Chaos in a "bouncing ball" electronic circuit

20275516
School of Physics and Astronomy
University of Nottingham

03/04/2022

Abstract

The aims of this investigation were to explore chaotic behaviour and patterns through a non-linear circuit, and using the data obtained, find a coefficient of restitution for this system, plot phase space maps and Poincaré sections to observe behaviour and find the Feigenbaum constant. The circuit used was analogous to that of a bouncing ball on an oscillating surface, with the two control parameters being the coefficient of restitution K and the ratio of the surface acceleration over the acceleration due to gravity α. A square wave was inputted into the circuit to observe damped oscillatory behaviour. This was used to find the coefficient of restitution for this circuit. The value calculated from the data was $K = 0.522 \pm 0.004$. Next, sinusoidal waveforms were inputted into the circuit with equal amplitudes, and different frequencies. Phase space maps and Poincaré sections were plotted for each waveform. At lower frequencies the orbits on the phase space maps were stable, however, as the frequency increased, the motion of the 'ball' underwent a series of period-doubling bifurcation which lead to chaotic behaviour of the system. This can be seen through the multiple non-repeating orbits in the phase space maps and the Poincaré sections converging towards a strange attractor. The frequency was then set to 450Hz and an amplitude that increased in increments was coded into the waveforms. The positions of the ball, on Poincaré sections, were plotted against surface amplitude. This produced a bifurcation diagram which was used to find the Feigenbaum constant. The value of the Feigenbaum constant calculated was $\delta_{expm} = 4.4 \pm 0.5$ with the accepted value of $\delta = 4.669$ lying within one standard deviation[1]. Due to the accepted Feigenbaum constant lying within the experimental uncertainty, the experimental value can be assumed to be equal to the accepted value $\delta_e xpm = \delta$, this implies all non-linear chaotic systems bifurcate at the same rate.

1 Introduction

Chaotic systems appear everywhere in nature, some examples of this are the weather, ocean turbulence and tree branches. 'A weather prediction cannot be perfectly verified after a few days of integration due to the non-linearity of the equations of the hydrodynamic models'[2], this statement signifies, as the system progresses, it becomes more unpredictable and seemingly random due to the non-linearity of its deterministic laws. In accordance with this, the latter examples follow the the same unpredictable and erratic nature due to their individual non-linear deterministic laws. These systems may seem random in nature due to their high unpredictability, however, this is often mistaken due to their extreme sensitivity to initial conditions. As these non-linear systems progress, they becomes more unpredictable and seemingly random as there are continuous bifurcations in there state, with the point at which the bifurcation occurs being dependent on the initial conditions. The system then takes the state of one of the two bifurcations which can be predicted if the initial state of the system is known. However, it is impossible to know, precisely, to the infinite decimal point the initial state of the system[3], which is where the confusion of the random nature arises from.

There have been many studies of different chaotic systems which have lead to potential applications. A fairly obvious application of chaotic systems is 'random' number generation. As stated previously, chaos is not truly a random process, however, due to its nature it may appear random and therefore can be used as a pseudo-random number generator[4]. Although this generator may not truly be random, the bias to it will be negligible, essentially having no difference to a truly random number generator. A less obvious application of chaos is audio encryption. A chaotic system can be coded to confuse and diffuse audio data, with the initial value of the chaotic system being controlled by the hash value of the audio, this makes the chaotic trajectory unpredictable[5]. Additionally, 3D image encryption through chaotic systems has also been explored[6] in a similar way to the audio encryption stated previously.

This investigation explores chaotic behaviour through a non-linear circuit, analogous to that of a bouncing ball on an oscillating surface. The investigation could have been carried out mechanically, using an actual ball, oscillating surface and equipment to track the state of the ball, however, the equipment required for this is very complex and only the first bifurcations will be visible due to noise[7]. Bifurcations of this system occur when the period doubles, this is when a slight change in the systems parameters cause a new periodic orbit to appear from the existing periodic orbit. The circuit used considers all aspects of a mechanically bouncing ball, it considers: free fall, dissipative and elastic impacts, and energy exchange with the movable surface. As these can be simulated, a coefficient of restitution K will also be present. This value is a measure of the elasticity of collisions and can vary from system to system. It takes values of $0 < K < 1$ with 1 being a perfectly elastic collision and 0 being a perfectly inelastic collision. K is one of the two control parameters of the system and will remain constant throughout, however, the second control parameter, that being the ratio of the acceleration of the oscillating 'surface' and gravity, α can be changed. The acceleration of the 'surface' is dependent on the waveform inputted into it, meaning it fundamentally is dependent on the frequency and amplitude. By outputting different waveforms to the 'surface', with different amplitudes and frequencies, it was possible to take advantage of the period doubling to explore chaotic behaviour.

Phase space maps and Poincaré sections were used as plots for analysis in the investigation. Phase space is useful in tracking the states of systems. Due to its multidimensional space, each axis corresponds to one state of the system and therefore each point in phase space is a different state of the system. The behaviour in phase space can be tracked in phase space maps which have successfully been used to analyse a range of complex systems, including microscopic one-dimensional neuron models[8] and macroscopic six-dimensional galaxy potentials[9]. These maps are easier to use when the motion of the mechanical system is restricted to certain axis. The bouncing ball is restricted to the axis parallel to the oscillation of the surface and gravity, and due to the analogies between the mechanical motion and circuital voltages, phase space maps are very convenient for this investigation. Poincaré sections are almost exclusively used in analysing systems that appear to have periodic behaviour, due to this it makes identifying chaotic systems very convenient. When exhibiting chaotic behaviour, a Poincaré section will converge to a curve known as a strange attractor. This perfectly describes chaotic behaviour as it is impossible to tell where on the attractor the state of the system may be in at any time. For example, two points on the attractor may be close at one point then relatively further away after time has passed. This emphasises the chaotic and seemingly random nature of chaos.

Another plot used for analysis in this investigation was the bifurcation diagram which shows the systems descent into chaos. In general, the first bifurcations are easy to see as they are fairly spread out, however, as a system progresses, bifurcations gradually become closer and due to the number of routes increasing, individual bifurcations will become nearly impossible to see. A constant called the Feigenbaum constant, with a value of $\delta = 4.669$ can be found from this graph by using the locations of successive bifurcations[1]. As this constant always arises from bifurcation diagrams, this implies that all chaotic systems bifurcate at the same rate.

The objectives of this investigation were to find the coefficient of restitution of the circuit, plot phase space maps and Poincaré sections for different frequencies and plot a bifurcation diagram which will be used to calculate the Feigenbaum constant.

2 Theory

The circuit shown in figure 1 inputs velocity v_c and position v_b of the 'ball' in volts. Likewise, the surface output v_t is given equivalently. All elements of the mechanical setup have been accounted for in the circuit. When the diode shown in figure 1 is reverse biased, and the current exceeds the intrinsic reverse current of the diode, the ball will analogously be in free fall. When the diode begins to conduct, it will provide a current large enough to invert the charge C_2, that being the capacitor in parallel with the v_b voltage input. This is analogous to the change in velocity when the ball impacts the surface elastically. To simulate the dissipative element the capacitor C_f and the resistor R_f have been included in the circuit.

