Examples of derived equivalences

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Rickard. Morita theory

R, S: rings,

$$\begin{array}{ccc} D^b(R\text{-Mod}) & \overset{F}{\cong} & D^b(S\text{-Mod}) \\ \cup\!| & & \cup\!| \\ K^b(R\text{-proj}) & \cong & K^b(S\text{-proj}) \\ \\ T^\bullet & \longmapsto & S \end{array}$$

Construct a derived equivalence ?

tilting complex
$$\left\{ \begin{array}{ll} (1) & \mathrm{Hom}(T^\bullet, T^\bullet[n]) = 0 \quad \forall\, n \neq 0 \\ (2) & \mathrm{add}\, T^\bullet \text{ generates } K^b(R\text{-proj}) \\ & \text{as a triangulated category} \end{array} \right.$$

(3) $\mathrm{End}(T^\bullet) \cong S$

$R \overset{der}{\sim} S \overset{def}{\Longleftrightarrow} \mathcal{D}^b(R\text{-Mod}) \cong \mathcal{D}^b(S\text{-Mod})$ as triangulated categories

$R \overset{der}{\sim} S \Longleftrightarrow \exists$ tilting complex $T^\bullet \in K^b(R\text{-proj})$ s.th.

$$\mathrm{End}_{K^b(R\text{-proj})}(T^\bullet) \cong S.$$

<u>Task</u>: Find a "suitable" tilting complex!

<u>Examples of tilting complexes</u>

(1) Tilting modules. R: ring, $T \in R\text{-Mod}$ satisfies

a) $0 \to \underbrace{P_n \to \cdots \to P_1 \to P_0}_{\text{f.g. projective}} \to T \to 0$

(2) $\mathrm{Ext}_R^i(T, T) = 0 \quad \forall i > 0$

(3) $\exists$ exact seq

$$0 \to R \to \underbrace{T_0 \to T_1 \to \cdots \to T_n}_{\in \operatorname{add} T} \to 0$$

$\Rightarrow$

$$R \overset{\text{der}}{\sim} \mathrm{End}_R(T)$$

In $\mathcal{D}^b(R\text{-Mod})$, $\quad T \cong T^{\bullet}: \; 0 \to P_n \to P_{n-1} \to \cdots \to P_0 \to 0$

$\in K^b(R\text{-proj})$

- $\mathrm{Hom}_{\mathcal{D}^b(R\text{-Mod})}(T^{\bullet}, T^{\bullet}[n]) \cong \mathrm{Hom}_{\mathcal{D}^b(R\text{-Mod})}(T, T[n])$

$$\cong \mathrm{Ext}_R^n(T, T) = 0 \quad \forall n \neq 0$$

- By (3) $\quad 0 \to R \to T_0 \to K_0 \to 0, \quad 0 \to K_0 \to T_1 \to K_1 \to 0 \cdots$

$$0 \to K_{n-2} \to T_{n-1} \to T_n \to 0$$

$\leadsto \mathcal{D}^b(R\text{-Mod})$

$$K_{n-2} \to \underbrace{T_{n-1} \to T_n}_{\in \operatorname{add} T = \operatorname{add} T^\bullet} \to K_{n-1}[1] \Rightarrow K_{n-2} \in \langle \operatorname{add} T^\bullet \rangle$$

triangulated category generated by $\operatorname{add} T^\bullet$

$$K_{n-3} \to \underbrace{T_{n-2}}_{\in \langle \operatorname{add} T^\bullet \rangle} \to K_{n-2} \to K_{n-3}[1]$$

$$\Rightarrow K_{n-3} \in \langle \operatorname{add} T^\bullet \rangle$$

......

$R \in \langle \operatorname{add} T^\bullet \rangle, \quad \Rightarrow \quad \langle \operatorname{add} T^\bullet \rangle = K^b(R\text{-proj})$

Other examples?

[Rickard]: A: f.d. $\underbrace{\text{self-injective}}_{\text{proj} = \text{inj}}$ algebra / K: field

$L \in A$-mod.

$P \xrightarrow{f} Q \to L \to 0$: minimal projective presentation.

Satisfying (1) $\exists$ projective A-mod W s.th.

