DISCRETE AND CONTINUOUS
DYNAMICAL SYSTEMS
Volume **33**, Number **5**, May **2013**

doi:10.3934/dcds.2013.33.1905

pp. **1905–1926**

GLOBAL WELL-POSEDNESS AND SCATTERING FOR THE DEFOCUSING, CUBIC NONLINEAR SCHRÖDINGER EQUATION WHEN $n = 3$ VIA A LINEAR-NONLINEAR DECOMPOSITION

Benjamin Dodson

970 Evans Hall, number 3840
UC Berkeley mathematics
Berkeley, CA 94720-3840, USA

(Communicated by Alessio Figalli)

Abstract. In this paper, we prove global well-posedness and scattering for the defocusing, cubic nonlinear Schrödinger equation when $n = 3$ and $u_0 \in H^s(\mathbf{R}^3)$, $s > 5/7$. To this end, we utilize a linear-nonlinear decomposition, similar to the decomposition used in [20] for the wave equation.

1. Introduction.

In this paper we study the three-dimensional defocusing, cubic nonlinear Schrödinger equation,

$$\begin{aligned} iu_t + \Delta u &= |u|^2 u, \\ u(0,x) &= u_0(x) \in H^s(\mathbf{R}^3). \end{aligned} \tag{1}$$

$H^s(\mathbf{R}^3)$ denotes the usual inhomogeneous Sobolev space. We use the standard definition for well - posedness used in the study of (1).

Definition 1.1. (1) is said to be well - posed on an interval $0 \in I \subset \mathbf{R}$, for $u_0 \in H^s(\mathbf{R}^3)$ if,

1. A strong solution to (1) exists,

$$u(t) = e^{it\Delta} u_0 - i \int_0^t e^{i(t-\tau)\Delta} (|u|^2 u)(\tau) d\tau, \tag{2}$$

2. $u(t)$ is unique,
3. For any compact $J \subset I$, the map from initial data u_0 to the solution $u(t)$ to (1),

$$H^s(\mathbf{R}^3) \to L_t^\infty H_x^s(J \times \mathbf{R}^3) \cap L_{t,x}^5(J \times \mathbf{R}^3) \tag{3}$$

is continuous.

2010 *Mathematics Subject Classification.* Primary: 35Q55.
Key words and phrases. Nonlinear Schrödinger equation.
The author was supported by NSF postdoctoral fellowship 1103914.

Theorem 1.2. *If $u_0 \in H^s(\mathbf{R}^3)$, $s > \frac{1}{2}$, then there exists $T(\|u_0\|_{H^s(\mathbf{R}^3)}) > 0$ such that (1) is locally well - posed on $[-T, T]$. Furthermore, if I is the maximal interval of well - posedness of a solution to (1), then I is an open interval. If* $\sup(I) = T_* < \infty$ *then for all $s > 1/2$,*

$$\lim_{t \nearrow T_*} \|u(t)\|_{H^s(\mathbf{R}^3)} = \infty. \tag{4}$$

An identical result holds if $\inf(I) > -\infty$.

Proof. See [6]. □

[5] showed that solutions to (1) enjoy the conservation of both mass,

$$M(u(t)) = \int |u(t,x)|^2 dx = M(u(0)), \tag{5}$$

and energy

$$E(u(t)) = \frac{1}{2} \int |\nabla u(t,x)|^2 dx + \frac{1}{4} \int |u(t,x)|^4 dx = E(u(0)). \tag{6}$$

Thus (1) is globally well-posed in the defocusing case when $s = 1$. The work of [5] and [6] considered (1) in the context of $\dot{H}^s$ - critical problems for $0 \leq s \leq 1$. In this case (1) is $\dot{H}^{1/2}$ - critical since if u solves (1), then for any $\lambda > 0$,

$$\frac{1}{\lambda} u(\frac{t}{\lambda^2}, \frac{x}{\lambda}) \tag{7}$$

also solves (1).

$$\|\frac{1}{\lambda} u(\frac{x}{\lambda})\|_{\dot{H}^{1/2}(\mathbf{R}^3)} = \|u(x)\|_{\dot{H}^{1/2}(\mathbf{R}^3)}. \tag{8}$$

Conjecture 1. (1) *is globally well - posed and scattering for $u_0 \in \dot{H}^{1/2}(\mathbf{R}^3)$.*

Remark 1. (1) also has a local solution on $[0, T)$, $T(u_0) > 0$ when $u_0 \in \dot{H}^{1/2}(\mathbf{R}^3)$. In this case $T > 0$ depends on the profile of the initial data, not just its size. If $\|u_0\|_{\dot{H}^{1/2}(\mathbf{R}^3)}$ is small, then (1) is globally well-posed and scatters to a free solution.

Definition 1.3. (Scattering) Define the wave operators

$$u^{\pm} = u_0 - i \int_0^t e^{-i\tau\Delta}(|u|^2 u)(\tau) d\tau. \tag{9}$$

$$\begin{aligned} \Omega^{\pm} &: H^s(\mathbf{R}^3) \to H^s(\mathbf{R}^3), \\ \Omega^{\pm} u^{\pm} &= u_0. \end{aligned} \tag{10}$$

We say (1) is asymptotically complete, or scattering, if the wave operators $\Omega^{\pm}$ are surjective.

[14] proved (1) is scattering when $u_0 \in H^1(\mathbf{R}^3)$.

The main difficulty in moving from the results of [5] and [14] for $u_0 \in H^1(\mathbf{R}^3)$ to the conjectured results of global well - posedness and scattering lies in the fact that there are no known controlled quantities that control the $\dot{H}^s$ norm of a solution to (1) for $0 < s < 1$. Indeed,

Theorem 1.4. *If $[0, T_*)$ is a maximal interval of existence for (1), $T_* < \infty$, then*

$$\limsup_{t \nearrow T_*} \|u(t)\|_{\dot{H}^{1/2}(\mathbf{R}^3)} = +\infty. \tag{11}$$

Additionally, if

$$\sup_{t \in \mathbf{R}} \|u(t)\|_{\dot{H}^{1/2}(\mathbf{R}^3)} < \infty \tag{12}$$

then (1) is globally well - posed and scattering.

Proof. This was proved by [17] using the concentration compactness / rigidity method. □

Remark 2. In two dimensions,

$$\|\frac{1}{\lambda} u(\frac{x}{\lambda})\|_{L^2(\mathbf{R}^2)} = \|u\|_{L^2(\mathbf{R}^2)}. \tag{13}$$

Since $\|u(t)\|_{L^2(\mathbf{R}^2)}$ is a conserved quantity, [12] was able to use the concentration compactness / rigidity method to prove $iu_t + \Delta u = |u|^2 u$ is globally well - posed and scattering for $u_0 \in L^2(\mathbf{R}^d)$.

Despite the advantages that the cubic problem in $d = 2$ has over the cubic problem when $d = 3$, many techniques used to study the cubic problem when $d = 2$ can also be applied to the cubic problem when $d = 3$. Since there are no known conserved quantities that control $\|u\|_{\dot{H}^{1/2}(\mathbf{R}^3)}$ many have attempted to prove global well - posedness and scattering for $u_0 \in H^s$, $s < 1$, by means of producing an almost conserved quantity.

[2] utilized the Fourier truncation method to prove that the cubic problem is globally well - posed for $u_0 \in H^s$, $s > \frac{3}{5}$ when $d = 2$. When $d = 3$ [1] proved global well - posedness for $s > \frac{11}{13}$ and global well-posedness and scattering for u_0 radial, $s > 5/7$. In this method $u = u_h + u_l$ is split into a low frequency piece and a high frequency piece. The evolution of the high frequency part is approximated by linear evolution and the evolution of the low frequency part is approximated by the evolution of the cubic nonlinear Schrödinger equation with finite H^1 norm. See also [3] for additional information.

In fact, [1] and [2] proved something more, namely for s in the appropriate range,

$$u(t) - e^{it\Delta} u_0 \in H^1(\mathbf{R}^d). \tag{14}$$

It was precisely (14) that lead to the development of the I - method, since (14) is false for many dispersive partial differential equations. See [15] for example. Instead, [7] utilized the modified energy $E(Iu(t))$, where $I : H^s(\mathbf{R}^3) \to H^1(\mathbf{R}^3)$ is the Fourier multiplier given by the smooth, radial, decreasing function

$$m(\xi) = \begin{cases} 1, & |\xi| \leq N; \\ \frac{N^{1-s}}{|\xi|^{1-s}}, & |\xi| > 2N. \end{cases} \tag{15}$$

Tracking the change of $E(Iu(t))$, [7] proved global well - posedness without scattering for the cubic problem when $s > 4/7$, $d = 2$ and for $s > 5/6$, $d = 3$.

[8] introduced the interaction Morawetz estimate for a solution to (1),

$$\|u\|_{L^4_{t,x}([0,T]\times\mathbf{R}^3)}^4 \lesssim \|u\|_{L_t^\infty L_x^2([0,T]\times\mathbf{R}^3)}^2 \|u\|_{L_t^\infty \dot{H}^{1/2}([0,T]\times\mathbf{R}^3)}^2. \tag{16}$$

This estimate is based on the standard Morawetz estimate of [19] combined with a tensor product and an interaction potential. Combining (16) with a bootstrap argument [8] proved (1) is globally well - posed and scattering for $s > \frac{4}{5}$. Scattering follows from well - known arguments of [4] and [23]. [8] also improved the scattering result of [14] to give bounds on the scattering size that depend polynomially on the mass and energy.

In this paper we prove

Theorem 1.5. (1) *is globally well-posed for* $s > 5/7$. *Additionally,*

$$\|u(t)\|_{H^s(\mathbf{R}^3)} \leq C(\|u_0\|_{H^s(\mathbf{R}^3)}), \tag{17}$$

and the solution scatters. There exist $u_\pm \in H^s(\mathbf{R}^3)$ *such that*

$$\begin{aligned} \lim_{t\to\infty} \|u(t) - e^{it\Delta}u_+\|_{H^s(\mathbf{R}^3)} = 0, \\ \lim_{t\to\infty} \|u(-t) - e^{-it\Delta}u_-\|_{H^s(\mathbf{R}^3)} = 0. \end{aligned} \tag{18}$$

Remark 3. [1] proved this result for radial data.