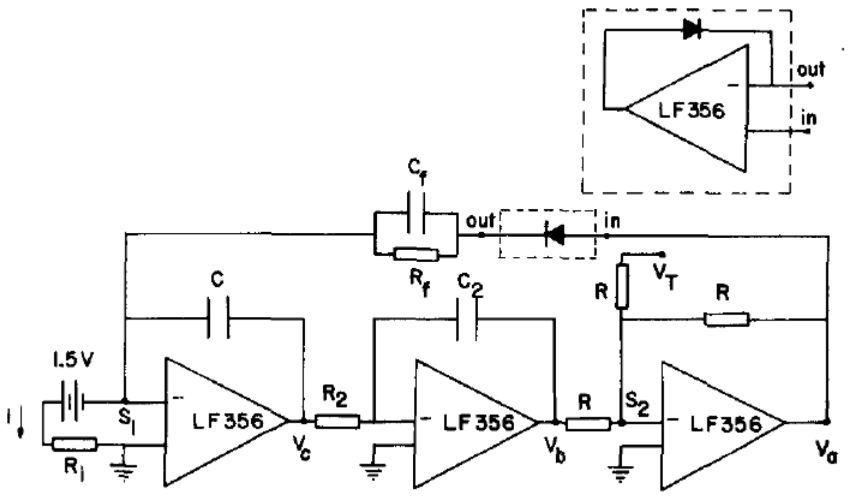

Figure 1: Diagram showing a similar circuit to that used in the investigation[7].

A square wave outputted to v_t will create a damped oscillation of the ball. The three graphs plotted in figure 2 are predictions of how the circuit will respond to the outputted square wave, these being a signal against time (the square wave outputted), ball position against time and ball velocity against time. The two latter graphs can be used to find the coefficient of restitution using,

$$K = \sqrt{\frac{x_{N+1}^{max}}{x_N{}^{max}}}, \tag{1}$$

with $x_N{}^{max}$ being the initial peak of the balls position or velocity in volts and x_{N+1}^{max} being the subsequent peak.

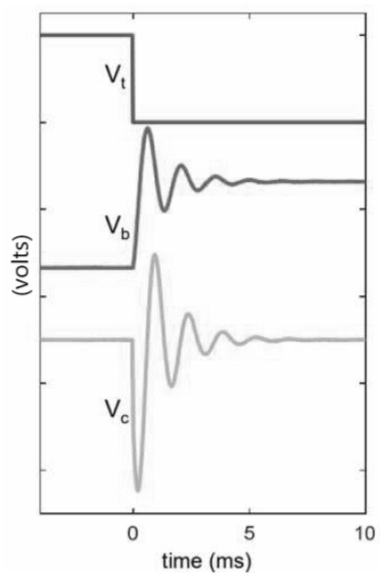

Figure 2: Expected circuit response to a square wave.

The phase space most useful for this investigation is the velocity-position phase space due to these making up the state of the system. When a sinusoidal wave is outputted to v_t with a small amplitude and or frequency (small α), the orbit of the ball should be stable. This can be seen in the predicted stable orbits in figure 3. These orbit(s) are said to be stable due to there distinct number of orbit(s) and predictability of future states. When the amplitude and or frequency of the oscillation is increased (large α) the system should enter the chaotic regime as shown in figure 3. This can be said to be chaotic due to multiple non-repeating orbits. This also makes predicting the future state of the system near to impossible which follows the unpredictable nature of chaos.

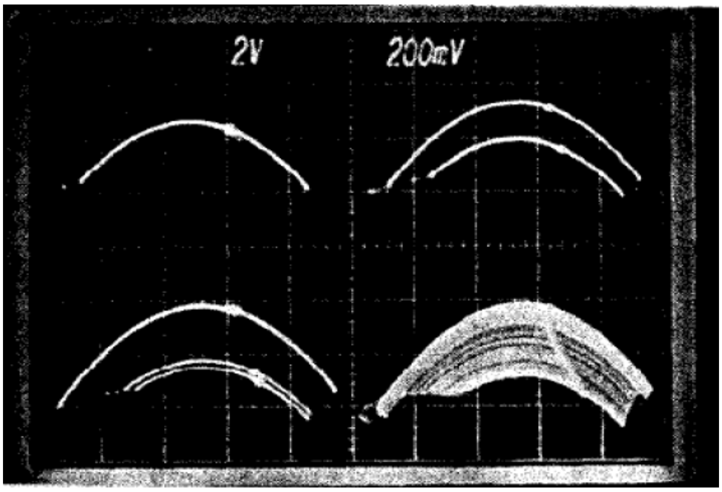

Figure 3: The expected phase space maps for the investigation. Period-1 orbit (top left). Period-2 orbit (top right). Period-3 orbit (bottom left). Non-repeating orbits in the chaotic regime (bottom right)[7].

A period-1 orbit will oscillate with the same period of the surface; for a period-2 orbit, the original period-1 orbits' period would have doubled once, creating an additional orbit, the ball will now have two orbits with different periods. Each separate orbit will have a period of one surface oscillation, therefore if the system is in a period-3 orbit, it will take 3 oscillations of the surface to return to the first orbit. The period will continue to double until the system enters the chaotic regime and when this occurs the orbits should become non-repeating.

Poincaré sections can be plotted as an addition to phase space maps. To plot the Poincaré section for this system, one point per period of the external drive, from the phase space map is plotted. The period of the external drive being one over the frequency of the waveform outputted, $T = \frac{1}{f}$. For a period one orbit this should plot a single point due to the Poincaré section taking a data point from the same place each orbit on the phase space map. Similarly, for a period-two orbit, the Poincaré section will take a point from each orbit, hence there should be two points plotted. The Poincaré section should exhibit n distinct points when the period has doubled n-1 times. Once the system has entered the chaotic regime, the Poincaré section will converge to a curve known as a strange attractor, shown in figure 4, which essentially is made up of points from each of the multiple non-repeating orbits in the chaotic phase space map.

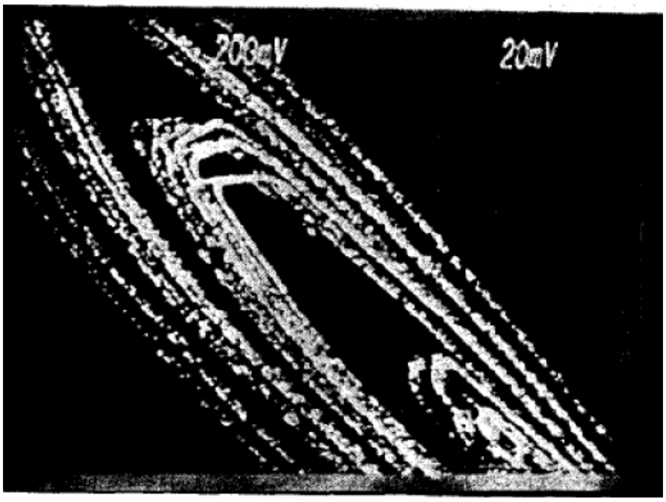

Figure 4: Poincaré section of a chaotic system showing a strange attractor[7]

A bifurcation diagram can be plotted by taking the values of v_b or v_c from each Poincaré section and plotting it against its frequency, while keeping the amplitude constant, and vice versa. In figure 5, bifurcations can be seen, as stated previously the points at which the diagram bifurcates are points when the period doubles. In addition, the curved lines that can be seen running through the whole graph are curved routes the state of the system takes. There are multiple routes the system can take, however, these are hidden within the others and are not clear due to not being smooth. The system is predicted to quickly fall into chaos, with bifurcations occurring closer the more the period has doubled. This can be seen in figure 5 where the first bifurcation occurs approximately 0.45 away from the second and the second bifurcation occurring approximately 0.1 away from the third. The x value of these bifurcation points can be used to find the Feigenbaum constant using,

$$\delta = \frac{\mu_{n+1} - \mu_n}{\mu_{n+2} - \mu_{n+1}} \tag{2}$$

Where μ_n would be the amplitude or frequency of the first bifurcations, μ_{n+1} being the next and μ_{n+2} being the third. The accepted value of the Feigenbaum constant is $\delta = 4.669$ and should appear from all bifurcation plots due to the theory that all chaotic systems bifurcate at the same rate[1].