$\text{add}(W \oplus Q) = A\text{-proj}$

$\text{add}\, W \cap \text{add}\, Q = \{0\}$

$P \in \text{add}\, W$.

(2) $\text{Hom}_A(W, L) = 0 = \text{Hom}_A(L, W)$

$T_L^\bullet: \quad 0 \to P \xrightarrow{f} Q \to 0$

$T_W^\bullet: \quad 0 \to W \to Q$

Then $T^\bullet = T_L^\bullet \oplus T_W^\bullet$ is a tilting complex.

$B_0(A_5) \overset{der}{\sim} B_0(A_4)$ $\quad$ char $k = 2$

$W \in \langle \mathrm{add}\, T^\bullet \rangle$

$$\begin{array}{ccccc} & & Q & & \\ & & \| & & \\ 0 \to & P & \to Q & \to 0 \\ & \| & & & \\ & P & & & \end{array}$$

$\Rightarrow \underbrace{W \oplus Q}_{\in \langle \mathrm{add}\, T^\bullet \rangle}$

$\Rightarrow Q \in \langle \mathrm{add}\, T^\bullet \rangle$

$$Q \to \underbrace{T_L^\bullet}_{\in \langle \mathrm{add}(T^\bullet) \rangle} \to \underbrace{P[1]}_{\langle \mathrm{add}\, T^\bullet \rangle} \to Q[1]$$

$\Rightarrow \mathrm{add}\, T^\bullet$ generates $K^b(A\text{-proj})$ as Δ-cat.

- A: $\begin{tikzcd} 1 \arrow[r] & 2 \arrow[dl] \\ 3 \arrow[u] & \end{tikzcd}$ $/ \operatorname{rad}^4$, $\begin{smallmatrix}1\\2\\3\\1\end{smallmatrix}$, $\begin{smallmatrix}2\\3\\1\\2\end{smallmatrix}$, $\begin{smallmatrix}3\\1\\2\\3\end{smallmatrix}$
  $\quad P_1 \quad P_2 \quad P_3$

$e = e_1 + e_2$

$L = 3$

$$\underbrace{\begin{smallmatrix}1\\2\\3\\1\end{smallmatrix}}_{P = P_1} \to \underbrace{\begin{smallmatrix}3\\1\\2\\3\end{smallmatrix}}_{Q = P_3} \to 3 \to 0 \ : \text{minimal proj presentation}$$

$3 = A/AeA$.

$$\operatorname{Hom}(W, \underset{3}{L}) = 0 = \operatorname{Hom}(\underset{3}{L}, W)$$

$W = P_1 \oplus P_2$

$\begin{smallmatrix}1\\2\\3\\1\end{smallmatrix} \quad \begin{smallmatrix}2\\3\\1\\2\end{smallmatrix}$

$0 \to P_1 \to P_3 \to 0 \ : T_3^\bullet$

$0 \to P_1 \to 0 \ : T_1^\bullet$

$0 \to P_2 \to 0 \ : T_2^\bullet$

$T^\bullet = T_3^\bullet \oplus T_1^\bullet \oplus T_2^\bullet$ is a tilting complex.

How does $\mathrm{End}(T^\cdot)$ look like?

<u>**Q**</u>: $\dim_k \mathrm{End}(T^\cdot) = ?$

<u>Lemma</u>: A: f.d. $P^\cdot, Q^\cdot \in K^b(A\text{-proj})$,

$$\sum_{i \in \mathbb{Z}} (-1)^i \dim_k \mathrm{Hom}_{K(A)}(P^\cdot, Q^\cdot[i])$$

$$= \sum_{r,s \in \mathbb{Z}} (-1)^{r-s} \dim_k \mathrm{Hom}_A(P^r, Q^s)$$

If $\mathrm{Hom}_{K(A)}(P^\cdot, Q[i]) = 0$ $\forall i \neq 0$, then

the left hand side is $\dim_k \mathrm{Hom}_{K(A)}(P^\cdot, Q^\cdot)$

$$0 \to P_1 \to P_3 \to 0 \quad : T_3^\bullet$$

$$0 \to P_1 \to 0 \quad : T_1^\bullet$$

$$0 \to P_2 \to 0 \quad : T_2^\bullet$$

$$\begin{aligned} C_{33} &= \dim \operatorname{Hom}_{K(A)}(T_3^\bullet, T_3^\bullet) \\ &= \dim \operatorname{Hom}_A(P_1, P_1) \\ &\quad + \dim \operatorname{Hom}_A(P_3, P_3) \\ &\quad - \dim \operatorname{Hom}_A(P_1, P_3) \\ &\quad - \dim \operatorname{Hom}_A(P_3, P_1) \\ &= 2 \end{aligned}$$

