ to $\mathscr{C}(C, D)$ for all A, B, C, D in $\mathscr{C}$ is an element of $\mathfrak{U}'$. Thus $\mathscr{Set}(\mathscr{C})$, the "category" whose objects are $\mathscr{C}(A, B)$ for all A and B in $\mathscr{C}$ with ordinary set maps for maps is a category in the sense of this paper, i.e. is a $\mathfrak{U}'$-category.

Denoting the opposite category of $\mathscr{C}$ by $\mathscr{C}^0$, we have the usual functor $\mathscr{C}^0 \times \mathscr{C} \to \mathscr{S}(\mathscr{C})$ given by $(A, B) \mapsto \mathscr{C}(A, B)$.

We shall say that a full subcategory $\mathscr{I}$ of $\mathscr{C}$ is a generator of $\mathscr{C}$ if

a) The objects of $\mathscr{I}$ can be indexed by a set in $\mathfrak{U}$.

b) The functor $F: \mathscr{I}^0 \times \mathscr{C} \to \mathscr{S}ets(\mathscr{C})$ defined by $F(A, B) = \mathscr{C}(A, B)$ is isomorphic to a functor $G: \mathscr{I}^0 \times \mathscr{C} \to \mathscr{S}$ (the category of $\mathfrak{U}$-sets), i.e. for each pair A, B we are given isomorphisms $F(A, B) \approx G(A, B)$ which are functorial in A and B.

c) For all A and B in $\mathscr{C}$ the map

$$\mathscr{C}(A, B) \to \prod_{Y \in \mathscr{I}} \operatorname{Hom}(\mathscr{C}(Y, A), \mathscr{C}(Y, B))$$

is a monomorphism where Hom indicates set maps.

Suppose $\mathscr{I}$ is a generator for the category $\mathscr{C}$. Then it is not difficult to see that for each A in $\mathscr{C}$ the set of maps $f: Y \to A$ where Y ranges through $\mathscr{I}$ and f ranges through $\mathscr{C}(Y, A)$ is a family of maps which is an epimorphism. Also there exists a functor $\mathscr{C}^0 \times \mathscr{C} \to \mathscr{S}$ which is isomorphic to the functor $\mathscr{C}^0 \times \mathscr{C} \to \mathscr{S}ets(\mathscr{C})$ given by $(A, B) \mapsto \mathscr{C}(A, B)$. Thus if $\mathscr{C}'$ is a subcategory of $\mathscr{C}$ whose objects can be indexed by a set in $\mathfrak{U}$, then $\mathscr{C}'$ is isomorphic to a $\mathfrak{U}$-category.

If $\mathscr{C}$ is an abelian category, then for $\mathscr{S}ets(\mathscr{C})$ we substitute $\mathscr{Ab}(\mathscr{C})$, the additive category of abelian groups whose objects are $\mathscr{C}(A, B)$ for all A and B in $\mathscr{C}$ and maps group homomorphisms. We shall say that a full subcategory $\mathscr{I}$ of $\mathscr{C}$ is a generator for $\mathscr{C}$ if it satisfies conditions a, b, c with $\mathscr{Ab}(\mathscr{C})$ and $\mathscr{Ab}$ substituted for $\mathscr{S}ets(\mathscr{C})$ and $\mathscr{S}$ respectively and group homomorphisms substituted for set maps. It is clear that if $\mathscr{I}$ satisfies these conditions, then the category $\mathscr{I}'$ whose objects are all finite direct sums of objects in $\mathscr{I}$ also satisfies the same conditions. For sake of convenience we shall assume in addition to the conditions a, b, c, that a generator for an abelian category has finite direct sums, i.e. is a full additive subcategory of $\mathscr{C}$ satisfying a, b, und c. If $\mathscr{C}$ is an abelian category with a generator, then $\mathscr{C}$ has, in addition to the properties sited above, the feature that the set of subobjects of an object in $\mathscr{C}$ can be indexed by a set in $\mathfrak{U}$ (see [*1*, pg. 336]).

Suppose $\mathscr{C}$ is an abelian category with exact direct limits and a generator $\mathscr{I}$ consisting of finitely generated objects in $\mathscr{C}$. Then an object A in $\mathscr{C}$ is finitely generated if and only if there exists an exact sequence $B \to A \to 0$ with B in $\mathscr{I}$. It follows from proposition 1.2 that if such an exact sequence exists, then A is finitely generated. On the other hand, since $\mathscr{I}$ is a generator, we know that there exists an exact sequence $\sum_{j \in J} B_j \to A \to 0$ with the B_j in $\mathscr{I}$. If A is finitely generated we know that there is a finite subset $J' \subset J$ such that $\sum_{j \in J} B_j \to A \to 0$ is exact, which gives us the other implication. Finally defining an object A in $\mathscr{C}$ to be finitely presented if it is finitely generated and given any exact sequence $0 \to X' \to X \to A \to 0$ with X finitely generated, we have that X' is finitely generated, we obtain the following usual result.

Proposition 1.5. *Let $\mathscr{C}$ be an abelian category. Suppose that $\mathscr{I}$ is a generator of $\mathscr{C}$ such that each object in $\mathscr{I}$ is a finitely generated projective object in $\mathscr{C}$. Also suppose $\mathscr{C}$ has exact direct limits. Then we have:*

a) An object X in $\mathscr{C}$ is finitely generated if and only if there exists an epimorphism $A \to X$ for some A in $\mathscr{I}$.

b) If X is an object in $\mathscr{C}$, then the following statements are equivalent:

i) X is finitely generated;

ii) there is an exact sequence $A_1 \to A_2 \to X \to 0$ with the A_i in $\mathscr{I}$.

iii) there is an exact sequence $A + A \to A \to X \to 0$ with A in $\mathscr{I}$.

Proof. a) Obvious.

b) i) $\Rightarrow$ ii) Obvious.

ii) $\Rightarrow$ iii) We imbed the exact sequence $A_1 \to A_2 \to X \to 0$ in the exact sequence $A_1 + A_1 \to A_1 + A_2 \to X \to 0$ by defining the map on the extra $A_1 \to A_1$ to be the identity. In turn we imbed this exact sequence in $A_1 + A_1 + A_2 + A_2 \to A_1 + A_2 \to X \to 0$ by sending $A_2 + A_2$ to zero. Thus letting $A = A_1 + A_2$ we obtain our desired exact sequence.

iii) $\Rightarrow$ ii) Obvious.

ii) $\Rightarrow$ i) Same as for modules and left as an excercise.

Corollary 1.6. *Let $\mathscr{C}$ and $\mathscr{I}$ be as in proposition 1.5 and let $\mathscr{C}'$ be the full subcategory of $\mathscr{C}$ whose objects are the finitely presented objects of $\mathscr{C}$. Then*

a) $\mathscr{C}'$ is equivalent to a $\mathfrak{U}$-category.

b) Suppose $\mathscr{I}$ is a $\mathfrak{U}$-category which is a directed set and $F : \mathscr{I} \to \mathscr{C}$, a functor. If A is in $\mathscr{C}'$, then $\varinjlim \mathscr{C}(A, F(i)) = \mathscr{C}(A, \varinjlim F)$.

Proof. a) Let $\mathscr{D}$ be the full subcategory of $\mathscr{C}$ consisting of representatives of the isomorphism classes of objects in $\mathscr{C}'$. For each object D in $\mathscr{D}$ choose a representation $A_1 \to A_2 \to D \to 0$ with A_1 and A_2 in $\mathscr{I}$. Then the objects of $\mathscr{D}$ are in one to one correspondence with a subset of $\bigcup \mathscr{C}(A_1, A_2)$ where A_1 and A_2 range through $\mathscr{I}$. Since the set of objects of $\mathscr{I}$ can be indexed by a set in $\mathfrak{U}$, so can the objects in $\mathscr{I}$. Since each $\mathscr{C}(A_1, A_2)$ is isomorphic to a $\mathfrak{U}$-set, we have that $\bigcup \mathscr{C}(A_1, A_2)$ is isomorphic to a $\mathfrak{U}$-set. Thus the objects in $\mathscr{D}$ can be indexed by a $\mathfrak{U}$-set. Therefore it follows from the remarks following the definition of a generator for a category, that the category $\mathscr{D}$ is isomorphic to a $\mathfrak{U}$-category. Since $\mathscr{D}$ is equivalent to $\mathscr{C}$, we are done.

b) Same proof as for modules.

2. Coherent Functors

Throughout the rest of this paper we will assume (unless stated to the contrary) that all categories and functors are additive. Let $\mathscr{C}$ denote a fixed abelian $\mathfrak{U}$-category. We shall denote by $\widecheck{\mathscr{C}}$ the category $\mathscr{H}om(\mathscr{C}, \mathscr{A}b)$ of covariant (additive) functors from $\mathscr{C}$ to $\mathscr{A}b$, the category of abelian groups in $\mathfrak{U}$. We shall denote by $\widehat{\mathscr{C}}$ the category $\mathscr{H}om(\mathscr{C}^0, \mathscr{A}b)$ of contravariant functors from $\mathscr{C}$ to $\mathscr{A}b$. It should be noted that since $\mathscr{A}b$ has exact direct limits, so do $\widehat{\mathscr{C}}$ and $\widehat{\mathscr{C}^0}$. The rest of this paper is devoted to studying the categories $\check{\mathscr{C}}$ and $\hat{\mathscr{C}}$, the full subcategories of finitely presented objects in $\widecheck{\mathscr{C}}$ and $\widehat{\mathscr{C}}$, respectively. Since $\widehat{\mathscr{C}^0} = \widecheck{\mathscr{C}}$ and $\widecheck{\mathscr{C}^0} = \widehat{\mathscr{C}}$ it suffices to state and prove theorems for one or the other of these categories. The analogous statements for the other category is left as an excercise for the reader, although such statements will be used freely in the paper when convenient.

Let $\mathscr{P}$ be the full subcategory of $\widehat{\mathscr{C}}$ consisting of those functors of the form $\mathscr{C}(\_, A)$ or more simply $(\_, A)$ for each A in $\mathscr{C}$. Suppose G is in $\widehat{\mathscr{C}}$ and $\alpha : (\_, A) \to G$ in $\widehat{\mathscr{C}}$. Then $\alpha_A : (A, A) \to G(A)$ and thus $\alpha_A(1_A)$ is an element in $G(A)$. It is well known that $\varphi : \widehat{\mathscr{C}}((\_, A), G) \to G(A)$ given by $\varphi(\alpha) = \alpha_A(1_A)$ is an isomorphism which is functorial in A and G. From this it readily follows that

a) $(\_, A)$ is projective for each A in $\mathscr{C}$;

b) $\widehat{\mathscr{C}}((\_, A), \_)$ commutes with direct limits and thus by proposition 1.4, each $(\_, A)$ is finitely generated;

c) $(\_, A) + (\_, B) = (\_, A + B)$ and thus $\mathscr{P}$ is closed under finite direct sums;

d) $\mathscr{P}$ is a generator for $\widehat{\mathscr{C}}$.

Thus the generator $\mathscr{P}$ of $\widehat{\mathscr{C}}$ satisfies the hypothesis of proposition 1.5. Therefore a functor $F \in \widehat{\mathscr{C}}$ is finitely presented if and only if there exists an exact sequence $(\_, A) \to (\_, B) \to F \to 0$. Suppose $(\_, A) \to (\_, B) \to F \to 0$ is exact. Then the map $(\_, A) \to (\_, B)$ is induced by a map $A \to B$. If we let $A' = \operatorname{Ker}(A \to B)$, then we obtain the exact sequence

$$0 \to (\_, A') \to (\_, A) \to (\_, B) \to F \to 0 .$$

Thus the projective dimension of F (notation: pd F) is at most 2. Therefore we have shown that if $P_1 \to P_2$ is a map in $\mathscr{P}$, then the $\operatorname{Ker}(P_1 \to P_2)$ is in $\mathscr{P}$. Also that each $F \in \hat{\mathscr{C}}$ the category of finitely presented functors, has a projective resolution which is also in $\widehat{\mathscr{C}}$ and that the pd $F \leqq 2$. Some additional properties of the category $\hat{\mathscr{C}}$ of finitely presented functors will become clear from the following general proposition.

Proposition 2.1. *Let $\mathscr{A}$ be an abelian category and $\mathscr{P}$ a full additive subcategory of $\mathscr{A}$ whose objects are projective objects in $\mathscr{A}$. Let $\mathscr{P}(\mathscr{A})$ be the full subcategory of $\mathscr{C}$ consisting of those objects A in $\mathscr{A}$ such that there exists an exact sequence $P_1 \to P_0 \to A \to 0$ in $\mathscr{A}$ with the P_i in $\mathscr{P}$. Then $\mathscr{P}(\mathscr{A})$ has the following properties:*

a) If $0 \to C_1 \to C_2 \to C_3 \to 0$ is exact in $\mathscr{A}$ with C_1 and C_3 in $\mathscr{P}(\mathscr{A})$, then C_2 is in $\mathscr{P}(\mathscr{A})$.

b) Suppose $0 \to C_1 \to C_2 \to C_3 \to C_4 \to 0$ is exact with C_2 and C_3 in $\mathscr{P}(\mathscr{A})$, then C_4 is in $\mathscr{P}(\mathscr{A})$. If $\mathscr{P}$ has the additional property that given $P_2 \to P_3$ in $\mathscr{P}$, there is an exact sequence $P_1 \to P_2 \to P_3$ with P_1 in $\mathscr{P}$, then C_1 is also in $\mathscr{P}(\mathscr{A})$.

c) The inclusion functor $u : \mathscr{P} \to \mathscr{P}(\mathscr{A})$ has the property that given any additive category $\mathscr{D}$ with cokernels, then the induced functor

$$u : \mathscr{H}om(\mathscr{P}(\mathscr{A}), \mathscr{D}) \to \mathscr{H}om(\mathscr{P}, \mathscr{D})$$

has a left adjoint $u^{\cdot} : \mathscr{H}om(\mathscr{P}, \mathscr{D}) \to \mathscr{H}om(\mathscr{P}(\mathscr{A}), \mathscr{D})$. This $u^{\cdot}$ and the functorial isomorphisms $(u^{\cdot}F, G) \approx (F, u_{.}G)$ for F in $\mathscr{H}om(\mathscr{P}, \mathscr{D})$ and G in $\mathscr{H}om(\mathscr{P}(\mathscr{A}), \mathscr{D})$ can be chosen so that:

i) $u^{\cdot}u.F = F$ for all F in $\mathscr{H}om(\mathscr{P}, \mathscr{D})$;

ii) $u^{\cdot}F$ is right exact for all F in $\mathscr{H}om(\mathscr{P}, \mathscr{D})$ (i.e. given an exact sequence $C_1 \to C_2 \to C_3 \to 0$ in $\mathscr{P}(\mathscr{A})$, then $F(C_1) \to F(C_2) \to F(C_3) \to 0$ is exact);

iii) $u.u^{\cdot}u.G = u.G$ for all $G \in \mathscr{H}om(\mathscr{P}(\mathscr{A}), \mathscr{D})$ and the usual map $u^{\cdot}u.G \to G$ is an isomorphism if and only if G is right exact, in which case $u^{\cdot}u.G = G$.

Proof. a) Follows easily from standard techniques of building resolutions for exact sequences.

b) Suppose $P_1 \to P_0 \to C_2 \to 0$ and $Q_1 \to Q_0 \to C_3 \to 0$ are exact with the P_i and Q_i in $\mathscr{P}$. Given the map $g : C_2 \to C_3$ we know that it can be lifted to a map of the complexes $g : P \to Q$ where P is the complex $\ldots 0 \to P_1 \to P_0 \to 0 \ldots$ and Q is the complex $\ldots 0 \to Q_1 \to Q_0 \to 0 \ldots$ Let $M(g)$ be the mapping cone of g, i.e. $M(g)_i = P_{i-1} + Q_i$ and $d : M(g)_i \to M(g)_{i-1}$ is $(-d_P, d_Q + g)$ where d_P and d_Q are the boundary maps in P and Q respectively. Then as is well known (see [*4*, ch. II, §4] for instance) we obtain an exact sequence

$(\star)\ H_1(P) \to H_1(Q) \to H_1(M(g)) \to H_0(P) \to H_0(Q) \to H_0(M(g)) \to 0$

with $H_0(P) = C_2$ and $H_0(Q) = C_3$ and the map $H_0(P) \to H_0(Q)$ the map $g : C_2 \to C_3$. Thus we have that $H_0(M(g)) \approx \operatorname{Coker} g \approx C_4$. But we have the exact sequence $P_0 + Q_1 \to Q_0 \to H_0(M(g)) \to 0$ which shows that $H_0(M(g)) \in \mathscr{P}(\mathscr{A})$ since $\mathscr{P}$ is closed under finite direct sums. Thus the first part of b) is proven.

