Statistics

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Contents

1 Lecture 1: 2022-09-08

1.1 Introduction to Dispersive PDEs

Definition 1 (Dispersive PDE)**.** *Informally, a PDE is characterized as* ***dispersive*** *if, when the the boundary conditions are dropped, its wave solutions are going to spread out in space as time evovles. Like a rock thrown into water.*

For now we will focus on *linear* dispersive PDEs. Associated with a dispersive PDE:

1. Dispersive Estimates
2. On the Fourier side, different frequencies at different speedss in different directions.

We first consider the simplest example.

Example 1 (Linear dispersive PDE with constant coefficients)**.** *Suppose* $u(t,x) : \mathbb{R} \times \mathbb{R}^d \to V \in \mathbb{R}^d$ *such that*

$$\begin{cases} i\partial_t u(t,x) = Lu(t,x) \\ u(0,x) = u_0(x) \end{cases} \tag{1}$$

where L *is a skew-adjoint constant coeffieicent differential operator of order* k*. In symbols, there exists constants* $\left\{c_\alpha : \alpha \in \mathbb{Z}^d_{\geq 0}\right\}$ *such that*

$$Lu(t,\cdot) = \sum_{|\alpha| \leq k} c_\alpha \partial_x^\alpha u(t,\cdot)$$

where $|\alpha| := \alpha_1 + \ldots + \alpha_d$ *and for* $x \in \mathbb{R}^d$*, the symbol* ∂_x^α *denotes the operator defined by*

$$\partial_x^\alpha u := \prod_{i=1}^{d} \partial_{x_i}^{\alpha_i} u \tag{2}$$

So, for example, if $d = 2$*,* $c_{(0,1)} = 2$*,* $c_{(2,5)} = -3$*, and* $c_\alpha = 0$ *for all other choices of* α*, then* L *would be an order 7 differential operator taking the following form:*

$$\begin{aligned} Lu(t,\cdot) &= 2\partial_x^{(0,1)} u(t,\cdot) - 3\partial_x^{(2,5)} u(t,\cdot) \\ &= 2\partial_{x_1}^0 u(t,\cdot)\partial_{x_2}^1 u(t,\cdot) - 3\partial_{x_1}^2 u(t,\cdot)\partial_{x_2}^5 u(t,\cdot) \end{aligned}$$

or equivalently,

$$Lu(t,x_1,x_2) = 2u(t,x_1,x_2)u_{x_2}(t,x_1,x_2) - 3u_{x_1x_1}(t,x_1,x_2)u_{x_2x_2x_2x_2x_2}(t,x_1,x_2)$$

Writing x, y *in place of* x_1, x_2*, we get the nicer-looking formulation:*

$$Lu(t,x) = 2u(t,x,y)u_y(t,x,y) - 3u_{xx}(t,x,y)u_{yyyyy}(t,x,y).$$

Since the operator defined in Eq. (2) *does not compose different partial differentiation operators together,* Lu *does not involve any mixed partial derivatives of* u *(e.g. you'll never see terms like* u_{xy} *or* u_{xxy}*).*

Remark 1. *The operator* L *from Example 1 is defined classically (i.e. pointwise) only if* $u \in C^k(\mathbb{R} \times \mathbb{R}^d)$*, that is only if* u *is* k*-times continuously differentiable. But of course we may extend* L *in the usual manner so that it is defined in a distributional sense.)*

1.2 Dispersion Relation

Definition 2 (Dispersion Relation and Frequency Operator)**.** *We can write* L *in the form*

$$L = ia(D)$$

where a *is some function, called the* ***dispersion relation****, and* D *is the* ***frequency operator*** *defined as*

$$D = \frac{1}{i}\nabla := \left(\frac{1}{i}\partial_{x_1}, \ldots, \frac{1}{i}\partial_{x_d}\right)$$

Remark 2 (Polynomial Dispersion Relations)**.** *It turns out that the form of the dispersion relation is important. When the operator L is of the form*

$$Lu = \partial_t u$$

(or something), then the dispersion relation takes the form

$$a(\xi_1, \ldots, \xi_d) = \sum_{|\alpha| \leq k} i^{|\alpha|-1} c_\alpha \xi_1^{\alpha_1} \cdots \xi_d^{\alpha_d} \tag{3}$$

A large number of PDEs are governed by dispersion relations of the form given in Eq. (3), as we show in the next example.

Example 2 (Examples for Remark 2)**.** *Here we list some examples of PDEs whose dispersion relations take the form shown in Eq.* (3)*.*

- ***A degenerate (i.e. nondispersive) example.*** *Suppose Eq.* (1) *takes the form*

$$\begin{cases} \partial_t u(t,x) = iwu(t,x) \\ u(0,x) = u_0(x) \end{cases}$$

  where $u : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ and $w \in \mathbb{R}$. In this case the dispersion relation is

$$a(\xi) = w.$$

  The solution to this system is $u(t,x) = u_0 e^{itw}$.

- ***Another degenerate equation.*** *Fix $\nu \in \mathbb{R}^d$ and consider the*

$$\begin{cases} \partial_t u(t,x) = -\nu \cdot \nabla_x u(t,x) \\ u_0(0,x) = u_0(x) \end{cases}$$

  An example solution to this is

$$u(t,x) = u_0(x - \nu t)$$

  something something transport equation.