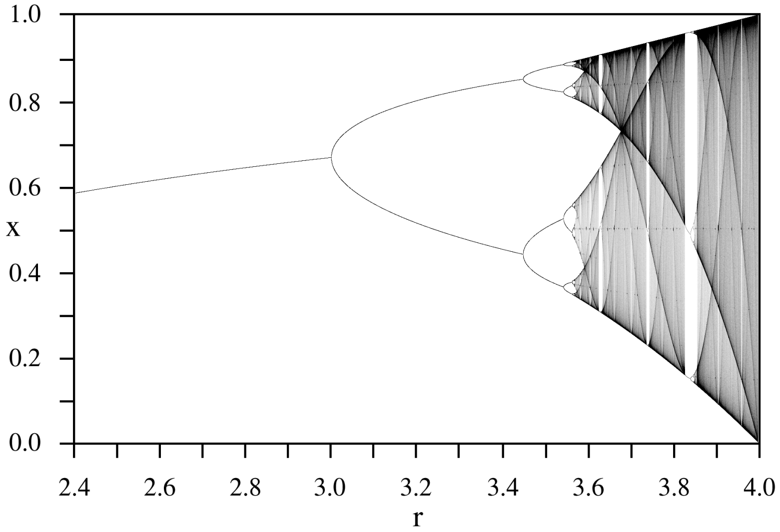

Figure 5: An ideal expected bifurcation diagram [10]

As stated in the introduction, period doubling will occur through a slight change in the systems parameters. The parameter that will be changed is α and the behaviour will be monitored through phase space maps and Poincaré sections. Transient behaviour will be present at the start of every waveform output, this will need to be removed as it can be mistaken for a random additional orbit that is not there.

3 Experimental method

The non-linear circuit used in this investigation was similar to that shown in figure 1. A computer next to this circuit was connected to it through a break out box. This break out box was used to output signal voltages into the circuit through the computer and receive input voltages. Analog DAQ input and output channels were coded for the states of the system and the control parameter. Input 0 was connected to v_b on the circuit, input 1 was connected to v_c and output 0 was connected to v_t. v_t was outputted to the circuit using the a.run() command and the input received was a 2D array containing values for v_c and v_b, and another array containing the time these data points occurred, from the time the code was run.

To gather the data needed to find the coefficient of restitution, a square wave with a maximum gain of $1V$ and a minimum of $0V$ was outputted into the surface v_t. A sampling rate of 25000 was set due to the system response occurring in a small time interval. The inputs and output were plotted in graphs against time to see how the system reacted. Transient data and other insignificant data were removed from the data set by array slicing, leaving only the data where the system reacts to the square wave. A value of 0.8445 was added to the v_b data set to bring the equilibrium to zero which made it easier to use the relative positions of the peaks when calculating K. This value was found by finding the average of the points furthest down the data set, when the system had a negligible oscillation. The peaks were qualitatively found from the data set and set to two decimal places as the data was moving in significant increments of about 0.01. This gave a more fitting uncertainty rather than using up to the 10s.f given in the input data. The graphs of v_b and v_c were then re-plotted with the values of their peaks which made it more simple to calculate values of K. Three values of K were found from each graph, using equation 1, and these were all averaged to find a final value for the coefficient of restitution for this system.

Sinusoidal waveforms were outputted to v_t to see how the system would respond to a change in the control parameter α. The waveforms were outputted with constant amplitudes of 1V and different frequencies of 450Hz, 600Hz, 650Hz, 750Hz and 1000Hz. The sampling rate was dependent on the frequency, with a higher sampling rate for higher frequencies, due to orbits being faster which may make non smooth orbits if not large enough. Initial transient behaviour was removed from the array leaving the non-transient parts of the data to be plotted. v_b was plotted against v_c to form a phase space map. The same sliced array that was used to plot the phase space map was used to plot the Poincaré section, in the way described in the theory section. This was done through code, by appending the selected data points required to new arrays. These new arrays were then plotted against each other, v_b against v_c. The same axis limits were used for the phase space map and Poincaré section as it somewhat made it clear to see where the individual orbits were. The response of the circuit was also plotted in graphs of v_b against time, v_c against time and v_t against time, in subplots stacked on top of one another so it was easy to see how the position and velocity varied with the signal.

In a new code, data was taken to create Poincaré sections for a constant frequency of 450Hz and an amplitude that increased in steps of 0.005V, from 0.1V to 3V. The values of v_b, from the Poincaré section were plotted against there equivalent amplitude to form a bifurcation diagram. The data was then qualitatively analysed to find the points at which period doubling bifurcations occurred and when found these points were added to the bifurcation diagram. These values were then used to calculate the Feigenbaum constant, using equation 2. In figure 6 a flow chart shows how the code worked to form the bifurcation diagram.

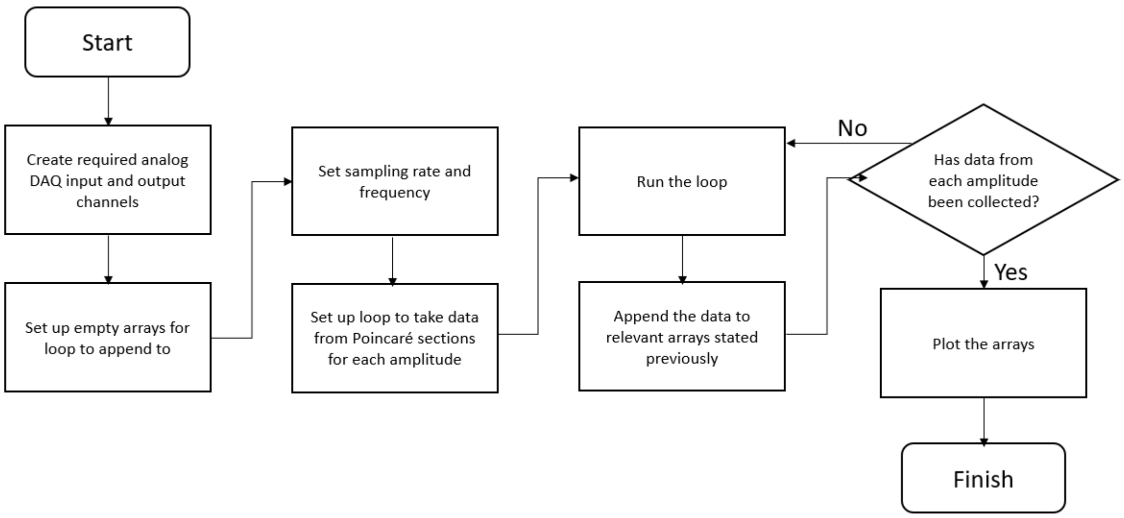

Figure 6: Flow chart showing how the code that formed the bifurcation diagram worked.

4 Results

Figure 7, shows the response of the circuit to an output square wave. This was produced as expected.

