从代数三角范畴到预三角微分分次范畴

2021年5月3日 9:54

§1. Frobenius categories

§2. Algebraic triangulated categories

§3. DG categories

§4. DG enhancements

DG = differential graded

§1. Frobenius categories

1.1. Exact categories

$\mathcal{A}$ an add. cat.

a <u>kernel-cokernel pair</u> (i, p)

$$X \xrightarrow{i} Y \xrightarrow{p} Z$$

$$i = \operatorname{Ker} p \text{ and } p = \operatorname{Cok} i.$$

<u>Def</u>: (Quillen 1973) An <u>exact structure</u> $\mathcal{E}$ on $\mathcal{A}$ is a chosen class of kernel-cokernel pairs,

[$(i, p) \in \mathcal{E}$ <u>conflation</u>, i <u>inflation</u>, p <u>deflation</u>]

add. mono. add. epi

subject to

(Ex0) Id_0 deflation $\quad \mathrm{Id}_0 : 0 \to 0$

(Ex1) def $\circ$ def $\subseteq$ def

$(\mathrm{Ex1})^{op}$ inf $\circ$ inf $\subseteq$ inf

(Ex2) def closed under pullback

$$\begin{tikzcd} X' \arrow[r, dashed, two heads, "p'"] \arrow[d, dashed, "t'"'] \arrow[dr, phantom, "P.B."] & Y' \arrow[d, "t"] \\ X \arrow[r, two heads, "p"] & Y \end{tikzcd}$$

p def. $\Rightarrow p'$ def.

$(\mathrm{Ex2})^{op}$ inf closed under pushout

i inf.
$$\begin{tikzcd} X \arrow[r, "t"] \arrow[d, "i"'] \arrow[dr, phantom, "P.O."] & X' \arrow[d, dashed, "i'"] \\ Y \arrow[r, dashed, "t'"] & Y' \end{tikzcd}$$
i' inf.

<u>RMK</u>: • [Keller 1991, Appendix A]

• [Bühler, Exact categories, Expo. Math. 2010]

<u>eg</u>: $\mathcal{B}$ abelian, $\mathcal{A} \subseteq \mathcal{B}$ ext-closed $\Rightarrow \mathcal{A}$ is exact cat.

<u>RMK</u>: Gabriel-Quillen's embedding theorem: $\mathcal{A}$ <u>small</u> exact cat, $\exists\ \mathcal{A} \simeq \mathcal{A}'$. $\mathcal{A}' \subseteq \mathcal{B}$ abelian, ext-closed,

→ as exact functor

1-2

2021年5月3日 10:08

fix an exact category $(\mathcal{A}, \mathcal{E})$

(1) projective obj. $\Leftrightarrow$ $\mathrm{Hom}_{\mathcal{A}}(P, -)$ sends conflation to s.e.s.

(2) have enough proj. : $\forall X$, $\exists$ deflation $f: P \twoheadrightarrow X$, P projective

Def (Heller 1960) $(\mathcal{A}, \mathcal{E})$ Frobenius category if

(1) $\exists$ enough proj., enough inj.

(2) $\boxed{\text{proj} = \text{inj}}$. $\square$

eg. ① A-mod, A self injective

② $C(\mathcal{A})$, $\mathcal{A}$ additive

conflations: $0 \to X \to Y \to Z \to 0$ $\quad$ $\forall n$ $\; 0 \to X^n \to Y^n \to Z^n \to 0$ split exact in $\mathcal{A}$

$C(\mathcal{A})$ is Frobenius! (proj = inj = contractible complex)

③ $\mathcal{P}$ an additive cat.

$GP(\mathcal{P}) \supseteq \mathrm{add}(\mathcal{P})$ $\quad$ Gorenstein-projective $\mathcal{P}$-mod

[mod-$\mathcal{P}$, GP. objects]

$\Rightarrow GP(\mathcal{P})$ is Frobenius

(proj = inj = $\mathrm{add}(\mathcal{P})$, proj. mod. over $\mathcal{P}$)

Theorem (C. 2012) Assume that $\mathcal{A}$ is Frobenius. Then

$\mathcal{A} \xrightarrow{\theta} GP(\mathcal{P})$ $\quad X \longmapsto \mathrm{Hom}(-, X)|_{\mathcal{P}}$

s.t. $\mathcal{A} \simeq \mathrm{Im}\,\theta$ as exact cat. $\square$

$\mathcal{P}$ = the full subcat. of proj. in $\mathcal{A}$

1-3

2021年5月3日　　15:32

<u>Theorem</u>: $(\mathcal{A}, \mathcal{E})$ Frobenius

The stable category $\underline{\mathcal{A}}$ is canonically tr.:

① $\Sigma : \underline{\mathcal{A}} \xrightarrow{\sim} \underline{\mathcal{A}}$ ($\forall X$

$$\begin{tikzcd} 0 \arrow[r] & X \arrow[r] \arrow[d] & I(X) \arrow[r] \arrow[d, dotted] & \Sigma(X) \arrow[r] \arrow[d, dotted] & 0 \\ 0 \arrow[r] & Y \arrow[r] & I(Y) \arrow[r] & \Sigma(Y) \arrow[r] & 0 \end{tikzcd}$$

)

② conflations in $\mathcal{A}$ induce Δ in $\underline{\mathcal{A}}$

$$\begin{tikzcd} X \arrow[r, "i"] \arrow[d, equal] & Y \arrow[r, "p"] \arrow[d, dotted, "x"] & Z \arrow[d, dotted, "w"] \\ X \arrow[r] & I(X) \arrow[r, "p_X"] & \Sigma(X) \end{tikzcd}$$

conflation

a brief history:

① Heller 1968 (without TR4)

② Happel 1987 (received in 1985)

③ Linckelmann (1987), NOT well-known

④ Keller - Vossieck 1987 (in French, no detail, <u>one-sided tri. cat.</u>)

⑤ Gelfand - Manin 1988 (in Russian, transl. 1996)

1-4

2021年5月3日 11:45

<u>Def</u>: (Keller 2006 ICM) a tri. cat. $\mathcal{T}$ is <u>algebraic</u> if $\mathcal{T} \simeq \underline{\mathcal{A}}$, $\mathcal{A}$ Frobenius. $\square$

Δ-equ.

<u>RMK</u>: ① <u>in algebra</u>, all tri. cat. are algebraic

② more "topological" tri. cat. are NOT algebraic.

[Muro - Schwede - Strickland, Invent. Math. 2007]

[Rizzardo - Van den Bergh, Ann. Math. 2020]

§2. Algebraic tri. cat.

Fact: $\mathcal{A}$ Frobenius. $\mathcal{P} \subseteq \mathcal{A}$ ← proj. obj

$$C^{ac}(\mathcal{P}) \longrightarrow \mathcal{A} \qquad P \longrightarrow Z^1(P)$$

$(P^0 \twoheadrightarrow P^1$, image $f)$

induces

$$K^{ac}(\mathcal{P}) \overset{\text{tri. equi}}{\simeq} \mathcal{A}$$

$\therefore$ Frobenius "enhancements" are NOT unique!

Theorem (Keller 1994): Let $\mathcal{T}$ be an algebraic compactly generated tri. cat. Then $\exists$ a small DG category $\mathcal{C}$, s.t.,

$$\mathcal{T} \simeq D(\mathcal{C}^{op})$$

→ algebraic, compact. gen.

Proof: $\mathcal{T} \simeq \mathcal{A} \simeq K^{ac}(\mathcal{P})$

a set of generators $\mathcal{C} \subseteq C^{ac}_{dg}(\mathcal{P})$ ← dg structure

$\forall\, T \in \mathcal{T}$, $\mathrm{Hom}(-,T)|_{\mathcal{C}}$ a right dg $\mathcal{C}$-module (Hom$^{\bullet}$)

$$\Phi : \mathcal{T} \xrightarrow{\sim} D(\mathcal{C}^{op})$$
$$T \longmapsto \mathrm{Hom}(-,T)|_{\mathcal{C}}$$

- Φ preserves $\coprod$
- Φ sends compact to compact! $\square$

a $\boxed{\text{tilting}}$ version

Thm: $\mathcal{T} \simeq D(R\text{-Mod})$ for some ring R

$\iff$ $\mathcal{T}$ is algebraic, compactly generated with a tilting object (i.e., $T \in \mathcal{T}$
- compact
- T generates $\mathcal{T}$
- $\mathrm{Hom}_{\mathcal{T}}(T, \Sigma^i(T)) = 0 \quad \forall i \neq 0$)