Suppose now that $\mathscr{P}$ has the additional property that given $Y \to Z$ in $\mathscr{P}$ there is an exact sequence $X \to Y \to Z$ with X in $\mathscr{P}$. This is the same thing as assuming that if $Y \to Z$ is in $\mathscr{P}$ then Ker $(Y \to Z)$ is in $\mathscr{P}(\mathscr{A})$. Therefore it follows that $H_1(P)$ and $H_1(Q)$ are in $\mathscr{P}(\mathscr{A})$. Since we have shown that $\mathscr{P}(\mathscr{A})$ is closed under cokernels, we obtain from the exact sequence $(\star)$, the exact sequence

$$0 \to B \to H_1(M(g)) \to H_0(P) \to H_0(Q)$$

with B in $\mathscr{P}(\mathscr{A})$. Thus if we show that $H_1(M(g))$ is in $\mathscr{P}(\mathscr{A})$, then we will have again by part a), that $\mathrm{Ker}(H_0(P) \to H_0(Q)) \approx C_1$ is in $\mathscr{P}(\mathscr{A})$. But $Z_1(M(g)) = \mathrm{Ker}\ (M(g)_1 \to M(g)_0)$ which is in $\mathscr{P}(\mathscr{A})$ since $M(g)_1$ and $M(g)_0$ are in $\mathscr{P}$. Therefore $H_1(M(g)) = \mathrm{Coker}(M(g)_2 \to Z_1(M(g))$ is in $\mathscr{P}(\mathscr{A})$, which concludes the proof of b).

c) We first have to describe the functor

$$u^{\cdot} \colon \mathscr{H}om(\mathscr{P}, \mathscr{D}) \to \mathscr{H}om(\mathscr{P}(\mathscr{A}), \mathscr{D}).$$

For each C in $\mathscr{A}$ we pick a fixed exact sequence $P_1 \to P_0 \to C \to 0$ subject to the condition that if C is in $\mathscr{P}$, then we pick the sequence $0 \to C = C \to 0$. Suppose $F \colon \mathscr{P} \to \mathscr{D}$. Then define

$$u^{\cdot} F(C) = \mathrm{Coker}(F(P_1) \to F(P_0)).$$

If we have $C \xrightarrow{g} C'$, then we lift this to a map of $P \to P'$ which gives us a map $u^{\cdot}(F)(g) \colon F(C) \to F(C')$ which is well known to be independent of the lifting used. Thus $u^{\cdot}$ is defined.

Suppose F is in $\mathscr{H}om(\mathscr{P}, \mathscr{D})$ and G is in $\mathscr{H}om(\mathscr{P}(\mathscr{C}), \mathscr{D})$. If P is in $\mathscr{P}$, then $u^{\cdot}(F(P) = F(P)$ and $u.\,G(P) = G(P)$. Thus given h in $(u^{\cdot} F, G)$, we have for each P in $\mathscr{P}$ a map $h_P \colon u^{\cdot} F(P) \to G(P)$ which is the same thing as a map $h_P \colon F(P) \to u.\,G(P)$. Thus we have a map $(u^{\cdot} F, G) \to (F, u.\,G)$ which is easily seen to be an isomorphism which is functorial in F and G. The properties i), ii), and iii) are easy to check and are left as excercises.

It should be observed that in the case $\mathscr{P}(\mathscr{A}) = \mathscr{A}$ and G is in $\mathscr{H}om(\mathscr{A}, \mathscr{D})$, then $u^{\cdot} u.\,G$ is usually denoted by $L_0\,G$ and is called the 0-th left derived functor of G.

We know return to the categories $\widehat{\mathscr{C}}$ and $\hat{\mathscr{C}}$. Letting $\mathscr{P}$ be the full subcategory of $\hat{\mathscr{C}}$ whose objects are $(\_, A)$ for all A in $\mathscr{C}$, we know that $\hat{\mathscr{C}} = \mathscr{P}(\hat{\mathscr{C}})$ and that if $P_2 \to P_3$ is in $\mathscr{P}$, then $\mathrm{Ker}(P_2 \to P_3) \in \mathscr{P}$. It then follows from a) and b) of proposition 2.1, that if $0 \to F_1 \to F_2 \to F_3 \to 0$ is exact with F_1 and F_2 finitely presented, then F_2 is finitely presented. Also if $0 \to F_1 \to F_2 \to F_3 \to F_4 \to 0$ is exact and F_2 and F_3 are finitely presented, then F_1 and F_4 are finitely presented. Thus $\hat{\mathscr{C}}$ is an abelian category and the inclusion functor $\hat{\mathscr{C}} \to \hat{\mathscr{C}}$ is fully faithful, exact and preserves projective objects.

Suppose $0 \to F' \to F \to F'' \to 0$ is exact with F finitely presented and F' finitely generated. Then we have an exact sequence $(-, A) \to F' \to 0$ and thus an exact sequence $(-, A) \to F \to F'' \to 0$. Since $(-, A)$ and F are finitely presented, we have that F'' is finitely presented and thus that F' is finitely presented. Thus every finitely generated subobject of a finitely presented functor is finitely presented. Such objects in a category are usually called coherent objects. Thus all finitely presented functors are coherent functors.

We now give some examples of coherent functors. Suppose X is a complex in $\mathscr{C}$. Then $(-, X)$ is a complex in $\hat{\mathscr{C}}$ and thus for each i, the functors $Z_i((-, X))$, $B_i((-, X))$ and $H_i((-,))$ are in $\hat{\mathscr{C}}$. Thus if $\mathscr{C}$ has sufficiently many injective objects, then $\mathrm{Ext}^i(-, A)$ is in $\hat{\mathscr{C}}$ for all i and all A in $\mathscr{C}$. On the other hand, suppose $\mathscr{C}$ has sufficiently many projective objects. If $P \to A \to 0$ is exact with P projective, then the functor $\underline{\Pi}(-, A) = \mathrm{Coker}((-, P) \to (-, A))$ introduced by ECKMANN-HILTON is independent of the choice of P and is clearly in $\hat{\mathscr{C}}$. More generally, $\underline{\Pi}_n(-, A)$ is in $\hat{\mathscr{C}}$ for all n, where $\underline{\Pi}_{n+1}(-, A) = H_n((-, P))$ (for $n > 0$) and P is a projective resolution of A (see [2] for further details).

Similarly if X is a complex in $\mathscr{C}$, then the complex $(X, -)$ is in $\check{\mathscr{C}}$ and thus $Z_i((X, -))$, $B_i((X, -))$ and $H_i((X, -))$ are in $\check{\mathscr{C}}$. Thus if $\mathscr{C}$ has sufficiently many projective objects, then $\mathrm{Ext}^i(A, -)$ is in $\check{\mathscr{C}}$ for all i and A in $\mathscr{C}$. If $\mathscr{C}$ has sufficiently many injectives, then the injective homotopy functors $\overline{\Pi}_n(A, -)$ are also in $\check{\mathscr{C}}$ where for $n > 0$ the functor $\overline{\Pi}_{n+1}(A, -) = H_n((Q, -)$ where Q is an injective resolution of A. We shall give other examples later on.

We now return to applying part c) of proposition 2.1 to the categories of coherent functors. First we consider the category $\hat{\mathscr{C}}$. The canonical functor $\mathscr{C} \to \hat{\mathscr{C}}$ given by $A \to (-, A)$ gives us an isomorphism between the categories $\mathscr{C}$ and $\mathscr{P}$ which we will often consider an identification. Thus by proposition 2.1 c) we have that given any abelian category $\mathscr{D}$, the functor $u.: \mathscr{H}om(\hat{\mathscr{C}}, \mathscr{D}) \to \mathscr{H}om(\mathscr{C}, \mathscr{D})$ induced by $u: \mathscr{C} \to \hat{\mathscr{C}}$ has a left adjoint $u^{\cdot}: \mathscr{H}om(\mathscr{C}, \mathscr{D}) \to \mathscr{H}om(\hat{\mathscr{C}}, \mathscr{D})$ which has the following properties:

1. Suppose F is in $\mathscr{H}om(\mathscr{C}, \mathscr{D})$ and G is in $\mathscr{H}om(\hat{\mathscr{C}}, \mathscr{D})$. Then $u^{\cdot} F((-, A)) = F(A)$ and $u.\, G(A) = G((-, A))$ for all A in $\mathscr{C}$. Further, if $\alpha: u^{\cdot} F \to G$ is a map in $\mathscr{H}om(\hat{\mathscr{C}}, \mathscr{D})$, then for each A in $\mathscr{C}$, we get a map $F(A) \to u.\, G(A)$, namely $\alpha_A: u^{\cdot} F((-, A)) \to G((-, A))$. Thus we get an isomorphism $(u^{\cdot} F, G) \to (F, u.\, G)$ which is functorial in F and G.

2. $u.\,u^{\cdot} F = F$ for all F in $\mathscr{H}om(\mathscr{C}, \mathscr{D})$.

3. $u^{\cdot} F$ is right exact for all F in $\mathscr{H}om(\mathscr{C}, \mathscr{D})$.

4. The usual map $u^{\cdot}\, u.\, G \to G$ for all G in $\mathscr{H}om(\hat{\mathscr{C}}, \mathscr{D})$ has the properties that a) $u^{\cdot}\, u.\, G((-, A)) \to G((-, A))$ is the identity for A in $\mathscr{C}$ and b) $u^{\cdot}\, u.\, G \to G$ is an isomorphism if and only if G is right exact in which case $u^{\cdot}\, u.\, G = G$.

We have also one more property that is not simply a rephrasing of proposition 2.1.

5. Suppose F in $\mathscr{H}om(\mathscr{C}, \mathscr{D})$ is left exact. Then $u^{\cdot} F : \hat{\mathscr{C}} \to \mathscr{D}$ is exact. For let T be in $\hat{\mathscr{C}}$ and let $0 \to A_1 \to A_2 \to A_3$ be an exact sequence in $\mathscr{C}$ such that $0 \to (-, A_1) \to (-, A_2) \to (-, A_3) \to T \to 0$ is exact. Since $u^{\cdot} F((-, A_i)) = F(A_i)$ and F is left exact we have that

$$0 \to u^{\cdot} F((-, A_i)) \to u^{\cdot} F((-, A_2)) \to u^{\cdot} F((-, A_3))$$

is exact. Thus the derived functors $L^i u^{\cdot} F(T) = 0$ for $i > 0$. Since this is true for all T in $\hat{\mathscr{C}}$, it follows that $u^{\cdot} F$ is exact.

Suppose now that $F : \mathscr{C} \to \mathscr{D}$ is left exact and X is a complex in $\mathscr{C}$. Then $F(X) = u^{\cdot} F((-, X))$ and thus $H_i(F(X)) = u^{\cdot} F(H_i(-, X))$ for all i since $u^{\cdot} F$ is exact. In particular, suppose $\mathscr{C}$ has enough injectives and $F : \mathscr{C} \to \mathscr{D}$ is left exact. Then $R^i F(A) = u^{\cdot} F(\mathrm{Ext}^i(-, A))$ for all i and all A in $\mathscr{C}$.

We end this section by summarizing in the following theorems some of the more important facts concerning the categories of coherent functors established so far.

Theorem 2.2. *Let $\mathscr{C}$ be an abelian $\mathfrak{U}$-category and $\hat{\mathscr{C}}$ the category of contravariant finitely presented functors from $\mathscr{C}$ to $\mathscr{A}b$.*

a) A functor F in $\widehat{\mathscr{C}}$ is in $\hat{\mathscr{C}}$ if and only if there is an exact sequence $(-, A) \to (-, B) \to F \to 0$.

b) $\hat{\mathscr{C}}$ is an abelian category with enough projectives of global dimension at most 2. Also the inclusion functor from $\hat{\mathscr{C}}$ to $\widehat{\mathscr{C}}$ is a fully faithful, exact functor.

c) The functor $u : \mathscr{C} \to \hat{\mathscr{C}}$ given by $u(A) = (-, A)$ is a fully faithful, left exact functor.

d) If $\mathscr{D}$ is an abelian category, then the exact functor $u. : \mathscr{H}om(\hat{\mathscr{C}}, \mathscr{D}) \to \mathscr{H}om(\mathscr{C}, \mathscr{D})$ has a left adjoint $u^{\cdot}$ with the following properties:

1. If $F : \mathscr{C} \to \mathscr{D}$, then $(u^{\cdot} F)\, u = F$ and $u^{\cdot} F$ is right exact.

2. $F : \mathscr{C} \to \mathscr{D}$ is left exact if and only if $u^{\cdot} F$ is exact.

3. If $\mathscr{C}$ has enough injectives and $F : \mathscr{C} \to \mathscr{D}$ is left exact, then $(R^i F)(A) = u^{\cdot} F(\mathrm{Ext}^i(-, A))$ for all $i \geqq 0$ and all A in $\mathscr{C}$.

We state for the convenience of the reader the analogous theorem for $\check{\mathscr{C}}$.

Theorem 2.3. *Let $\mathscr{C}$ be an abelian $\mathfrak{U}$-category and $\check{\mathscr{C}}$ the category of finitely presented covariant functors from $\mathscr{C}$ to $\mathscr{A}b$.*

a) A functor F in $\check{\mathscr{C}}$ is in $\check{\mathscr{C}}$ if and only if there exists an exact sequence $(A, \_) \to (B, \_) \to F \to 0$.

b) $(\check{\mathscr{C}})^0$ *is an abelian category with enough injectives of global dimension at most 2. Further, the inclusion functor* $(\check{\mathscr{C}})^0 \to (\check{\mathscr{C}})^0$ *is a fully faithful, exact functor.*

c) The functor $u: \mathscr{C} \to (\check{\mathscr{C}})^0$ *given by* $u(A) = (A, \_)$ *is a fully faithful, right exact functor.*

d) If $\mathscr{D}$ is an abelian category, then the exact functor $u_.: \mathscr{H}om((\hat{\mathscr{C}})^0, \mathscr{D}) \to \mathscr{H}om(\mathscr{C}, \mathscr{D})$ *has a left adjoint* $u^{\cdot}: \mathscr{H}om(\mathscr{C}, \mathscr{D}) \to \mathscr{H}om((\check{\mathscr{C}})^0, \mathscr{D})$ *with the following properties:*

1. If $F: \mathscr{C} \to \mathscr{D}$, *then* $(u^{\cdot} F)\, u = F$ *and* $u^{\cdot} F$ *is a left exact.*

2. $F: \mathscr{C} \to \mathscr{D}$ *is right exact if and only if* $u^{\cdot} F$ *is exact.*

3. If $\mathscr{C}$ has enough projectives and F is right exact, then $L^i F(A) = u^{\cdot} F(\mathrm{Ext}^i(A, \_))$ *for all* $i \geqq 0$ *and all A in $\mathscr{C}$.*

3. The Categories $\hat{\mathscr{C}}_0$ and $\hat{\mathscr{C}}/\hat{\mathscr{C}}_0$

As in section § 2, we assume that $\mathscr{C}$ is a fixed abelian $\mathfrak{U}$-category and we continue our study of the categories $\hat{\mathscr{C}}$ and $\check{\mathscr{C}}$. Letting $u: \mathscr{C} \to \hat{\mathscr{C}}$ be the usual functor $C \mapsto (\_, C)$ we know by theorem 2.2, that the functor $u_.: \mathrm{Hom}(\hat{\mathscr{C}}, \mathscr{C}) \to \mathrm{Hom}(\mathscr{C}, \mathscr{C})$ has a left adjoint $u^{\cdot}$ which has certain properties. In particular, since the identity functor $I: \mathscr{C} \to \mathscr{C}$ is exact, we know that $u^{\cdot} I: \hat{\mathscr{C}} \to \mathscr{C}$ is exact and $(u^{\cdot} I)\, u = u$. We shall denote $u_. I$ by v. It follows from the exactness of v, that $\hat{\mathscr{C}}_0$, the full subcategory of $\hat{\mathscr{C}}$ consisting of all F such that $v(F) = 0$, is a dense subcategory of $\hat{\mathscr{C}}$, i.e. if $0 \to F_1 \to F_2 \to F_3 \to 0$ is exact in $\hat{\mathscr{C}}$, then F_2 is in $\hat{\mathscr{C}}_0$ if and only if F_1 and F_2 are in $\hat{\mathscr{C}}_0$. Since the category $\hat{\mathscr{C}}_0$ plays an important role in our general theory, we will give various descriptions of the objects in $\hat{\mathscr{C}}_0$. However, before doing this we make the following trivial but useful observation.