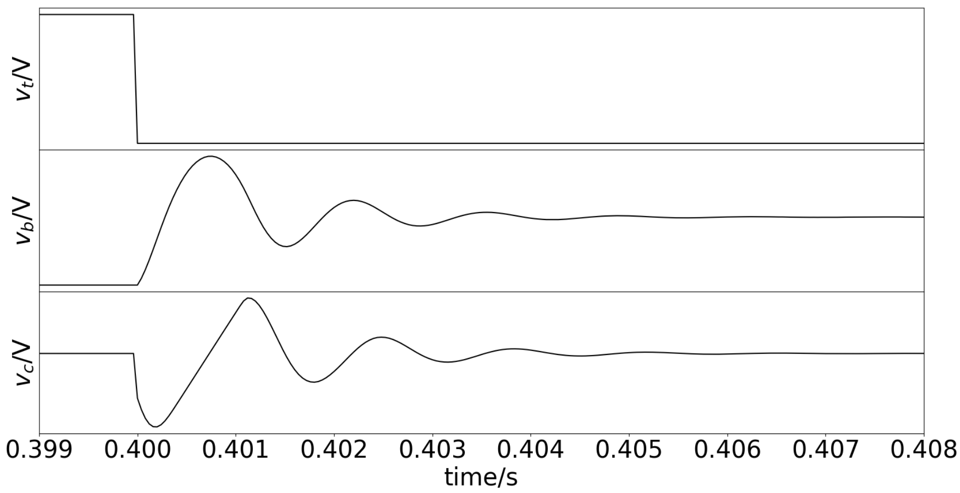

Figure 7: Response of the circuit to an output square wave.

The value for the coefficient of restitution calculated was $K = 0.503 \pm 0.004$, this was made easier through the use of figure 8 as it clearly shows the peaks relative to the equilibrium. An uncertainty propagation equation was derived for equation 1 and the uncertainties for all the values of K were found through this. All the uncertainties were equal, so when finding a mean calculated value of K, the standard error equation was used. The mean average of K was then calculated using the 6 values found previously. The average value found can be said to be fairly accurate due to the number of K values found from the readings.

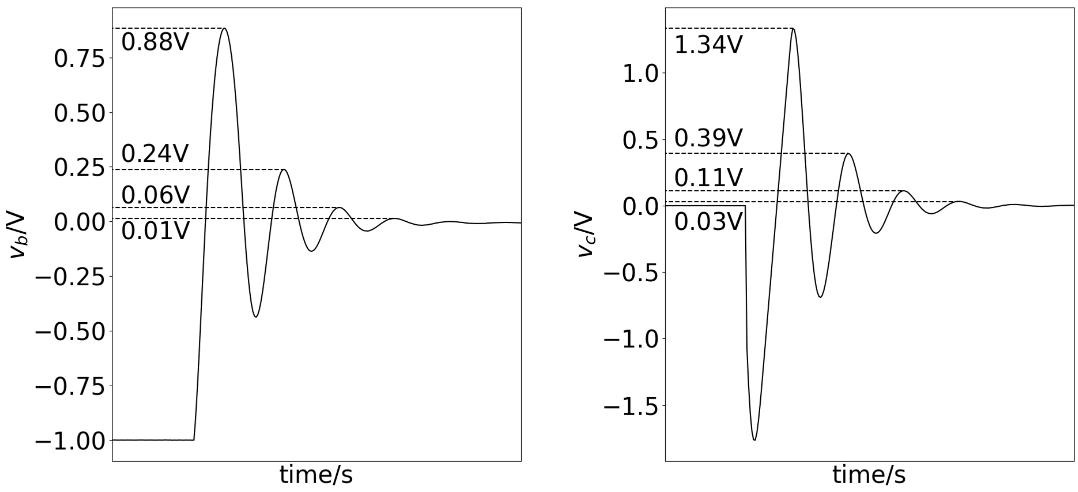

Figure 8: Re-plotted graphs, setting the equilibrium to zero and values of the peaks shown.

The phase space map shown in figure 9 displays a single very clear orbit. The ball, when the external drive is 450Hz, can be deduced as having a period-1 orbit. More evidence to support this claim is the Poincaré section having a single point plotted. The single point lies on the orbit of the ball, as expected. This orbit can be assumed to be stable due to this and as no chaotic behaviour is present. This phase space map and Poincaré section is exactly as expected for this period-1 orbit, however, the phase space map is not a perfect circle which may be an effect of the coefficient of restitution.

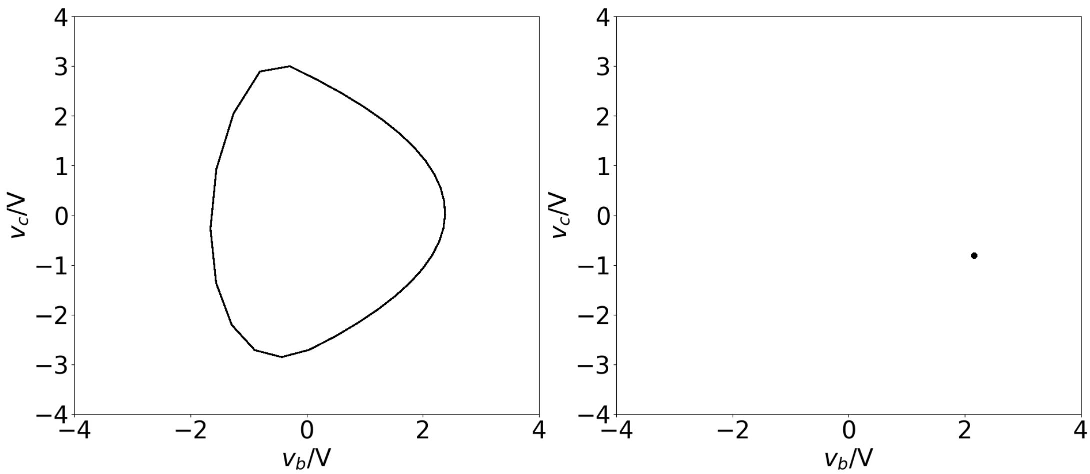

Figure 9: Phase space map (left) and Poincaré section (right) produced by a surface oscillation with a frequency of 450Hz and an amplitude of 1V

Figure 10 shows three plots which display normal oscillations. This is more evidence to prove the system is stable as each state is repeating.

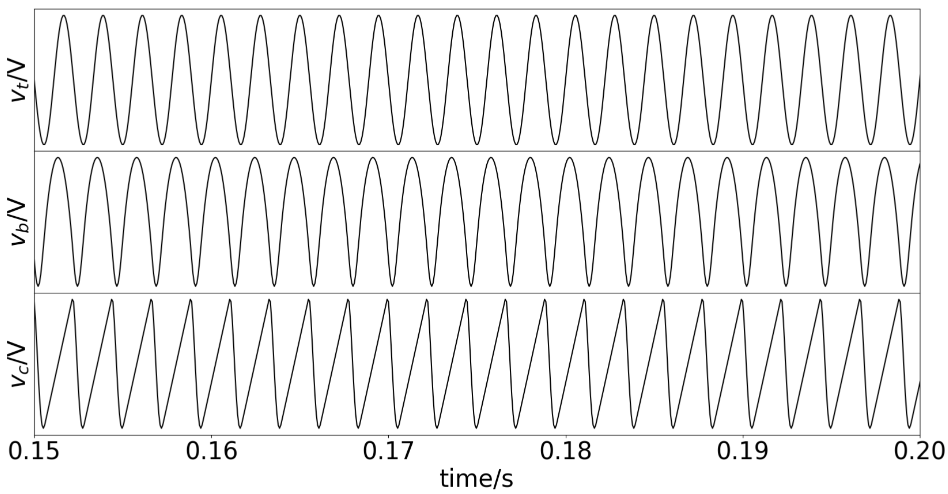

Figure 10: Response of the circuit to a 450Hz, 1V amplitude sinusoidal waveform.