- ***The Airy Equation.*** *Suppose $d = 1$, $u = u(t,x) : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ such that*

$$\partial_t u + \partial_{xxx} u = 0.$$

  In this case, the dispersion relation is

$$a(\xi) = \xi^3, \quad (\xi \in \mathbb{R}).$$

- ***Schrodinger (?) equation.*** *Suppose*

$$i\partial_t u + \Delta u = 0.$$

  In this case, $L = -\Delta$ and $a(\xi) = |\xi|^2$.

- ***Wave Equation.*** *Suppose*

$$-u_{tt} + \Delta u = 0. \tag{4}$$

  in any dimnensions. This is dispersive.

- ***Klein-Gordon Equation.*** *Suppose*

$$-u_{tt} + \Delta u - u = 0.$$

  This is dispersive.

It is not always the case that dispersion relations are polynomials of the form given in Eq. (3). Indedd, for deep water gravity waves, $a(\xi) = |\xi|^{1/2}$. Also for capillary waves, something something. Also the BO equation

$$u_t + H\partial_x^2 u = 0$$

has dispersion relation $a(\xi) = \xi|\xi|$.

1.3 Group Velocity

Definition 3 (Wave-Plane Solution)**.** *A **wave-plane solution** is the name for functions of the form*

$$u(t, v) = e^{i(kx-wt)}. \tag{5}$$

Definition 4 (Wave Number and Angular Frequency)**.** *The parameter k in Eq.* (5) *is called the **wave number**. The parameter w is called the **angular frequency**.*

Example 3 (Wave-plane solutions for the wave equation)**.** *If we plug the function u from Eq.* (5) *into the wave equation shown in Eq.* (4)*, we obtain*

$$-(-iw)^2 e^{(i(kx-wt))} + (ik)^2 e^{(i(kx-wt))} = 0$$

which implies that $w^2 = k^2$, and hence that $w(k) = \pm|k|$.

Definition 5 (Group Velocity)**.** *The derivative of the dispersion relation with respect to the dwave number is called the **group velocity**.*

Definition 6 (Dispersive Equation)**.** *A PDE is said to be dispersive if $a''(\xi) \neq 0$.*

1.4 Symmetries for Linear Dispersive PDEs

1. All are invariant under time and space translations. That is, suppose τ is the translation operator defined by
$$\tau u(t, x) = u(t - t_0, x - x_0)$$
for some fixed t_0 and x_0. Then τu is a solution provided that u is a solution.
2. Scaling symmetries. For any $\lambda > 0$, the function
$$u(t/\lambda^k, x/\lambda)$$
is a solution whenever $u(t, x)$ is a solution.
3. I think there were more, but my hand got tired and I stopped taking notes.

2 Lecture 2: 2022-09-13

2.1 Full Dispersion

Recall the dispersion relation $a(\xi)$. If $a''(\xi) \neq 0$ then the PDE is said to be **fully dispersive**. If $\xi \in \mathbb{R}^d$ then $\nabla^2 a(\xi)$ is the hessian. The appropriate thing to do in that case is to diagonalize... we'll get to that.

Example 4 (Wave Equation)**.** *Suppose we have $\Box u = 0$ with $d = 1$. Then $a(\xi) = |\xi|$ so that $a''(\xi) = 0$ which is not dispersive. But if $d > 1$ then since $\alpha(\xi) = |\xi|$. So that $a'(\xi) = -\nabla a(\xi) = \frac{\xi}{|\xi|}$. Then the hessian is*

$$\nabla_\xi^2 a(\xi) = \frac{1}{|\xi|}\left[I_n - \frac{\xi}{|\xi|} \otimes \frac{\xi}{|\xi|}\right]$$

where I_n is the $n \times n$ identity matrix. The thing in brackets is the orthogonal projection onto the plane perpendicular to $\xi/|\xi|$ (i.e. tangent to the sphere).

For example, consider $d = 2$. Then we can diagonalize the Hessian matrix $\nabla_\xi^2 a$. You get $\lambda = 0$ and $\lambda = \frac{1}{|\xi|}$ as eigenvalues and the diagonalized matrix is

$$\begin{pmatrix} 0 & 0 \\ 0 & \frac{1}{|\xi|} \end{pmatrix}$$

The interpretation is that since one of the eigenvalues is zero, so you don't have full dispersion. It's like degenerate dispersion. In particular, we don't have disperion in the radian direction. Try playing with this in the 2D.

Remark 3. *The wave equation in $d > 1$ has dispersion (degenerate). Also, last lecture had something false in it. The false claim was that finite speed of propagation is impied by bounded group velocity. (Recall that the* ***group velocity*** *is the negative of the derivative of the dispersion relation: $-\nabla a(\xi)$.)*

Example 5. *Suppose $u : \mathbb{R} \times \mathbb{R}^3 \to \mathbb{R}$ satisfying*

$$\begin{cases} i\partial_t u + A(D)u = 0 \\ u_0 = u(0, \lambda) \end{cases} \tag{6}$$

We claim that this equation with $A(D) = |D|$ does not have finite speed of propagation.