To prove this we utilize the linear - nonlinear decomposition of [20] combined with the interaction Morawetz estimate of [8]. [20] applied the I - method to the semilinear wave equation,

$$\begin{aligned} \partial_{tt}u - \Delta u = -u^3, \\ u(0,x) \in H^s(\mathbf{R}^3), \\ u_t(0,x) \in H^{s-1}(\mathbf{R}^3). \end{aligned} \tag{19}$$

The linear - nonlinear decomposition more effectively estimates the energy change for large times. For the initial value problem (1), the change in energy over an interval I_j is estimated by

$$\frac{1}{N^{1-}}\|\nabla P_{>cN} Iu\|_{L_t^2 L_x^6(I_j\times\mathbf{R}^3)}^2 + O(\frac{1}{N^{2-}}), \tag{20}$$

which sums very nicely over a collection of consecutive intervals.

Remark 4. After this result was posted, [25] improved theorem (1.5) to

Theorem 1.6. (1) *is globally well - posed and scattering for* $s > 2/3$.

This was proved by combining the linear - nonlinear decomposition with the resonant decomposition of [10].

In §2, some preliminary facts from harmonic analysis will be mentioned. In §3, a local well-posedness result will be proved. In §4, a formula for the energy increment will be computed. In §5 a smoothing estimate using a bilinear estimate will be proved. In §6, the double-layer I-decomposition will be used to prove the theorem.

2. **Preliminaries.** This section will serve to introduce some preliminary information that will be needed throughout the paper. None of the results in this section are new, and the proofs can be found in many places. Let $\phi(x)$ be a smooth, radial function,

$$\phi(x) = \begin{cases} 1, & |x| \leq 1; \\ 0, & |x| > 2. \end{cases} \tag{21}$$

Let

$$\begin{aligned} \mathcal{F}(P_{\leq N} u) &= \hat{u}(\xi) \phi(\frac{\xi}{N}), \\ \mathcal{F}(P_{> N} u) &= \hat{u}(\xi)(1 - \phi(\frac{\xi}{N})). \end{aligned} \tag{22}$$

Then define the standard Littlewood - Paley decomposition,

$$P_N f = u_{\leq 2N} - u_{\leq N}. \tag{23}$$

We let $u_{<N} = P_{<N} u$, similarly for u_N and $u_{>N}$. If $N = 2^j$ for some integer j let $u_j = u_{2^j}$. The Littlewood - Paley decomposition obeys the embedding

$$\|u_N\|_{L^p(\mathbf{R}^3)}, \|u_{<N}\|_{L^p(\mathbf{R}^3)}, \|u_{>N}\|_{L^p(\mathbf{R}^3)} \lesssim_p \|u\|_{L^p(\mathbf{R}^3)} \tag{24}$$

for all $1 \leq p \leq \infty$.

The L^p norms obey the l^2 summation rule for $1 < p < \infty$,

$$\|(\sum_j |u_j|^2)^{1/2}\|_{L^p(\mathbf{R}^3)}^2 \sim_p \|u\|_{L^p(\mathbf{R}^3)}. \tag{25}$$

Additionally Bernstein's inequality holds. For $1 < p < \infty$,

$$\|P_N u\|_{L^p(\mathbf{R}^3)} \lesssim_p \frac{1}{N^s} \|u\|_{\dot{H}^{s,p}(\mathbf{R}^3)}, \tag{26}$$

where $\dot{H}^{s,p}$ is the p - based Sobolev space of order s. See [21], [22], [27], [28], or [29] for more details on the Littlewood - Paley decomposition.

Make a high-low frequency decomposition.

$$u = P_{\leq N} u + P_{>N} u = u_b + u_s. \tag{27}$$

Remark 5. Since we will also make a linear-nonlinear decomposition, to avoid any potential confusion we will write u_b for low frequencies (b for bass), rather than u_l, and u_s (s for soprano) for high frequencies.

The I-operator is a Fourier multiplier given by a smooth, decreasing, radially symmetric symbol,

$$I_N : H^s(\mathbf{R}^3) \to H^1(\mathbf{R}^3), \tag{28}$$

$$(I_N f)(\xi) = m_N(\xi) \hat{f}(\xi), \tag{29}$$

$$m_N(\xi) = \begin{cases} 1, & |\xi| \leq N; \\ (\frac{N}{|\xi|})^{1-s}, & |\xi| > 2N. \end{cases} \tag{30}$$

For the rest of the paper, we understand that If refers to the function $I_N f$. We have the estimates,

$$\begin{aligned} \|\nabla I u\|_{L_x^2(\mathbf{R}^3)} &\lesssim N^{1-s}\|u\|_{H^s(\mathbf{R}^3)}, \\ \|u\|_{H^s(\mathbf{R}^3)} &\lesssim \|Iu\|_{H^1(\mathbf{R}^3)}. \end{aligned} \tag{31}$$

Remark 6. If $E(Iu(t))$ was a conserved quantity then (31) would imply (1) is globally well - posed for all $s > 1/2$. Sadly this is not true. Instead, to prove theorem (1.5) we will be content to merely estimate the change of $E(Iu(t))$. This estimate occupies §4.

By Bernstein's inequality we have

$$\|P_{>M}u\|_{L_t^p L_x^q(J\times\mathbf{R}^3)} \lesssim \left(\frac{1}{M} + \frac{1}{N^{1-s}M^s}\right)\|\nabla I u\|_{L_t^p L_x^q(J\times\mathbf{R}^3)}, \tag{32}$$

and

$$\||\nabla|^{1/2}P_{>M}u\|_{L_t^p L_x^q(J\times\mathbf{R}^3)} \lesssim \left(\frac{1}{M^{1/2}} + \frac{1}{N^{1-s}M^{s-1/2}}\right)\|\nabla I u\|_{L_t^p L_x^q(J\times\mathbf{R}^3)}. \tag{33}$$

We also have the Sobolev embedding theorem, for $1 \leq p < q \leq \infty$,

$$\|P_N u\|_{L^q(\mathbf{R}^3)} \lesssim N^{\frac{3}{p}-\frac{3}{q}}\|P_N u\|_{L^p(\mathbf{R}^3)}. \tag{34}$$

2.1. **Strichartz estimates.** A pair (p,q) will be called an admissible pair in three dimensions if

$$\frac{2}{p} = 3\left(\frac{1}{2} - \frac{1}{q}\right). \tag{35}$$

We will also use the Strichartz space,

$$\|u\|_{S^0(J\times\mathbf{R}^3)} = \sup_{(p,q)\text{ admissible}} \|u\|_{L_t^p L_x^q(J\times\mathbf{R}^3)}, \tag{36}$$

as well as its dual,

$$\|F\|_{N^0(J\times\mathbf{R}^3)} = \|F\|_{L_t^1 L_x^2 + L_t^2 L_x^{\frac{6}{5}}(J\times\mathbf{R}^3)}. \tag{37}$$

Theorem 2.1. *If $u(t,x)$ solves the equation*

$$\begin{aligned} iu_t + \Delta u &= F(t), \\ u(0,x) &= u_0, \end{aligned} \tag{38}$$

$$\|u\|_{S^0(J\times\mathbf{R}^3)} \lesssim \|u_0\|_{L^2(\mathbf{R}^3)} + \|F\|_{N^0(J\times\mathbf{R}^3)}. \tag{39}$$

Proof. These estimates were first introduced in [24] using Fourier restriction theory. [13] proved the non - endpoint version and [16] the endpoint version. See also [26] for a good description of these estimates. □

2.2. **Bilinear estimate.** We will also make use of the bilinear Strichartz estimate,

Lemma 2.2. *Suppose*

$$u(t,x) = e^{it\Delta}u_0 - i\int_0^t e^{i(t-\tau)\Delta}F(\tau)d\tau, \tag{40}$$

and

$$v(t,x) = e^{it\Delta}v_0 - i\int_0^t e^{i(t-\tau)\Delta}G(\tau)d\tau, \tag{41}$$

with u_0, F *supported on* $N \leq |\xi| \leq 2N$ *and* v_0, G *supported on* $M \leq |\xi| \leq 2M$, $N << M$. *Then,*

$$\begin{aligned}\|uv\|_{L^2_{t,x}(J\times\mathbf{R}^3)} \lesssim \frac{N}{M^{1/2}}&(\|u_0\|_{L^2_x(\mathbf{R}^3)} + \|F\|_{L^1_tL^2_x(J\times\mathbf{R}^3)})\\ &\times(\|v_0\|_{L^2_x(\mathbf{R}^3)} + \|G\|_{L^1_tL^2_x(J\times\mathbf{R}^3)}).\end{aligned} \tag{42}$$

Proof. See [9] for a proof of the non - endpoint result, [18] in the endpoint case. □

2.3. **Interaction Morawetz estimate.**

Theorem 2.3. *If* $u(t,x)$ *solves* (1), *then*

$$\|u\|^4_{L^4_{t,x}(J\times\mathbf{R}^3)} \lesssim \|u\|^2_{L^\infty_tL^2_x(J\times\mathbf{R}^3)}\|u\|^2_{L^\infty_t\dot{H}^{1/2}_x(J\times\mathbf{R}^3)}. \tag{43}$$

Proof. See §2 of [8]. □

3. **Local well-posedness.** In this section we prove local well-posedness on an interval J when $\|u\|_{L^4_{t,x}(J\times\mathbf{R}^3)}$ is small and $E(Iu(t)) \leq 1$ for $t \in J$. To that end, we prove that the norm of u is controlled by the norm of Iu.