Pf $\Rightarrow$ ✓ $\Leftarrow$ $\mathcal{T} \simeq D(A)$ A dg-alg $= \mathrm{End}^\bullet(T)$ $\simeq D(R\text{-Mod})$ quasi $\sim R = \mathrm{End}_{\mathcal{T}}(T)$ $\square$

a finite version

Prop: $\mathcal{T} \simeq K^b(R\text{-proj})$ for some ring R

(Bondal-Kapranov) $\iff$ $\mathcal{T}$ is idempotent-split, algebraic and has a tilting object (i.e. $T \in \mathcal{T}$
- $\mathrm{thick}\langle T\rangle = \mathcal{T}$
- $\mathrm{Hom}_{\mathcal{T}}(T, \Sigma^i(T)) = 0 \quad \forall i \neq 0$)

Rmk: · "tilting" originated from the reflection functors of BGP 1973

· a relevant paper by [Keller-Krause, 2020] CRM

3-1

2021年5月3日 15:46

§3. DG categories

k - a comm. ring

a dg k-category $\mathcal{C}$ is a category s.t.

① $\forall X, Y$. $\mathcal{C}(X,Y) = (\bigoplus_{p \in \mathbb{Z}} \mathcal{C}(X,Y)^p, d_{\mathcal{C}})$ $d_{\mathcal{C}}^2 = 0$

a complex

② Composition

$$\mathcal{C}(Y,Z) \otimes \mathcal{C}(X,Y) \longrightarrow \mathcal{C}(X,Z)$$
$$g \otimes f \longmapsto g \circ f$$

is a cochain map

- $|g \circ f| = |g| + |f|$
- $d_{\mathcal{C}}(g \circ f) = (d_{\mathcal{C}} g) \circ f + \underbrace{(-1)^{|g|}} g \circ (d_{\mathcal{C}} f)$

RMK: Kelly 1965, Kleiner-Roiter 1974, [Bondal-Kapranov 1990, Keller 1994, Drinfeld 2004 ...]

RMK 1. $\mathcal{C}(X,X)$ is a dg alg. $d_{\mathcal{C}}$

2. $\forall X, Y \in \mathrm{obj}\, \mathcal{C}$

${}_{\mathcal{C}(Y,Y)}\mathcal{C}(X,Y)_{\mathcal{C}(X,X)}$ $\mathcal{C}(Y,Y)$-$\mathcal{C}(X,X)$-bimodule

a dg

RMK: the ordinary cat. $Z^0(\mathcal{C})$ $\left\{ \begin{array}{l} \mathrm{obj} = \mathrm{obj}\,\mathcal{C} \\ Z^0\mathcal{C}(X,Y) = Z^0(\mathcal{C}(X,Y)) \end{array} \right.$ k-linear

the homotopy cat $H^0(\mathcal{C}) \twoheadleftarrow Z^0(\mathcal{C})$

$H^{0}(e)(x, y) = H^{0}(e(x, y))$

3-2

2021年5月3日 16:16

Example: $C_{dg}(k)$

- objects: cochain complex $X, Y, \dots$
- the Hom-complex

$Hom(X,Y)$:

$$Hom(X,Y)^n = \prod_{p\in\mathbb{Z}} Hom(X^p, Y^{n+p})$$

$$\ni f = (f_p)_{p\in\mathbb{Z}}$$

$$d: Hom(X,Y)^n \longrightarrow Hom(X,Y)^{n+1}$$

$$f \longmapsto df$$

$|f|$

$$(df)_p = d_Y^{n+p}\circ f_p - (-1)^n f_{p+1}\circ d_X^p : X^p \longrightarrow Y^{n+p+1} \quad \forall p\in\mathbb{Z}$$

Key Fact:

$$Z^0\, \mathcal{C}_{dg}(k) = C(k\text{-Mod})$$

$$H^0\, C_{dg}(k) = K(k\text{-Mod})$$

$|f|=0$, $df=0$

$\boxed{\text{coboundary} = 0}$

Example: $\mathcal{C}$ a small dg category

a left dg $\mathcal{C}$-module $M: \mathcal{C} \longrightarrow C_{dg}(k)$ a dg functor

$$M = \bigoplus_{X\in obj(\mathcal{C})} M(X) \ni m$$

$f: X\to Y$

$\boxed{f.m \triangleq M(f)(X)}$

$$\begin{array}{ccc} X & \longmapsto & M(X) \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle M(f)} \\ Y & \longmapsto & M(Y) \end{array}$$

the dg cat. $\mathcal{C}$-DGMod of left dg $\mathcal{C}$-modules

$$Hom_{\mathcal{C}}(M,N)^n = \{\, \xi = (\xi_X)_{X\in obj(\mathcal{C})}$$

s.t. $\xi_X : M(X) \longrightarrow N(X)$ $|\xi_X| = n$

$Hom_{\mathcal{C}}(M,N)$ complex!