Finite speed propagation is if u_0 is supported on $B(a, R)$ then $t > 0$, $u(x, t)$ is going to be supported by $B(a, R + ct)$. The infimum of all such c's satifying the above condition is called ***the finite speed of propagation.***

Recall that we can take the Fourier transform of Eq. (6) and solve for the fundamental solution:

$$e^{it|D|} = \cos(t|D|) + i \sin(t|D|)$$

here D is the derivative with radial.

$$|D| \frac{\sin(t|D|)}{|D|}$$

Recall the fundamental solution to the wave equation is $K_{\Box} = c\frac{1}{t}\delta$. Give this a little bit of thought. Do this example to bush up. This example shows why the finite group velocity does not imply finite speed of propagation. This example looks like the example from last time that had a star with it. The operatior $|D|$ is the operator with symbol $|\xi|$. In one dimension, this is the Hilbert transform.

2.2 Long-Time Dynamics for Linear Dispersive Waves

Question: What are the long time dynamics for linear dispersive waves?

Scalar case. Suppose we have

$$\begin{cases} i\partial_t u + A(D)u = 0 \\ u_0(x) = u(0, x) \in \mathcal{S} \end{cases}$$

where $\mathcal{S}$ denotes the Schwarz space. What happens with this wave as $t \to \infty$?

Then the solution is

$$\hat{u}(t, \xi) = \hat{u}_0 e^{ita(\xi)}$$

where $a(\xi)$ is the symbol of the operator A. Remember, dispersivity is that every frequency moves in its own direction and with its own velocity. We want to quantify this. My solution:

$$\hat{u}(t, \xi) = \hat{u}_0(\xi) e^{ita(\xi)} \tag{7}$$

now let's go back and write it on the spatial side:

$$u(t, x) = \int e^{ix\cdot\xi} e^{ita(\xi)} \hat{u}_0(\xi) d\xi$$

where $\hat{u}_0(\xi)$ is a Schwarz function (since $u_0 \in \mathcal{S}$ and the Fourier Transform maps Schwarz functions to Schwarz functions). Therefore

$$u(t, x) = \int e^{i(x\cdot\xi + ta(\xi))} \hat{u}_0(\xi) d\xi$$

Heuristically, we make the assumption that the waves (each localized a different frequency) travel and go out in a linear fasion (at least when time is big). When t is very large, we write

$$u(t, x) = \int e^{it\left[\frac{x}{t}\xi + a(\xi)\right]} \hat{u}_0(\xi) d\xi$$

and we think of $v := x/t$ as velocity. Thus

$$u = u(t, v) = \int e^{it[v\xi + a(\xi)]} \hat{u}_0(\xi) d\xi$$

and this is an oscillatory integral. This is oscillate rapidly when t is large. One nice thing about it is that it has a complex phase, which is what makes it oscillate. We are interested in the long-time dynamics. That will result in cancellations, which can be seen when integrating by parts. There is a complication when doing integration by parts however because we don't know that the quantity

$$\frac{\partial}{\partial \xi} \left[itv\xi + a(\xi) \right]$$

is nonzero and integration by parts would require it to be on the denominator. So we need to introduce something called stationary/nonstationary phase. In $d = 1$, this is from Stein. The main understanding is the following:

Let

$$I = \int e^{i\lambda\phi(\xi)} a(\xi) d\xi$$

First, suppose that $\phi(\xi) \neq 0$. This is called the nonstationary phase argument. Take

$$I = \int \partial_\xi \left(e^{i\lambda\phi} \right) \frac{1}{i\lambda\phi'(\xi)} a(\xi) d\xi$$

Integrating by parts N times will give a λ^{-N} in front (e.g. if a is a polynomial then N would be the degree of a). Therefore as $t \to \infty$,

$$I = O(\lambda^{-N})$$

which decays very quickly. Therefore the solution of Eq. (7), understood as a function $u = u(t, v)$ disperses very fast—it decays rapidly.

On the other hand, suppose that $\phi'(\xi) = 0$. For example, if $\phi(\xi)$ is a constant. But that doesn't makeif ξ_0 is a critical point sense for our problem. So let's correct for that by requiring that

$$\phi''(\xi) \neq 0.$$

Thus any zeros are when we have critical points of ϕ. Around those points, we cannot integrate by parts. But what can we do? Another heuristic: if ξ_0 is a critical point, assume that when $\xi \approx \xi_0$, we can Taylor expand:

$$\begin{aligned} \phi(\xi) &= \phi(\xi_0) + (\xi - \xi_0)\phi(\xi_0) + \frac{1}{2}(\xi - \xi_0)^2 \phi''(\xi) \\ &= \phi(\xi_0) + \frac{1}{2}(\xi - \xi_0)^2 \phi''(\xi). \end{aligned}$$

so that

$$\begin{aligned} I &\approx \int e^{i\lambda\left[\phi(\xi_0) + \frac{1}{2}(\xi - \xi_0)^2 \phi''(\xi_0)\right]} a(\xi_0) d\xi \\ &= e^{i\lambda\phi(\xi_0)} a(\xi_0) \int e^{\frac{1}{2} i\lambda(\xi - \xi_0)^2 \phi''(\xi_0)} d\xi \end{aligned}$$

and the integral is a complex Gaussian which can be computed with the rule

$$\int_0^\infty e^{i\alpha x^2} dx = e^{i\pi \operatorname{sign}(|\alpha|)/4} \sqrt{\frac{\pi}{4\alpha}}, \quad \alpha \neq 0.$$

with $\alpha = \frac{1}{2}\lambda\phi''(\xi_0)$. So we get

$$I \approx e^{i\lambda\phi(\xi_0)} a(\xi_0) \frac{1}{\sqrt{\lambda}} \cdot \frac{1}{\sqrt{\phi''(\xi_0)}}$$

multiplied by some other thing that aren't important. Taking $\phi = v\xi + a(\xi)$ and $\lambda = t$ gives

$$u(t, vt) \approx e^{it[v_\xi \xi_0 + a(\xi_0)]} \frac{1}{\sqrt{t}} \hat{u}_0(\xi_v)$$

with some other things.