Lemma 3.1. *Let F be in $\hat{\mathscr{C}}$ and G in $\hat{\mathscr{C}}$. Suppose* $0 \to A_0 \to A_1 \to A_2$ *is an exact sequence in $\mathscr{C}$ such that* $0 \to (\_, A_0) \to (\_, A_1) \to (\_, A_2) \to F \to 0$ *is exact. Then* $\mathrm{Ext}^i(F, G) \approx H_i(G(A))$ *where $G(A)$ is the complex*

$$\ldots 0 \to G(A_2) \to G(A_1) \to G(A_0) \to 0 \ldots$$

Proof. Follows immediately from the fact that $((\_, B), G) \approx G(B)$ for all B in $\mathscr{C}$.

Proposition 3.2. *Let F be in $\hat{\mathscr{C}}$. Then the following statements are equivalent:*

a) F is in $\hat{\mathscr{C}}_0$.

b) If $0 \to A_0 \to A_1 \to A_2$ *is exact in* $\mathscr{C}$ *such that* $0 \to (-, A_0) \to (-, A_1) \to (-, A_2) \to F \to 0$ is *exact, then* $0 \to A_0 \to A_1 \to A_2 \to 0$ *is exact.*

c) There exists an exact sequence $0 \to A_0 \to A_1 \to A_2 \to 0$ *in* $\mathscr{C}$ *such that* $0 \to (-, A_0) \to (-, A_1) \to (-, A_2) \to F \to 0$ *is exact.*

d) $\mathrm{Ext}^i(F, G) = 0$ *for* $i = 0,1$ *and any* G *in* $\hat{\mathscr{C}}$ *which is left exact, i.e. if* $X \to Y \to Z \to 0$ *is exact in* $\mathscr{C}$*, then* $0 \to G(Z) \to G(Y) \to G(X)$ *is exact.*

e) $(F, G) = 0$ *for any* G *in* $\hat{\mathscr{C}}$ *which is left exact.*

f) $(F, (-, A)) = 0$ *for all* A *in* $\mathscr{C}$.

Proof. a) $\Leftrightarrow$ b). Suppose $0 \to A_0 \to A_1 \to A_2$ is exact such that $0 \to (-, A_0) \to (-, A_1) \to (-, A_2) \to F \to 0$ is exact. Then applying v to this exact sequence, we obtain the exact sequence $0 \to A_0 \to A_1 \to A_2 \to v(F) \to 0$. Thus $v(F) = 0$ if and only if $0 \to A_0 \to A_1 \to A_2 \to 0$ is exact, which gives us our desired result.

b) $\Rightarrow$ c) Trivial.

c) $\Rightarrow$ d) Follows immediately from Lemma 3.1.

d) $\Rightarrow$ e) and e) $\Rightarrow$ f) are trivial.

f) $\Rightarrow$ b) Suppose $0 \to A_0 \to A_1 \to A_2$ is exact such that $0 \to (-, A_0) \to (-, A_1) \to (-, A_2) \to F \to 0$ is exact. Then it follows from Lemma 3.1, that $(F, (-, A)) = \mathrm{Ker}((A_2, A) \to (A_1, A))$. Thus $(F, (-, A)) = 0$ for all A if and only if $0 \to (A_2, A) \to (A_1, A)$ is exact for all A, i.e. if and only if $A_1 \to A_2 \to 0$ is exact. Thus f) $\Rightarrow$ b), which finishes the proof.

We now recall the definition of the $v\colon \hat{\mathscr{C}} \to \mathscr{C}$. For each F in $\hat{\mathscr{C}}$ we chose a fixed exact sequence $0 \to A_0 \to A_1 \to A_2$ and a fixed map $(-, A_i) \to F$ such that $0 \to (-, A_0) \to (-, A_1) \to (-, A_2) \to F \to 0$ is exact, subject only to the condition that if $F = (-, X)$, then we choose $A_0 = 0 = A_1$ and $A_2 = X$ with the map $(-, A_2) \to F$ the identity. Then $v(F) = A_3 = \mathrm{Coker}(A_1 \to A_2)$. Now if we let $B = \mathrm{Coker}(A_0 \to A_1) = \mathrm{Ker}(A_2 \to A_3)$, then from the exact sequence $0 \to A_0 \to A_1 \to B \to 0$ and $0 \to B \to A_2 \to A_3 \to 0$ we deduce the following commutative diagram with exact rows and columns

$$\begin{tikzcd}
 & & & 0 \arrow[d] & & \\
0 \arrow[r] & (-, A_0) \arrow[r] \arrow[d, equal] & (-, A_1) \arrow[r] \arrow[d, equal] & (-, B) \arrow[r] \arrow[d] & F_0 \arrow[r] & 0 \\
0 \arrow[r] & (-, A_0) \arrow[r] & (-, A_1) \arrow[r] & (-, A_2) \arrow[r] \arrow[d] & F \arrow[r] & 0 \\
 & & & (-, A_3) \arrow[d] & & \\
 & & & F_1 \arrow[d] & & \\
 & & & 0 & &
\end{tikzcd} \tag{3.3}$$

where F_0 and F_1 are defined by the exact sequences and are obviously in $\hat{\mathscr{C}}^0$. From this it follows that there is a unique sequence of maps

$$F_0 \to F \to (\_, A_3) \to F_1$$

which makes the above diagram commutative. Further elementary diagram chasing shows that $0 \to F_0 \to F \to (\_, A_3) \to F_1 \to 0$ is exact. Since $A_3 = v(F)$, this exact sequence can be rewritten as $0 \to F_0 \to F \to (\_, v(F)) \to F_1 \to 0$. It is not difficult to see that: a) the map $F \to (\_, v(F))$ is functorial in F; b) F_0 and F_1 are functorial in F; c) the exact sequence $0 \to F_0 \to F \to (\_, v(F)) \to F_1 \to 0$ is functorial in F.

Let G be a left exact functor in $\hat{\mathscr{C}}$. Since F_i is in $\hat{\mathscr{C}}_0$ for $i = 0$ and 1, we know by proposition 3.2, that $\operatorname{Ext}^j(F_i, G) = 0$ for $j = 0$ and 1 and $i = 0$ and 1. From this it follows by easy direct computations that the map $((\_, v(F)), G) \to (F, G)$ is an isomorphism. Thus given any map $F \to G$ there exists one and only one map $(\_, v(F)) \to G$ which makes the diagram

$$\begin{array}{ccc} F & \to & (\_, v(F)) \\ \| & & \downarrow \\ F & \to & G \end{array}$$

commutative.

As an application of the above observations, we obtain:

Proposition 3.4. *Let F be in $\hat{\mathscr{C}}$ and let $0 \to F' \to F \to G \to F'' \to 0$ be an exact sequence in $\hat{\mathscr{C}}$ with G left exact and F' and F'' in $\hat{\mathscr{C}}_0$. Then there exists one and only one map of exact sequences*

$$\begin{array}{ccccccccc} 0 \to & F_0 & \to & F & \to & (\_, v(F)) & \to & F_1 & \to 0 \\ & \downarrow & & \| & & \downarrow & & \downarrow & \\ 0 \to & F' & \to & F & \to & G & \to & F'' & \to 0 \end{array}$$

and this is an isomorphism.

Proof. The existence and uniqueness of the map has already been shown. A similar argument shows that since F' and F'' are in $\hat{\mathscr{C}}_0$, then the induced map $(G, G') \to (F, G')$ is an isomorphism whenever G' is left exact. Thus we obtain that there is one and only one map of exact sequences

$$\begin{array}{ccccccccc} 0 \to & F' & \to & F & \to & G & \to & F'' & \to 0 \\ & \downarrow & & \| & & \downarrow & & \downarrow & \\ 0 \to & F_0 & \to & F & \to & (\_, v(F)) & \to & F_1 & \to 0 \end{array}$$

and this map is easily shown to be the inverse af the previous map.

We now briefly recall the definition of the category $\hat{\mathscr{C}}/\hat{\mathscr{C}}_0$, the quotient category of $\hat{\mathscr{C}}$ by $\hat{\mathscr{C}}_0$ (see [*1*] for details). The objects of $\hat{\mathscr{C}}/\hat{\mathscr{C}}_0$ are the same as the objects in $\hat{\mathscr{C}}$. Given F and G in $\hat{\mathscr{C}}$ we define $\hat{\mathscr{C}}/\hat{\mathscr{C}}_0(F, G) =$

$\varinjlim_{F',G'} \hat{\mathscr{C}}(F', G/G')$ when F' and G' run through all subobjects of F and G respectively such that F/F' and G' are in $\hat{\mathscr{C}}_0$. Then it is well known that $\hat{\mathscr{C}}/\hat{\mathscr{C}}_0$ is an abelian category and that the canonical functor $\hat{\mathscr{C}} \to \hat{\mathscr{C}}/\hat{\mathscr{C}}_0$ given by $F \to F$ is exact. Since $v(\hat{\mathscr{C}}_0) = 0$, it follows from general facts that there is one and only one functor $v': \hat{\mathscr{C}}/\hat{\mathscr{C}}_0 \to \mathscr{C}$ such that $\hat{\mathscr{C}} \to \hat{\mathscr{C}}/\hat{\mathscr{C}}_0 \to \hat{\mathscr{C}}$ is v and that v' is exact. From the definition of v' (see [*1*, page 369]) and the above discussion we see that $v': \hat{\mathscr{C}}/\hat{\mathscr{C}}_0\,(F, G) \to \mathscr{C}(v'(F), v'(G))$ is an isomorphism for all F and G in $\hat{\mathscr{C}}/\hat{\mathscr{C}}_0$. Thus the category $\hat{\mathscr{C}}/\hat{\mathscr{C}}_0$ is equivalent to $\mathscr{C}$.

It should be observed that the category $\hat{\mathscr{C}}_0$ can be described without reference to the category $\mathscr{C}$ or the functor $v: \hat{\mathscr{C}} \to \mathscr{C}$. For we have seen that F is in $\hat{\mathscr{C}}_0$ if and only if $(F, (\_, A)) = 0$ for all A in $\mathscr{C}$. But since the projective objects in $\hat{\mathscr{C}}$ are precisely those isomorphic to $(\_, A)$ for some A in $\mathscr{C}$ (i.e. they are the representable functors), we have that F is in $\hat{\mathscr{C}}_0$ if and only if $(F, G) = 0$ for all projective objects G in $\hat{\mathscr{C}}$. Thus if $\hat{\mathscr{C}}$ is equivalent to $\hat{\mathscr{D}}$, it then follows that $\hat{\mathscr{C}}_0$ is equivalent to $\hat{\mathscr{D}}_0$ under this equivalence and thus we get an induced equivalence of $\hat{\mathscr{C}}/\hat{\mathscr{C}}_0$ with $\hat{\mathscr{D}}/\hat{\mathscr{D}}_0$. Since $\mathscr{C}$ is equivalent to $\hat{\mathscr{C}}/\hat{\mathscr{C}}_0$ and $\mathscr{D}$ is equivalent to $\hat{\mathscr{D}}/\hat{\mathscr{D}}_0$, it follows that $\mathscr{C}$ is equivalent to $\mathscr{D}$. Thus, up to equivalence the category $\hat{\mathscr{C}}$ determines the category $\mathscr{C}$.

Finally we end this section by observing that the gl. dim $\hat{\mathscr{C}} = 0$ or 2 or in other words, the gl. dim $\hat{\mathscr{C}} \neq 1$. For suppose the gl.dim $\hat{\mathscr{C}} \leqq 1$. Let $0 \to A' \to A \to A'' \to 0$ be an exact sequence in $\mathscr{C}$ and consider the exact sequence $0 \to (\_, A') \to (\_, A) \to (\_, A'') \to F \to 0$. Now by Lemma 3.1 we know that $\mathrm{Ext}^2(F, (\_, A')) = \mathrm{Coker}((A, A') \to (A', A'))$. Since we are assuming that the gl.dim $\hat{\mathscr{C}} \leqq 1$, we know that

$$\mathrm{Ext}^2\,(F, (\_, A')) = 0 \quad \text{or that} \quad (A, A') \to (A', A') \to 0$$

is exact. Thus the sequence $0 \to A' \to A \to A'' \to 0$ splits. Since this is true for any exact sequence, it follows that if the gl.dim $\hat{\mathscr{C}} \leqq 1$, then all exact sequences in $\mathscr{C}$ split, i.e. the gl.dim $\mathscr{C} = 0$. But from this it follows trivially that gl.dim $\hat{\mathscr{C}} = 0$ also, concluding our proof. Thus the gl.dim $\hat{\mathscr{C}} = 0$ if and only if gl.dim $\mathscr{C} = 0$. Further if the gl.dim $\mathscr{C} = 0$, then $\hat{\mathscr{C}}_0 = 0$ or $\hat{\mathscr{C}}$ is equivalent to $\mathscr{C}$.

Of course a similar discussion can be carried through for the category $\check{\mathscr{C}}$ and it is left as an excercise to the reader to actually do so.

4. Half-Exact Functors

As usual, we shall assume that $\mathscr{C}$ is an abelian $\mathfrak{U}$-category. In this section we shall be concerned with classifying the half exact coherent functors. Since most of the applications we have in mind are for covariant functors, we shall be concerned mainly with the category $\check{\mathscr{C}}$. For the sake of convenience we shall make the blanket assumption that $\mathscr{C}$ has enough projective objects, although there are some results which do not need this assumption.

We start by investigating the structure of the half exact functors in $\check{\mathscr{C}}_0$.

Lemma 4.1. *All functors which are isomorphic to* $\mathrm{Ext}^1(C, \_)$ *for some* C *in* $\mathscr{C}$ *are in* $\check{\mathscr{C}}_0$. *Also if* G *is in* $\check{\mathscr{C}}_0$, *then there is a monomorphism* $G \to \mathrm{Ext}^1(C, \_)$ *for some* C *in* $\mathscr{C}$.

Proof. Let $0 \to K \to P \to C \to 0$ be an exact sequence in $\mathscr{C}$ with P a projective object in $\mathscr{C}$. Then we have the exact sequence

$$0 \to (C, \_) \to (P, C) \to (K, \_) \to \mathrm{Ext}^1(C, \_) \to 0\,,$$

which shows that $\mathrm{Ext}^1(C, \_)$ is in $\check{\mathscr{C}}_0$ (see proposition 3.2).

Since G in $\check{C}_0$, we know there is an exact sequence

$$0 \to C_0 \to C_1 \to C_2 \to 0$$

in $\mathscr{C}$ such that

$$0 \to (C_2, \_) \to (C_1, \_) \to (C_0, \_) \to G \to 0$$

is exact. But we also have the exact sequence

$$0 \to (C_2, \_) \to (C_1, \_) \to (C_0, \_) \to \mathrm{Ext}^1(C_2, \_) \to \cdots,$$

which shows that there is a monomorphism $G \to \mathrm{Ext}^1(C_2, \_)$.

Lemma 4.2. *Let* F *be in* $\check{\mathscr{C}}$. *Then the following statements are equivalent:*

a) F *is half exact;*

b) $\mathrm{Ext}^1(G, F) = 0$ *for all* G *in* $\check{\mathscr{C}}_0$. *Thus if* F *is half exact, then the functor* $(\_, F)\colon \check{\mathscr{C}}_0 \to \mathscr{A}b$ *given by* $G \mapsto (G, F)$ *is exact.*

Proof. a) $\Leftrightarrow$ b). Let C be the exact sequence $0 \to C_0 \to C_1 \to C_2 \to 0$ and let $0 \to (C_2, \_) \to (C_1, \_) \to (C_0, \_) \to G \to 0$ be exact. Then by lemma 3.1, we know that $\mathrm{Ext}^1(G, F) = H_1(F(C))$. From this it follows that $\mathrm{Ext}^1(G, F) = 0$ if and only if $F(C_0) \to F(C_1) \to F(C_2)$ is exact. Since as C runs through all possible short exact sequences in $\mathscr{C}$ we get all objects G in $\hat{\mathscr{C}}_0$ and the other way around, we have that a) $\Leftrightarrow$ b).