In figure 11, two stable orbits can be seen. This increase in frequency has caused the period to double. Again, to support this, the Poincaré section has two very distinct points that lie on the two orbits of the phase space map. The system ,when the external drive has these parameters, can be assumed to be in a period-2 orbit. This period-2 orbit has turned out as expected.

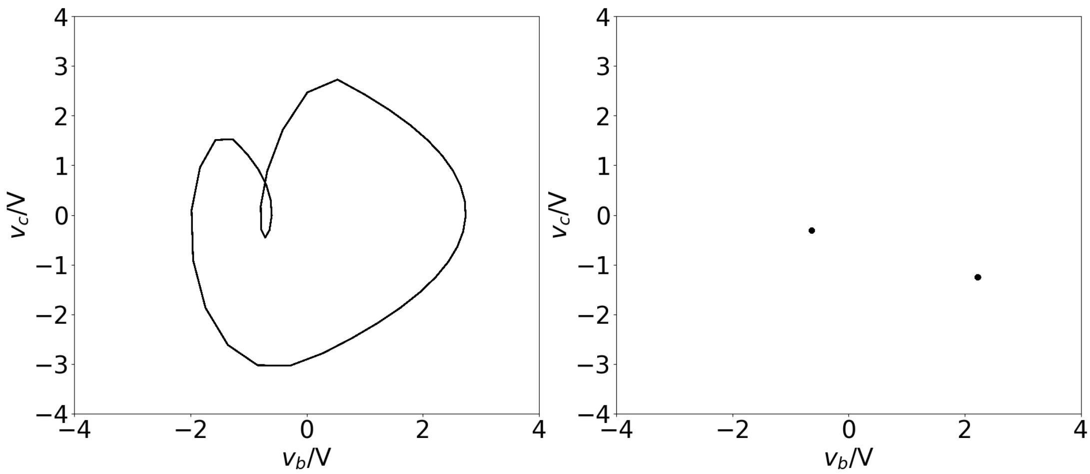

Figure 11: Phase space map (left) and Poincaré section (right) produced by a surface oscillation with a frequency of 600Hz and an amplitude of 1V.

Two different oscillations can be seen in the lower two subplots in figure 12 with the smaller one integrated into the original one. This is still a stable orbit as ball position and velocity are repeating.

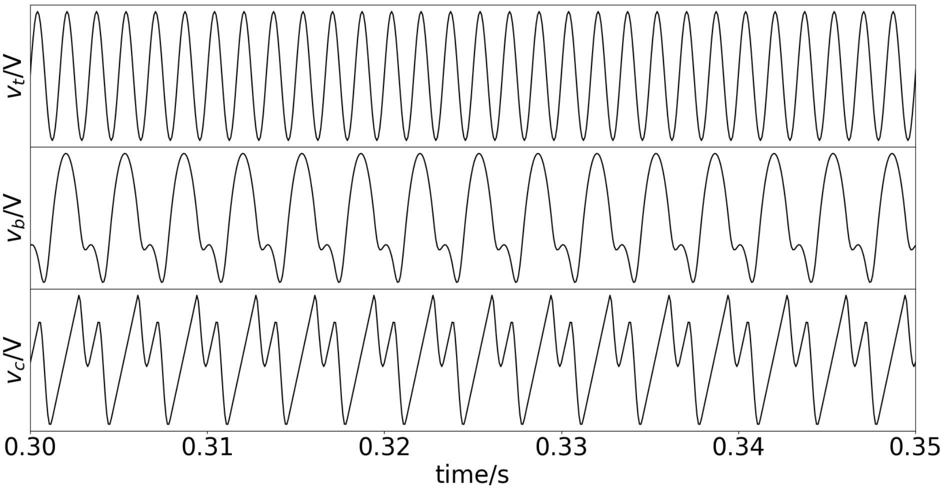

Figure 12: Response of the circuit to a 600Hz, 1V amplitude sinusoidal waveform.

Figure 13 appears to show three stable orbits, however, due to the Poincaré section showing four distinct points, this can be assumed to be a period-four orbit, with the forth being hidden within one of the others. When zooming into an orbit, multiple orbits can be seen, however these may be of the same one which are offset due to random errors in the system. For this period-4 orbit, all the orbits that can be seen when zoomed in are all muddled together which makes it appear as a single orbit, even though there may be two present.

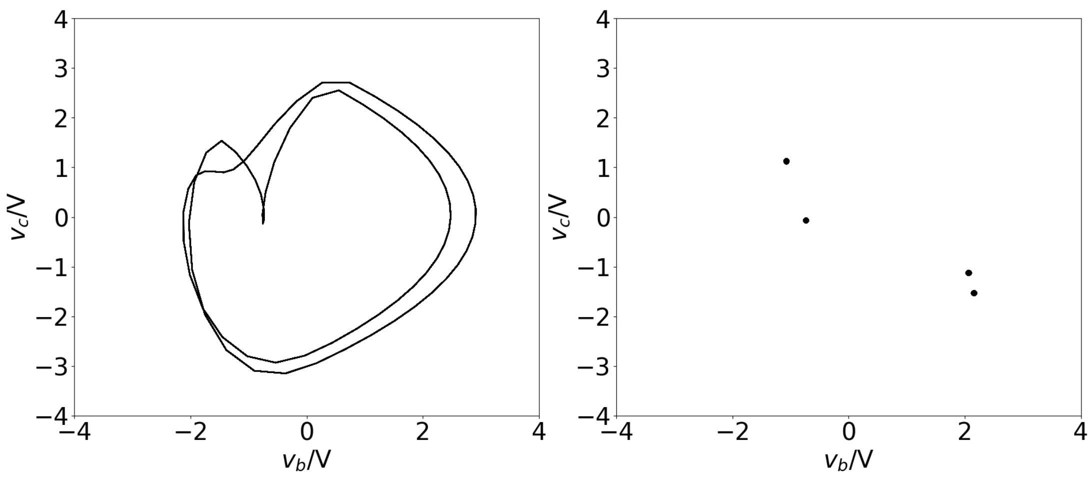

Figure 13: Phase space map and Poincaré section produced by a surface oscillation with a frequency of 650Hz and an amplitude of 1V.

This was unexpected, however, in figure 14 4 orbits can be seen which supports that this a period-4 orbit. This means there is a hidden orbit in the phase space map that cannot be seen due to the precision of the equipment.

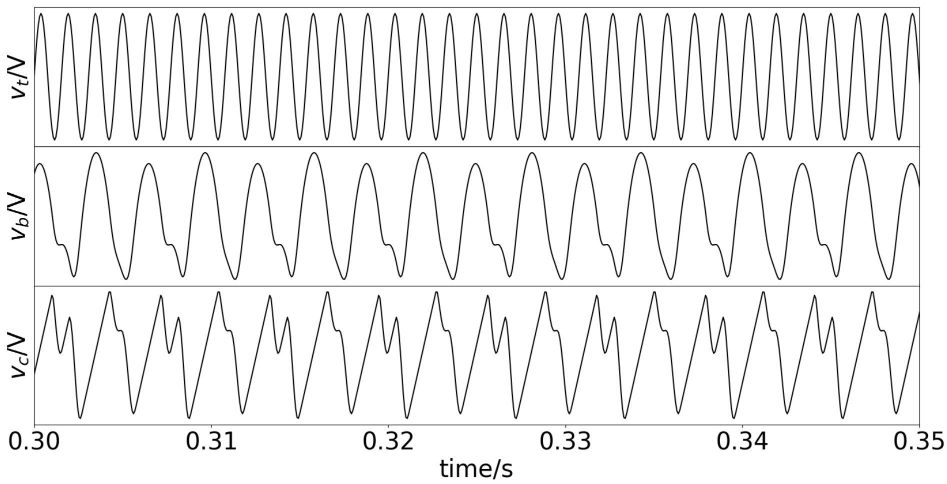

Figure 14: Response of the circuit to a 650Hz, 1V amplitude sinusoidal waveform.