3 2022-09-15: Lecture Notes

3.1 Littlewood-Payley Decomposition

Let ϕ be a smooth radial[1] function such that

$$\begin{cases} \operatorname{supp}(\phi\ ) \subseteq \{\xi \in \mathbb{R}^n : 0 \leq |\xi| \leq 2\} \\ \phi \equiv 1 \text{ in } B(0,1/2) \end{cases}$$

where $B(0,1) \subset \mathbb{R}^n$ denotes the unit ball centered at the origin. In addition, for each $\xi \in \mathbb{R}^n$ define

$$\psi(\xi) := \phi(\xi) - \phi(2\xi).$$

It is easy to see that ψ is a radial function with

$$\operatorname{supp}(\psi) \subseteq \left\{\xi \in \mathbb{R}^n : \frac{1}{2} \leq |\xi| \leq 2\right\}. \tag{8}$$

We then construct a sequence of functions in the following manner. For each $k \in \mathbb{Z}$, define ϕ_k by

$$\psi_k(\xi) := \psi\left(\frac{\xi}{2^k}\right), \quad \xi \in \mathbb{R}^n.$$

Then the sequence of functions $(\psi_k)_{k\in\mathbb{Z}}$ is a **partition of unity**. To see this, observe that

$$\begin{aligned} J(\xi) &:= \{k \in \mathbb{Z} : \psi_k(\xi) \neq 0\} \\ &= \left\{k \in \mathbb{Z} : \frac{1}{2} \leq |\xi/2^k| \leq 2\right\} \\ &= \{k \in \mathbb{Z} : -1 \leq \log_2|\xi| - k \leq 1\} \\ &= \{k : \log_2|\xi| - 1 \leq k \leq \log_2|\xi| + 1\} \end{aligned}$$

which is clearly a finite set, and, letting $j := \lceil \log_2|\xi| \rceil - 1$, observe that

$$\begin{aligned} \sum_{k\in\mathbb{Z}} \psi_k(\xi) &= \sum_{k\in J} \psi_k(\xi) \\ &= \psi(\xi/2^j) + \psi(\xi/2^{j+1}) \\ &= \phi(\xi/2^j) - \phi(\xi/2^{j-1}) \\ &= 1 - 0 \\ &= 1 \end{aligned}$$

Definition 7 (Littlewood-Paley Projection)**.** *For each $k \in \mathbb{Z}$, let P_k be the Fourier multiplication operator defined on $L^2(\mathbb{R}^n)$ by the formula*

$$\widehat{P_k f}(\xi) := \psi\left(\xi/2^k\right) \hat{f}(\xi)$$

and define $P_{\leq k}$ by

$$\widehat{P_{\leq k} f}(\xi) := \sum_{\ell \leq k} \widehat{P_\ell f}(\xi) = \phi(\xi/2^k)\hat{f}(\xi)$$

for all $f \in L^2(\mathbb{R}^n)$.

Remark 4. *The operators P_k and $P_{\leq k}$ are **almost** projections, since $\psi(\xi/2^k)$ is a smooth approxmation of an indicator function (but the tail turns out not to be an issue).*

Lemma 1 (Littlewood-Paley Projection Properties)**.** *The following properties hold for all $f \in L^2(\mathbb{R}^n)$:*

- *(i)* $P_k = P_{\leq k} - P_{\leq k-1}$
- *(i)* $\lim_{k\to-\infty} P_{\leq k} = 0$ *and* $\lim_{k\to\infty} P_{\leq k} f = f$ *in* L^2
- *(i)* $\sum_{k\in\mathbb{Z}} P_k f = f$ *in* L^2

Remark 5. *Property (iii) of Lemma 1 holds if $f \in L^p$ but not in general if $f \in L^1_{loc}$. Consider the case in which $f = 1$ and $\hat{f} = \delta_0$. Then supp$(\hat{f}) = \{0\}$.*

[1] A function ϕ is **radial** iff $\phi(x) = \phi(|x|)$ for all x.

3.2 Physical Space

How are we to interpret P_k in physical space?

Definition 8 (Dilation Operator)**.** *For each $\lambda > 1$ and each $1 \le p \le \infty$, define an operator $\mathrm{Di}|_\lambda^p$ on L^2 by*

$$\mathrm{Di}|_\lambda^p (f)(x) := \lambda^{-n/p} f(x/\lambda)$$

for each $x \in \mathbb{R}^n$ and each $f \in L^2(\mathbb{R}^n)$.