Lemma 3.1. *If* $\|u\|_{L^4_{t,x}(J\times\mathbf{R}^3)} \leq \epsilon$, *and* $I : H^s(\mathbf{R}^3) \to H^1(\mathbf{R}^3)$, $1/2 < s < 1$, *then*

$$\|u\|_{L^6_tL^{9/2}_x(J\times\mathbf{R}^3)} \lesssim (\epsilon^{2/3} + \frac{1}{N^{1/2}})(1 + \|\nabla Iu\|_{S^0(J\times\mathbf{R}^3)}). \tag{44}$$

Proof. Make a Littlewood-Paley decomposition. By Sobolev embedding

$$\|P_{\leq N}u\|_{L^\infty_tL^6_x(J\times\mathbf{R}^3)} \lesssim \|\nabla P_{\leq N}u\|_{L^\infty_tL^2_x(J\times\mathbf{R}^3)} \leq \|\nabla Iu\|_{S^0(J\times\mathbf{R}^3)}. \tag{45}$$

Interpolating this with $\|P_{\leq N}u\|_{L^4_{t,x}(J\times\mathbf{R}^3)} \leq \epsilon$,

$$\|P_{\leq N}u\|_{L^6_tL^{9/2}_x(J\times\mathbf{R}^3)} \lesssim \epsilon^{2/3}\|\nabla Iu\|^{1/3}_{S^0(J\times\mathbf{R}^3)}.$$

This takes care of the $P_{\leq N}$ part. On the other hand, when $N_j \geq N$,

$$\|P_{N_j}u\|_{L^6_tL^{9/2}_x(J\times\mathbf{R}^3)} \lesssim N_j^{1/2}\|u\|_{L^6_tL^{18/7}_x(J\times\mathbf{R}^3)} \lesssim \frac{1}{N^{1-s}}\frac{1}{N_j^{s-1/2}}\|\nabla Iu\|_{S^0(J\times\mathbf{R}^3)}. \tag{46}$$

Summing over $N_j \gtrsim N$ gives the bound for $P_{>N}u$. □

Theorem 3.2. *Suppose J is an interval such that*

$$\|u\|_{L^4_{t,x}(J\times\mathbf{R}^3)} \leq \epsilon,$$

and $E(Iu_0) \leq 1$. Then (1) is locally well-posed on J, and

$$\|\nabla Iu\|_{S^0(J\times\mathbf{R}^3)} \lesssim 1. \tag{47}$$

Proof. A solution to (1) satisfies Duhamel's formula,

$$Iu(t,x) = e^{it\Delta}Iu_0 - i\int_0^t e^{i(t-\tau)\Delta}I(|u|^2u)(\tau)d\tau. \tag{48}$$

Since the symbol of ∇I is strictly increasing as $|\xi| \to \infty$, $\nabla I(|u|^2u)$ obeys the product rule. Therefore, by (39),

$$\|\nabla Iu\|_{S^0(J\times\mathbf{R}^3)} \lesssim \|\nabla Iu_0\|_{L^2(\mathbf{R}^3)} + \|\nabla Iu\|_{L^2_tL^6_x(J\times\mathbf{R}^3)}\|u\|^2_{L^6_tL^{9/2}_x(J\times\mathbf{R}^3)} \tag{49}$$

$$\lesssim \|\nabla Iu_0\|_{L^2(\mathbf{R}^3)} + (\epsilon^{4/3} + \frac{1}{N})(\|\nabla Iu\|_{S^0(J\times\mathbf{R}^3)} + \|\nabla Iu\|^3_{S^0(J\times\mathbf{R}^3)}). \tag{50}$$

Applying the continuity method proves the theorem. □

4. **Energy increment.** In this section we prove an estimate on the energy increment which is well suited to making long time estimates on the change of the modified energy.

Theorem 4.1. *If u is a solution to (1), and $J = [a,b]$ is an interval with*

$$\|u\|^4_{L^4_{t,x}(J\times\mathbf{R}^3)} \leq \epsilon, \tag{51}$$

and $E(Iu(a)) \leq 1$, then

$$\sup_{t_1,t_2\in J} |E(Iu(t_1)) - E(Iu(t_2))| \lesssim \frac{1}{N^{1-}}\|\nabla IP_{>cN}u\|^2_{L^2_tL^6_x(J\times\mathbf{R}^3)} + O(\frac{1}{N^{2-}}), \tag{52}$$

where $c > 0$ is some constant.

Remark 7. The energy increment in [7] and [8] was

$$\sup_{t_1,t_2\in J} |E(Iu(t_1)) - E(Iu(t_2))| \lesssim \frac{1}{N^{1-}}.$$

(52) does not offer any advantage for one single interval. However, (52) can be summed over many disjoint intervals much more effectively than the estimate in [7].

Proof. To simplify notation let $F(u) = |u|^2u$. Recall (6).

$$E(Iu(t)) = \frac{1}{2}\int |\nabla Iu(t,x)|^2dx + \frac{1}{4}\int |Iu(t,x)|^4dx. \tag{53}$$

$$\frac{d}{dt}E(Iu(t)) = Re\int (I\partial_tu(t,x))[F(\overline{Iu}) - IF(\bar{u})](t,x)dx. \tag{54}$$

We estimate

$$Re\int_{t_1}^{t_2}\int (i\Delta Iu(t,x))[IF(\bar{u}) - F(\overline{Iu})](t,x)dxdt, \tag{55}$$

and

$$Re \int_{t_1}^{t_2} \int (iIF(u)(t,x))[IF(\overline{u}) - F(\overline{Iu})](t,x)dxdt \tag{56}$$

separately.

4.1. **The term (55).** By Parseval's theorem

$$\begin{aligned}(55) = Re \int_{t_1}^{t_2} \int_{\Sigma} (i|\xi_1|^2 \widehat{Iu}(t,\xi_1))[\frac{m(\xi_2+\xi_3+\xi_4)}{m(\xi_2)m(\xi_3)m(\xi_4)} - 1] \\ \times \widehat{\overline{Iu}}(t,\xi_2)\widehat{Iu}(t,\xi_3)\widehat{\overline{Iu}}(t,\xi_4)d\xi dt,\end{aligned} \tag{57}$$

where $\Sigma = \{\xi_1+\xi_2+\xi_3+\xi_4 = 0\}$ and $d\xi$ is the Lebesgue measure on the hyperplane Σ. Make a Littlewood-Paley decomposition. Without loss of generality let $N_2 \geq N_3 \geq N_4$. Consider a number of cases separately.

Case 1, $N_2 << N$: In this case

$$\frac{m(\xi_2+\xi_3+\xi_4)}{m(\xi_2)m(\xi_3)m(\xi_4)} - 1 \equiv 0.$$

Case 2, $N_1 \sim N_2 \gtrsim N$, $N_3 << N$:

Case 2(a): $N_4 \geq \frac{1}{N^2}$ In this case, apply the fundamental theorem of calculus.

$$|\frac{m(N_2+N_3+N_4)}{m(N_2)} - 1| \lesssim \frac{|\nabla m(N_2)|}{m(N_2)} N_3 \lesssim \frac{N_3}{N_2}.$$

Therefore,

$$\begin{aligned}(55) \lesssim \sum_{N \lesssim N_1 \sim N_2} \frac{N_1}{N_2^2} \|P_{N_1}\nabla Iu\|_{L_t^2 L_x^6(J\times\mathbf{R}^3)} \|P_{N_2}\nabla Iu\|_{L_t^2 L_x^6(J\times\mathbf{R}^3)} \\ \times \sum_{\frac{1}{N^2} \leq N_4 \leq N_3 << N} N_3 \|P_{N_3} Iu\|_{L_t^\infty L_x^2(J\times\mathbf{R}^3)} \|P_{N_4} Iu\|_{L_t^\infty L_x^6(J\times\mathbf{R}^3)},\end{aligned} \tag{58}$$

$$\begin{aligned}&\lesssim \|\nabla Iu\|_{S^0(J\times\mathbf{R}^3)}^2 \sum_{N \lesssim N_1 \sim N_2} \frac{\ln(N)}{N_1} \|P_{N_1}\nabla Iu\|_{L_t^2 L_x^6(J\times\mathbf{R}^3)} \|P_{N_2}\nabla Iu\|_{L_t^2 L_x^6(J\times\mathbf{R}^3)}, \\ &\lesssim \frac{1}{N^{1-}} \|P_{>cN}\nabla Iu\|_{L_t^2 L_x^6(J\times\mathbf{R}^3)}^2 \|\nabla Iu\|_{S^0(J\times\mathbf{R}^3)}^2.\end{aligned} \tag{59}$$

Case 2(b), $N_4 \leq \frac{1}{N^2}$: In this case, combine the Sobolev embedding theorem with (51),

$$\|P_{N_4} u\|_{L_t^4 L_x^\infty(J\times\mathbf{R}^3)} \lesssim N_4^{3/4} \|P_{N_4} u\|_{L_{t,x}^4(J\times\mathbf{R}^3)} \lesssim \epsilon N_4^{3/4}. \tag{60}$$

$$\begin{aligned}(55) &\lesssim \sum_{N \lesssim N_1 \sim N_2} \frac{N_1}{N_2^2} \|P_{N_1} \nabla Iu\|_{L_t^4 L_x^3 (J \times \mathbf{R}^3)} \|P_{N_2} \nabla Iu\|_{L_t^4 L_x^3 (J \times \mathbf{R}^3)} \\ &\times \sum_{N_4 \leq \frac{1}{N^2}; N_4 \leq N_3 << N} N_3 \|P_{N_3} Iu\|_{L_t^4 L_x^3 (J \times \mathbf{R}^3)} \|P_{N_4} Iu\|_{L_t^4 L_x^\infty (J \times \mathbf{R}^3)} \\ &\lesssim \frac{\epsilon}{N^{5/2-}} \|\nabla Iu\|^3_{S^0(J \times \mathbf{R}^3)}.\end{aligned} \tag{61}$$

Case 3, $N_2 \gtrsim N$, $N_3 \gtrsim N$, $N_2 \sim N_1$:

Case 3(a), $N_4 \geq \frac{1}{N^2}$: In this case make the crude estimate

$$\left|\frac{m(N_2 + N_3 + N_4)}{m(N_2)m(N_3)m(N_4)} - 1\right| \lesssim \frac{1}{m(N_3)m(N_4)}. \tag{62}$$

$$\begin{aligned}(55) &\lesssim \Big\| \sum_{N_1 \sim N_2} \frac{N_1}{N_2} (P_{N_1} \nabla Iu)(P_{N_2} \nabla Iu) \Big\|_{L_t^1 L_x^3 (J \times \mathbf{R}^3)} \\ &\times \sum_{N_3 \gtrsim N; N_4 \geq \frac{1}{N^2}} \frac{1}{N_3 m(N_3) m(N_4)} \|P_{N_3} \nabla Iu\|_{L_t^\infty L_x^2 (J \times \mathbf{R}^3)} \|P_{N_4} Iu\|_{L_t^\infty L_x^6 (J \times \mathbf{R}^3)},\end{aligned} \tag{63}$$

$$\begin{aligned}&\sum_{\frac{1}{N^2} \leq N_4 \leq N_3; N \lesssim N_3 \leq N_2} \frac{1}{N_3 m(N_3) m(N_4)} \\ &\lesssim \sum_{N \lesssim N_3 \leq N_2} \frac{1}{N_3 m(N_3)} (\ln(N) + \frac{N_3^{1-s}}{N^{1-s}}) \lesssim \frac{1}{N^{1-}}.\end{aligned} \tag{64}$$