Figure 15 shows the system exhibiting chaotic behaviour. This is due to the multiple non-repeating orbits seen in the phase space map and the Poincaré section converging towards a strange attractor.

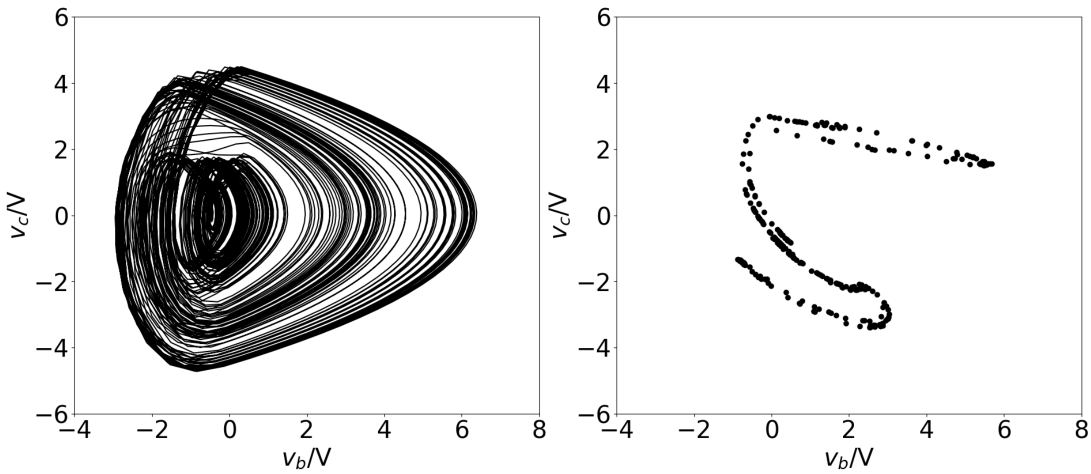

Figure 15: Phase space map and Poincaré section produced by a surface oscillation with a frequency of 750Hz and an amplitude of 1V.

In figure 15, no repeating orbits can be seen in these plots of the response of the circuit, therefore the system at this frequency evidently is exhibiting chaotic behaviour. The response of the circuit for the 1000Hz waveform is very similar to that shown in figure 15.

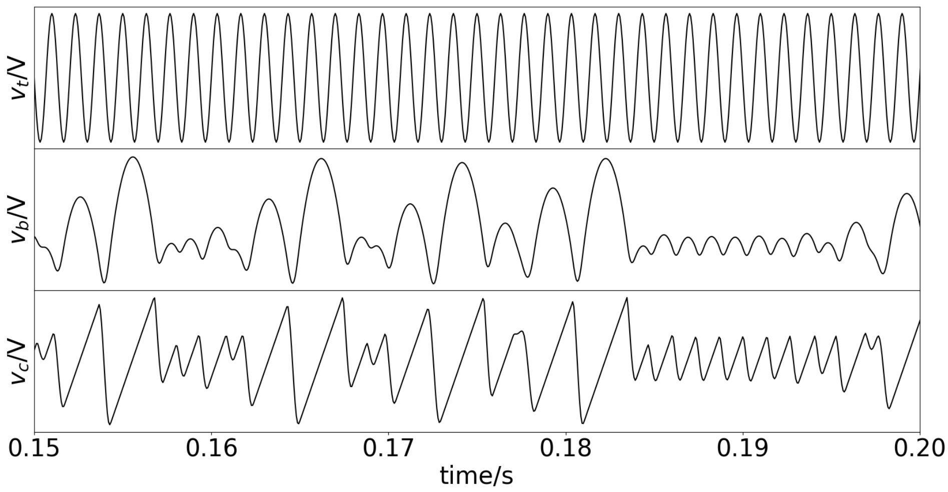

Figure 16: Response of the circuit to a 750Hz, 1V amplitude sinusoidal waveform.

A phase space map and Poincaré section were plotted for an 1000Hz frequency waveform, as shown in 17, to emphasise the rapid descent into chaos. In comparison to the 750Hz frequency phase space map, there appears to be many more orbits in the 1000Hz frequency and also the orbits seem to be much wider. In addition the Poincaré section can be seen more clearly as a strange attractor.

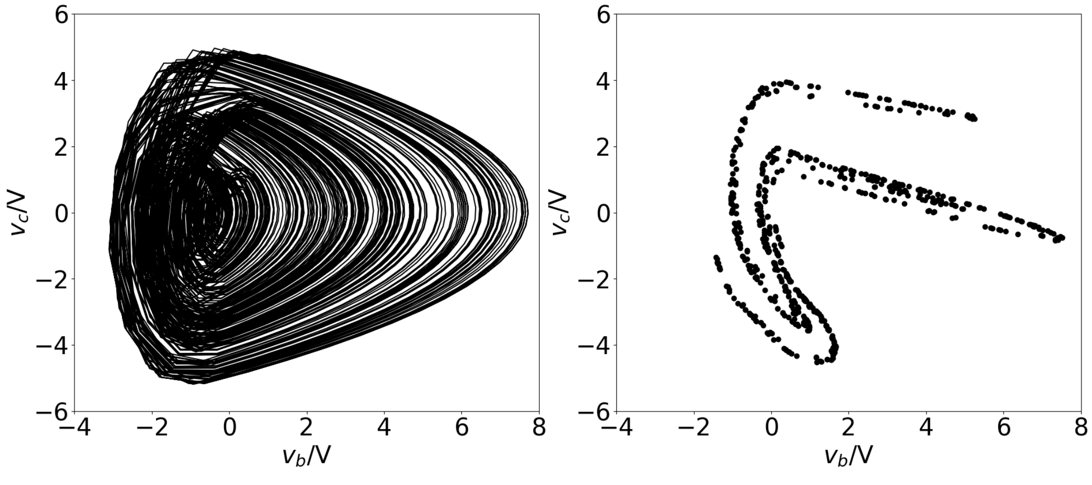

Figure 17: Phase space map and Poincaré section produced by a surface oscillation with a frequency of 1000Hz and an amplitude of 1V.

From figure 9, the 450Hz frequency up to figure 17 the 1000Hz frequency, the orbits can be seen to be increasing in size. This would make sense as the ball would have more energy kinetic energy when the surface is oscillating with a larger frequency.

The bifurcation diagram plotted had very visible bifurcations as shown in figure 18. The amplitude these bifurcations occurred at was used to find a value for the Feigenbaum constant. These bifurcations were given uncertainties of 0.005 as this was the step size the amplitude increased in. The experimentally found value of the Feigenbaum constant was calculated using equation 2 and was calculated to be $\delta_{expm} = 4.4 \pm 0.5$. An uncertainty was found by deriving an error propagation equation for equation 2. At around 2.4V, the data points are then restricted to one line. This appears in the expected diagram and may be due to the surface oscillating faster than the ball can fall which will make it seem like the ball is still on a non-oscillating surface. This would essentially create a stable system which would explain why the system is not exhibiting chaotic behaviour.