naturalness

$\forall f: X\to Y$ in $\mathcal{C}$

up to $(-1)^{|M(f)|\,|\xi|}$

$$\left.\begin{array}{ccc} M(X) & \xrightarrow{\xi_X} & N(X) \\ {\scriptstyle M(f)}\downarrow & \wr & \downarrow{\scriptstyle N(f)} \\ M(Y) & \xrightarrow{\xi_Y} & N(Y) \end{array}\right\}$$

$\forall f: x \to y$ in $\mathcal{C}$ $\quad$ $\mathcal{M} \dashv \mathcal{N}$ $\quad$ $M(y) \xrightarrow[\cong]{f_y} N(y)$ $\quad$ $\mathrm{id}_{\mathcal{M}}(f)$

up to $(-1)^{|M||f|}$!!

3-3

2021年5月5日 13:22

The dg cat. $\mathcal{C}$-DGMod has $\boxed{\text{rich}}$ structures.

① $\Sigma : \mathcal{C}\text{-DGMod} \xrightarrow{\sim} \mathcal{C}\text{-DGMod}$ a dg functor

$$\begin{tikzcd} M \arrow[r, mapsto] \arrow[d, "\xi"'] & \Sigma M \arrow[d, "\Sigma\xi"] \\ N \arrow[r, mapsto] & \Sigma N \end{tikzcd}$$

the definition of ΣM:

$$\Sigma M : \mathcal{C} \longrightarrow C_{dg}(k)$$

$$\begin{tikzcd} x \arrow[r, mapsto] \arrow[d, "f"'] & \Sigma(M(x)) \arrow[d, "\Sigma(M(f)) \leftarrow (-1)^{|f|}"] \\ y \arrow[r, mapsto] & \Sigma(M(y)) \end{tikzcd}$$

② $\forall\, \xi : M \longrightarrow N \quad |\xi| = 0,\ d\xi = 0$

$\exists \quad \mathrm{Con}(\xi) = N \oplus \Sigma M$ as graded modules

cone of ξ!

$\forall\, x \in \mathrm{obj}\,\mathcal{C}$

$$\mathrm{Con}(\xi)(x) = N(x) \oplus \Sigma M(x)$$

$$d_{\mathrm{Con}(\xi)(x)} = \begin{pmatrix} d_{N(x)} & 0 \\ \xi_x & d_{\Sigma M(x)} \end{pmatrix}$$

Fact:

- $Z^0(\mathcal{C}\text{-DGMod})$ is a Frobenius cat. (conflation = $\boxed{\text{graded-split}}$ s.e.s.)
- $H^0(\mathcal{C}\text{-DGMod}) = \underline{Z^0(\mathcal{C}\text{-DGMod})}$ is tri. cat.

  $\quad = K(\mathcal{C})$
- $D(\mathcal{C}) = K(\mathcal{C})[\text{quas-iso.}^{-1}]$, derived cat. of dg $\mathcal{C}$-modules.

a dg-iso in $\mathcal{C}$: a morphism $f: X \to Y$

$$\begin{cases} |f| = 0, \ df = 0 \\ \exists\, g: Y \to X \quad g\circ f = Id_X, \ f \circ g = Id_Y \end{cases}$$

a homotopy equivalence $f: X \to Y$ in $\mathcal{C}$ if
- $|f| = 0$, $df = 0$
- f is an iso. in $H^0(\mathcal{C})$.

"$\mathcal{C}$-DGMod"

BK, strongly

pretri.

Def (Keller 1999) a dg cat. $\mathcal{C}$ is exact if

① $\forall X$, $\exists$ closed iso. of degree one $X \xrightarrow{\sim} X_1$, $X_2 \xrightarrow{\sim} X$ for some X_1, X_2

② $\forall f: X \to Y$, $df = 0$, $|f| = 0$, $\exists$ a diagram

$$\begin{tikzcd} Y \arrow[r, shift left, "i"] & C \arrow[l, dashed, shift left, "s"] \arrow[r, shift left, "p"] & X \arrow[l, dashed, shift left, "t"] \end{tikzcd}$$

s.t.
- $|i| = 0 = |s|$, $|p| = 1$, $|t| = -1$
- $di = 0 = dp$
- $p \circ i = 0$, $t \circ s = 0$, $Id_C = i \circ s + t \circ p$
- $\boxed{s \circ (dt) = f}$

$i \circ s = Id_Y$, $Id_X = p \circ t$ $\square$

RMK X_i the internal shifts of X $[X_2 = \Sigma(X)]$

C the internal cone of f

3-5

2021年5月3日 16:15

Proposition (Keller): $\mathcal{C}$ exact dg cat.