Summing $N_1 \sim N_2$ by Cauchy - Schwartz and (25),

$$(55) \lesssim \frac{1}{N^{1-}} \|P_{>cN} \nabla Iu\|^2_{L_t^2 L_x^6 (J \times \mathbf{R}^3)} \|\nabla Iu\|^2_{S^0(J \times \mathbf{R}^3)}. \tag{65}$$

Case 3(b), $N_4 \leq \frac{1}{N^2}$: Here,

$$\left|\frac{m(N_2 + N_3 + N_4)}{m(N_2)m(N_3)m(N_4)} - 1\right| \lesssim \frac{1}{m(N_3)}.$$

Once again use the Sobolev embedding theorem combined with $\|u\|_{L^4_{t,x}(J \times \mathbf{R}^3)} \leq \epsilon$.

$$\begin{aligned}(55) &\lesssim \Big\| \sum_{N \lesssim N_1 \sim N_2} \frac{N_1}{N_2} (P_{N_1} \nabla Iu)(P_{N_2} \nabla Iu) \Big\|_{L_t^2 L_x^{3/2} (J \times \mathbf{R}^3)} \\ &\times \sum_{N_4 \leq \frac{1}{N^2}; N_3 \gtrsim N} \frac{1}{N_3 m(N_3)} \|P_{N_3} \nabla Iu\|_{L_t^4 L_x^3 (J \times \mathbf{R}^3)} \|P_{N_4} Iu\|_{L_t^4 L_x^\infty (J \times \mathbf{R}^3)} \\ &\lesssim \frac{\epsilon}{N^{5/2-}} \|\nabla Iu\|^3_{S^0(J \times \mathbf{R}^3)}.\end{aligned} \tag{66}$$

Case 4, $N_2 \gtrsim N$, $N_2 \sim N_3$, $N_1 \lesssim N_2$:

In this case

$$|\frac{m(\xi_2+\xi_3+\xi_4)}{m(\xi_2)m(\xi_3)m(\xi_4)} - 1| \lesssim \frac{1}{m(\xi_2)m(\xi_3)m(\xi_4)}. \tag{67}$$

Case 4(a), $N_4 \geq \frac{1}{N^2}$*:*

$$\begin{aligned}(55) \lesssim & \sum_{N \lesssim N_2 \sim N_3} \frac{1}{m(N_2)m(N_3)N_3N_2} \|P_{N_2}\nabla Iu\|_{L_t^2 L_x^6(J\times\mathbf{R}^3)} \|P_{N_3}\nabla Iu\|_{L_t^2 L_x^6(J\times\mathbf{R}^3)} \\ \times & \sum_{N_1 \lesssim N_2; \frac{1}{N^2} \leq N_4 \leq N_3} \frac{N_1}{m(N_4)} \|P_{N_1}\nabla Iu\|_{L_t^\infty L_x^2(J\times\mathbf{R}^3)} \|P_{N_4} Iu\|_{L_t^\infty L_x^6(J\times\mathbf{R}^3)}.\end{aligned} \tag{68}$$

$$\sum_{\frac{1}{N^2} \leq N_4 \leq N_3} \frac{1}{m(N_4)} \|P_{N_4} Iu\|_{L_t^\infty L_x^6(I\times\mathbf{R}^3)} \lesssim (\ln(N) + \frac{N_3^{1-s}}{N^{1-s}}) \|\nabla Iu\|_{S^0(J\times\mathbf{R}^3)}. \tag{69}$$

Because $s > \frac{1}{2}$,

$$\sum_{N_1 \lesssim N_2 \sim N_3} \frac{N_1}{N_2^s N_3^s N^{2(1-s)}} \lesssim \frac{1}{N^{1-}}. \tag{70}$$

Therefore,

$$(68) \lesssim \frac{1}{N^{1-}} \|P_{>cN}\nabla Iu\|_{L_t^2 L_x^6(J\times\mathbf{R}^3)}^2 \|\nabla Iu\|_{S^0(J\times\mathbf{R}^3)}^2. \tag{71}$$

Case 4(b), $N_4 \leq \frac{1}{N^2}$*:* As usual use the Sobolev embedding.

$$\begin{aligned}(55) \lesssim & \sum_{N \lesssim N_2 \sim N_3} \frac{1}{N_2 m(N_3) N_3} \|P_{N_2}\nabla Iu\|_{L_t^4 L_x^3(J\times\mathbf{R}^3)} \|P_{N_3}\nabla Iu\|_{L_t^4 L_x^3(J\times\mathbf{R}^3)} \\ \times & \sum_{N_4 \leq \frac{1}{N^2}; N_1 \lesssim N_2} N_1 \|P_{N_1}\nabla Iu\|_{L_t^4 L_x^3(J\times\mathbf{R}^3)} \|P_{N_4} Iu\|_{L_t^4 L_x^\infty(J\times\mathbf{R}^3)} \\ \lesssim & \frac{\epsilon}{N^{5/2-}} \|\nabla Iu\|_{S^0(J\times\mathbf{R}^3)}^3.\end{aligned} \tag{72}$$

Combining all these cases with theorem (3.2) proves (55) satisfies theorem (4.1).

4.2. **The term** (56)**.** To estimate this term we use a lemma.

Lemma 4.2.

$$\|P_M I(|u|^2 u)\|_{L_{t,x}^2(J\times\mathbf{R}^3)} \lesssim (\frac{1}{M} + \frac{1}{N}) \|\nabla Iu\|_{S^0(J\times\mathbf{R}^3)}^3. \tag{73}$$

Proof. Make a high-low decomposition of u.

$$\|\nabla I(|u_b|^2 u_b)\|_{L_{t,x}^2(J\times\mathbf{R}^3)} \lesssim \|\nabla Iu\|_{L_t^2 L_x^6(J\times\mathbf{R}^3)} \|u_b\|_{L_t^\infty L_x^6(J\times\mathbf{R}^3)}^2 \lesssim \|\nabla Iu\|_{S^0(J\times\mathbf{R}^3)}^3. \tag{74}$$

$$\|\nabla I(|u_b|^2 u_s)\|_{L_{t,x}^2(J\times\mathbf{R}^3)} \lesssim \|\nabla Iu\|_{L_t^2 L_x^6(J\times\mathbf{R}^3)} \|u_b\|_{L_t^\infty L_x^6(J\times\mathbf{R}^3)}^2 \lesssim \|\nabla Iu\|_{S^0(J\times\mathbf{R}^3)}^3. \tag{75}$$

Make a similar argument for $u_b^2\bar{u}_s$. Next, by the Sobolev embedding theorem

$$\begin{aligned}&\|I(|u_s|^2u_b)\|_{L^2_{t,x}(J\times\mathbf{R}^3)}\lesssim\|\nabla I(|u_s|^2u_b)\|_{L_t^2L_x^{6/5}(J\times\mathbf{R}^3)}\\ \lesssim&\|\nabla Iu\|_{L_t^2L_x^6(J\times\mathbf{R}^3)}\|u_b\|_{L_t^\infty L_x^6(J\times\mathbf{R}^3)}\|u_s\|_{L_t^\infty L_x^2(J\times\mathbf{R}^3)}\lesssim\frac{1}{N}\|\nabla Iu\|^3_{S^0(J\times\mathbf{R}^3)}.\end{aligned}\tag{76}$$

Make a similar argument for $u_s^2\bar{u}_b$. Here we applied (32) and (33) to show

$$\|P_{>N}u\|_{L_t^\infty L_x^2(J\times\mathbf{R}^3)}\lesssim\frac{1}{N}\|\nabla Iu\|_{L_t^\infty L_x^2(J\times\mathbf{R}^3)}.\tag{77}$$

Similarly, by the Sobolev embedding, (32), and (33),

$$\begin{aligned}\|\nabla I(|u_s|^2u_s)\|_{L_t^2L_x^{6/5}(J\times\mathbf{R}^3)}&\lesssim\|\nabla Iu\|_{L_t^2L_x^6(J\times\mathbf{R}^3)}\|u_s\|^2_{L_t^\infty\dot{H}_x^{1/2}(J\times\mathbf{R}^3)}\\&\lesssim\frac{1}{N}\|\nabla Iu\|^3_{S^0(J\times\mathbf{R}^3)}.\end{aligned}\tag{78}$$

Applying Bernstein's inequality to (74) and (75) proves the lemma. □

The nonlinear term is a 6-linear term. Let $\xi_{123}=\xi_1+\xi_2+\xi_3$ and let N_{123} be the corresponding dyadic frequency such that $N_{123}\sim|\xi_{123}|$.

$$\begin{aligned}(56)=-\int_{t_1}^{t_2}\int_\Sigma i\widehat{I(|u|^2u)}(t,\xi_{123})[\frac{m(\xi_4+\xi_5+\xi_6)}{m(\xi_4)m(\xi_5)m(\xi_6)}-1]&\\ \times\widehat{\overline{Iu}}(t,\xi_4)\widehat{Iu}(t,\xi_5)\widehat{\overline{Iu}}(t,\xi_6)d\xi dt,&\end{aligned}\tag{79}$$

where $\Sigma=\{\xi_1+\xi_2+\xi_3+\xi_4+\xi_5+\xi_6=0\}$ and $d\xi$ is the Lebesgue measure on the hyperplane. Make a Littlewood-Paley decomposition and assume without loss of generality that $N_4\geq N_5\geq N_6$.

Case 1, $N_4<<N$: In this case the multiplier

$$\frac{m(\xi_4+\xi_5+\xi_6)}{m(\xi_4)m(\xi_5)m(\xi_6)}-1\equiv 0.\tag{80}$$

Case 2, $N_4\gtrsim N$, $N_5<<N$: Again use the fundamental theorem of calculus. Because $N_5,N_6<<N_4$, $N_{123}\sim N_4$.