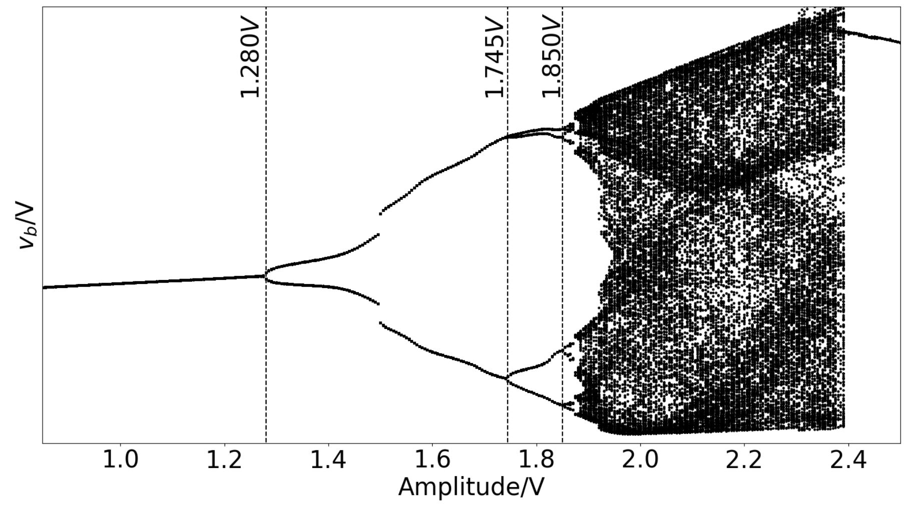

Figure 18: Plot of v_b against amplitude, showing bifurcation diagram.

5 Discussion of Results

A potential systematic error that may have been within the circuit and affected all data could include internal resistance not accounted for, this may have had an impact on the input or output values in the data sets.

There was no true value of the coefficient of restitution as it can vary in different systems, however, the value calculated is believed to be accurate due to the number of values of K that were used to calculate it. One of the values of K calculated could have been omitted from the data due to it clearly lying out of the range of the other values found. This error may be a result of a systematic error, this being, the value added to the data set to set the equilibrium position at zero. The potential error can be avoided by omitting the value from the data set or preferably, adding to the original code, something that sets the equilibrium to zero quantitatively rather than qualitatively. To reduce random errors in the calculation of the coefficient of restitution, the code could have been ran multiple times to get more data sets. Multiple coefficients of restitution could then be found for each data set, in the same way as stated in the experimental method section, and a mean value for these could be found. This would greatly decrease the uncertainty in the value due to the amount of values averaged over, making it more precise. To increase the number of K values per data set, the square wave that was outputted could have a greater gain. This would create more noticeable peaks in the plotted graphs which would allow more values of K to be calculated, again decreasing the uncertainty and random errors of the system. To get more accurate values for the peaks, rather than qualitatively looking for these points, a code could have been created that quantitatively finds them. This may not be needed for smaller data sets, however, increasing the sampling rate will increase the accuracy but also many more data points will be in the data set. So, a code may be useful in a situation similar to this.

The phase space maps and Poincaré sections appeared exactly as expected, however, for the 650Hz frequency, that being shown in figure 13, the number of orbits may easily be miscounted without the Poincaré section and circuit response subplots, in figure 14. As stated this orbit is hidden within another orbit and the random errors make it hard to distinguish between the two. This orbit may be visible with more accurate equipment that has smaller random errors. What may also help make the orbit more visible, is decreasing the widths of the lines plotted as distinct orbits will be seen clearer. Period-doubling bifurcations occurred more often as the frequency increased, which may potentially be why a period-three orbit was missed. A noticeable error visibly seen in figure 9 is rather than points being plotted in a continuous line, that being small increments, they were plotted with larger increments creating pointy corners, this may be a result of the sampling rate being too small.

The bifurcation diagram mostly came out as expected. There is a split in the continuous line between the first and second bifurcation which may be a coding error or the amplitude was not set to increase in small enough increments. However, this does not effect the position of the period-doubling bifurcations which is most important. The value for the Feigenbaum constant calculated was $\delta_{expm} = 4.4 \pm 0.5$, with the accepted value being $\delta = 4.669$ which lies within one standard deviation of the calculated value[1]. This shows that this chaotic system bifurcates at the same rate as every other, which implies that all chaotic systems bifurcate at the same rate. Each of the points plotted had a large number of data points making it up. This means that each of the values of v_b plotted on the y axis were highly accurate, however, due to the x axis increments being potentially too large, the amplitude at which the curve bifurcates is accurate up to $\pm 0.05V$, that being the size of the increment. The uncertainty in this could have been reduced by making the increments that the amplitude increased in smaller, although this may take longer to plot, the number of data points for v_b can be decreased to account for this. Although this may make v_b less accurate, it will plot the graph faster and make the point at which the systems period doubles more accurate which will lead to a more accurate value of the Feigenbaum constant. With smaller steps of amplitude, more data points will be plotted which may make it hard to find period-doubling bifurcations qualitatively. For experiments with larger data sets, as stated previously, it may be easier to code a program that finds the points at which the system bifurcates quantitatively.

6 Conclusions

The coefficient of restitution found for this circuit was $K = 0.503 \pm 0.004$, this was found to explore how the system reacted to a damped oscillation and was also one of the control parameter of the system. The phase space maps and Poincaré sections were as expected with chaotic behaviour appearing when a system control parameter was pushed to an extreme. The bifurcation diagram was plotted successfully and a value of $\delta_{expm} = 4.4 \pm 0.5$ was found for the Feigenbaum constant. The accepted value for the Feigenbaum constant lay within one standard deviation of this value and therefore the value calculated can be said to be equal to the accepted Feigenbaum constant, if random and systematic errors were not present. This shows that all chaotic systems bifurcate at the same rate.

References

[1] Keith Briggs. A precise calculation of the Feigenbaum constants. *Mathematics of Computation*, 57(195), 1991. ISSN 0025-5718. doi:10.1090/s0025-5718-1991-1079009-6.

[2] I. A. SANTOS and J. BUCHMANN. Would be the Atmosphere Chaotic? *Anuário do Instituto de Geociências - UFRJ*, 36_2(1), 2013. ISSN 01019759. doi:10.11137/2013_2_40_44.

[3] Paul Sutter. Chaos theory explained: A deep dive into an unpredictable universe, 3 2022. URL `https://www.space.com/chaos-theory-explainer-unpredictable-systems.html`.

[4] Luyao Wang and Hai Cheng. Pseudo-random number generator based on logistic chaotic system. *Entropy*, 21(10), 2019. ISSN 10994300. doi:10.3390/e21100960.

[5] Xingyuan Wang and Yining Su. An Audio Encryption Algorithm Based on DNA Coding and Chaotic System. *IEEE Access*, 8, 2020. ISSN 21693536. doi:10.1109/ACCESS.2019.2963329.

[6] Ji Xu, Chen Zhao, and Jun Mou. A 3D Image Encryption Algorithm Based on the Chaotic System and the Image Segmentation. *IEEE Access*, 8, 2020. ISSN 21693536. doi:10.1109/ACCESS.2020.3005925.

[7] Robert L. Zimmerman, Sergio Celaschi, and Luis G. Neto. The electronic bouncing ball. *American Journal of Physics*, 60(4), 1992. ISSN 0002-9505. doi:10.1119/1.17134.