The last assertion follows trivially from the first.

Note. The fact that $\mathscr{C}$ has sufficiently many projectives was not used in this lemma.

As an immediate consequence of lemmas 4.1 and 4.2 we have

Proposition 4.3. *Let F be in $\check{\mathscr{C}}_0$. Then the following statements are equivalent:*

a) F is half exact;

b) $\operatorname{Ext}^1(G, F) = 0$ for all G in $\check{\mathscr{C}}_0$;

c) F is injective in $\check{\mathscr{C}}_0$;

d) F is a direct summand of $\operatorname{Ext}^1(C, \_)$ for some C in $\mathscr{C}$.

Thus we see that the problem of finding the half exact functors in $\check{\mathscr{C}}_0$ is the same as finding the direct summands of $\operatorname{Ext}^1(C, \_)$ for all C in $\mathscr{C}$. It seems reasonable to conjecture that if G is a direct summand of $\operatorname{Ext}^1(C, \_)$ for some C in $\mathscr{C}$, then $G \approx \operatorname{Ext}^1(C', \_)$ for some C' in $\mathscr{C}$.

To the best knowledge of the author, this is still an open question. We briefly summarize what is known at the present time.

It is a well known result of Hilton and Eckmann that the natural map

$$(C, D) \to (\operatorname{Ext}^1(D, \_), \operatorname{Ext}^1(C, \_))$$

induces an isomorphism

$$\underline{\Pi}(C, D) \approx (\operatorname{Ext}^1(D, \_), \operatorname{Ext}^1(C, \_)).$$

Thus we have that

$$\underline{\Pi}(C, C) \approx (\operatorname{Ext}^1(C, \_), \operatorname{Ext}^1(C, \_)).$$

If $(C, P) = 0$ for all projective objects P in $\mathscr{C}$, then $\underline{\Pi}(C, C) = (C, C)$ and thus the direct summands of $\operatorname{Ext}^1(C, \_)$ are given by $\operatorname{Ext}^1(D, \_)$ where D ranges over all direct summands of C and the conjecture holds in this case.

P. Freyd has shown that if $\mathscr{C}$ is closed under denumerable direct sums, then the conjecture is true for arbitrary C in $\mathscr{C}$. The reader is referred to his article in this publication for the proof.

We present now an independent proof in the case $\mathscr{C}$ has finite projective dimension, which does not require the existence of denumerable direct sums.

Lemma 4.4. *Let C be an arbitrary object in $\mathscr{C}$ and suppose $\operatorname{Ext}^1(C, \_) \approx G_1 + G_2$ (direct sum). If $G_1 \approx \operatorname{Ext}^1(D_1, \_)$ for some D_1 in $\mathscr{C}$, then $G_2 \approx \operatorname{Ext}^1(D_2, \_)$ for some D_2 in $\mathscr{C}$.*

Proof. This is essentially a result of Hilton and Rees [*3*, Theorem 2.4]. They show that since we have a monomorphism $0 \to \operatorname{Ext}^1(D, \_) \to \operatorname{Ext}^1(C, \_)$, then there is a projective object P in $\mathscr{C}$ and an epimorphism $C + P \xrightarrow{f} D_1 \to 0$ which splits such that the induced map $\operatorname{Ext}^1(D_1, \_) \to \operatorname{Ext}^1(C + P, \_) = \operatorname{Ext}^1(C, \_)$ is the original imbedding of G_1 in $\operatorname{Ext}^1(C, \_)$. If we let $D_2 = \operatorname{Ker} f$, then it follows that $G_2 \approx \operatorname{Ext}^1(D, \_)$.

Proposition 4.5. *Let* $0 \to P_1 \to P_0 \to A \to 0$ *be an exact sequence in* $\mathscr{C}$ *with* P_1 *a projective object in* $\mathscr{C}$. *Also suppose that* $(P_0, \_) \to (P_1, \_) \to G \to 0$ *is exact. Further suppose that we are given an exact sequence*

$$G \to \operatorname{Ext}^1(B, \_) \to G' \to 0.$$

Then there is a D *in* $\mathscr{C}$ *such that* $\operatorname{Ext}^1(D, \_) \approx G'$ *and* $\operatorname{Ext}^i(D, \_) \approx \approx \operatorname{Ext}^i(B, \_)$ *for* $i > 1$.

Proof. Let $0 \to Q_1 \to Q_0 \to B \to 0$ be exact with Q_0 projective. Then the map $G \to \operatorname{Ext}^1(B, \_)$ can be extended to a map of the exact sequences

$$\begin{array}{ccccccc} (P_0, \_) & \to & (P_1, \_) & \to & G & \to & 0 \\ \downarrow & & \downarrow & & \downarrow & & \\ (Q_0, \_) & \to & (Q_1, \_) & \to & \operatorname{Ext}^1(B, \_) & \to & 0 \end{array}$$

Applying the mapping cone we obtain an exact sequence

$$(Q_0, \_) + (P_1, \_) \to (Q_1, \_) \to G' \to 0.$$

Thus we have an exact sequence $Q_1 \to Q_0 + P_1 \to D \to 0$. Since G' is in $\hat{\mathscr{C}}_0$, it follows that $0 \to Q_1 \to Q_0 + P_1 \to D \to 0$ is exact (see proposition 3.2). Since Q_0 and P_1 are projective, we have that $G' \approx \operatorname{Ext}'(D, \_)$. Since $\operatorname{Ext}^{i+1}(B, \_) \approx \operatorname{Ext}^i(Q_1, \_) \approx \operatorname{Ext}^{i+1}(D, \_)$ for all $i \geqq 1$, we have the last assertion.

As an immediate consequence we have:

Corollary 4.6. *Let* A *and* B *be in* $\mathscr{C}$ *with the* pd $A \leqq 1$. *If* $\operatorname{Ext}^1(A, \_) \to \to \operatorname{Ext}^1(B, \_) \to G \to 0$ *is exact, then there is a* D *in* $\mathscr{C}$ *such that* $G \approx \approx \operatorname{Ext}^1(D, \_)$ *and* $\operatorname{Ext}^i(B, \_) \approx \operatorname{Ext}^i(D, \_)$ *for* $i > 1$.

We now combine the above remarks to obtain our desired result.

Proposition 4.7. *Let* C *be in* $\mathscr{C}$ *with the* pd $C < \infty$. *If* G *is a direct summand of* $\operatorname{Ext}^1(C, \_)$, *then there is a* D *in* $\mathscr{C}$ *such that* $G \approx \operatorname{Ext}'(D, \_)$.

Proof. By induction on $n =$ pd C. Suppose $n = 1$. Then there is an exact sequence $0 \to \operatorname{Ext}^1(C, \_) \to \operatorname{Ext}^1(C, \_) \to G \to 0$. Thus $G \approx \approx \operatorname{Ext}^1(D, \_)$ for some D in $\mathscr{C}$ by corollary 4.6.

Suppose true for $n = k \geqq 1$ and let $n = k + 1$. Since G is a direct summand of $\operatorname{Ext}^1(C, \_)$ we know that G is in $\hat{\mathscr{C}}_0$. Let $0 \to X \to Y \to Z \to 0$ be an exact sequence in $\mathscr{C}$ such that $0 \to (Z, \_) \to (Y, \_) \to (X, \_) \to G \to 0$ is exact and let $G \to \operatorname{Ext}^1(C, \_)$ be the imbedding of G as a direct summand. Thus we obtain the map $(X, \_) \to \operatorname{Ext}^1(C, \_)$. Let $0 \to X \to E \to C$ be the image of the identity $X \to X$ in $\operatorname{Ext}^1(C, X)$.

Then we obtain a commutative diagram with exact rows and columns

$$\begin{array}{ccccccccc} & & & & 0 & & & & \\ & & & & \downarrow & & & & \\ (Y, \_) & \to & (X, \_) & \to & G & \to & 0 & & \\ \downarrow & & \| & & \downarrow & & & & \\ (E, \_) & \to & (X, \_) & \to & \operatorname{Ext}^1(C, \_) & \to & \operatorname{Ext}^1(E, \_) & \to & \cdots \end{array}$$

from which it follows that

$$0 \to G \to \operatorname{Ext}^1(C, \_) \to \operatorname{Ext}^1(E, \_) \to \operatorname{Ext}^1(X, \_) \to \operatorname{Ext}^2(C, \_)$$

is exact. Since G is a direct summand of $\operatorname{Ext}^1(C, \_)$ it is half exact and thus injective in $\check{\mathscr{C}}_0$ (see proposition 4.3). Since each of the objects appearing in this exact sequence in $\check{\mathscr{C}}_0$ are half exact and thus injective in $\check{\mathscr{C}}_0$, it follows that the $\operatorname{Im}(\operatorname{Ext}^1(X, \_) \to \operatorname{Ext}^2(C, \_))$ is a direct summand of $\operatorname{Ext}^2(C, \_)$. But $\operatorname{Ext}^2(C, \_) \approx \operatorname{Ext}^1(C', \_)$ for some C' in $\mathscr{C}$ with $\operatorname{pd} C' = k$. Thus by the inductive hypothesis we have that

$$\operatorname{Ext}^1(E, \_) \to \operatorname{Ext}^1(X, \_) \to \operatorname{Ext}^1(X', \_) \to 0$$

is exact for some X' in $\mathscr{C}$. But then by lemma 4.4, we have that the $\operatorname{Ker}(\operatorname{Ext}^1(X, \_) \to \operatorname{Ext}^1(X', \_))$ is isomorphic to $\operatorname{Ext}^1(E', \_)$ for some E', since

$$\operatorname{Ext}^1(X, \_) \approx \operatorname{Ext}^1(X', \_) + \operatorname{Ker}(\operatorname{Ext}^1(X, \_) \to \operatorname{Ext}^1(X', \_)).$$

Proceeding in this way we conclude that $G \approx \operatorname{Ext}^1(D, \_)$ for some D in $\mathscr{C}$.

We summarize our remarks in

Theorem. 4.8. *Let $\mathscr{C}$ be an abelian $\mathfrak{U}$-category which either has denumerable sums or else each object in $\mathscr{C}$ has finite projective dimension. For a G in $\check{\mathscr{C}}$, the following statements are equivalent:*

a) $G \approx \operatorname{Ext}^1(D, \_)$ *for some D in $\mathscr{C}$.*

b) G is in $\check{\mathscr{C}}_0$ and is half exact.

c) G is in $\check{\mathscr{C}}_0$ and is injective in $\check{\mathscr{C}}_0$.

For the rest of this section we will assume that $\mathscr{C}$ not only has enough projectives, but also statisfies the conclusions of Theorem 4.8. In view of Theorem 4.8 we see that this is not a serious restriction.

We now turn our attention to arbitrary half exact functors in $\check{\mathscr{C}}$. If F is an arbitrary object in $\check{\mathscr{C}}$ not necessarily half exact, we have the exact sequence $0 \to F_0 \to F \to (w(F), \_) \to F_1 \to 0$ where $w : (\check{\mathscr{C}})^0 \to \mathscr{C}$ is the analogue of the functor $v : \hat{\mathscr{C}} \to \mathscr{C}$. From the results of § 3, it follows that $(w(F), \_) \approx R^0 F$, the 0-th right derived functor of F and that $F \to (w(F), \_)$ is the usual functor from F to $R^0 F$.

If we now assume that F is half exact, then it easily follows from the exact sequence $0 \to F_0 \to F \to (w(F), \_) \to F_1 \to 0$ that F_0 is also half exact and thus of the form $\operatorname{Ext}^1(B, \_)$ for some B in $\mathscr{C}$. Thus given a half exact functor F, associated with it is an exact sequence

$$0 \to \operatorname{Ext}^1(B, \_) \to F \to (w(F), \_) \to F_1 \to 0$$

with the F_1 in $\hat{\mathscr{C}}_0$. Therefore, if given two objects A and B in $\mathscr{C}$, we can classify the half exact functors F in $\hat{\mathscr{C}}$ such that there is an exact sequence $0 \to \operatorname{Ext}^1(B, \_) \to F \to (\mathrm{A}, \_) \to F_1 \to 0$ with F_1 in $\check{\mathscr{C}}_0$, we will in effect have given a complete classification of all half exact functors F in $\check{\mathscr{C}}$.

Let $0 \to \operatorname{Ext}^1(B, \_) \to F \to (A, \_) \to F_1 \to 0$ and $0 \to \operatorname{Ext}^1(B, \_) \to$ $\to G \to (A, \_) \to G_1 \to 0$ be exact with the F_1 and G_1 in $\check{\mathscr{C}}_0$. By a map from the first sequence to the second, we mean a map $F \to G$ which gives a commutative diagram

$$\begin{array}{ccccccccc} 0 \to & \operatorname{Ext}^1(B, \_) & \to & F & \to & (A, \_) & \to & F_1 \to 0 \\ & \| & & \downarrow & & \| & & \\ 0 \to & \operatorname{Ext}^1(B, \_) & \to & G & \to & (A, \_) & \to & G_1 \to 0. \end{array}$$

Given this commutative diagram it is trivial to verify: a) there is a unique map $F_1 \to G_1$ which makes the diagram commutative and this map is an epimorphism; b) $0 \to F \to G$ is exact; c) there is a map $G \to F_1$ which gives an exact sequence $0 \to F \to G \to F_1 \to G_1 \to 0$.

We shall say that the two sequences $0 \to \operatorname{Ext}^1 (B, \_) \to F \to (A, \_)$ $\to F_1 \to 0$ and $0 \to \operatorname{Ext}^1(B, \_) \to G \to (A, \_) \to G_1 \to 0$ are isomorphic if there is a map of sequences which is an isomorphism on F to G and thus on the whole sequence. Our primary interest will be in the isomorphism classes of these exact sequences.

Our first step in the classification of half exact functors is to characterize those exact sequences $0 \to \operatorname{Ext}^1(B, \_) \to F \to (A, \_) \to F_1 \to 0$ such that the F is a half exact functor. Before doing this we need the following technically important lemma.

Lemma 4.9. *Suppose we are given an exact sequence* $0 \to \operatorname{Ext}^1(B, \_) \to$ $\to F \to (A, \_) \to F_1 \to 0$ *(with* F_1 *in* $\check{\mathscr{C}}_0$ *of course). Then*

a) There is an exact sequence $0 \to A \to X \to Y \to B \to 0$ *in* $\mathscr{C}$ *and exact sequences* $0 \to (B, \_) \to (Y, \_) \to (X, \_) \to F \to 0$ *and* $0 \to (B, \_) \to$ $\to (Y, \_) \to (Z, \_) \to \operatorname{Ext}^1(B, \_) \to 0$, *where* $Z = \operatorname{Im}(X \to Y)$, *with the property that our given maps* $0 \to \operatorname{Ext}^1(B, \_) \to F \to (A, \_)$ *are the unique* *maps which complete the following commutative diagram*

$$\begin{array}{ccccccccc}
 & & & & 0 & & & & \\
 & & & & \downarrow & & & & \\
0 \to (B, _) & \to & (Y, _) & \to & (Z, _) & \to & \mathrm{Ext}^1(B, _) & \to & 0 \\
\| & & \| & & \downarrow & & & & \\
0 \to (B, _) & \to & (Y, _) & \to & (X, _) & \to & F & \to & 0 \\
 & & & & \downarrow & & & & \\
 & & & & (A, _) & = & (A, _) & & \\
 & & & & \downarrow & & & & \\
 & & & & 0 & & & &
\end{array}$$

b) If the Y in part a) is projective then F is half exact.

c) There exists a map of exact sequences

$$\begin{array}{ccccccccc}
0 \to \mathrm{Ext}^1(B, _) & \to & F & \to & (A, _) & \to & F_1 & \to & 0 \\
\| & & \downarrow & & \| & & \downarrow & & \\
0 \to \mathrm{Ext}^1(B, _) & \to & G & \to & (A, _) & \to & G_1 & \to & 0
\end{array}$$

with G half exact.