Case 2(a), $N_6\geq\frac{1}{N^2}$:

$$|\frac{m(\xi_4+\xi_5+\xi_6)}{m(\xi_4)m(\xi_5)m(\xi_6)}-1|\lesssim\frac{|\xi_5|}{|\xi_4|}.\tag{81}$$

$$\begin{aligned}(56)\lesssim&\sum_{N\lesssim N_4\sim N_{123}}\frac{1}{N_4}\|P_{N_{123}}I(|u|^2u)\|_{L^2_{t,x}(J\times\mathbf{R}^3)}\|P_{N_4}Iu\|_{L_t^2L_x^6(J\times\mathbf{R}^3)}\\ \times&\sum_{\frac{1}{N^2}\leq N_6\leq N_5<<N}N_5\|P_{N_5}Iu\|_{L_t^\infty L_x^6(J\times\mathbf{R}^3)}\|P_{N_6}Iu\|_{L_t^\infty L_x^6(J\times\mathbf{R}^3)}\end{aligned}\tag{82}$$

$$\lesssim\ln(N)N\|\nabla Iu\|^6_{S^0(J\times\mathbf{R}^3)}\sum_{N\lesssim N_4\sim N_{123}}\frac{1}{N_4}(\frac{1}{N_{123}}+\frac{1}{N})\lesssim\frac{1}{N^{2-}}\|\nabla Iu\|^6_{S^0(J\times\mathbf{R}^3)}.\tag{83}$$

Case 2(b): $N_6\leq\frac{1}{N^2}$: As before use the Sobolev embedding

$$\|P_{N_6} Iu\|_{L_t^4 L_x^\infty(J\times\mathbf{R}^3)} \lesssim N_6^{3/4} \|P_{N_6} Iu\|_{L_{t,x}^4(J\times\mathbf{R}^3)} \lesssim \epsilon N_6^{3/4}. \tag{84}$$

$$\begin{aligned}(56) \lesssim & \sum_{N \lesssim N_{123} \sim N_4} \frac{1}{N_4} \|P_{N_{123}} I(|u|^2 u)\|_{L_{t,x}^2(J\times\mathbf{R}^3)} \|P_{N_4} Iu\|_{L_t^4 L_x^3(J\times\mathbf{R}^3)} \\ & \times \sum_{N_6 \leq N_5 << N; N_6 \leq \frac{1}{N^2}} N_5 \|P_{N_5} Iu\|_{L_t^\infty L_x^6(J\times\mathbf{R}^3)} \|P_{N_6} Iu\|_{L_t^4 L_x^\infty(J\times\mathbf{R}^3)}\end{aligned} \tag{85}$$

$$\lesssim \epsilon \sum_{N \lesssim N_{123} \sim N_4} (\frac{1}{N} + \frac{1}{N_{123}}) \frac{N}{N_4^2} \frac{1}{N^{3/2}} \|\nabla Iu\|_{S^0(J\times\mathbf{R}^3)}^5 \tag{86}$$

$$\lesssim \frac{\epsilon}{N^{7/2-}} \|\nabla Iu\|_{S^0(J\times\mathbf{R}^3)}^5. \tag{87}$$

Case 3, $N_5 \gtrsim N$, $N_4 \sim N_{123}$: Here make the crude estimate,

$$|\frac{m(\xi_4+\xi_5+\xi_6)}{m(\xi_4)m(\xi_5)m(\xi_6)} - 1| \lesssim \frac{1}{m(\xi_5)m(\xi_6)}. \tag{88}$$

Case 3(a), $N_6 \geq \frac{1}{N^2}$:

$$\begin{aligned}(56) \lesssim & \sum_{N \lesssim N_4 \sim N_{123}} \|P_{N_{123}} I(|u|^2 u)\|_{L_{t,x}^2(J\times\mathbf{R}^3)} \|P_{N_4} Iu\|_{L_t^2 L_x^6(J\times\mathbf{R}^3)} \\ & \times \sum_{\frac{1}{N^2} \leq N_6 \leq N_5; N \lesssim N_5} \frac{1}{m(N_5)m(N_6)} \|P_{N_5} Iu\|_{L_t^\infty L_x^6(J\times\mathbf{R}^3)} \|P_{N_6} Iu\|_{L_t^\infty L_x^6(J\times\mathbf{R}^3)}.\end{aligned} \tag{89}$$

$$\begin{aligned}& \sum_{\frac{1}{N^2} \leq N_6 \leq N_5} \frac{\|P_{N_6} Iu\|_{L_t^\infty L_x^6(J\times\mathbf{R}^3)}}{m(N_6)} \lesssim (\ln(N) + \frac{N_5^{1-s}}{N^{1-s}}) \|\nabla Iu\|_{S^0(J\times\mathbf{R}^3)}. \\ & \|\nabla Iu\|_{S^0(J\times\mathbf{R}^3)} \sum_{N \lesssim N_5 \lesssim N_4} \frac{1}{m(N_5)} \|P_{N_5} Iu\|_{L_t^\infty L_x^6(J\times\mathbf{R}^3)} (\ln(N) + \frac{N_5^{1-s}}{N^{1-s}}) \\ & \lesssim (\ln(N)^2 + \frac{N_4^{2(1-s)}}{N^{2(1-s)}}) \|\nabla Iu\|_{S^0(J\times\mathbf{R}^3)}^2.\end{aligned} \tag{90}$$

Since $2(1-s) < 1$, by lemma (4.2)

$$(56) \lesssim \|\nabla Iu\|_{S^0(J\times\mathbf{R}^3)}^6 \sum_{N \lesssim N_4 \sim N_{123}} \frac{1}{N} (\ln(N)^2 + \frac{N_4^{2(1-s)}}{N^{2(1-s)}}) \frac{1}{N_4} \lesssim \frac{1}{N^{2-}} \|\nabla Iu\|_{S^0(J\times\mathbf{R}^3)}^6. \tag{91}$$

Case 3(b), $N_6 \leq \frac{1}{N^2}$: In this case

$$|\frac{m(\xi_4+\xi_5+\xi_6)}{m(\xi_4)m(\xi_5)m(\xi_6)} - 1| \lesssim \frac{1}{m(\xi_5)}. \tag{92}$$

$$(56) \lesssim \sum_{N \lesssim N_{123} \sim N_4} \|P_{N_{123}} I(|u|^2 u)\|_{L^2_{t,x}(J \times \mathbf{R}^3)} \|P_{N_4} Iu\|_{L^4_t L^3_x(J \times \mathbf{R}^3)} \times \sum_{N_5 \gtrsim N; N_6 \leq \frac{1}{N^2}} \frac{1}{m(N_5)} \|P_{N_5} Iu\|_{L^\infty_t L^6_x(J \times \mathbf{R}^3)} \|P_{N_6} Iu\|_{L^4_t L^\infty_x(J \times \mathbf{R}^3)} \tag{93}$$

$$\lesssim \sum_{N \lesssim N_{123} \sim N_4} \epsilon(\frac{1}{N} + \frac{1}{N_{123}}) \frac{1}{N_4} \frac{N_4^{1-s}}{N^{1-s}} \frac{1}{N^{3/2}} \|\nabla Iu\|^5_{S^0(J \times \mathbf{R}^3)} \tag{94}$$

$$\lesssim \frac{\epsilon}{N^{7/2-}} \|\nabla Iu\|^5_{S^0(J \times \mathbf{R}^3)}. \tag{95}$$

Case 4, $N_5 \gtrsim N$, $N_4 \sim N_5$, $N_{123} \lesssim N_4$: Make the crude estimate

$$|\frac{m(\xi_4 + \xi_5 + \xi_6)}{m(\xi_4) m(\xi_5) m(\xi_6)} - 1| \lesssim \frac{1}{m(\xi_4) m(\xi_5) m(\xi_6)}. \tag{96}$$

Case 4(a), $N_6 \geq \frac{1}{N^2}$: In this case we rely on the fact that $s > \frac{2}{3}$.

$$(56) \lesssim \| \sum_{N \lesssim N_4 \sim N_5} (P_{N_4} u)(P_{N_5} u)\|_{L^2_t L^{3/2}_x(J \times \mathbf{R}^3)} \times \sum_{\frac{1}{N^2} \leq N_6 \leq N_5; \frac{1}{N^2} \leq N_{123} \leq N} \|P_{N_{123}} I(|u|^2 u)\|_{L^2_t L^6_x(J \times \mathbf{R}^3)} \|P_{N_6} u\|_{L^\infty_t L^6_x(J \times \mathbf{R}^3)} \tag{97}$$

$$+ \| \sum_{N_4 \sim N_5} (P_{N_4} u)(P_{N_5} u)\|_{L^2_t L^{3/2}_x(J \times \mathbf{R}^3)} \times \sum_{\frac{1}{N^2} \leq N_6 \leq N_5; N_{123} \leq \frac{1}{N^2}} \|P_{N_{123}} I(|u|^2 u)\|_{L^2_t L^\infty_x(J \times \mathbf{R}^3)} \|P_{N_6} u\|_{L^\infty_t L^3_x(J \times \mathbf{R}^3)} \tag{98}$$

$$+ \| \sum_{N_4 \sim N_5} (P_{N_4} u)(P_{N_5} u)\|_{L^2_t L^{3/2}_x(J \times \mathbf{R}^3)} \times \sum_{\frac{1}{N^2} \leq N_6 \leq N; N_{123} \geq N} \|P_{N_{123}} I(|u|^2 u)\|_{L^2_t L^2_x(J \times \mathbf{R}^3)} \|P_{N_6} u\|_{L^\infty_t L^6_x(J \times \mathbf{R}^3)} \tag{99}$$

$$+ \| \sum_{N_4 \sim N_5} (P_{N_4} u)(P_{N_5} u)\|_{L^2_t L^{9/2}_x(J \times \mathbf{R}^3)} \times \sum_{N_6 \geq N; N_{123} \geq N} \|P_{N_{123}} I(|u|^2 u)\|_{L^2_t L^2_x(J \times \mathbf{R}^3)} \|P_{N_6} u\|_{L^\infty_t L^{18/5}_x(J \times \mathbf{R}^3)} \tag{100}$$

$$\lesssim \frac{\ln(N)}{N^2} \lesssim \frac{1}{N^{2-}}. \tag{101}$$

Since $s > 2/3$, for $N_4 \gtrsim N$,

$$\||\nabla|^{2/3} P_{N_4} u\|_{L^4_t L^3_x(J \times \mathbf{R}^3)} \lesssim N_4^{2/3-s} N^{s-1} E(Iu(t)), \tag{102}$$

$$\|P_{N_4} u\|_{L^4_t L^3_x(J \times \mathbf{R}^3)} \lesssim N_4^{-s} N^{s-1} E(Iu(t)). \tag{103}$$

Plugging (102) and (103) into (97) to (100) along with the fact that $I = 1$ for $|\xi| \leq N$ gives (101).