$$(C_{ij}) = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 0 \\ 1 & 0 & 2 \end{bmatrix}$$

$$\dim \operatorname{End}(T) = 10$$

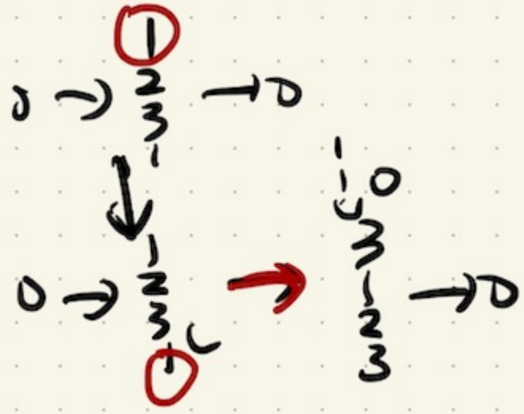
In particular $\operatorname{End}(T) \not\cong A$.

$$\begin{tikzcd} T_3^\bullet \arrow[r, "\alpha"] & T_1^\bullet \arrow[l, bend left, "\gamma"] \arrow[r, "\beta"] & T_2^\bullet \arrow[l, bend left, "\delta"] \end{tikzcd}$$

$$\begin{tikzcd} 0 \arrow[r] & P_1 \arrow[r] \arrow[d, equal] & P_3 \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & P_1 \arrow[r] & 0 \end{tikzcd} \qquad \alpha$$

$$\begin{smallmatrix} 1 \\ 2 \\ 3 \\ 1 \end{smallmatrix} \xrightarrow{\beta} \begin{smallmatrix} 2 \\ 3 \\ 1 \\ 2 \end{smallmatrix} \qquad\qquad \begin{smallmatrix} 2 \\ 3 \\ 1 \\ 2 \end{smallmatrix} \longrightarrow \begin{smallmatrix} 1 \\ 2 \\ 3 \\ 1 \end{smallmatrix}$$

γ



$$K\left(3 \underset{\gamma}{\overset{\alpha}{\leftrightarrows}} 1 \underset{\delta}{\overset{\beta}{\leftrightarrows}} 2\right) \Big/ \begin{matrix}\alpha\beta = 0 \\ \gamma\alpha\gamma = 0 \\ \cdots ?\end{matrix} \;\xrightarrow[\cong]{}\!\!\!\!\rightarrow \underbrace{\mathrm{End}(T)}_{\dim = 10}$$

$\dim = 10$

Okuyama: A: symmetric algebra.

$P_1, \cdots, P_n$: indec projective A-modules.

$\quad \| \qquad$ provides

$\mathcal{X} \cup \mathcal{Y} \rightsquigarrow$ a tilting complex over A.

$\forall\ P \in \mathcal{X}$

$$\begin{tikzcd} 0 \arrow[r] & Q' \arrow[r, "f"] & P \arrow[r] & 0 \\ & & Q \arrow[ul, dashed, "\exists h"] \arrow[u, "\forall g"'] \arrow[ul, phantom, "\circlearrowleft"] & \end{tikzcd} \quad , \ Q' \in \operatorname{add} \mathcal{Y}$$

$\forall Q \in \mathcal{Y}$

$\overset{\text{def}}{\Longleftrightarrow}$ f is right $\operatorname{add}\mathcal{Y}$-approximation.