Lemma 2 (Dilation Properties)**.** *The following properties hold:*

- *If $\mathcal{F}$ is the Fourier transform operator, then*

$$\mathcal{F}\, \mathrm{Di}|_\lambda^p = \mathrm{Di}|_{\lambda^{-1}}^q\, \mathcal{F}$$

  where $\frac{1}{p} + \frac{1}{q} = 1$.
- $\phi(\xi/2^k) = \mathrm{Di}|_{2^k}^\infty\, \phi(\xi)$
- $P_{\le k} f(x) = \mathrm{Di}|_{2^{-k}}^1\, \hat{\phi} * f(x) = \int_{\mathbb{R}^m} f(x-y) 2^k \hat{\phi}(2^k y) dy$

Remark 6. *I didn't really understand this remark.*

Lemma 3. *Let $k \in \mathbb{Z}$ and let f be a function such that supp$(\hat{f}) \subseteq \left\{\xi : 2^{k-1} \le |\xi| \le 2^{k+1}\right\}$. Then*

$$\|\nabla f\|_{L^p} \sim 2^k \|f\|_{L^p}, \quad 1 \le p \le \infty$$

and in particular,

$$\|P_k f\|_{L^p} \sim 2^k \|f\|_{L^p}$$

in particular we have

$$\|\nabla P_k f\|_{L^p} \sim 2^k \|P_k f\|_{L^p} \tag{9}$$

4 2022-10-04: Lecture Notes

Proof of Lemma 3. From the notes of last time, we have

$$\|\nabla f(x)\|_p \le 2^k \|f\|_p$$

It remains to prove an inequality in the other direction. Intuitively, we need to "invert" ∇. Recall we have

$$\widehat{\partial_{x_j} f}(\xi) = 2\pi i \xi_j \widehat{f}(\xi)$$

Therefore if $|\xi| \le 2^{k+2}$, we have

$$\phi\left(\frac{\xi}{2^{k+2}}\right) \widehat{\partial_{x_j} f}(\xi) = 2\pi i \xi_j \widehat{f}(\xi)$$

then we multiply both sides by ξ_i and sum over j:

$$\sum_{j=1}^m \xi_j \phi\left(\frac{\xi}{2^{k+2}}\right) \widehat{\partial_{x_j} f}(\xi) = \sum_{j=1}^m 2\pi i \widehat{f}(\xi) \xi_j^2 |\xi|^2 = 2\pi i \hat{f}(\xi)|\xi|^2$$

here m is the dimension ($\xi \in \mathbb{R}^m$). Then (why)

$$\widehat{f}(\xi) = \sum_{j=1}^m \frac{\xi_j \phi\left(\frac{\xi}{2^{k+2}}\right) \widehat{\partial_{x_j} f}(\xi)}{2\pi i |\xi|^2}$$

Then taking the inverse Fourier transform and using that $\widehat{f} \cdot \widehat{g} = \widehat{f * g}$

$$f = 2^{-k} \sum_{j=1}^{m} K_{k,j} * \delta_{x_j} f \tag{10}$$

where

$$\begin{aligned} K_{k,j} &= 2^k \int \phi\left(\frac{\xi}{2^{k+2}}\right) \frac{\xi_j}{2\pi i |\xi|^2} e^{2\pi i x \cdot \xi} d\xi \\ &= 2^{nk} \int \phi\left(\frac{\xi}{2^2}\right) \frac{\xi_j}{2\pi i |\xi|^2} e^{\pi i 2^k \cdot x \cdot \xi} d\xi \end{aligned}$$

where the second equality follows by a change of variable. In particular, we have

$$|K_{k,j}(x)| \lesssim 2^{nk}$$

where the squiggly inequality hides some constants. Note that we can write our formula as

$$K_{k,j}(x) = 2^{nk} \int \phi\left(\frac{\xi}{4}\right) \frac{\xi_j}{2\pi i |\xi|^2} \partial_\xi \left(\frac{e^{2\pi i 2^k} x \cdot \xi}{2\pi i 2^k x}\right) ds$$

which suggests an integration by parts. Integrating by parts s times, we obtain

$$|K_{k,j}(x)| \lesssim 2^{nk} \left|2^k x\right|^{-s}$$

for $s > 0$. Here $K_{k,j}$ is like an approximation of identity. Finally, applying the Minkowski inequality to Eq. (10), we get

$$\|\nabla f\|_{L^p} \geq 2^k \|f\|_{L^p},$$

as required. □

Remark 7 (Singularity at zero)**.** *There is some question about a singularity at zero. But this is not an issue since under the hypotheses of Lemma 3, the function $\hat{f}$ is zero in a neighborhood of zero.*

Remark 8. *We will use the notation $f_k := P_k f$. Morally at the level of L^p, $\|\nabla f_k\|_p \sim \|f_k\|_{L^p}$ so heuristically, $\nabla \sim \sum_k 2^k P_k$. This might become clear later.*

Now we want to connect $P_k(f)$ on $P_{\leq k} f$ to f itself. We have

1. $\|P_{\leq k} f\|_{L^p} \leq \int_{\mathbb{R}^m} \left\|f(x - 2^{-k} y)\right\|_{L^p} \left|\widehat{\phi}(y)\right| dy \lesssim \|f\|_{L^p}$ also [check this] $\widehat{P_{\leq k}\hat{f}} = \phi(\xi/2^k)\hat{f}(\xi)$ implies

$$\|P_{\leq k} f\|_{L^p} \leq \|f\|_{L^p} \tag{11}$$

2. (Cheap LP inequality): By Minkowski inequality, since

$$f = \sum_k P_k f$$

implies

$$\sup_k \|P_k f\|_p \lesssim \|f\|_p \leq \sum_k \|P_k f\|_p$$

There is a way to elegantly prove the Sobolev embeddings using Littlewood-Paley. The idea will be to prove it for a dyadic piece.