Proof. a) Since $w(F) \approx A$ we know we can find an exact sequence $0 \to A \to X \to V \to D \to 0$ such that $(V, \_) \to (X, \_) \to F \to 0$ is exact. Since the $\mathrm{Im}(\mathrm{Ext}^1(B, \_) \to F) = F_0$ an argument similar to that used in § 3 for deriving (3.3) shows that we have an exact sequence

$$0 \to (D, \_) \to (V, \_) \to (V', \_) \to \mathrm{Ext}^1(B, \_) \to 0$$

where $V' = \mathrm{Im}(X \to V)$, which gives us a commutative diagram

$$(*) \quad \begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
0 \to (D, _) & \to & (V, _) & \to & (V', _) & \to & \mathrm{Ext}^1(B, _) & \to & 0 \\
\| & & \| & & \downarrow & & \downarrow & & \\
0 \to (D, _) & \to & (V, _) & \to & (X, _) & \to & F & \to & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & (A, _) & = & (A, _). & &
\end{array}$$

Now we also have the commutative diagram

$$\begin{array}{ccccccccc}
 & & & & & & 0 & & \\
 & & & & & & \downarrow & & \\
0 \to (D, _) & \to & (U, _) & \to & (V', _) & \to & \mathrm{Ext}^1(B, _) & \to & 0 \\
\| & & \| & & \| & & \downarrow & & \\
0 \to (D, _) & \to & (U, _) & \to & (V', _) & \to & \mathrm{Ext}^1(D, _) & \to & \cdots .
\end{array}$$

From this it follows that there is a commutative diagram

$$\begin{array}{ccccccc}
0 \to V' & \to & V & \to & D & \to & 0 \\
\| & & \downarrow & & \downarrow & & \\
0 \to V' & \to & Y & \to & B & \to & 0
\end{array}$$

(with exact rows and columns), such that the induced map $\mathrm{Ext}^1(B, \_) \to \to \mathrm{Ext}^1(D, \_)$ agrees with our original one. Thus we see that we have a commutative diagram

$$\begin{array}{ccccccccc} 0 \to & (B,\_) & \to & (Y,\_) & \to & (V',\_) & \to & \mathrm{Ext}^1(B,\_) & \to 0 \\ & \downarrow & & \downarrow & & \| & & \| & \\ 0 \to & (D,\_) & \to & (V,\_) & \to & (V',\_) & \to & \mathrm{Ext}^1(B,\_) & \to 0 \end{array}$$

with exact rows and columns. It is now a straight forward matter to check from (*) that the exact sequence $0 \to A \to X \to Y \to B \to 0$ together with the maps $(V', \_) \to \mathrm{Ext}^1(B, \_)$ and $(X, \_) \to F$ given in (*) have our desired properties.

b) Follows trivially by diagram chasing from the exact sequence $(Y, \_) \to (X, \_) \to F \to 0$.

c) Let $0 \to A \to X \to Y \to B \to 0$ be an exact sequence having the properties described in a). Let $P \to Y \to 0$ be exact with P projective. Then it is well known that we have a commutative diagram

$$\begin{array}{ccccccccc} 0 \to & A & \to & S & \to & P & \to & B & \to 0 \\ & \| & & \downarrow & & \downarrow & & \| & \\ 0 \to & A & \to & X & \to & Y & \to & B & \to 0 \end{array}$$

with exact rows and columns. Thus we have the commutative diagrams

$$\begin{array}{ccccccccc} 0 \to & (B,\_) & \to & (Y,\_) & \to & (Y',\_) & \to & \mathrm{Ext}^1(B,\_) & \to 0 \\ & \| & & \downarrow & & \downarrow & & \| & \\ 0 \to & (B,\_) & \to & (P,\_) & \to & (P',\_) & \to & \mathrm{Ext}^1(B,\_) & \to 0 \end{array} \tag{1}$$

(where $P' = \mathrm{Im}(S \to P)$ and $Y' = \mathrm{Im}(X \to Y)$);

$$\begin{array}{ccccccccc} 0 \to & (B,\_) & \to & (Y,\_) & \to & (X,\_) & \to & F & \to 0 \\ & \| & & \downarrow & & \downarrow & & \downarrow & \\ 0 \to & (B,\_) & \to & (P,\_) & \to & (S,\_) & \to & G & \to 0. \end{array} \tag{2}$$

From this it easily follows that we have an exact sequence

$$0 \to \mathrm{Ext}^1(B, \_) \to G \to (A, \_) \to G_1 \to 0$$

with G_1 in $\check{\mathscr{C}}_0$ such that the map $F \to G$ described in (2) gives a map

$$\begin{array}{ccccccccc} 0 \to & \mathrm{Ext}^1(B,\_) & \to & F & \to & (A,\_) & \to & F_1 & \to 0 \\ & \| & & \downarrow & & \| & & \downarrow & \\ 0 \to & \mathrm{Ext}^1(B,\_) & \to & G & \to & (A,\_) & \to & G_1 & \to 0. \end{array}$$

Since P is projective we know by part b) that G is half exact, which finishes the proof of c).

Proposition 4.10. *Let* $0 \to \mathrm{Ext}^1(B, \_) \to F \to (A, \_) \to F_1 \to 0$ *exact (with* F_1 *in* $\check{\mathscr{C}}_0$*). Then the following statements are equivalent:*

a) F is half exact;

b) any map

$$\begin{array}{ccccccccc} 0 \to & \mathrm{Ext}^1(B, \_) & \to & F & \to & (A, \_) & \to & F_1 & \to 0 \\ & \| & & \downarrow & & \| & & \downarrow & \\ 0 \to & \mathrm{Ext}^1(B, \_) & \to & G & \to & (A, \_) & \to & G_1 & \to 0 \end{array}$$

is an isomorphism;

c) there exists an exact sequence $0 \to A \to X \to Y \to B \to 0$ *in* $\mathscr{C}$ *with Y projective which satisfies the properties of part a) of lemma 4.9.*

Proof. a) $\Rightarrow$ b). If we have such a map then we have an exact sequence $0 \to F \to G \to F_1 \to G_1 \to 0$ by our initial remarks concerning maps of sequences. Thus we have an exact sequence $0 \to F \to G \to H \to 0$ with H in $\check{\mathscr{C}}_0$. Since F is half exact, we know that $\mathrm{Ext}^1(H, F) = 0$ (see proposition 4.3). Thus the sequence $0 \to F \to G \to H \to 0$ splits, i.e. $G = F + H'$ with H' in $\check{\mathscr{C}}_0$. But then the

$$\mathrm{Ker}(G \to (A, \_)) \supset \mathrm{Im}(\mathrm{Ext}^1(B, \_) \to G) + H'.$$

Since the

$$\mathrm{Ker}(G \to (A, \_)) = \mathrm{Im}(\mathrm{Ext}^1(B, \_) \to G),$$

we have that $H' = 0$. Thus H is zero or $F \to G$ is an isomorphism.

b) $\Rightarrow$ c). The same construction used in proving part c) of lemma 4.9 shows that b) $\Rightarrow$ c).

c) $\Rightarrow$ a). Follows trivially from part b) of lemma 4.9.

Our classification of half exact functors will consist of showing that there is a natural one to one correspondence between the isomorphism classes of exact sequences $0 \to \mathrm{Ext}^1(B, \_) \to F \to (A, \_) \to F_1 \to 0$ (with F_1 in $\check{\mathscr{C}}_0$) and the elements of $\mathrm{Ext}^2(B, A)$. To achieve this, we need

Proposition 4.11. *Let* $0 \to G \to (A, \_) \to G_1 \to 0$ *be exact with* G_1 *in* $\check{\mathscr{C}}_0$. *Then we have an exact sequence*

$$0 \to \mathrm{Ext}^1(G, \mathrm{Ext}^i(B, \_)) \to \mathrm{Ext}^{i+1}(B, A) \quad \textit{for all} \quad i \geqq 0.$$

Proof. We first define the map $\mathrm{Ext}^1(G, \mathrm{Ext}^i(B, \_)) \to \mathrm{Ext}^{i+1}(B, A)$. We know that there is an exact sequence $C_0 \to C_1 \to 0$ in $\mathscr{C}$ such that $0 \to (C_1, \_) \to (C_0, \_) \to G \to 0$ is exact. Thus we have the exact sequence $0 \to (C_1, \_) \to (C_0, \_) \to (A, \_) \to G_1 \to 0$. Since G_1 is in $\check{\mathscr{C}}_0$, we know that the associated sequence $0 \to A \to C_0 \to C_1 \to 0$ is exact (see proposition 3.2). Thus we have the exact sequence

$$\mathrm{Ext}^i(B, C_0) \to \mathrm{Ext}^i(B, C_1) \to \mathrm{Ext}^{i+1}(B, A).$$

But by lemma 3.1 we know that

$$\mathrm{Ext}^1(B, C_0) \to \mathrm{Ext}^1(B, C_1) \to \mathrm{Ext}^1(G, \mathrm{Ext}^i(B, \_)) \to 0$$

is exact. Thus we have our desired monomorphism

$$\mathrm{Ext}^1(G, \mathrm{Ext}^i(B, \_)) \to \mathrm{Ext}^{i+1}(B, A).$$

It is easily seen that this map does not depend on the particular resolution of G which was chosen.

Now suppose that given a map $F \to (A, \_)$ we denote by $\bar{F}$ the $\mathrm{Im}(F \to (A, \_))$. Now if $0 \to \mathrm{Ext}^1(B, \_) \to F \to (A, \_) \to F_1 \to 0$ is exact (with F_1 in $\check{\mathscr{C}}_0$), then $0 \to \mathrm{Ext}^1(B, \_) \to F \to \bar{F} \to 0$ is an element of $\mathrm{Ext}^1(\bar{F}, \mathrm{Ext}^1(B, \_))$ and this element depends only on the isomorphism class of $0 \to \mathrm{Ext}^1(B, \_) \to F \to (A, \_) \to F_1 \to 0$. Combining this map with the map of $\mathrm{Ext}^1(\bar{F}, \mathrm{Ext}^1(B, \_)) \to \mathrm{Ext}^2(B, A)$ given in proposition 4.11 we obtain a map of the isomorphism classes of exact sequences to the elements of $\mathrm{Ext}^2(B, A)$. Thus if we denote by $H(B, A)$ the set of isomorphism classes of exact sequences $0 \to \mathrm{Ext}^1(B, \_) \to \to F \to (A, \_) \to F_1 \to 0$ with F half exact and F_1 in $\check{\mathscr{C}}_0$, we obtain a map $H(B, A) \to \mathrm{Ext}^2(B, A)$ which we wish to show is one to one and onto.

We now define a map $\mathrm{Ext}^2(B, A) \to H(B, A)$. It is well known and easily seen that any element in $\mathrm{Ext}^2(B, A)$ can be represented by an exact sequence $0 \to A \to X \to Y \to B \to 0$ with Y projective. From the commutative diagram

$$\begin{array}{ccccccccc}
 & & & & & & 0 & & \\
 & & & & & & \downarrow & & \\
0 \to & (B,_) & \to & (Y,_) & \to & (Y',_) & \to & \mathrm{Ext}^1(B,_) & \\
 & \| & & \| & & \downarrow & & & \\
0 \to & (B,_) & \to & (Y,_) & \to & (X,_) & \to & G \to 0 & \\
 & & & & & \downarrow & & & \\
 & & & & & (A,_) & & & \\
 & & & & & \downarrow & & & \\
 & & & & & \mathrm{Ext}^2(B,_) & & &
\end{array} \tag{4.12}$$

with exact rows and columns where $Y' = \mathrm{Im}(X \to Y)$. From this, as usual, we deduce the unique exact sequence

$$0 \to \mathrm{Ext}^1(B, \_) \to G \to (A, \_) \to \mathrm{Ext}^2(B, \_)$$

which makes the above diagram commutative.

If we choose another representative $0 \to A \to X_1 \to Y_1 \to B \to 0$ of the element in $\mathrm{Ext}^2(B, A)$ represented by $0 \to A \to X \to Y \to B \to 0$, then it is not hard to see that we will have a map of the new sequence into the old which must be an isomorphism since all the functors are half exact (see proposition 4.10).

Thus we have obtained a map $\mathrm{Ext}^2(B, A) \to H(B, A)$. It is not difficult to see that the composite maps $H(B, A) \to \mathrm{Ext}^2(B, A) \to \to H(B, A)$ and $\mathrm{Ext}^2(B, A) \to H(B, A) \to \mathrm{Ext}^2(B, A)$ are identity maps which shows that the map $H(B, A) \to \mathrm{Ext}^2(B, A)$ is indeed a one to one correspondence.

The diagram 4.12 shows that if $0 \to \mathrm{Ext}^1(B, \_) \to F \to (A, \_) \to \to F_1 \to 0$ is exact with F half exact then the map $(A, \_) \to \mathrm{Ext}^2(B, \_)$ given by sending the identity of A to the element of $\mathrm{Ext}^2(B, A)$ determined by the original exact sequence gives us an exact sequence

$$0 \to \mathrm{Ext}^1(B, \_) \to F \to (A, \_) \to \mathrm{Ext}^2(B, A).$$

Thus we have established:

Theorem 4.13. *Let $\mathscr{C}$ be an abelian $\mathfrak{U}$-category with enough projectives and such that all half exact functors in $\check{\mathscr{C}}_0$ are of the form $\mathrm{Ext}^1(B, \_)$ for some B in $\mathscr{C}$. Let F be a half exact coherent functor. Then for some B and A in $\mathscr{C}$ we have an exact sequence*

$$0 \to \mathrm{Ext}^1(B, \_) \to F \to (A, \_) \to \mathrm{Ext}^2(B, \_).$$

Further the map which sends this exact sequence to the element of $\mathrm{Ext}^2(B, A)$ determined by the map $(A, \_) \to \mathrm{Ext}^2(B, \_)$ gives a one to one correspondence between $H(B, A)$ and $\mathrm{Ext}^2(B, A)$.

Corollary 4.14. *Suppose $\mathscr{C}$ is an abelian $\mathfrak{U}$-category with enough projectives and with $\mathrm{gl.dim}\,\mathscr{C} \leqq 1$. Then if F is a coherent functor and F_0 is half exact, then $F_0 \approx \mathrm{Ext}^1(B, \_)$ for some B in $\mathscr{C}$ and F_0 is a direct summand of F. Thus $F \approx F_0 + \bar{F}$ where $\bar{F} = \mathrm{Im}(F \to (w(F), \_)$. Further, F is half exact if and only if $F_0 \approx \mathrm{Ext}^1(B, \_)$ for some B and $\bar{F} = = (w(F), \_)$, in which case $F \approx \mathrm{Ext}^1(B, \_) + (w(F), \_)$ for some B in $\mathscr{C}$.*

Proof. We know we have an exact sequence

$$0 \to F_0 \to F \to (w(F), \_) \to F_1 \to 0$$

with F_1 in $\check{\mathscr{C}}_0$. Suppose $F_0 \approx \mathrm{Ext}^1(B, \_)$. Then we know that

$$\mathrm{Ext}^1(\bar{F}, \mathrm{Ext}^1(B, \_)) \subset \mathrm{Ext}^2(B, w(F))$$

by proposition 4.11. Since the $\mathrm{gl.dim}\,\mathscr{C} \leqq 1$, we know that $\mathrm{Ext}^2(B, \_) = 0$ and thus the extension $0 \to F_0 \to F \to \bar{F} \to 0$ splits. Thus the first part of the theorem is established.

The rest is a trivial consequence of theorem 4.13.

5. Complexes

In the last three sections of this paper we give some applications and illustrations of the general theory of coherent functors. In this section we will be concerned with complexes in an abelian $\mathfrak{U}$-category $\mathscr{C}$.

We recall that a covariant functor $F: \mathscr{C} \to \mathscr{A}b$ is right exact if given any exact sequence $0 \to C' \to C \to C'' \to 0$, the sequence $F(C') \to \to F(C) \to F(C'') \to 0$ is exact. In section two we have shown that if $\mathscr{C}$ has enough projectives, then there is a functor $L^0: \check{\mathscr{C}} \to \check{\mathscr{C}}$ and a map $L^0 \to I$ (the identity on $\check{\mathscr{C}}$) such that L^0F is right exact for all F in $\check{\mathscr{C}}$ and $L^0F(P) \to F(P)$ is an isomorphism for all projective objects P in $\mathscr{C}$ (see proposition 2.1 and the discussion following it). Also $L^0F \to F$ is an isomorphism if and only if F is right exact.