$T_P^\bullet$ $\quad 0 \to Q' \xrightarrow{f} P \to 0$. $P \in \mathcal{X}$

$T_Q^\bullet$ $\quad 0 \to Q \to 0$, $\quad Q \in \mathcal{Y}$

$$T^\bullet = \bigoplus_{P \in \mathcal{X}} T_P^\bullet \oplus \bigoplus_{Q \in \mathcal{Y}} T_Q^\bullet \quad \text{is a tilting complex.}$$

General form: "Hoshino-Kato" 2002

A: ring, $e \in A$, $e = e^2$

s.th A/AeA is finitely presented.

$$\boxed{\underbrace{P_1 \to P_0}_{\text{f.g. proj.}} \to A/AeA \to 0}$$

$$\begin{tikzcd}
& Ae \arrow[dl, dashed] \arrow[d, "\forall g"] & & \\
Q \arrow[r, "f"] & P \arrow[r, two heads] & A(1-e)/AeA(1-e) \arrow[r] & 0
\end{tikzcd}$$

$\underbrace{Q}$ $\in \operatorname{add} Ae$. f is right $\operatorname{add} Ae$-approximation.

If $\operatorname{Hom}_A(A/AeA, Ae) = 0$, then

$$\begin{array}{l} 0 \to Q \xrightarrow{f} P \to 0 \\ \quad\ \ \oplus \\ 0 \to Ae \to 0 \end{array}$$ is a tilting complex.

<u>proof:</u>

$$\begin{tikzcd}
T: & 0 \arrow[r] & Q \oplus Ae \arrow[r, "{[f,0]}"] & P \arrow[r] \arrow[d, "g"'] \arrow[dl, dashed, "0="'] \arrow[rr, "\pi"] & 0 & A(1-e)/AeA(1-e) \arrow[dll, dashed, "h"] \\
T[-1] & & 0 \arrow[r] & Q \oplus Ae \arrow[r] & P \arrow[r] & 0
\end{tikzcd}$$

$h = 0$ by assumption

$\Rightarrow g = 0$

$$\begin{tikzcd}
T^{\bullet} & & 0 \arrow[r] & \underbrace{Q \oplus Ae}_{\text{take } Ae} \arrow[r, "{[f,0]}"] \arrow[d, "g"] \arrow[dl, dashed, "0"'] & P \arrow[r] \arrow[dl, dashed, "0"] & 0 \\
T^{\bullet}[1] & 0 \arrow[r] & Q \oplus Ae \arrow[r, "{[f,0]}"'] & P \arrow[r] & 0 &
\end{tikzcd}$$

$\Rightarrow g$ is null-homotopic.

i.e. $g = 0$ in $\mathrm{Hom}_{K(A)}(T^{\bullet}, T^{\bullet}[1])$

Hence $\mathrm{Hom}_{K(A)}(T^\cdot, T^\cdot[n]) = 0$. $\forall n \neq 0$.

It is easy to check that $\mathrm{add}(T^\cdot)$ generates $K^b(A\text{-proj})$ as a triangulated cat. $\square$

More examples

[Happel - Unger]. A: f.d. algebra.

$T = M \oplus X$: tilting module s.th $X \in \mathrm{Fac}\, M$.

indec. $[\exists$ epi. $M^r \twoheadrightarrow X]$

$$\begin{tikzcd} 0 \arrow[r] & Y \arrow[r, "g"] \arrow[d, "\alpha"'] & M' \arrow[r, "f"] \arrow[dl, dashed, "\exists h"] & X \arrow[r] & 0 \\ & M & & & \end{tikzcd}$$

minimal addM-approximation.

$\Rightarrow$ $T' = M \oplus Y$ is again a tilting module

$$\begin{array}{ccc} A & \overset{der}{\sim} & \mathrm{End}_A(T) \\ der\wr & & \\ \mathrm{End}_A(T') & & \end{array} \Rightarrow \mathrm{End}_A(T) \overset{der}{\sim} \mathrm{End}_A(T')$$

Remark: g is left add$M$-approximation

Applying $\mathrm{Hom}_A(-, M)$.

$$0 \to (X, M) \to (M', M) \xrightarrow{\mathrm{Hom}(g,M)} (Y, M) \to \underset{\substack{\| \\ 0}}{\mathrm{Ext}_A^1}(X, M) \quad \underbrace{X, M}_{\in \,\mathrm{add}\, T}.$$

Similar sequences ....