Then $Z^0(\mathcal{C})$ has a natural Frobenius exact structure s.t.

$$H^0(\mathcal{C}) = \underline{Z^0(\mathcal{C})}.$$

$\therefore$ $H^0(\mathcal{C})$ is canonically triangulated! $\square$

Problem: exact dg categories are NOT invariant under quasi-equivalences!!

Def: a dg functor $F: \mathcal{C} \to \mathcal{D}$ is quasi-fully faithful, if $\forall\, x, y \in \mathrm{Obj}\,\mathcal{C}$

$$\mathcal{C}(x, y) \xrightarrow[\sim]{F} \mathcal{D}(Fx, Fy) \quad \text{quasi-iso.}$$

a dg functor $F: \mathcal{C} \to \mathcal{D}$ is quasi-equivalence if it is quasi-f.f. and $H^0(F): H^0(\mathcal{C}) \to H^0(\mathcal{D})$ is dense. $\square$

equivalence

Slogan: $\boxed{\text{quasi-equivalent dg cat. should be identified!}}$

3-6

2021年5月5日 13:33

Fact: $\forall\, \mathcal{C}$, $\exists$ a universal dg functor
$$\mathcal{C} \hookrightarrow \mathcal{C}^{ex}$$
into an exact dg cat, the exact hull

RMK: ① explicit construction by Bondal-Kapranov 1990

② the Yoneda embedding
$$\mathcal{C} \hookrightarrow \mathcal{C}^{op}\text{-DGMod}$$
$$X \longmapsto \mathcal{C}(-,X)$$

$\mathcal{C}^{ex} = \mathcal{C}(-,X)$, closure under $\Sigma^{\pm 1}$, Cone.

Def: (Bondal-Kapranov 1990) $\mathcal{C}$ is pretriangulated if $\mathcal{C} \hookrightarrow \mathcal{C}^{ex}$ is a quasi-equivalence.

$\Rightarrow H^0(\mathcal{C})$ is triangulated ! $\square$

$D(\mathcal{C}^{op})$ Δ-subcat.

Fact: $\forall\, \mathcal{C} \xrightarrow{F} \mathcal{D}$ $\forall$ dg functor between pretri. dg cat.
Then $H^0(F): H^0(\mathcal{C}) \longrightarrow H^0(\mathcal{D})$ is naturally a triangle functor ! $\square$

Fact: $\forall\, F: \mathcal{C} \to \mathcal{D}$ a quasi-equivalence.
Then $\mathcal{C}$ is pretri. $\Leftrightarrow$ so is $\mathcal{D}$. $\square$

4-1

2021年5月5日 13:40

§4. DG enhancements

dgcat = the cat. of small dg cat.

$\mathrm{Hodgcat}$ = $\mathrm{dgcat}[\text{quasi-equivalence}^{-1}]$ → the homotopy cat. of dg cat.

RMK. · dgcat has a model structure [Tabuada 2005]

· a morphism $\mathcal{C} \dashrightarrow \mathcal{D}$ in Hodgcat,

$$\mathcal{C} \xleftarrow{F} \mathcal{C}' \longrightarrow \mathcal{D}$$

F quasi-equivalence

called a <u>dg quasi-functor</u>

<u>Def</u>: $\mathcal{T}$ a tri. cat.

a <u>dg enhancement</u> $(\mathcal{C}, E)$, $\mathcal{C}$ pretri. dg. cat.