Similarly we say that a contravariant functor $G: \mathscr{C} \to \mathscr{A}b$ is right exact if given any exact sequence $0 \to C' \to C \to C'' \to \to 0$ the sequence $G(C'') \to G(C) \to G(C') \to 0$ is exact. By duality, if $\mathscr{C}$ has enough injectives then there is a functor $L^0: \hat{\mathscr{C}} \to \hat{\mathscr{C}}$ and a map $L^0 \to I$ (where I is the identity on $\hat{\mathscr{C}}$), such that L^0G is right exact and $L^0G(Q) \to G(Q)$ is an isomorphism for all injective objetcs Q in $\mathscr{C}$. Also $L^0G \to G$ is an isomorphism if and only if G is right exact.

As an immediate consequence of these definitions and lemma 3.1 we have

Lemma 5.1. *A covariant functor $F: \mathscr{C} \to \mathscr{A}b$ is right exact if and only if the functor $\check{\mathscr{C}} \to \mathscr{A}b$ given by $F' \mapsto (F', F)$ is exact.*

Similarly a contravariant functor $G: \mathscr{C} \to \mathscr{A}b$ is right exact if and only if the functor $\hat{\mathscr{C}} \to \mathscr{A}b$ given by $G' \mapsto (G', G)$ is exact.

We now state and prove the main result of the first part of this section. It can be viewed as being a sort of universal coefficient theorem.

Theorem 5.2. *Let X be a complex in $\mathscr{C}$ and $F: \mathscr{C} \to \mathscr{A}b$ a covariant right exact functor. Then we have functorial isomorphisms $H_i(F(X)) \approx \approx \check{\mathscr{C}}(H_i((X, \_)), F)$ for all i.*

Similarly if $G: \mathscr{C} \to \mathscr{A}b$ is a contravariant right exact functor, then we have functorial isomorphisms $H_i(G(X)) \approx \hat{\mathscr{C}}(H_i((\_, X)), G)$ for all i.

Proof. Since for all j we have that $F(X_j) = ((X, \_), F)$, we have that $H_i(F(X)) \approx H_i(\hat{\mathscr{C}}((X, \_), F)$. But since F is right exact, we know by lemma 5.1 that the functor $(\_, F)$ is exact. Thus $H_i(\hat{\mathscr{C}}((X, \_), F) \approx \approx \hat{\mathscr{C}}(H_i((X, \_), F)$, which gives the desired result. The other half of the theorem is proved similarly.

Suppose now that $\mathscr{C}$ has enough projectives and that X is a projective complex in $\mathscr{C}$. Now let F be an arbitrary covariant functor from $\mathscr{C}$ to $\mathscr{Ab}$. Then we know that the map $L^0F \to F$ has the property that $L^0F(P) \approx F(P)$ for all projectives in $\mathscr{C}$. Thus we have that

$$H_i(F((X)) \approx H_i(L^0F(X)) \approx \check{\mathscr{C}}(H_i((X, \_)), L^0F),$$

the last isomorphism being the one given in theorem 5.2. Thus we obtain

Corollary 5.3. *Suppose $\mathscr{C}$ has enough projectives and X is a projective complex. Let F be in $\check{\mathscr{C}}$. Then we have functorial isomorphisms $H_i(F(X)) \approx \approx \check{\mathscr{C}}(H_i(X, \_), L^0F)$ for all i.*

Similarly, suppose $\mathscr{C}$ has enough injectives and that X is an injective complex. Let G be in $\hat{\mathscr{C}}$. Then we have functorial isomorphisms $H_i(G(X)) \approx \approx \check{\mathscr{C}}(H_i((\_, X)), L^0G)$ for all i.

In particular we have the following well known result of Yoneda.

Corollary 5.4. *Suppose $\mathscr{C}$ has enough projectives and C is an object in $\mathscr{C}$. Let F be in $\check{\mathscr{C}}$. Then we have functorial isomorphisms*

$$(L^iF)(C) \approx \check{\mathscr{C}}(\mathrm{Ext}^i(C, \_), L^0F)$$

for all i.

Similarly, suppose $\mathscr{C}$ has enough injectives and C is an object in $\mathscr{C}$. Let G be in $\hat{\mathscr{C}}$. Then we have functorial isomorphisms

$$(L^iF(C) \approx \hat{\mathscr{C}}(\mathrm{Ext}^i(\_, C), L^0G)$$

for all i.

Another immediate consequence of corollaries 5.3 and 5.4 is

Corollary 5.5. *Suppose $\mathscr{C}$ has enough projectives and A and C are objects in $\mathscr{C}$. Then we have for all $i > 0$, the functorial isomorphisms*

$$\begin{aligned} \underline{\Pi}_{i+1}(C, A) &\approx \check{\mathscr{C}}(\mathrm{Ext}^i(A, \_), L^0(C, \_)) \\ &\approx (L^i(C, \_))(A). \end{aligned}$$

Similarly, suppose $\mathscr{C}$ has enough injectives. Then for all $i > 0$, we have functorial isomorphisms

$$\begin{aligned} \overline{\Pi}_{i+1}(A, C) &\approx \mathscr{C}(\mathrm{Ext}^i(\_, A), L^0(\_, C)) \\ &\approx (L^i(\_, C))(A). \end{aligned}$$

We now give some interpretation for these formulae in the case $\mathscr{C}$ is the category of modules over a ring.

We shall say that a ring R is a ring in $\mathfrak{U}$ if the underlying set of R is an element in $\mathfrak{U}$ and $\mathfrak{U}$ is not the smallest universe containing R as an element. If R is a ring in $\mathfrak{U}$, we shall denote by ${}_R\mathscr{M}$ and $\mathscr{M}_R$ the category of left and right R-modules respectively whose underlying sets are elements of the same universe in $\mathfrak{U}$ containing R. Thus ${}_R\mathscr{M}$ and $\mathscr{M}_R$ are $\mathfrak{U}$-categories if R is a ring in $\mathfrak{U}$ which we shall call the categories of left and right R-modules in $\mathfrak{U}$.

Suppose R is a ring in $\mathfrak{U}$ and ${}_R\mathscr{M}$ and $\mathscr{M}_R$ the categories of left and right R-modules in $\mathfrak{U}$. Suppose C is a finitely presented left R-module. We wish to find $L^0(C, \_)$. Let (C, R) be denoted by C^*. Then we have the usual map $C^* \otimes \_ \to (C, \_)$ where for each B in ${}_R\mathscr{M}$ the map $C^* \otimes B \to (C, B)$ is given by $(f \otimes b)(c) = f(c)b$ for all f in C^*, b in B and c in C. It is well known that $C^* \otimes P \to (C, P)$ is an isomorphism for all finitely generated free R-modules P. Since both $C^* \otimes \_$ and $(C, \_)$ commute with direct limits (remember C is finitely presented), we have that $C^* \otimes P \to (C, P)$ is an isomorphism for all free R-modules P and thus for all projective R-modules P. Since $C^* \otimes \_$ is also right exact, it easily follows that $L^0(C, \_)$ is isomorphic to $C^* \otimes \_$ in such way that we have a commutative diagram

$$\begin{array}{ccc} C^* \otimes \_ & \longrightarrow & (C, \_) \\ \wr\wr & & \| \\ L^0(C, \_) & \to & (C, \_) . \end{array}$$

Now if A is left R-module, we have by corollary 5.5, that

$$\underline{\Pi}_{i+1}(C, A) \approx L^i((C, \_))(A)$$

for all $i > 0$. Since $L^0(C, \_) \approx C^* \otimes \_$, we obtain that

$$\underline{\Pi}_{i+1}(C, A) \approx \operatorname{Tor}_i^R(C^*, A) .$$

Summarizing we have

Proposition 5.6. *Let R be a ring and C is a finitely presented R-module. Then $L^0(C, \_) \approx C^* \otimes \_$ and for each $i > 0$ the functors $\underline{\Pi}_{i+1}(C, \_)$ and $\operatorname{Tor}_i^R(C^*, \_)$ are functorially isomorphic.*

The last application in this direction that we give is the following:

Proposition 5.7. *Let R be a ring, X a complex of finitely generated projective left R-modules. If A is a left R-module, then for all i we have functorial isomorphisms*

$$H_i((X, A)) \approx \check{\mathscr{M}}_R(H_i(\_ \otimes X), \_ \otimes A) .$$

In particular, if B is a left R-module which has a projective resolution consisting of finitely generated R-modules, then for all left mogules A we have functorial isomorphisms

$$\operatorname{Ext}^i(B, A) \approx \check{\mathscr{M}}_R(\operatorname{Tor}_i(\_, B), \_ \otimes A)$$

for all i.

Proof. Since each of the components X_i in X is a finitely generated projective R-module, we know that we have maps $X_i^* \otimes \_ \to (X_i, \_)$ which are isomorphisms for all i. Thus the complexes $(X, \_)$ and $X^* \otimes \_$ are isomorphic. Thus for a particular A, we know that $(X, A) \approx X^* \otimes A$. Since $\_ \otimes A$ is a right exact functor we know by theorem 5.2, that

$$H_i(X^* \otimes A) \approx \check{\mathcal{M}}_R(H_i(X^*, \_), \_ \otimes A)$$

for all i. But the complex $(X^*, \_)$ and $\_ \otimes X$ are isomorphic. Thus we obtain our desired result that

$$H_i(X, A) \approx \check{\mathcal{M}}_R(H_i(\_ \otimes X), \_ \otimes A).$$

The second part of the proposition follows trivially from the first part.

We now return to the situation of an arbitrary abelian $\mathfrak{U}$-category $\mathcal{C}$ and give some results in different directions. These results are given in terms of covariant functors since this is the form we shall use them in the next section. Of course analogous results hold for contravariant functors.

Let X be a complex $\cdots \to X_{i+1} \to X_i \to X_{i-1} \to \cdots$ in $\mathcal{C}$. We shall denote the Coker $(X_{i+1} \to X_i)$ by $Z_i'(X)$ or, more simply, by Z_i'. Thus for each i we have an exact sequence

$$0 \to H_i(X) \to Z_i' \to X_{i-1} \to Z_{i-1}' \to 0.$$

From this we obtain the exact sequence

$$0 \to (Z_{i-1}', \_) \to (X_{i-1}, \_) \to (Z_i', \_) \to H^i(X, \_) \to 0$$

where $H^i(X, \_) = H_i((X, \_))$. Then by the results of section 4 following Theorem 4.8 we have that $w(H^i(X, \_)) = H_i(X)$, that $R^0(H^i(X, \_)) \approx$ $\approx (H_i(X), \_)$ and that the map $H^i(X, \_) \to (H_i(X), \_)$ has the usual properties of the map of a functor into its 0-th right derived functor.

If we assume further that $\mathcal{C}$ has enough projective objects and that X_{i-1} is a projective object, then it follows from the commutative diagram (3.3) and the definition of the functors $H^i(X, \_)_0$ and $H^i(X, \_)_1$, that $H^i(X, \_)_0 = \mathrm{Ext}^1(Z_{i-1}', \_)$ and that $H^i(X, \_)_1$ is a subfunctor of $\mathrm{Ext}^2(Z_{i-1}', \_)$. Further, if X_i is projective and $X_{i+1} = 0$, then we have that $Z_i' = X_i$ and thus that $H^i(X, \_)_1 = \mathrm{Ext}^2(Z_{i-1}', \_)$. Summarizing we have,

Proposition 5.8. *Let $\mathcal{C}$ be a $\mathfrak{U}$-category and $\check{\mathcal{C}}$ the category of coherent covariant functors.*

a) If F is in $\check{\mathcal{C}}$, then $R^0 F \approx (w(F), \_)$ and $F \to (w(F), \_)$ is the usual map $F \to R^0 F$.

b) If X is a complex in $\mathcal{C}$, then $w(H^i(X, \_ = H_i(X)$ and thus $R^0(H^i(X, \_)) = (H_i(X), \_)$.

c) If X is a projective complex, then for each i we have that

$$H^i(X,\_)_0 = \mathrm{Ext}^1(Z'_{i-1},\_) \text{ and that } H^i(X,\_)_1 \subset \mathrm{Ext}^2(Z'_{i-1},\_).$$

Thus we have exact sequence

$$0 \to \mathrm{Ext}^1(Z'_{i-1},\_) \to H^i(X,\_) \to (H_i(X),\_) \to \mathrm{Ext}^2(Z'_{i-1},\_)$$

which is functorial in X with X in the category of projective complexes. Further, if $X_{i+1} = 0$, then $H^i(X,\_)_1 = \mathrm{Ext}^2(Z'_{i-1},\_)$.

As a special case of proposition 5.7, we obtain the classical result,

Corollary 5.9. *If* $\mathrm{gl.dim}\,\mathscr{C} \leqq 1$ *and X is a projective complex in $\mathscr{C}$, then we have exact sequences*

$$0 \to \mathrm{Ext}^1(H_{i-1}(X),\_) \to H^i(X,\_) \to (H_i(X),\_) \to 0$$

which of course split.

Proof. Since we have the exact sequence $0 \to H_{i-1}(X) \to Z'_{i-1} \to X_{i-2}$ and the $\mathrm{Im}(Z'_{i-1} \to X_{i-2})$ is projective (remember that the $\mathrm{gl.dim}\,\mathscr{C} \leqq 1$), we have that $Z'_{i-1} = H_{i-1}(X) + B_{i-2}$ where B_{i-2} is projective. Thus $\mathrm{Ext}^1(Z'_{i-1},\_) = \mathrm{Ext}^1(H_{i-1}(X),\_)$. Since the $\mathrm{gl.dim}\,\mathscr{C} \leqq 1$, we know that $\mathrm{Ext}^2(Z'_{i-1},\_) = 0$, which gives us our desired exact sequence. They split since $(H_i(X),\_)$ is projective for each i.

It is a well known result of homological algebra that if X and Y are two projective resolutions of the same object, then for each i we have that $Z'_i(X) + P_i \approx Z'_i(Y) + Q_i$ where P_i and Q_i are projective objects in $\mathscr{C}$. We now give a generalization of this result.

Proposition 5.10. *Let X and Y be two projective complexes in $\mathscr{C}$ and $f\colon X \to Y$ a map and i an integer such that the induced map*

$$f^i\colon H^i(Y,\_) \to H^i(X,\_)$$

is an isomorphism. Then $f_i\colon H_i(X) \to H_i(Y)$ is an isomorphism and the induced map $g\colon Z'_{i-1}(X) \to Z'_{i-1}(Y)$ gives an isomorphism

$$\mathrm{Ext}^1(Z'_{i-1}(Y),\_) \to \mathrm{Ext}^1(Z'_{i-1}(X),\_).$$

Thus there exist projective objects P and Q such that

$$Z'_{i-1}(X) + P \approx Z'_{i-1}(Y) + Q.$$

Proof. The map $f\colon X \to Y$ induces a map of exact sequences

$$\begin{array}{ccccccccc}
0 \to & \mathrm{Ext}^1(Z'_{i-1}(Y),_) & \to & H^i(Y,_) & \to & (H_i(Y),_) & \to & \mathrm{Ext}^2(Z'_{i-1}(Y),_) \\
& \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 \to & \mathrm{Ext}^1(Z'_{i-1}(X),_) & \to & H^i(X,_) & \to & (H_i(X),_) & \to & \mathrm{Ext}^2(Z'_{i-1}(X),_)
\end{array}$$

since it induces maps of $Z'_{i-1}(X) \to Z'_{i-1}(Y)$ and $H_i(X) \to H_i(Y)$. Now since the left hand and right hand functors are in $\check{\mathscr{C}}_0$, we know by proposition 3.4 that this is the only map of exact sequences compatible with the given map $H^i(Y,\_) \to H^i(X,\_)$. Since the map $H^i(Y,\_) \to H^i(X,\_)$ is an isomorphism, it follows from proposition 3.4 that the rest of the maps are isomorphisms, which gives us our first results.