- $\mathcal{C}$: 2-CY Δ-cat. $D\mathrm{Hom}_{\mathcal{C}}(X, Y) \cong \mathrm{Hom}_{\mathcal{C}}(Y, X[2])$

$T = M \oplus X$, $\mathrm{Hom}_{\mathcal{C}}(T, T[1]) = 0$

$$Y \xrightarrow{g} M' \xrightarrow{f} X \to Y[1]$$

right add M-approximation

$\Rightarrow$ g is left add M-approximation.

- $\mathcal{C}$: Δ-cat. T: presilting. $\mathrm{Hom}_{\mathcal{C}}(T, T[n]) = 0$ $\forall n > 0$

$T = M \oplus X$

$$Y \xrightarrow{g} M' \xrightarrow{f} X \to Y[1]$$

right add M-approximation.

- A = f.d. alg. $X \in A\text{-mod}$. indec non-proj

$$0 \to D\mathrm{Tr}X \xrightarrow{g} M \xrightarrow{f} X \to 0$$: almost split seq.

right add M-approximation.

- **Q** $Y \xrightarrow{g} M' \xrightarrow{f} X$: short exact seq or triangle

right add M-approx.

left add M-approxi.

$$\mathrm{End}(M \oplus X) \overset{\mathrm{der}}{\sim} \mathrm{End}(M \oplus Y) \ ?$$

$\underline{\underline{A}}$:

[H. Xi 2011]

$0 \to Y \xrightarrow{g} M' \xrightarrow{f} X \to 0$ short exact seq in an abelian cat. $\mathcal{A}$

called add$M$-split sequence. $M \in \mathcal{A}$

s.th.

(1) f is right add$M$-approximation

(2) g is left add$M$-approximation

$$\Longrightarrow \mathrm{End}_{\mathcal{A}}(M \oplus X) \overset{\text{der}}{\simeq} \mathrm{End}_{\mathcal{A}}(M \oplus Y)$$

proof: $W = M \oplus X$. $V = M \oplus Y$

Show that $\mathrm{Hom}(V, W)$ is a tilting module over $\mathrm{End}(V)$. with endomorphism ring

iso to $\mathrm{End}(W)$ $\square$

$Y \xrightarrow{g} M' \xrightarrow{f} X \to Y[1]$, $\mathcal{C}$: Δ-cat , $M \in \mathcal{C}$

s.th (1) f : right add$M$-approximati

(2) g : left add$M$-approximatis

$$\Longrightarrow \quad \frac{\mathrm{End}_{\mathcal{C}}(M\oplus X)}{\left\{ M\oplus X \xrightarrow{\theta} M\oplus X \right\}} \quad , \quad \frac{\mathrm{End}_{\mathcal{C}}(M\oplus Y)}{\left\{ M\oplus Y \xrightarrow{\theta} M\oplus Y \right\}}$$

$$\begin{tikzcd} M\oplus X \arrow[r, "\theta"] \arrow[dr, dashed, "0"'] & M\oplus X \arrow[d, "\alpha"] \\ & M \end{tikzcd} \qquad \begin{tikzcd} M\oplus Y \arrow[r, "\theta"] & M\oplus Y \\ M \arrow[u, "\forall\alpha"] \arrow[ur, dashed, "0"'] & \end{tikzcd}$$

$$\mathrm{cogh}_M(M\oplus X) \qquad\qquad \mathrm{gh}_M(M\oplus Y)$$

$$gh_M(A,B) \overset{\text{def}}{=} \{ f: A \to B \mid \mathrm{Hom}(M,f)=0 \}$$

$$cogh_M(A,B) \overset{\text{def}}{=} \{ f: A \to B \mid \mathrm{Hom}(f,M)=0 \}$$

- A: selfinj f.d. $\quad X \in A\text{-mod}.$

$$\begin{tikzcd}
 & & & A \arrow[dl, dashed] \arrow[d, "f"] & \\
0 \arrow[r] & \Omega X \arrow[r, "g"] \arrow[d] & P_X \arrow[r] \arrow[dl, dashed] & X \arrow[r] & 0 \\
 & A & & &
\end{tikzcd}$$

proj = inj $\qquad$ addA-split seq.

$$\Longrightarrow \underbrace{\mathrm{End}_A(A \oplus X)}_{\Lambda\ \shortparallel} \overset{\text{der.}}{\sim} \underbrace{\mathrm{End}_A(A \oplus \Omega X)}_{\shortparallel\ \Gamma}$$

[Liu-Xi] $\quad \widetilde{st.M}$

$$\Lambda\text{-}\underline{\mathrm{mod}} \cong \Gamma\text{-}\underline{\mathrm{mod}}$$

$\underline{\underline{Q}}$: Given $F: \mathcal{D}^b(R\text{-Mod}) \xrightarrow{\sim} \mathcal{D}^b(S\text{-Mod})$

$T^\bullet \longmapsto S$

$\rightsquigarrow$ $R\text{-Mod}$ VS $S\text{-Mod}$?