Case 4(b), $N_6 \leq \frac{1}{N^2}$:

$$\begin{aligned}(56) \lesssim & \sum_{N \lesssim N_4 \sim N_5} \frac{1}{m(N_4)m(N_5)} \|P_{N_4} Iu\|_{L_t^\infty L_x^2(J \times \mathbf{R}^3)} \|P_{N_5} Iu\|_{L_t^4 L_x^3(J \times \mathbf{R}^3)} \\ & \times [\sum_{N_{123} \leq \frac{1}{N^2}; N_6 \leq \frac{1}{N^2}} \|P_{N_{123}} Iu\|_{L_t^2 L_x^\infty(J \times \mathbf{R}^3)} \|P_{N_6} Iu\|_{L_t^4 L_x^6(J \times \mathbf{R}^3)} \\ & + \sum_{\frac{1}{N^2} \leq N_{123} \lesssim N_4; N_6 \leq \frac{1}{N^2}} \|P_{N_{123}} Iu\|_{L_t^2 L_x^6(J \times \mathbf{R}^3)} \|P_{N_6} Iu\|_{L_t^4 L_x^\infty(J \times \mathbf{R}^3)}] \end{aligned} \tag{104}$$

$$\lesssim \|\nabla Iu\|_{S^0(J \times \mathbf{R}^3)}^5 \sum_{N \lesssim N_4 \sim N_5} \frac{1}{N_4 N_5 m(N_4) m(N_5)} [\frac{\epsilon}{N^{3/2}} + (\ln(N) + \frac{N_4}{N}) \frac{\epsilon}{N^{3/2}}] \tag{105}$$

$$\lesssim \frac{\epsilon}{N^{7/2-}} \|\nabla Iu\|_{S^0(J \times \mathbf{R}^3)}^5. \tag{106}$$

This concludes the proof of theorem (4.1). □

5. **A smoothing estimate.** In this section we take advantage of lemma (2.2) to prove a smoothing estimate for the Duhamel term.

Lemma 5.1. *Take $N_j \leq N$. If*

$$\|u\|_{L_{t,x}^4(J \times \mathbf{R}^3)} \leq \epsilon, \tag{107}$$

then

$$\|P_{N_j}(|u|^2 u)\|_{L_t^1 L_x^2(J \times \mathbf{R}^3)} \lesssim \frac{1}{N_j} \|P_{N_j} \nabla I(|u|^2 u)\|_{L_t^1 L_x^2(J \times \mathbf{R}^3)} \lesssim \frac{1}{N_j} \|\nabla Iu\|_{S^0(J \times \mathbf{R}^3)}^3. \tag{108}$$

Proof. The first inequality is Bernstein's inequality. Because $m(\xi)|\xi|$ is increasing,

$$\begin{aligned}\|P_{N_j} \nabla I(|u|^2 u)\|_{L_t^1 L_x^2(J \times \mathbf{R}^3)} \lesssim & \|\nabla Iu\|_{L_t^2 L_x^6(J \times \mathbf{R}^3)} \\ & \times (\|P_{\leq 1} u\|_{L_t^4 L_x^6(J \times \mathbf{R}^3)}^2 + \|P_{>1} u\|_{L_t^4 L_x^6(J \times \mathbf{R}^3)}^2). \end{aligned} \tag{109}$$

By the Sobolev embedding theorem and (107),

$$\|P_{\leq 1} u\|_{L_t^4 L_x^6(J \times \mathbf{R}^3)} \lesssim \|u\|_{L_{t,x}^4(J \times \mathbf{R}^3)} \leq \epsilon. \tag{110}$$

On the other hand,

$$\|P_{N_k} u\|_{L_t^4 L_x^6(J \times \mathbf{R}^3)} \lesssim N_k^{1/2} \|P_{N_k} u\|_{S^0(J \times \mathbf{R}^3)}. \tag{111}$$

Therefore,

$$\begin{aligned}\|P_{>1} u\|_{L_t^4 L_x^6(J \times \mathbf{R}^3)} \lesssim & \sum_{1 \leq N_k \leq N} \frac{1}{N_k^{1/2}} \|\nabla Iu\|_{S^0(J \times \mathbf{R}^3)} \\ & + \sum_{N_k > N} \frac{1}{N_k^{s-1/2} N^{1-s}} \|\nabla Iu\|_{S^0(J \times \mathbf{R}^3)} \lesssim \|\nabla Iu\|_{S^0(J \times \mathbf{R}^3)}. \end{aligned} \tag{112}$$

□

Theorem 5.2. *Suppose* $J = [0, T]$ *is an interval with*

$$\|u\|_{L^4_{t,x}(J \times \mathbf{R}^3)} \leq \epsilon, \tag{113}$$

and $\|\nabla I u_0\|_{L^2(\mathbf{R}^3)} \leq 1$. *The solution to* (21) *on* $[0, T]$ *can be split into a linear piece and a nonlinear piece,*

$$u(t) = e^{it\Delta} u_0 + \int_0^t e^{i(t-\tau)\Delta}(|u|^2 u)(\tau) d\tau = u^l(t) + u^{nl}(t), \tag{114}$$

with

$$\|P_{>N} \nabla I u^{nl}\|_{S^0(J \times \mathbf{R}^3)} \lesssim \frac{1}{N^{1/2-}}(1 + \|\nabla I u\|^7_{S^0(J \times \mathbf{R}^3)}), \tag{115}$$

and

$$\|P_{>N} \nabla I u^{nl}\|_{L^\infty_t L^2_x(J \times \mathbf{R}^3)} \lesssim \frac{1}{N^{1-}}(1 + \|\nabla I u\|^9_{S^0(J \times \mathbf{R}^3)}). \tag{116}$$

Proof. Make a high-low decomposition of u, $u = u_b + u_s$ with $u_b = P_{\leq N/20} u$.

Since $P_{>N}(|u_b|^2 u_b) \equiv 0$, it suffices to consider $O(u^2 u_s)$. Because $|\xi| m(|\xi|)$ is increasing as $|\xi| \to \infty$,

$$\|\nabla I(|u_b|^2 u_s)\|_{N^0(J \times \mathbf{R}^3)} \lesssim \|(\nabla I u_s)|u_b|^2\|_{L^{4/3}_t L^{3/2}_x(J \times \mathbf{R}^3)} \tag{117}$$

$$\lesssim \|(\nabla I u_s) u_b\|_{L^2_{t,x}(J \times \mathbf{R}^3)} \|u_b\|_{L^4_t L^6_x(J \times \mathbf{R}^3)}. \tag{118}$$

By Sobolev embedding, (113), Strichartz estimates, and $\dot{H}^{1/2} \subset \dot{H}^1$ when $|\xi| \geq 1$,

$$\|u_b\|_{L^4_t L^6_x(J \times \mathbf{R}^3)} \leq \|u_{\leq 1}\|_{L^4_t L^6_x(J \times \mathbf{R}^3)} + \|u_{\geq 1}\|_{L^4_t L^6_x(J \times \mathbf{R}^3)} \tag{119}$$

$$\lesssim \epsilon + \|\nabla I u\|_{S^0(J \times \mathbf{R}^3)}. \tag{120}$$

Next,

$$\|(\nabla I u_s)(P_{\leq N^{-2}} u_b)\|_{L^2_{t,x}(J \times \mathbf{R}^3)} \lesssim \|P_{\leq N^{-2}} u_b\|_{L^4_t L^6_x(J \times \mathbf{R}^3)} \|\nabla I u_s\|_{L^4_t L^3_x(J \times \mathbf{R}^3)} \tag{121}$$

$$\lesssim \epsilon N^{-1/2} \|\nabla I u\|_{S^0(J \times \mathbf{R}^3)}. \tag{122}$$

Finally, estimate

$$\|(\nabla I u_s)(P_{>N^{-2}} u_b)\|_{L^2_{t,x}(J \times \mathbf{R}^3)} \tag{123}$$

using the bilinear estimates in (42) and lemma (5.1),

$$\|(\nabla I u_s) P_{>N^{-2}} u_b\|_{L^2_{t,x}(J \times \mathbf{R}^3)} \lesssim \Big(\sum_{N^{-2} \leq N_k \leq N/20} \frac{1}{N_k} \frac{N_k}{N^{1/2}} \Big)(\|\nabla I u\|^2_{S^0(J \times \mathbf{R}^3)} + \|\nabla I u\|^6_{S^0(J \times \mathbf{R}^3)}) \tag{124}$$

$$\lesssim \frac{1}{N^{1/2-}}(\|\nabla I u\|^2_{S^0(J \times \mathbf{R}^3)} + \|\nabla I u\|^6_{S^0(J \times \mathbf{R}^3)}). \tag{125}$$

Therefore

$$\|(\nabla I u_s) u_b\|_{L^2_{t,x}(J \times \mathbf{R}^3)} \lesssim \frac{1}{N^{1/2-}} (1 + \|\nabla I u\|^6_{S^0(J \times \mathbf{R}^3)}), \tag{126}$$

which combined with (120) takes care of the term $I(|u_b|^2 u_s)$. The term $I(u_b^2 \bar{u}_s)$ can be estimated in a similar manner.