It is a well known result of Eckmann and Hilton (see [2]), that $\mathrm{Ext}^1(D,\_) \approx \mathrm{Ext}^1(D',\_)$ for any objects D and D' if and only if there are projective objects P and Q such that $P + D \approx Q + D'$. Applying this to our special case gives the last result.

6. Tensor Products

We assume throughout this section that R is a ring in $\mathfrak{U}$ and that ${}_R\mathscr{M}$ and $\mathscr{M}_R$ are the categories of left and right R-modules in $\mathfrak{U}$, as defined in section 5. Since we will not change the ring R, we will suppress the R and denote the additive, subcategories of finitely presented left and right R-modules by $\ell(\mathscr{F})$ and $r(\mathscr{F})$ respectively. Also, we shall denote by $\mathscr{E}(\ell(\mathscr{F}))$ the additive full subcategory of ${}_R\check{\mathscr{M}}_0$ whose objects consist of all functors $G \approx \mathrm{Ext}^1(M,\_)$ for some finitely presented left R-module M. A similar definition can be given for $\mathscr{E}(r(\mathscr{F}))$.

Now we have the usual contravariant functor $\ell(\mathscr{F}) \to \mathscr{E}(\ell(\mathscr{F}))$ given by $M \mapsto \mathrm{Ext}^1(M,\_)$. It is well known that the maps

$$(M, N) \to (\mathrm{Ext}^1(N,\_), \mathrm{Ext}^1(M,\_))$$

are epimorphisms and that $M \to N$ gives the zero map

$$\mathrm{Ext}^1(N,\_) \to \mathrm{Ext}^1(M,\_)$$

if and only if it can be factored through a projective. Thus the maps

$$(M, N) \to (\mathrm{Ext}^1(N,\_), \mathrm{Ext}^1(M,\_))$$

induce maps

$$\Pi(M, N) \to (\mathrm{Ext}^1(N,\_), \mathrm{Ext}^1(M,\_))$$

which are isomorphisms. Therefore if we denote by $\Pi(\ell(\mathscr{F}))$, the additive category whose objects are the same as those of $\ell(\mathscr{F})$ and whose maps are $\Pi(M, N)$ for M and N in $\ell(\mathscr{F})$, we obtain a functor $\Pi(\ell(\mathscr{F})) \to$ $\to E(\ell(\mathscr{F}))$ which is an equivalence of categories. Having established these notational matters, we now turn our attention to tensor products.

Lemma 6.1. *Let A be a left R-module. Then the functor $\_\otimes A : \mathscr{M}_R \to \mathscr{Ab}$ is coherent if and only if A is a finitely presented module. Further, if $P_1 \to P_0 \to A \to 0$ is exact with P_i finitely presented projective R-modules, then $(P_1^*,\_) \to (P_0^*,\_) \to \_\otimes A \to 0$ is exact, where $P_i^* = (P_i, R)$.*

Proof. Suppose A is a finitely presented R-module and $P_1 \to P_0 \to A \to 0$ is exact with the P_i finitely generated projective R-modules. Now we know that there exists a commutative diagram with exact rows and columns

$$\begin{array}{ccccc}
0 & & 0 & & \\
\downarrow & & \downarrow & & \\
-\otimes P_1 & \rightarrow & -\otimes P_0 & \rightarrow -\otimes A \rightarrow 0 & \\
\downarrow & & \downarrow & & \\
(P_1^*, -) & \rightarrow & (P_0^*, -) & & \\
\downarrow & & \downarrow & & \\
0 & & 0 & &
\end{array}$$

where $B \otimes P_i \rightarrow (P_i^*, B)$ is the usual map given by $(b \otimes p)(f) = f(p)(b)$ for all b in B, $p \in P_i$ and f in P_i^*. Thus we obtain the last part of the lemma.

Suppose $-\otimes A$ is coherent and let $F \rightarrow A \rightarrow 0$ be exact with a free R-module. Then $-\otimes F \rightarrow -\otimes A \rightarrow 0$ is exact. Since $-\otimes F$ is isomorphic to a direct sum of copies of $-\otimes R$ and $-\otimes A$ is coherent and thus finitely generated, we know that we can find a finitely generated free R-module G_0 such that $-\otimes G_0 \rightarrow -\otimes A \rightarrow 0$ is exact. Thus we obtain an epimorphism $G_0 \rightarrow A \rightarrow 0$ with G_0 a finitely generated free R-module. Suppose $G \rightarrow G_0 \rightarrow A \rightarrow 0$ is exact with G a free R-module. Then we know that $-\otimes G \rightarrow -\otimes G_0 \rightarrow -\otimes A \rightarrow 0$ is exact. Since $-\otimes A$ is coherent, we know that $H = \operatorname{Im}(-\otimes G \rightarrow -\otimes G_0)$ is finitely generated. Since $-\otimes G \rightarrow H \rightarrow 0$ is exact and $-\otimes G$ is isomorphic to a direct sum of copies of $-\otimes R$, we know there is a finitely generated free module G_1 such that $-\otimes G_1 \rightarrow H \rightarrow 0$ is exact, or what is the same thing,

$$-\otimes G_1 \rightarrow -\otimes G_0 \rightarrow -\otimes A \rightarrow 0$$

is exact. Thus we obtain an exact sequence $G_1 \rightarrow G_0 \rightarrow A \rightarrow 0$ which shows that A is finitely presented.

Thus for each M in $\ell(\mathscr{F})$, we know that the functor $-\otimes M$ is a coherent functor. Since $-\otimes M$ is certainly half exact we know by theorem 4.13, that $(-\otimes M)_0$ is in $\mathscr{E}(r(\mathscr{F}))$, i.e. for some N in $r(\mathscr{F})$ we have that $(-\otimes M)_0 \approx \operatorname{Ext}^1(N, -)$. Thus we obtain a functor $\ell(\mathscr{F}) \rightarrow \mathscr{E}(r(\mathscr{F}))$ by sending M to $(-\otimes M)_0$. It is our purpose to show next that this induces an equivalence of categories $\Pi\ell((\mathscr{F})) \rightarrow \mathscr{E}(r(\mathscr{F}))$. To this end, we first make some general observations.

Proposition 6.2. *Let $\mathscr{C}$ be an abelian $\mathfrak{U}$-category and let $\check{\mathscr{C}} \rightarrow \check{\mathscr{C}}_0$ be the functor $F \rightarrow F_0$. Then*

a) If $F \rightarrow G$ can be factored through a projective in $\check{\mathscr{C}}$, then the corresponding map $F_0 \rightarrow G_0$ is the zero map.

b) The functor $\check{\mathscr{C}} \rightarrow \check{\mathscr{C}}_0$ induces a functor $\Pi(\check{\mathscr{C}}) \rightarrow \check{\mathscr{C}}_0$.

c) If G is right exact, then the map $\Pi(\check{\mathscr{C}})(F, G) \rightarrow (F_0, G_0)$ is a monomorphism for all F in $\check{\mathscr{C}}$.

d) If G preserves epimorphisms, then the map $\Pi(\check{\mathscr{C}})\ (F, G) \to (F_0, G_0)$ is an epimorphism for all F in $\check{\mathscr{C}}$.

e) If G is right exact, then the map $\Pi(\check{\mathscr{C}})\ (F, G) \to (F_0, G_0)$ is an isomorphism for all F in $\check{\mathscr{C}}$.

Proof. The only projective objects in $\check{\mathscr{C}}$ are the representable functors. Since $(A, \_)_0 = 0$ for all A in $\mathscr{C}$, we have established a).

b) Follows trivially from a).

c) Suppose G is half exact and suppose we have a map $F \to G$ such that the induced map $F_0 \to G_0$ is the zero map. Suppose

$$0 \to F_0 \to F \to (A, \_) \to F_1 \to 0$$

is our usual exact sequence with the F_i in $\check{\mathscr{C}}_0$. Since $F_0 \to G_0$ is the zero map, we have a map $\bar{F} \to G$ such that $F \to \bar{F} \to G$ is the map $F \to G$ where $\bar{F} = \operatorname{Im}(F \to (A, \_)$. From the exact sequence

$$0 \to \bar{F} \to (A, \_) \to F_1 \to 0$$

we deduce the exact sequence $((A, \_), G) \to (\bar{F}, G) \to \operatorname{Ext}^1(F_1, G)$. Since F_1 is in $\check{\mathscr{C}}_0$ and G is half exact, we know by proposition 3.2, that $\operatorname{Ext}^1(F_1, G) = 0$. Thus the map $\bar{F} \to G$ can be extended to $(A, \_) \to G$ which shows that the map $F \to G$ can be factored through a projective. Thus we have that if G is half exact, then the map $\Pi(\check{\mathscr{C}}_0)\ (F, G) \to (F_0, G_0)$ is a monomorphism for all F in $\check{\mathscr{C}}$.

d) Suppose G preserves epimorphisms and let $0 \to F_0 \to F \to \bar{F} \to 0$ be as in c). Then we have the exact sequence $(F, G) \to (F_0, G) \to \operatorname{Ext}^1(\bar{F}, G)$. But it is easily seen that $\bar{F}$ has a resolution $0 \to (B, \_) \to (A, \_) \to \bar{F} \to 0$ where $A \to B \to 0$ is exact. From this and lemma 3.1, it follows that since G preserves epimorphisms, then $\operatorname{Ext}^1(\bar{F}, G) = 0$. Thus

$$(F, G) \to (F_0, G) \to 0$$

is exact. But the map $0 \to G_0 \to G$ induces an isomorphism

$$(F_0, G_0) \to (F_0, G).$$

From this it follows that the map $(F, G) \to (F_0, G_0)$ is an epimorphism, giving our desired result.

c) Trivial.

Returning to the category $\ell(\mathscr{F})$, we see that for all M and N in $\ell(\mathscr{F})$, the map $(\_\otimes M, \_\otimes N) \to ((\_\otimes M)_0,\ (\_\otimes N)_0)$ induces an isomorphism $\Pi(\_\otimes M, \_\otimes N) \to ((\_\otimes M)_0,\ (\_\otimes N)_0)$.

Now we also have the obvious maps $(M, N) \to (\_\otimes M, \_\otimes N)$ which can be easily seen to be an isomorphism for all M and N in $\ell(\mathscr{F})$ (actually in ${}_R\mathcal{M}$). Claim this isomorphism induces an isomorphism

$$\Pi(M,N) \to \Pi(\_\otimes M, \_\otimes N).$$

For if $M \to N$ can be factored through any projective it can be factored through any exact sequence $P \to N \to 0$ with P projective. Since N is finitely presented, we can choose the P to be a finitely generated projective R-module. Thus we obtain that the induced map $\_\otimes M \to \_\otimes N$ can be factored through $\_\otimes P$. But $\_\otimes P \approx (P^*, \_)$ and is thus projective. Therefore we indeed have an epimorphism

$$\Pi(M,N) \to \Pi(\_\otimes M, \_\otimes N).$$

The fact that it is an isomorphism is essentially the same argument as above.

Combining the above observations we see that the functor

$$\ell(\mathscr{F}) \to \mathscr{E}(r(\mathscr{F})) \quad \text{given by} \quad M \mapsto (\_\otimes M)_0$$

does indeed induce a functor $\Pi\,\ell((\mathscr{F})) \to \mathscr{E}(r(\mathscr{F})$ which is fully faithful. In order to establish that it is an equivalence, we have to show that given any finitely presented right R-module N, there is a left R-module M such that $\mathrm{Ext}^1(N, \_) \approx (\_\otimes M)_0$.

This is a trivial consequence of

Proposition 6.3. *Let $P_1 \to P_0 \to M \to 0$ be exact with the P_i finitely generated projective R-modules (left). Let $N = \mathrm{Coker}(P_0^* \to P_1^*)$. Then N is a finitely presented right R-module and $\mathrm{Ext}^1(N, \_) \approx (\_\otimes M)_0$. Further $R^0(\_\otimes M) \approx (M^*, \_)$ and we have an exact sequence*

$$0 \to \mathrm{Ext}^1(N, \_) \to (\_\otimes M) \to (M^*, \_) \to \mathrm{Ext}^2(N, \_) \to 0$$

where $(\_\otimes M) \to (M^*, \_)$ is the usual map.
Thus if $Q_1 \to Q_0 \to A$ is an exact sequence of finitely generated right R-modules with the Q_i projectives and $B = \mathrm{Coker}\ (Q_0^* \to Q_1^*)$, then $(\_\otimes B)_0 \approx \mathrm{Ext}^1(A, \_)$.

Proof. Since each of the P_i are finitely generated projective right R-modules, it follows that N is a finitely presented right R-module. From the exact sequence $0 \to M^* \to P_0^* \to P_1^* \to N \to 0$, we deduce the exact sequence

$$0 \to (N, \_) \to (P_1^*, \_) \to (P_0^{**}, \_) \to \_\otimes M \to 0$$

(see lemma 6.1). The first part of the proposition now follows easily from proposition 5.8.

The rest of the proposition is an immediate consequence of the first part, once one observes that finitely generated projective modules are reflexive, i.e. $Q_i \approx Q_i^{**}$ under the natural map.

Remark. It should be observed that this proposition gives a proof of the fact that $(\_\otimes M)_0$ is in $\mathscr{E}(r(\mathscr{F}))$ which is independent of the results of section 4.

Thus we have proven.

Theorem 6.4. *The functor* $\Pi\ell((\mathscr{F}))\to\mathscr{E}(r(\mathscr{F}))$ *given by* $M\mapsto(\_\otimes M)_0$ *is an equivalence of categories. Since* $\mathscr{E}(r(\mathscr{F}))$ *is contravariantly equivalent to* $\Pi(r(\mathscr{F}))$, *by means of the functor* $\Pi(r(\mathscr{F})\to\mathscr{E}(r(\mathscr{F}))$, *we obtain a contravariant equivalence between* $\Pi(\ell(\mathscr{F}))$ *and* $\Pi(r(\mathscr{F}))$.

As an easy consequence of this theorem we obtain,

Proposition 6.5. *Suppose* $0\to A\to B\to C\to 0$ *is an exact sequence of right* R*-modules with* C *finitely presented. Then the following are equivalent:*

a) The exact sequence $0\to A\to B\to C\to 0$ *splits.*

b) For every left R*-module* M, *the sequence* $0\to A\otimes M\to B\otimes M$ *is exact.*

c) For every finitely presented left R*-module* M, *the sequence*

$$0\to A\otimes M\to B\otimes M$$

is exact.

d) If M *is a finitely presented left* R*-module such that*

$$\mathrm{Ext}^1(C,\_)\approx(\_\otimes M)_0, \quad \textit{then} \quad 0\to A\otimes M\to B\otimes M$$

is exact.

Proof. Clearly the implications a) $\Rightarrow$ b) $\Rightarrow$ c) $\Rightarrow$ d) are all trivial.

d) $\Rightarrow$ a). That such M exist we know from theorem 6.4. From the exact sequence $0\to\mathrm{Ext}^1(C,\_)\to\_\otimes M\to(M^*,\_)$ (see proposition 6.3), we deduce the commutative diagram with exact rows and columns

$$\begin{array}{ccccc}
 & 0 & & 0 \\
 & \downarrow & & \downarrow \\
 & \mathrm{Ext}^1(C,A) & \to & \mathrm{Ext}^1(C,B) \\
 & \downarrow & & \downarrow \\
 & A\otimes M & \longrightarrow & B\otimes M \\
 & \downarrow & & \downarrow \\
0\to & (M^*,A) & \longrightarrow & (M^*,B)
\end{array}$$

from which it easily follows that the

$$\mathrm{Ker}(\mathrm{Ext}^1(C,A)\to\mathrm{Ext}^1(C,B))=\mathrm{Ker}(A\otimes M\to B\otimes M).$$

Thus we have the exact sequence

$$\mathrm{Hom}(C,B)\to\mathrm{Hom}(C,C)\to A\otimes M\to B\otimes M.$$

Thus if $0\to A\otimes M\to B\otimes M$ is exact, then $\mathrm{Hom}(C,B)\to\mathrm{Hom}(C,C)\to 0$ is exact and thus $0\to A\to B\to C\to 0$ splits.