$\underline{\underline{A}}$: $\cdots \to T^i \to T^{i+1} \to \cdots$

$$\underbrace{(F[m])}^{-1}(S) : 0 \to T^{-n} \to \cdots \to T^{-1} \to T^0 \to 0$$

Assume that $F^{-1}S = 0 \to T^{-n} \to \cdots \to T^{-1} \to T^0 \to 0$

$\forall X \in R\text{-Mod}$

$$FX \cong 0 \to M_X \to \underbrace{P_X^1 \to \cdots \to P_X^n}_{\text{proj}} \to 0$$

$X \rightsquigarrow M_X$

$\exists$ functor $\overline{F}: R\text{-}\underline{\mathrm{Mod}} \to S\text{-}\underline{\mathrm{Mod}}$

$$X \longmapsto M_X.$$

(1) F, G derived equivalence $\quad \overline{F}\circ\overline{G} \cong \overline{FG}$

(2) $\overline{F}(\Omega^n X) \cong \Omega^n \overline{F}X$.

(3) $\mathrm{pd}_S \overline{F}X \leq \mathrm{pd}_R X \leq \mathrm{pd}_S \overline{F}X + n$

(4) $0 \to X \to Y \to Z \to 0$ in $R\text{-}\mathrm{Mod}$

$$0 \to \overline{F}X \to \underbrace{\overline{F}Y \oplus Q} \to \overline{F}Z \to 0 \quad \text{exact}$$

ISS

$\overline{F}Y$ in $S\text{-}\underline{\mathrm{Mod}}$

(5) $X \in {}^{\perp}(R\text{-}\mathrm{Proj}) \overset{\text{def}}{=\!=} \{Y \mid \mathrm{Ext}^i(Y, R\text{-}\mathrm{Proj}) = 0\ \forall\, i>0\}$

$\overline{F}X \in {}^{\perp}(S\text{-Proj})$

$\Rightarrow$ R, S : f.d. alg

$|\mathrm{gldim} R - \mathrm{gldim} S| \leq n$

(3)

$|\mathrm{findim} R - \mathrm{findim} S| \leq n$

(4) + (5) . $\overline{F}: R\text{-}\underline{\mathrm{GProj}} \longrightarrow S\text{-}\underline{\mathrm{GProj}}$

$0 \to X^i \to P^i \to X^{i+1} \to 0 \quad \overset{\overline{F}}{\rightsquigarrow}$

$\forall i \in \mathbb{Z} \quad X^i \in {}^{\perp}(R\text{-Proj})$

$M \cong X^0$

$\overline{F}M$ is Gorenstein proj.

$$\overline{F} : R\text{-}\underline{\mathrm{Mod}} \to S\text{-}\underline{\mathrm{Mod}}$$

$$\overline{F} \circ \overline{G\circ[n]} \cong \overline{F\circ G\circ[n]} \cong \overline{[n]} = \Omega_S^n$$

$$\Omega_S^n : S\text{-}\underline{\mathrm{GProj}} \cong S\text{-}\underline{\mathrm{GProj}}$$

$$\overline{G\circ[n]} \circ \overline{F} \cong \Omega_R^n : R\text{-}\underline{\mathrm{GProj}} \cong R\text{-}\underline{\mathrm{GProj}}$$

$$\Longrightarrow \quad R\text{-}\underline{\mathrm{GProj}} \cong S\text{-}\underline{\mathrm{GProj}} \quad ;$$

$$0 \to \Delta^0 \to \underbrace{\Delta^1 \to \cdots \to \Delta^n}_{\text{proj bimodu}} \to 0.$$

$$\Delta^0 \otimes_A -$$