[8] Nazanin Zandi-Mehran, Shirin Panahi, Zahra Hosseini, Seyed Mohammad Reza Hashemi Golpayegani, and Sajad Jafari. One dimensional map-based neuron model: A phase space interpretation. *Chaos, Solitons and Fractals*, 132, 2020. ISSN 09600779. doi:10.1016/j.chaos.2019.109558.

[9] Sergey E. Koposov, Hans Walter Rix, and David W. Hogg. Constraining the milky way potential with a six-dimensional phase-space map of the GD-1 stellar stream. *Astrophysical Journal*, 712(1), 2010. ISSN 15384357. doi:10.1088/0004-637X/712/1/260.

[10] https://en.wikipedia.org/wiki/File:LogisticMap_BifurcationDiagram.png.

7 Appendix

```
import numpy as np
import matplotlib.pyplot as plt
import sys
sys.path.insert(1,"C:\\python")
import y2daq

# Create analog daq object with input channel 0, output channel 0
a=y2daq.analog()
a.reset()
a.addInput(0) #v_b ball position
a.addInput(1) #v_c ball velocity
a.addOutput(0) #v_t table input

#set sampling rate
a.Rate=25000

#create square wave output
A=np.ones(10000)
B=np.zeros(10000)
square=np.append(A,B)

#Run the DAQ
data, timestamps=a.run(square)

vb=data[0,:]
vc=data[1,:]

#create figure and subplots
fig=plt.figure()
ax1=fig.add_axes([0.1,0.6,0.8,0.25],yticklabels=[])
ax2=fig.add_axes([0.1,0.35,0.8,0.25],yticklabels=[])
ax3=fig.add_axes([0.1,0.1,0.8,0.25],yticklabels=[])

#slice data where steps occur
timestamps=timestamps[9950:10205]
vb=vb[9950:10205]
vc=vc[9950:10205]
squareplot=square[9950:10205]


#plot data with these arrays
ax1.plot(timestamps,squareplot,'black')
ax2.plot(timestamps,vb,'black')
ax3.plot(timestamps,vc,'black')

ax1.set_ylabel('$v_t$/V')
ax2.set_ylabel('$v_b$/V')
ax3.set_ylabel('$v_c$/V')
ax3.set_xlabel('time/s')

#save data
save=np.column_stack([vb,vc,timestamps,squareplot])

np.savetxt('square wave.txt',save,fmt=['%g','%g','%g','%g'])


#reset device
a.reset()
```

Figure 19: Code used to output a square wave into the circuit. The input data was used to find the coefficient of restitution of the system.

```python
import numpy as np
import matplotlib.pyplot as plt
import sys
sys.path.insert(1,"C:\\python")
import y2daq

# Create analog daq object with input channel 0, output channel 0
a=y2daq.analog()
a.reset()
a.addInput(0) #v_b ball position
a.addInput(1) #v_c ball velocity
a.addOutput(0) #v_t table input

#frequency can be changed for different phase space maps. 450Hz,550Hz,600Hz,650Hz,750Hz
#set sampling rate
freq=1000
rate=10*freq
a.Rate=rate

duration=1
t=np.arange(0,duration+1/a.Rate,1/a.Rate)
amp=1
sinwave=amp*np.sin(2*np.pi*freq*t)

#run the DAQ
data, timestamps=a.run(sinwave)

vb=data[0,:]
vc=data[1,:]

#create figure and subplots
fig=plt.figure()
ax1=fig.add_axes([0.1,0.6,0.8,0.25])
ax2=fig.add_axes([0.1,0.35,0.8,0.25])
ax3=fig.add_axes([0.1,0.1,0.8,0.25])

#remove transient data

timestamps=timestamps[2000:8000]
vbt=vb[2000:8000]
vct=vc[2000:8000]
sinwave=sinwave[2000:8000]


#plot non-transient data
ax1.plot(timestamps,sinwave,'black')
ax2.plot(timestamps,vbt,'black')
ax3.plot(timestamps,vct,'black')
ax1.set_ylabel('$v_t$/V')
ax2.set_ylabel('$v_b$/V')
ax3.set_ylabel('$v_c$/V')
ax3.set_xlabel('time/s')

#save data
save=np.column_stack([vbt,vct,timestamps,sinwave])
np.savetxt('sinusodial wave 1000.txt',save,fmt=['%g','%g','%g','%g'])


#reset device
a.reset()
```

Figure 20: Code used to output a sinusoidal wave into the circuit. The frequency was changed to get a range of different waveforms outputted.

```python
#remove transient data
vbp=vb[2000:8000]
vcp=vc[2000:8000]

#plot phase space maps
plt.figure()
plt.plot(vbp,vcp,'black','.')
plt.xlabel('$v_b$/V')
plt.ylabel('$v_c$/V')

#set limits if necessary
'''
plt.xlim([-2.5,3])
plt.ylim([-4,3])
'''
#change name depending on frequency
save=np.column_stack([vbp,vcp])
np.savetxt('phase space map 1000.txt',save,fmt=['%g','%g'])


#reset device
a.reset()

#%%

#for frequencies in range
samples=int(rate/freq)

#data for poincare section
vbpoin=vb[2000:8000:samples]
vcpoin=vc[2000:8000:samples]

#plot poincare section
plt.figure()
plt.plot(vbpoin,vcpoin,'.',color='black')
plt.xlabel('$v_b$/V')
plt.ylabel('$v_c$/V')

#set limits if necessary
'''
plt.xlim([-2.5,3])
plt.ylim([-4,3])
'''

#change name depending on frequency
save=np.column_stack([vbpoin,vcpoin])
np.savetxt('poincare section 1000.txt',save,fmt=['%g','%g'])


#reset device
a.reset()
```

Figure 21: Code used to create a phase space map and Poincaré section using the input values from its complimentary waveform frequency.

```python
import numpy as np
import matplotlib.pyplot as plt
import sys
sys.path.insert(1,"C:\\python")
import y2daq

# Create analog daq object with input channel 0, output channel 0
a=y2daq.analog()
a.reset()
a.addInput(0) #v_b ball position
a.addInput(1) #v_c ball velocity
a.addOutput(0) #v_t table input

#set up arrays
amparray=np.array([])
vbarray=np.array([])

#set sampling rate and frequency
freq=450
rate=20*freq
a.Rate=rate
duration=1
t=np.arange(0,duration+1/a.Rate,1/a.Rate)
samples=int(rate/freq)

#set up loop to take data from poincare section for each amplitude
scale=np.arange(0.1,3,0.005)

for amp in scale:
    sinwave=amp*np.sin(2*np.pi*freq*t)

    #run the DAQ
    data,timestamps=a.run(sinwave)

    vb=data[0,:]
    vc=data[1,:]

    size=np.size(vb[2000:8000:samples])
    amp1=np.ones(size)*amp

    amparray=np.append(amparray,amp1)
    vbarray=np.append(vbarray,vb[2000:8000:samples])

    amp=amp+0.01

#plot the data
plt.figure()
plt.plot(amparray,vbarray,'.',color='black',markersize=3.1)
plt.xlabel('Amplitude/V')
plt.ylabel('$v_b$/V')
plt.xlim([1,2.25])


save=np.column_stack([amparray,vbarray])
np.savetxt('bifurcation diagram.txt',save,fmt=['%g','%g'])

#reset device
a.reset()
```

Figure 22: Code used to plot the bifurcation diagram.