Remark. It should be observed that the argument given above actually yields the following more general result. Let $\mathscr{C}$ be a $\mathfrak{U}$-category satisfying the hypothesis that every half exact functor in $\check{\mathscr{C}}_0$ is isomorphic to $\operatorname{Ext}^1(C, \_)$ for some C in $\mathscr{C}$. Let F be a half exact coherent functor. Then F_0 is half exact and thus $F_0 \approx \operatorname{Ext}^1(C, \_)$ for some C in $\mathscr{C}$. Let $0 \to C_1 \to C_2 \to C_3 \to 0$ be an exact sequence in $\mathscr{C}$. Then we have an exact sequence $(C, C_2) \to (C, C_3) \to F(C_1) \to F(C_2)$. Thus if $C_3 \approx C$ and $0 \to F(C_1) \to F(C_2)$ is exact, then the sequence $0 \to C_1 \to C_2 \to C_3 \to 0$ splits.

As another application of the results of this section we prove

Proposition 6.6. *Let C be a right R-module such that there exists an exact sequence $P_2 \to P_1 \to P_0 \to C \to 0$ with the P_i finitely genented projective R-modules. Then the following statements are equivalent:*

a) $\operatorname{Ext}^1(C, R) = 0$.

b) There is a finitely presented left R-module M such that

$$\operatorname{Tor}_1(\_, M) \approx \operatorname{Ext}^1(C, \_).$$

c) If $0 \to A \to B \to C \to 0$ is exact and $0 \to \operatorname{Tor}_1(A, M) \to \operatorname{Tor}_1(B, M)$ is exact for all finitely presented left R-modules M, then

$$0 \to A \to B \to C \to 0$$

splits.

Proof. a) $\Rightarrow$ b). Since $\operatorname{Ext}^1(C, R) = 0$, we know that

$$0 \to C^* \to P_0^* \to P_1^* \to P_2^*$$

is exact. Let $M = \operatorname{Coker}(P_1^* \to P_2^*)$. Then M is a finitely presented R-module. Also we have that $H_1(\_ \otimes P^*) = \operatorname{Tor}_1(\_, M)$. But the complex $\_ \otimes P^*$ is isomorphic to the complex $(P, \_)$. Thus

$$\operatorname{Tor}_1(\_, M) \approx \operatorname{Ext}^1(C, \_).$$

b) $\Rightarrow$ c). Trivial.

c) $\Rightarrow$ a). Let $0 \to R \to E \to C \to 0$ be exact. Then certainly

$$0 \to \operatorname{Tor}_1(R, M) \to \operatorname{Tor}_1(E, M)$$

is exact for all finitely presented left R-modules M, since $\operatorname{Tor}_1(R, \_) = 0$. Thus $0 \to R \to E \to C \to 0$ splits or, what is the same thing,

$$\operatorname{Ext}^1(C, R) = 0.$$

7. Coherent Functors which are Tor

We make the same assumptions on the ring R as in section 6. We now describe another property possessed by the functor $\Pi\ \ell\ ((\mathscr{F})) \to \mathscr{E}(r(\mathscr{F}))$, defined by $C \mapsto (\_ \otimes C)_0$.

Proposition 7.1. *Suppose C is a finitely presented left R-module and M is a finitely presented right R-module such that $(\_\otimes C)_0 \approx \mathrm{Ext}^1(M, \_)$. Let $C^* \otimes \_ \to (C, \_)$ be the usual map. Then $L^0((C, \_)) \approx C^* \otimes \_$ and we have an exact sequence*

$$0 \to \mathrm{Tor}_2(M, \_) \to C^* \otimes \_ \to (C, \_) \to \mathrm{Tor}_1(M, \_) \to 0\,.$$

Proof. Since C is finitely presented, we know that $(C, \_)$ commutes with direct sums. Thus $C^* \otimes P \to (C, P)$ is an isomorphism for all projective modules P. Combining this with the fact that $C^* \otimes \_$ is right exact, we see immediately that $L^0((C, \_)) \approx C^* \otimes \_$.

Now suppose that $P_1 \to P_0 \to C \to 0$ is exact with the P_i finitely generated projective R-modules. Then from the exact sequence

$$0 \to C^* \to P_0^* \to P_1^* \to N \to 0$$

we get a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Tor}_2(N, _) \arrow[r] & C^* \otimes _ \arrow[r] \arrow[d] & P_0^* \otimes _ \arrow[r] \arrow[d, "\wr"] & P_1^* \otimes _ \arrow[d, "\wr"] \\
& 0 \arrow[r] & (C, _) \arrow[r] & (P_0, _) \arrow[r] & (P_1, _)
\end{tikzcd}$$

with the obvious exactness properties. From this it easily follows that we have an exact sequence

$$0 \to \mathrm{Tor}_2(N, \_) \to C^* \otimes \_ \to (C, \_) \to \mathrm{Tor}_1(N, \_) \to 0\,.$$

But by it follows that $(\_ \otimes C)_0 \approx \mathrm{Ext}^1(N, \_)$ and thus that

$$\mathrm{Ext}^1(N, \_) \approx \mathrm{Ext}^1(M, \_)\,.$$

Therefore $\mathrm{Tor}_i(N, \_) \approx \mathrm{Tor}_i(M, \_)$ for all i, which completes the proof.

It should be observed that since we know that if M is a finitely presented right module we can find a finitely presented left module C such that $(\_\otimes C)_0 \approx \mathrm{Ext}^1(M, \_)$ (see theorem 6.4), it follows that given any such right module M there is a finitely presented left module C such that

$$0 \to \mathrm{Tor}_2(M, \_) \to C^* \otimes \_ \to (C, \_) \to \mathrm{Tor}_1(M, \_) \to 0$$

is exact.

In this section we are interested in generalizing proposition 7.1 to a larger class of functors in the case that R is a noetherian ring. For example, if $F\colon {}_R\mathscr{M} \to \mathscr{A}b$ is a functor which commutes with direct sums, then there is a unique map $F(R) \otimes \_ \to F$ such that $R \otimes F(R) \to F(R)$ is the identity. From this it easily follows that $L^0F \approx F(R) \otimes \_$. It is our aim to describe the kernel and cokernel of such maps $F(R) \otimes \_ \to F$ where F is a coherent functor which commutes with direct limits and R is a noetherian ring. However, before restricting ourselves to the case of R noetherian, we make some useful observations.

In the category $_R\check{\mathcal{M}}$ let $\mathcal{D}_1$ be the full subcategory whose objects F have the property that for each module M we have that $\varinjlim F(M_i)=F(M)$, where (M_i) is the family of finitely generated submodules of M. It is easily checked that if $0 \to F_1 \to F_2 \to F_3 \to F_4 \to 0$ is exact with F_2 and F_3 in $\mathcal{D}_1$, then F_1 and F_4 are in $\mathcal{D}_1$.

If $0 \to F_1 \to F_2 \to F_3 \to 0$ is exact and F_1 and F_3 are in $\mathcal{D}_1$ so is F_2 in $\mathcal{D}_1$. Also $\mathcal{D}_1$ is closed under direct limits over directed sets. We now define $\varphi\colon {}_R\check{\mathcal{M}} \to \mathcal{D}_1$ as follows: $\varphi(F)\,(M) = \varinjlim F(M_i)$ where (M_i) is the family of finitely generated submodules of M. Since for each $M_i \subset M$ we have maps $F(M_i) \to F(M)$, this gives a map of $\varphi(F)\,(M) \to$ $\to F(M)$. Thus we have the map $\varphi(F) \to F$. It is easily seen that φ and $\varphi(F) \to F$ have the properties given below,

Lemma 7.2. *Let $\varphi\colon {}_R\check{\mathcal{M}} \to \mathcal{D}_1$ be the functor described above. Then*

a) φ is exact and commutes with direct limits;

b) For all G in $\mathcal{D}_1$ and F in $_R\check{\mathcal{M}}$, the map $(G, \varphi(F)) \to (G, F)$ induced by the map $\varphi(F) \to F$, is an isomorphism.

c) $\varphi(F)\,(M) \to F(M)$ is an isomorphism if M is finitely generated.

d) $\varphi(F) \to F$ is an isomorphism if and only if F is in $\mathcal{D}_1$.

We shall denote the full subcategory of $_R\check{\mathcal{M}}$ consisting of those functors which commute with direct limits taken over directed sets by $\mathcal{D}_0$. It is obvious that $\mathcal{D}_0 \subset \mathcal{D}_1$. Also it is easily seen that if

$$0 \to F_1 \to F_2 \to F_3 \to F_4 \to 0$$

is an exact sequence with F_2 and F_3 in $\mathcal{D}_0$, then F_1 and F_4 are in $\mathcal{D}_0$. And if $0 \to F_1 \to F_2 \to F_3 \to 0$ is exact with F_1 and F_3 in $\mathcal{D}_0$, then F_2 is in $\mathcal{D}_0$. And $\mathcal{D}_0$ is closed under direct limits over directed sets.

Proposition 7.3. *If R is a noetherian ring, then $\mathcal{D}_0 = \mathcal{D}_1$.*

Proof. Let F be in $_R\check{\mathcal{M}}$. Then we know that there exists an exact sequence

$$\sum_{j\in J} (A_j, \_) \to \sum_{i\in I} (B_i, \_) \to F \to 0\,.$$

Thus

$$\sum_{j\in J} \varphi((A_j, 6)) \to \sum_{i\in I} \varphi((B_i, \_)) \to \varphi(F) \to 0$$

is exact since φ is exact and commutes with direct limits. Therefore if we show that $\varphi(B, \_)$ is in $\mathcal{D}_0$ for all R-modules B, then we will have shown that $\varphi(F)$ is in $\mathcal{D}_0$ for all F in $_R\check{\mathcal{M}}$. Since $\varphi(F) \approx F$ for F in $\mathcal{D}_1$ this will show that $\mathcal{D}_0 = \mathcal{D}_1$.

Suppose $P_1 \to P_0 \to B \to 0$ is exact with the P_i projective, then we have the exact sequence $0 \to (B, \_) \to (P_0, \_) \to (P_1, \_)$. From this it follows $0 \to \varphi((B, -)) \to \varphi((P_0, -) \to \varphi((P_1, -))$ is exact. Thus if we show that $\varphi((P_i, -)$ commutes with direct limits over directed sets, we will have that $\varphi((B, -))$ also does. Therefore in order to establish the proposition it suffices to show that $\varphi(P, -)$ is in $\mathscr{D}_0$ for projective modules P.

Suppose P is a projective module and $M = \varinjlim M_i$ (a directed direct limit). Now it is easily seen that $\varphi((P, -))\ (M)$ is nothing more than the maps from P to M whose images are contained in finitely generated submodules of M. Or, since R is noetherian, $\varphi((P, -))\ (M)$ is the set of maps from P to M whose images are finitely generated. Let $f\colon P \to M_i$ be a map such that the composite $P \to M_i \to M$ is zero. Since $f(P)$ is finitely generated and $f(P) \subset \operatorname{Ker}(M_i \to M)$, we know there is a $j \geqq i$ such that $f(P)$ is carried to zero under the map $M_i \to M_j$, i.e. the composite $P \to M_i \to M_j$ is zero. Thus we have shown that the map $\varinjlim \varphi(P, -)\ (M_i) \to \varphi((P, -))\ (M)$ is a monomorphism.

Suppose $f\colon P \to M$ and $f(P)$ is finitely generated. Then there is an M_i such that $f(P) \subset \operatorname{Im}(M_i \to M)$. Let N be a finitely generated submodule of M_i which goes onto $f(P)$. Then, since P is projective, there is a map $P \to N$ such that the composite $P \to N \to M$ is the map f. Thus we have shown that the map $\varinjlim \varphi((P, -))\ (M_i) \to \varphi((P, -))\ (M)$ is an epimorphism and thus an isomorphism. Therefore $\varphi((P, -))$ is in $\mathscr{D}_0$ which completes the proof of the proposition.

From now on we will assume that our ring R is a noetherian ring. It is a trivial matter to check

Lemma 7.4. *Let M be an R-module. Then,*

a) The map $\varphi((M, -)) \to (M, -)$ is a monomorphism.

b) $\varphi((M, -))$ is left exact and is exact if M is projective.

c) The map $M^* \otimes - \to (M, -)$ has a unique factorization $M^* \otimes - \to \to \varphi((M, -)) \to (M, -)$ and if M is projective, then $M^* \otimes - \to \varphi((M, -))$ is an isomorphism.

d) $L_0(\varphi((M, -)) \approx M^* \otimes -$.

Suppose F is a coherent half exact functor. Then by proposition 4.10 we know that there is an exact sequence $C \to D \to E \to 0$ with D a projective module such that $(D, -) \to (C, -) \to F \to 0$ is exact. Thus we have the exact sequence $\varphi((D, -)) \to \varphi((C, -)) \to \varphi(F) \to 0$. Therefore if we assume that F also commutes with direct limits over directed sets, then we have an exact sequence $\varphi((D, -)) \to \varphi((C, -)) \to F \to 0$ with the D a projective R-module. Since $\varphi((X, -))(R) = X^*$ we also obtain the exact sequence $D^* \to C^* \to F(R) \to 0$. From this we deduce the commutative diagram with exact rows and columns

$$\begin{tikzcd}
 & 0 \arrow[d] & & \\
 & G_2 \arrow[r, equal] \arrow[d] & G_2 \arrow[d] & \\
D^* \otimes _ \arrow[r] \arrow[d, "\wr"] & C^* \otimes _ \arrow[r] \arrow[d] & F(R) \otimes _ \arrow[r] \arrow[d] & 0 \\
\varphi((D, _)) \arrow[r] & \varphi((C, _)) \arrow[r] \arrow[d] & F \arrow[r] \arrow[d] & 0 \\
 & G_1 \arrow[r, equal] & G_1 \arrow[d] & \\
 & & 0 &
\end{tikzcd}$$

where G_1 and G_2 are the cokernel and kernel respectively of the map $C^* \otimes \_ \to \varphi((C, \_))$. Thus if we obtain a description of G_1 and G_2 then we will have also described the cokernel and kernel of $F(R) \otimes \_ \to F$.

Let $P_1 \to P_0 \to C \to 0$ be exact with the P_i projective R-modules. Then we have the exact sequence $0 \to C^* \to P_0^* \to P_1^* \to N \to 0$. Since we know that P_i^* are flat (see lemma 7.4) we have a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Tor}_2(N, _) \arrow[r] & C^* \otimes _ \arrow[r] \arrow[d] & P_0^* \otimes _ \arrow[r] \arrow[d, "\wr"] & P_1^* \otimes _ \arrow[d, "\wr"] \\
 & 0 \arrow[r] & \varphi((C, _)) \arrow[r] & \varphi((P_0, _)) \arrow[r] & \varphi((P_1, _)).
\end{tikzcd}$$

From this it follows as in the proof of proposition 7.1 that we have an exact sequence

$$0 \to \mathrm{Tor}_2(N, \_) \to C^* \otimes \_ \to \varphi((C, \_)) \to \mathrm{Tor}_1(N, \_) \to 0.$$

Combining the above we obtain the promised generalization of proposition 7.1.

Theorem 7.5. *Let R be a noetherian ring and $P_1 \to P_0 \to C \to 0$ an exact sequence of R-modules with the P_i projective. If the right module N is defined by $P_0^* \to P_1^* \to N \to 0$, then we have the exact sequence*

$$0 \to \mathrm{Tor}_2(N, \_) \to C^* \otimes \_ \to \varphi((C, \_)) \to \mathrm{Tor}_1(N, \_) \to 0.$$

More generally if F is a half exact coherent functor which commutes with directed direct limits then there is a right module N and an exact sequence

$$\mathrm{Tor}_2(N, \_) \to F(R) \otimes \_ \to F \to \mathrm{Tor}_1(N, \_) \to 0.$$

As an immediate consequence we have,

Corollary 7.6. *Let R be a noetherian ring and F a half exact functor* *which commutes with directed direct limits. Then* $F \approx \mathrm{Tor}_1(N, \_)$ *for some right* R*-module* N *if and only if* $F(R) = 0$.

References

[1] GABRIEL, P.: Des catégories abeliennes. Bull. Soc. Math. France 1962, pp. 323—448.

[2] HILTON, P. J.: Homotopy theory of modules and duality. Symposium internacional de Topologia Algebraica 1958, pp. 273—281.

[3] —, and D. REES: Natural maps of extension functors and a theorem of R. G. SWAN. Proc. Cambridge Philo. Soc. 1961, pp. 489—502.

[4] MACLANE, S.: Homology. New York: Academic Press 1963.

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