$$E: H^0(\mathcal{C}) \xrightarrow{\sim} \mathcal{T}. \qquad \square$$

<u>Prop</u>: $\mathcal{T}$ has a dg enhancement iff $\mathcal{T}$ is algebraic

<u>proof</u>: " $\Leftarrow$ " $\mathcal{T} \simeq \underline{\mathcal{A}} \simeq K^{ac}(\mathcal{P})$ ← exact

$$= H^0(\underline{C^{ac}_{dg}(\mathcal{P})})$$

" $\Rightarrow$ " $\mathcal{C} \xrightarrow{\sim} \mathcal{C}^{ex}$, $\underline{H^0(\mathcal{C}^{ex})} = \underline{Z^0(\mathcal{C}^{ex})}$

$$\mathcal{T} \simeq \underline{H^0(\mathcal{C})} \qquad \square$$

"$\Longrightarrow$" $\mathcal{C} \xrightarrow{\sim} \mathcal{C}^{ex}$. $\underline{H^0(\mathcal{C}^{ex})} = \underline{Z^0(\mathcal{C}^{ex})}$

$$\begin{array}{c} H^0(\mathcal{C}^{ex}) \\ \wr\!\wr \\ \mathcal{A} \simeq \underline{H^0(\mathcal{C})} \end{array}$$

$\square$

4-2

2021年5月6日 20:51

Theorem (Keller 1999 / Drinfeld 2004)

Assume that $\mathcal{T} = H^0(\mathcal{C})$ and $\mathcal{N} = H^0(\mathcal{D})$ with $\mathcal{D} \subseteq \mathcal{C}$

Then $\exists$ a dg cat $\mathcal{C}/\mathcal{D}$ s.t.

$$H^0(\mathcal{C}/\mathcal{D}) = \mathcal{T}/\mathcal{N}$$

$\to$ Verdier

$\mathcal{C}/\mathcal{D}$ = the dg quotient ($\Leftrightarrow$ $\mathcal{T}/\mathcal{N}$ algebraic !!) $\square$

Example:

$$D^b(\mathcal{A}) = K^b(\mathcal{A}) / K^{b,ac}(\mathcal{A}), \quad \mathcal{A} \text{ abelian}$$

$$D^b_{dg}(\mathcal{A}) = C^b_{dg}(\mathcal{A}) / C^{b,ac}_{dg}(\mathcal{A})$$

the bounded $\boxed{\text{dg}}$ derived category of $\mathcal{A}$

$$\boxed{H^0(D^b_{dg}(\mathcal{A})) = D^b(\mathcal{A})}.$$

RMK:

- the uniqueness of enhancements is important!
  - [Lunts - Orlov, JAMS 2010]
  - [Canonaco - Stellari, JEMS 2016]
  - [Canonaco - Neeman - Stellari, 2021]
- Related to an open question by Rickard 1991:
  is any derived equivalence standard?

Def: Assume that $F: D^b(\mathcal{A}) \to D^b(\mathcal{B})$ is a triangle functor. By a <u>dg lift</u> of F, we mean

$$\tilde{F}: D^b_{dg}(\mathcal{A}) \dashrightarrow D^b_{dg}(\mathcal{B}) \quad \text{in Hodgcat}$$

s.t. $H^0(\tilde{F}) \simeq F$.

RMK: $\exists$ a triangle functor, NOT liftable [Rizzardo – Van den Bergh – Neeman, Invent. Math. 2019]

Prop: a tri. functor $F: D^b(A\text{-mod}) \to D^b(B\text{-mod})$ is liftable ($\tilde{F}$) iff $F \simeq X \otimes^{\mathbb{L}}_A -$ (X a B-A-bimodule) $\to$ standard. $\square$

RMK: Keller 2005 / Toën 2007 / Chen-C. 2020 (B.LMS)

Geigle-Lenzing 1987

$\mathcal{T} \in D^b(\mathrm{coh}\,\mathbb{P}^1) \hookrightarrow D(\mathrm{Qcoh}\,\mathbb{P}^1)$

$\qquad\quad \wr\wr \qquad\qquad\qquad\qquad \wr\wr$

$\mathbb{R}\mathrm{Hom}(\mathcal{T},-)\quad D^b(\mathrm{mod}\text{-}\rightrightarrows) \qquad K(\mathrm{Inj}/\mathbb{P}^1)$

$\qquad\quad \wr\wr \qquad\qquad\qquad\qquad \|$

$K^b(\mathrm{proj}\rightrightarrows) \qquad H^0(C_{dg}(\mathrm{Inj}/\mathbb{P}^1))$

$$D_{sg}(A) = D^b(A\text{-mod})/\mathrm{perf}(A)$$

$$\boxed{S_{dg}(A)} \triangleq D^b_{dg}(A\text{-mod})/_{dg}\mathrm{perf}(A) = \text{"colimit"}$$