The other terms are easier to estimate.

$$\|\nabla I(|u_h|^2 u_l)\|_{L^2_t L^{6/5}_x(J \times \mathbf{R}^3)} \lesssim \|\nabla I u\|_{L^2_t L^6_x(J \times \mathbf{R}^3)} \|u_h\|_{L^\infty_t L^2_x(J \times \mathbf{R}^3)} \|u_l\|_{L^\infty_t L^6_x(J \times \mathbf{R}^3)} \tag{127}$$

$$\lesssim \frac{1}{N^{1-}} \|\nabla I u\|^3_{S^0(J \times \mathbf{R}^3)}. \tag{128}$$

A similar calculation can be made for $\bar{u}_b u_s^2$. Finally,

$$\begin{aligned} \|\nabla I(|u_h|^2 u_h)\|_{L^1_t L^2_x(J \times \mathbf{R}^3)} &\lesssim \|\nabla I u\|_{L^2_t L^6_x(J \times \mathbf{R}^3)} \|u_h\|^2_{L^4_t L^6_x(J \times \mathbf{R}^3)} \\ &\lesssim \frac{1}{N^{1-}} \|\nabla I u\|^3_{S^0(J \times \mathbf{R}^3)}. \end{aligned} \tag{129}$$

This finishes the proof of (115). To prove (116) it only remains to show

$$\| \int_0^t e^{i(t-\tau)\Delta} P_{>N}(\nabla I(u_b^2 u_s)(\tau)) d\tau \|_{L^\infty_t L^2_x(J \times \mathbf{R}^3)} \lesssim \frac{1}{N^{1-}} (1 + \|\nabla I u\|^9_{S^0(J \times \mathbf{R}^3)}). \tag{130}$$

Take a function $f(t,x)$ supported on $|\xi| \geq \frac{N}{4}$ such that

$$\|f(t,x)\|_{L^1_t L^2_x(J \times \mathbf{R}^3)} = 1.$$

By duality, estimating (130) is equivalent to estimating

$$\int_J \langle \int_0^t e^{i(t-\tau)\Delta} (\nabla I(|u_b|^2 u_s)(\tau)) d\tau, f(t,x) \rangle dt, \tag{131}$$

for all such $f(t,x)$. By Fubini's theorem,

$$(131) = \int_J \langle (\nabla I(|u_b|^2 u_s)(\tau)), \int_\tau^T e^{i(\tau - t)\Delta} f(t,x) dt \rangle d\tau. \tag{132}$$

Let

$$\int_\tau^T e^{i(\tau-t)\Delta} f(t,x) dt = v(\tau, x), \tag{133}$$

where $v(\tau,x)$ solves the partial differential equation

$$\begin{aligned} i v_\tau - \Delta v &= -f(\tau, x) \\ v(T) &= 0. \end{aligned} \tag{134}$$

$$\int_J \langle (\nabla I(|u_b|^2 u_s)(\tau)), v(\tau) \rangle d\tau \lesssim \|(\nabla I u_s) u_b\|_{L^2_{t,x}(J \times \mathbf{R}^3)} \|(v) u_b\|_{L^2_{t,x}(J \times \mathbf{R}^3)}. \tag{135}$$

By (126),

$$\|(\nabla I u_s) u_b\|_{L^2_{t,x}(J \times \mathbf{R}^3)} \lesssim \frac{1}{N^{1/2-}}(1 + \|\nabla I u\|^6_{S^0(J \times \mathbf{R}^3)}). \tag{136}$$

Similarly,

$$\|v u_b\|_{L^2_{t,x}(J \times \mathbf{R}^3)} \lesssim \frac{1}{N^{1/2-}}(1 + \|\nabla I u\|^3_{S^0(J \times \mathbf{R}^3)}). \tag{137}$$

□

6. **Double layer I-decomposition.** Now we finally have enough tools to prove the main theorem.

Theorem 6.1. *Suppose $s > 5/7$. Then (1) is globally well-posed on $[0, \infty)$. Moreover, $\|u(t)\|_{H^s(\mathbf{R}^3)} \leq C(s, \|u_0\|_{H^s(\mathbf{R}^3)})$, and the solution to (1) scatters to free solutions $u_\pm \in \dot{H}^s(\mathbf{R}^3)$ as $t \to \pm\infty$ respectively.*

Proof. If $u(t,x)$ solves (1) on $[0,T]$, then $\frac{1}{\lambda} u(\frac{t}{\lambda^2}, \frac{x}{\lambda})$ solves (1) on $[0, \lambda^2 T]$. This scaling leaves the $\dot{H}^{1/2}$ norm invariant. Let $u_\lambda(t,x)$ refer to the rescaled solution.

$$\|u_\lambda(0,x)\|_{L^2(\mathbf{R}^3)} = \lambda^{1/2} \|u_0\|_{L^2(\mathbf{R}^3)}, \tag{138}$$

$$\|u_\lambda(0,x)\|_{\dot{H}^1(\mathbf{R}^3)} = \lambda^{-1/2} \|u_0\|_{\dot{H}^1(\mathbf{R}^3)}. \tag{139}$$

Combining the scaling identities with the estimates on (31),

$$\int |\nabla I u_{0,\lambda}(x)|^2 dx \leq \frac{C N^{2(1-s)}}{\lambda^{2s-1}} \|u_0\|^2_{H^s(\mathbf{R}^3)}. \tag{140}$$

$$\int |I u_{0,\lambda}(x)|^4 dx \leq \frac{C N^{3-4s}}{\lambda^{4s-2}} \|u_0\|^4_{H^s(\mathbf{R}^3)}. \tag{141}$$

Choose $\lambda \sim N^{\frac{1-s}{s-1/2}}$ so that $E(I u_\lambda(0,x)) \leq \frac{1}{2}$. Define a set

$$W = \{t : E(I u_\lambda(t)) \leq \frac{9}{10}\}. \tag{142}$$

Since $0 \in W$, $W \neq \emptyset$. Also, by the dominated convergence theorem, W is closed. So it remains to prove W is open in $[0, \infty)$.

If $W = [0,T]$, then by continuity of $E(Iu(t))$ there exists $\delta > 0$ such that $E(I u_\lambda(t)) \leq 1$ on $[0, T+\delta]$.

$$\|P_{\leq N} u_\lambda\|_{L^\infty_t \dot{H}^{1/2}(J \times \mathbf{R}^3)} \leq \|u_\lambda\|^{1/2}_{L^2(\mathbf{R}^3)} \|\nabla I u_\lambda\|^{1/2}_{L^\infty_t L^2_x(J \times \mathbf{R}^3)}. \tag{143}$$

Also,

$$\|P_{>N} u_\lambda\|_{L^\infty_t \dot{H}^{1/2}_x(J \times \mathbf{R}^3)} \leq \frac{1}{N^{1/2}} \|\nabla I u_\lambda\|_{L^\infty_t L^2_x(J \times \mathbf{R}^3)}. \tag{144}$$

Combining the interaction Morawetz estimate (43), (143) and (144),

$$\|u_\lambda\|^4_{L^4_{t,x}([0,T+\delta] \times \mathbf{R}^3)} \leq C N^{\frac{3(1-s)}{2s-1}}. \tag{145}$$

Partition $[0, T+\delta]$ into $\lesssim N^{\frac{3(1-s)}{2s-1}}$ subintervals with $\|u_\lambda\|_{L^4_{t,x}(J_k \times \mathbf{R}^3)} \leq \epsilon$ for each J_k.

Now we will make use of the double-layered I-decomposition utilized in [11]. Subdivide $[0, T + \delta]$ into $\lesssim N^{\frac{3(1-s)}{2s-1} - 1+}$ subintervals J_k where each J_k is the union of N^{1-} subintervals $J_{k,m}$ with $\|u_\lambda\|_{L^4_{t,x}(J_{k,m} \times \mathbf{R}^3)} \leq \epsilon$. We will refer to the intervals J_k as the big intervals, and the subintervals $J_{k,m}$ as the little intervals.

Take the first big interval J_k. Recalling theorem (4.1),

$$\sup_{t_1, t_2 \in J_k} |E(Iu(t_1)) - E(Iu(t_2))| \lesssim \frac{N^{1-}}{N^{2-}} + \frac{1}{N^{1-}} \|P_{>cN} \nabla Iu\|^2_{L^2_t L^6_x(J_k \times \mathbf{R}^3)}. \tag{146}$$

(36) and theorem (3.2) imply the crude estimate $E(Iu(t)) \leq 1$ on this big interval. Let $J_{k,m} = [a_m, b_m]$, $a_0 = 0$, $a_{m+1} = b_m$. Following (114)

$$e^{i(t-a_m)\Delta} u(a_m) + u^{nl}_m(t) = e^{it\Delta} u_0 + \sum_{j=1}^{m} e^{i(t-a_j)\Delta} u^{nl}_{j-1}(a_j) + u^{nl}_m(t). \tag{147}$$

We are content with the first term in (146) so we turn to the second term. We seek to utilize (147) to prove $\|P_{>cN} \nabla Iu\|^2_{L^2_t L^6_x(J_k \times \mathbf{R}^3)} \lesssim 1$.

$$\begin{aligned} \|P_{>cN} \nabla Iu\|_{L^2_t L^6_x(J \times \mathbf{R}^3)} \leq \|P_{>cN} \nabla Iu_0\|_{L^2_x(\mathbf{R}^3)} + \sum_{m=1}^{N^{1-}} \|\nabla P_{>cN} Iu^{nl}_m(a_m)\|_{L^2_x(\mathbf{R}^3)} \\ + (\sum_{m=0}^{N^{1-}} \|P_{>cN} \nabla Iu^{nl}_m\|^2_{L^2_t L^6_x(J_{k,m} \times \mathbf{R}^3)})^{1/2}. \end{aligned} \tag{148}$$

$$\|\nabla Iu_0\|_{L^2_x(\mathbf{R}^3)} \lesssim 1, \tag{149}$$

which takes care of the first term. By (116) and $\|\nabla Iu\|_{S^0(J_{k,m} \times \mathbf{R}^3)} \lesssim 1$,

$$\sum_{m=1}^{N^{1-}} \|\nabla Iu^{nl}_m(a_m)\|_{L^2_x(\mathbf{R}^3)} \lesssim \frac{N^{1-}}{N^{1-}} = 1, \tag{150}$$

which takes care of the second term. Finally by (115)

$$(\sum_{m=0}^{N^{1-}} \|P_{>cN} \nabla Iu^{nl}_m\|^2_{L^2_t L^6_x(J_{k,m} \times \mathbf{R}^3)})^{1/2} \lesssim (\frac{N^{1-}}{N^{1-}})^{1/2} \lesssim 1. \tag{151}$$

Therefore,

$$\sup_{t_1, t_2 \in J_k} |E(Iu(t_1)) - E(Iu(t_2))| \lesssim \frac{1}{N^{1-}}. \tag{152}$$

Recall there are $\lesssim N^{\frac{3(1-s)}{2s-1} - 1+}$ big intervals J_k. When $s > 5/7$,

$$CN^{\frac{3(1-s)}{2s-1} - 1+} << N^{1-}, \tag{153}$$

so choosing N sufficiently large proves

$$\sup_{[0, T+\delta]} E(Iu_\lambda(t)) \leq \frac{9}{10}. \tag{154}$$

This proves W is both open and closed in $[0, \infty)$ so $W = [0, \infty)$.