Lemma 4 (Non-endpoint Sobolev Embedding). *Let $1 \leq p < q \leq \infty$ such that $\frac{1}{p} - \frac{1}{n} > \frac{1}{2}$. Then*

$$\|f\|_{L^q(\mathbb{R}^n)} \leq C_{p,q,n} \|f\|_{L^p(\mathbb{R}^n)} + \|\nabla f\|_{L^p(\mathbb{R}^n)} \tag{12}$$

for all $f \in L^p(\mathbb{R}^n)$ such that the right hand side is finite. Here, $C_{p,q,n}$ is a constant which depends only on $p, q,$ and n.

Remark 9. *The endpoint version is when $\frac{1}{p} - \frac{1}{n} = \frac{1}{2}$. The proof will be similar but with some added complications. See Evans.*

Proof of Lemma 4. Let f be a Schwarz function, and denote the right hand side of Eq. (12) by X. Then by Eq. (11),

$$\|P_k f\|_p \leq X \tag{13}$$

for all k, and also

$$\|\nabla P_k f\|_p \leq \|\nabla f\|_p \leq X.$$

Also, by Eq. (9) from Lemma 3, we have

$$\|P_k f\|_p \lesssim 2^{-k} X \tag{14}$$

By Eqs. (13) and (14), we havem

$$\|P_k f\|_{L^p} \lesssim \min\left\{1, 2^{-k}\right\} X$$

Note taht if $|\xi| \sim 2^k$ then $2^{k-1} \leq |\xi| \leq 2^{k+1}$. What is the L^q norm of f? Since $q > p$, we use Bernstein's inequality, which states that

$$\|P_k f\|_q \leq 2^{\left(\frac{1}{p} - \frac{1}{q}\right)kn} \|P_k f\|_{L^p}.$$

We will prove this proof later. It follows that

$$\|P_k f\|_q \lesssim 2^{\left(\frac{1}{p} - \frac{1}{q}\right)kn} \min\left\{1, 2^{-k}\right\} X$$

then consideration of the cases where $k \to \pm\infty$, we find that the coefficient decays and the maximum value is when $k = 0$. Then summing over k's gives the result. $\square$

5 Lecture: 2022-10-06

We start with the following inequality.

Theorem 1 (Bernstein's Inequality). *Let $f_k = P_k f$, and assume that $1 \leq p \leq q \leq \infty$. Then*

(a) The following inequality holds:

$$\|f\|_{L^q(\mathbb{R}^d)} \lesssim 2^{k(\frac{d}{p} - \frac{d}{2})} \|f_k\|_{L^p(\mathbb{R}^d)}.$$

(a) A similar inequality holds for $f_{\leq k}$

(a) For all $s \in \mathbb{R}$ and all $1 \leq p \leq \infty$,

$$\||\nabla|^s f_k\|_{L^p} \sim 2^{ks} \|f_k\|_{L^p}.$$

Proof of part (a) of Theorem 1. Integrating by parts,

$$\|f_k\|_q = \|P_k f\|_q = \left\|f * 2^{kd} \check{\psi}(2^k \cdot)\right\|_q$$

We'll use Young's inequality, which says that if $\frac{1}{p} + \frac{1}{r} = \frac{1}{q} + 1$, then

$$\|g * h\|_q \lesssim \|g\|_p \|h\|_r.$$

Thus we obtain

$$\begin{aligned}\|f_k\|_q &\lesssim \|f\|_p \left\| 2^{kd}\check{\psi}(2^k\cdot)\right\|_r \\ &\lesssim \|f\|_p\, 2^{k(d-\frac{d}{r})} \left\|\check{\psi}\right\|_r\end{aligned}$$

where the second step is by a change of variables $y \mapsto 2^k y$. Then

$$\|f_k\|_q \lesssim \|f\|_p\, 2^{kd(\frac{1}{p}-\frac{1}{q})}$$

To fix this, we will take a fatter Littlewood-Paley projection:

$$\widetilde{P}_k := P_{2^{k-2}\leq 2^j \leq 2^{k+2}}$$

(recall that P_k was a projection to frequency $[2^{k-1}, 2^{k+1}]$). In this case, $\widetilde{P}_k P_k = P_k$ and we do the same computations as before to obtain

$$\|f_k\|_q = \left\|\widetilde{P}_k f_k\right\|_q$$

Something:

$$\widehat{\widetilde{P}_k f(\xi)} = \left(\sum_{2^{k-2}\leq 2^j\leq 2^{k+2}} \psi_{2^j}\right)(\xi)\widehat{f}(\xi)$$

which implies that

$$\begin{aligned}\widetilde{P_k} f &= f * \left(\sum \psi_{2^j}\right)^{\vee} \\ &= f * \sum 2^{jd}\check{\psi}(2^j\cdot) \\ &\sim f * 2^j \sum \check{\psi}(2^j\cdot)\end{aligned}$$

this is simple but $\square$

I got distracted by latex not playing nice and missed something.

Recall we wanted to...

First the cheap LP inequality:

$$\sup_k \|P_k f\|_p \lesssim \|f\|_p \leq \sum_k \|P_k f\|_p$$

and for $p = 2$, using Plancherel we can get

$$\|f\|_2 \sim \left(\sum_k \|P_k f\|_2^2\right)^{\frac{1}{2}}$$

which we rewrite as

$$\|f\|_2 \sim \left\|\left(\sum_k |P_k f|^2\right)^{\frac{1}{2}}\right\|_2 .$$

We call this the "square function". Define

$$Sf = \left(\sum_k |P_k f|^2\right)^{\frac{1}{2}}$$

Then we have the following theorem

Theorem 2 (LP inequality). *Let $1 < p < \infty$. Then*

$$\|Sf\|_p \sim \|f\|_p$$

where the constant depends on p.