Finally, we prove scattering, following the argument in [8]. We have proved that there is some $N(s, \|u_0\|_{H^s(\mathbf{R}^3)}) < \infty$ such that

$$E(Iu_\lambda(t)) \leq 1 \tag{155}$$

on $[0, \infty)$. By the interaction Morawetz estimates, (145),

$$\|u_\lambda\|_{L^4_{t,x}([0,\infty)\times\mathbf{R}^3)} \leq C(s, \|u_0\|_{H^s(\mathbf{R}^3)}). \tag{156}$$

Recall that by lemma (3.1), if $\|u_\lambda\|_{L^4_{t,x}(J_{k,m}\times\mathbf{R}^3)} \leq \epsilon$ and $E(Iu_\lambda(t)) \leq 1$ on $J_{k,m}$, then

$$\|u\|_{L^6_t L^{9/2}_x(J_{k,m}\times\mathbf{R}^3)} \lesssim (\epsilon^{2/3} + \frac{1}{N^{1/2}}). \tag{157}$$

Let

$$S_s(t) = \sup_{(p,q) \text{ admissible}} \|\langle\nabla\rangle^s u\|_{L^p_t L^q_x([0,t]\times\mathbf{R}^3)}. \tag{158}$$

$$S_s(t) \lesssim \|\langle\nabla\rangle^s u_0\|_{L^2(\mathbf{R}^3)} + \|\langle\nabla\rangle^s u\|_{L^2_t L^6_x(J\times\mathbf{R}^3)} \|u\|^2_{L^6_t L^{9/2}_x(J\times\mathbf{R}^3)} \tag{159}$$

$$\lesssim \|\langle\nabla\rangle^s u_0\|_{L^2(\mathbf{R}^3)} + S_s(t)(\epsilon^{4/3} + \frac{1}{N}). \tag{160}$$

For $\epsilon > 0$ sufficiently small and N sufficiently large, this proves $S_s(t)$ is bounded on the first subinterval. Iterating over a finite number of subintervals proves $S_s(t) \leq C < \infty$ for $t \in [0, \infty)$. In particular, this proves

$$\|u\|_{H^s(\mathbf{R}^3)} \leq C(\|u_0\|_{H^s(\mathbf{R}^3)}). \tag{161}$$

Now set

$$u_+ = u_0 - i \int_0^\infty e^{-i\tau\Delta} |u(\tau)|^2 u(\tau) d\tau. \tag{162}$$

$$\begin{aligned} \|\langle\nabla\rangle^s (e^{it\Delta} u_+ - u(t,x))\|_{L^2_x(\mathbf{R}^3)} &= \| \int_t^\infty \langle\nabla\rangle^s e^{-i\tau\Delta} |u(\tau)|^2 u(\tau) d\tau \|_{L^2_x(\mathbf{R}^3)} \\ &\lesssim \|\langle\nabla\rangle^s u\|_{L^{10/3}_{t,x}([T,\infty)\times\mathbf{R}^3)} \|u\|^2_{L^5_{t,x}([T,\infty)\times\mathbf{R}^3)}. \end{aligned} \tag{163}$$

As $T \to \infty$, $\|u\|_{L^4_{t,x}([T,\infty)\times\mathbf{R}^3)} \to 0$, on the other hand,

$$\|u\|_{L^6_{t,x}([0,\infty)\times\mathbf{R}^3)} \lesssim \|\langle\nabla\rangle^{2/3} u\|_{L^6_t L^{18/7}_x([0,\infty)\times\mathbf{R}^3)} \lesssim S_{2/3}(t) < \infty, \tag{164}$$

by (158). Interpolation proves $\|u\|_{L^5_{t,x}([T,\infty)\times\mathbf{R}^3)} \to 0$ as $T \to \infty$. By Duhamel's principle

$$\begin{aligned} &\| \int_0^\infty \langle\nabla\rangle^s e^{-i\tau\Delta} |u(\tau)|^2 u(\tau) d\tau \|_{H^s(\mathbf{R}^3)} \\ \lesssim &\|u\|^2_{L^5_{t,x}([0,\infty)\times\mathbf{R}^3)} \|\langle\nabla\rangle^s u\|_{L^5_t L^{30/11}_x([0,\infty)\times\mathbf{R}^3)} \\ \lesssim &(\sup_{t\in[0,\infty)} S_s(t)) \|u\|^2_{L^5_{t,x}([0,\infty)\times\mathbf{R}^3)} < \infty. \end{aligned} \tag{165}$$

Also,

$$\begin{aligned}&\| \int_T^\infty e^{-i\tau\Delta} |u(\tau)|^2 u(\tau) d\tau \|_{L_x^2(\mathbf{R}^3)} \\ \lesssim & \|u\|^2_{L^5_{t,x}([T,\infty)\times\mathbf{R}^3)} \|\langle\nabla\rangle^s u\|_{L_t^5 L_x^{30/11}([T,\infty)\times\mathbf{R}^3)} \to 0 \end{aligned} \tag{166}$$

as $T \to \infty$. This completes the proof of theorem (1.5). □

REFERENCES

[1] J. Bourgain, *Scattering in the energy space and below in 3D NLS*, Journal d'Analyse Mathematique, **4** (1998), 267–297.

[2] J. Bourgain, *Refinements of Strichartz' inequality and applications to 2D-NLS with critical nonlinearity*, International Mathematical Research Notices, **5** (1998), 253–283.

[3] J. Bourgain, "Global Solutions of Nonlinear Schrödinger Equations," American Mathematical Society, American Mathematical Society Colloquium Publications, Providence, RI. 1999.

[4] T. Cazenave, "An Introduction to Nonlinear Schrödinger Equations," Instituto de Matematica - UFRJ - Rio de Janeiro, 1996.

[5] T. Cazenave and F. B. Weissler, *The Cauchy problem for the nonlinear Schrödinger equation in H^1*, Manuscripta Math., **61** (1988), 477–494.

[6] T. Cazenave and F. B. Weissler, *The Cauchy problem for the critical nonlinear Schrödinger equation in H^s*, Nonlinear Analysis, **14** (1990), 807–836.

[7] J. Colliander, M. Keel, G. Staffilani, H. Takaoka and T. Tao, *Almost conservation laws and global rough solutions to a nonlinear Schrödinger equation*, Mathematical Research Letters, **9** (2002), 659–682.

[8] J. Colliander, M. Keel, G. Staffilani, H. Takaoka and T. Tao, *Global existence and scattering for rough solutions of a nonlinear Schrödinger equation on $\mathbf{R}^3$*, Communications on Pure and Applied Mathematics, **21** (2004), 987–1014

[9] J. Colliander, M. Keel, G. Staffilani, H. Takaoka and T. Tao, *Global existence and scattering for the energy - critical nonlinear Schrödinger equation on $\mathbf{R}^3$*, Annals of Mathematics. Second Series, **167** (2008), 767–865

[10] J. Colliander, M. Keel, G. Staffilani, H. Takaoka and T. Tao, *Resonant decompositions and the I-method for cubic nonlinear Schrödinger equation on $\mathbf{R}^2$*, Discrete and Continuous Dynamical Systems A, **21** (2007), 665–686.

[11] J. Colliander and T. Roy, *Bootstrapped Morawetz estimates and resonant decomposition for low regularity global solutions of cubic NLS on $\mathbb{R}^2$*, Communications in Pure and Applied Analysis, **10** (2011), 397–414.

[12] B. Dodson, *Global well - posedness and scattering for the defocusing L^2 - critical nonlinear Schrödinger equation when $d = 2$*, preprint, `arXiv:1006.1375`,

[13] J. Ginibre and G. Velo, *Smoothing properties and retarded estimates for some dispersive evolution equations*, Communications in Mathematical Physics, **144** (1992), 163–188.

[14] J. Ginibre and G. Velo, *Scattering theory in the energy space for a class of nonlinear Schrödinger equations*, Journal de Mathmatiques Pures et Appliques, **9** (1985), 363–401.

[15] M. Keel and T. Tao, *Local and global well posedness of wave maps on $\mathbf{R}^{1+1}$ for rough data*, International Mathematics Research Notices, **21** (1998), 1117–1156.

[16] M. Keel and T. Tao, *Endpoint strichartz estimates*, American Journal of Mathematics, **120** (1998), 945–957.

[17] C. Kenig and F. Merle, *Scattering for $\dot{H}^{1/2}$ bounded solutions to the cubic, defocusing NLS in 3 dimensions*, Transactions of the American Mathematical Society, **362** (2010), 1937–1962.

[18] R. Killip and M. Visan, "Nonlinear Schrodinger Equations at Critical Regularity," Clay Lecture Notes 2009. Available from: `http://www.math.ucla.edu/~visan/lecturenotes.html`.

[19] J. Lin and W. Strauss, *Decay and scattering of solutions of a nonlinear Schrödinger equation*, Journal of Functional Analysis, **30** (1978), 245–263.

[20] T. Roy, *Adapted linear - nonlinear decomposition and global well - posedness for solutions to the defocusing cubic wave equation on $\mathbb{R}^3$*, Discrete and Continuous Dynamical Systems. Series A., **24** (2009), 1307–1323.

[21] E. M. Stein, "Singular Integrals and Differentiability Properties of Functions," Princeton University Press, Princeton, NJ, 1970.

[22] E. M. Stein, "Harmonic Analysis: Real-Variable Methods, Orthogonality, and Oscillatory Integrals," Princeton University Press, Princeton, NJ, 1993.

[23] W. Strauss, "Nonlinear Wave Equations," CBMS Regional Conference Series in Mathematics, **73**. American Mathematical Society, Providence, RI, 1989.

[24] R. S. Strichartz, *Restrictions of Fourier transforms to quadratic surfaces and decay of solutions of wave equations*, Duke Mathematical Journal, **44** (1977), 705–714.

[25] Q. Su, *Global well - posedness and scattering for the defocusing, cubic NLS in* $\mathbb{R}^3$, Mathematical Research Letters, **19** (2012), 431–451.

[26] T. Tao, "Nonlinear Dispersive Equations," CBMS Regional Conference Series in Mathematics, **106**. Published for the Conference Board of the Mathematical Sciences, Washington, DC; by the American Mathematical Society, Providence, RI, 2006

[27] M. E. Taylor, "Pseudodifferential Operators and Nonlinear PDE," Birkhäuser, Boston, 1991.

[28] M. E. Taylor, "Partial Differential Equations I - III," Springer-Verlag, New York, 1996.

[29] M. E. Taylor, "Tools for PDE," American Mathematical Society, Mathematical Surveys and Monographs, **31** Providence, RI, 2000.

Received October 2011; revised September 2012.

E-mail address: benjadod@math.berkeley.edu