Sobolev Spaces. Suppose $f \in W^{s,p}$, and define

$$\|f\|_{W^{s,p}} := \sum_{j=1}^{s} \left\|\nabla^j f\right\|_p < \infty$$

where s is an integer. Our goal is to replace ∇ in terms of P_k. We have the following lemma.

Lemma 5. *Let $j \geq 0$ and $1 < p < \infty$. Then*

$$\left\|\nabla^j f\right\|_p \sim \left\| \left(\sum_k \left|2^{jk} P_k f\right|^q \right)^{\frac{1}{2}} \right\|_p$$

Proof. Later. $\square$

Then we can rewrite the norm.

$$\|f\|_{W^{s,p}} \sim \left\| \left(\sum_k \left|(1+2^k)^s P_k f\right|^2 \right)^2 \right\|_p$$

actually in htis case, the thing is defined for any real number s, and not just when s is an integer.

Definition 9 (Japanese Bracket). *Define*

$$\langle x \rangle := \sqrt{1+|x|^2}.$$

*This is called the **Japanese bracket**. We define an operator*

$$\langle \nabla \rangle^s := \sqrt{1+\Delta^s}$$

which is defined in the Fourier side.

Remark 10. *The "plus 1" in Definition 9 will take on significance when we distinguish between homogenoeous and inhomogenous Sobolev spaces.*

Remark 11. *The operator $\langle \nabla \rangle \approx |\Delta|$ at frequencies $|\xi| > 1$, but treats differently low frequencies.*

We have $\widehat{\nabla f}(\xi) = 2\pi i \xi \hat{f}(\xi)$ and we define similarly, $\widehat{|\nabla| f}(\xi) = 2\pi i |\xi| \hat{f}(\xi)$ and the Japanese bracket notation is defined similarly.

6 2022-10-09: Make-Up Lecture

In this lecture, we will discuss Stricharz estimates. These will be particular to each dispersive equation. For example, nonlinear schrodinger equation has its own Strichartz estimates which ar edifferent from, say, those for the Benjamin-Ono equtions or the wave equations. For your own equations, you'll need to derive it.

Remark 12 (Notation). *For $t \in \mathbb{R}$ and $x \in \mathbb{R}^n$, denote*

$$\|u\|_{L_t^q L_X^r} := \left(\int_{\mathbb{R}} \left[\int_{\mathbb{R}^d} |u(t,x)|^r \, dx \right]^{\frac{1}{r} \cdot q} dt \right) \frac{1}{q}$$

Let's consider the Schordinger equation

$$\left\{ \begin{array}{l} i\partial_t u(t,x) + \Delta u(t,x) = 0 \\ u(0,x) = u_0(x) \end{array} \right. \tag{15}$$

and for simplicity we will assume that $u_0(x)$ is a Shwartz function so that $u(t,x)$ is Schwartz for all x, t.

Apply the Fourier transform in space to Eq. (15).

$$\left\{ \begin{array}{l} i\widehat{u}_t(t,\xi) = |\xi|^2 \widehat{u}(t,\xi) \\ \widehat{u}(0,\xi) = \widehat{u_0}(\xi) \end{array} \right.$$

This is a seperable equation with respect to t. Separating gives:

$$\frac{d\widehat{u}}{\widehat{u}} = -i|\xi|^2 dt.$$

and integrating gives

$$\log \widehat{u} = -i|\xi|^2 t + C$$

which ipmles that

$$\widehat{u}(t,\xi) = e^{-i|\xi|^2 t} \widehat{u_0}(\xi)$$

and inverting the Fourier transform gives

$$\begin{aligned} u(t,x) &= \mathcal{F}^{-1}\left[e^{-i|\xi|^2 t}\widehat{u_0}(\xi)\right] \\ &= (2\pi)^{d/2}\, \mathcal{F}^{-1}\left[e^{-i|\xi|^2 t}\right] * u_0(x) \end{aligned}$$

Therefore it remains to compute $\mathcal{F}^{-1}\left[e^{-i|\xi|^2 t}\right]$. We did this in a previous class.

We have a solution operator $S(t) = e^{it\Delta}$ [is this correct?] which is

$$\begin{aligned} S(t)u_0(x) &= (2\pi)^{d/2}\, \mathcal{F}\left[e^{-t|\xi|^2 i}\right] * u_0(x) \\ &= (4\pi i t)^{-d/2} \int_{\mathbb{R}^d} e^{\frac{i|x-y|^2}{4t}} u_0(y) dy \end{aligned} \tag{16}$$

The operator $e^{it\Delta}$ is the linear Schrodinger operator. Now we can derive the dispersive estimate. First, observe that

$$\begin{aligned} \|u\|_\infty &= \left\|e^{it\Delta}u_0\right\|_{L^\infty(\mathbb{R}^d)} \\ &\lesssim |t|^{-\frac{d}{2}} \int_{\mathbb{R}^d} |u_0(y)|\, dy \\ &= |t|^{-\frac{d}{2}} \|u_0\|_{L^1(\mathbb{R}^d)} \end{aligned} \tag{17}$$

Second, observe that,

$$\left\|e^{it\Delta}u_0\right\|_{L^2} = \|u\|^2_{L^2_x} = \|u_0\|_{L^2_x} \tag{18}$$

[somehow the second equality is obvious from Eq. (17)] so that

$$\|u\|_{H^s} = \|u_0\|_{H^s}$$

Interpolating Eqs. (17) and (18) gives, for all $2 \le p \le \infty$,

$$\left\|e^{it\Delta}\right\|_{L^p\mathbb{R}^d} \le |t|^{-\left(\frac{d}{2}-\frac{d}{p}\right)} \|u_0\|_{L^{p'}}$$

where p' is the Holder conjugate of p.

Difference between $\dot{H}^s$ and H^s.

7 2022-10-11: Lecture

Theorem 3 (Strichartz)**.** *Assume we have q, r a Stricharz admissible pair with $2 \le q, r \le \infty$. Then for (q, r) and $(\tilde{q}, \tilde{r})$ we have $(q, \infty, 2) \neq (q, r, d)$, $\frac{2}{q} + \frac{d}{r} = \frac{d}{2}$. Then*

- *From Str. Est*
$$\left\| e^{i\Delta t} u_0 \right\|_{L^q L^r(\mathbb{R}\times\mathbb{R}^d)} \lesssim C_{d,q,r} \left\| u_0 \right\|_{L^2_x}$$
- *From dual form of Str.:* $\left\| \int_{\mathbb{R}} e^{is\Delta} F(s) ds \right\|_{L^2_x} \lesssim_{d,\tilde{q},\tilde{r}} \|F\|_{L^{\tilde{q}} L^{\tilde{r}}}$
- *Inform str est*
$$\left\| \int_{t'<t} i(t-t')F(t')ds \right\|_{L^q L^r} \lesssim_{d,q,r,\tilde{q},\tilde{r}} \|F\|_{L^{\tilde{r}'} L^{\tilde{q}'}}$$

Proof. TT^* argument. Here $T : H \to B$, $TT^\infty : B' \to B$, $T^* : B' \to H'$ we have

$$\|T\| < \infty \iff \|TT^*\| < \infty \iff \|T^*\| < \infty$$

and here we have $T = e^{it\Delta}$. We want to show that (1) $T : L^2 \to L^q L^{r'}$. I will do the TT^* bound. Here

$$TT^* : L^{q'} L^{r'} \to L^q L^r$$

and we want to show this is truly the mapping domains and range. What is the adjoint of T^*? What do we usually do to find the adjoint? We look at the inner product and we move things. Our inner product is

$$\begin{aligned}
\langle f, T^*G\rangle_{L^2_t L^2_x} &= \langle Tf, G\rangle_{L^2_x} \\
&= \int \int e^{it\Delta} f \overline{G(t,x)} dx dt \\
&= \int f \int \overline{e^{-it\Delta} G(t,x)} dx dt
\end{aligned}$$

so that $T^* = \int e^{-it\Delta} G(t,x) dt$ $\square$

Using the dispersive estimate for the Schrodinger equation from earlier

$$\begin{aligned}
\|TT^*F\|_{L^q_t L^r_x} &= \left\| e^{it\Delta} e^{-is\Delta} F \right\| \\
&= \left\| e^{i(t-s)\Delta} F \right\|_{L^q L^r_x} \\
&\lesssim \left\| |t-s|^{-\left(\frac{d}{2} - \frac{d}{r}\right)} \|F(s)\|_{L^{r'}} ds \right\|_{L^q_t} \\
&= \left\| |t|^{-2/q} * \|F(t)\|_{L^{r'}} \right\|_{L^q_t}
\end{aligned}$$

at the end of the day, using some Littlewood Hardy Sobolev inequality, we get an upper bound

$$\|F\|_{L^{q'} L^{r'}}$$

and we get

$$\|TT^*\|_{L^{q'} L^{r'}} \to L^q L^r < \infty$$

this proves the second part of Theorem 3. And first part follow from something.

We will apply the following lemma with K as the schrodinger operator.

Lemma 6 (Christ-Kiselev)**.** *Suppose X, Y are Banach spaces and $T : L^p(\mathbb{R}, X) \to L^q(\mathbb{R}, Y)$ where $1 \le p < q < \infty$ which is given by the integral transform*

$$Tf(t) = \int_{\mathbb{R}} K(t,s) f(s) ds$$

where $K : \mathbb{R} \times \mathbb{R} \to \mathcal{S}(X, Y)$

[Need to fix this part.]

7.0.1 Bilinear Strichartz estimates

We have linear Sch equation ($i\partial_t u = -\Delta u$). Two intial data v_0, u_0 both of which are scharz functions. [missed the last 15 minutes of this lecture]

8 Lecture: 2022-10-18

At this point we all should know Duhamel's principle, which is what we did last time. In general, for any PDE of the form

$$\begin{cases} \partial_t u - Lu = N(u) \\ u_0 = u(0) \end{cases}$$

where $N(u)$ is nonlinear (and with the right L, which can be used for energy estimates), Duhamel's principle states that your solution will be of the form

$$u(t,x) = \underbrace{e^{tL}\mu_0}_{\text{linear PDE}} + \underbrace{\int_0^t e^{(t-s)L}N(u(s))ds}_{\text{The Duhamel term. Inhomogenous. Call it } DN(u).} \tag{19}$$

When studying local wellposedness (LWP) for NLS (Nonlinear Schrodinger - semilinear). There is a huge difference when studying LWP for quasilinear versus semilinear PDEs, and even more so when the LPW theory is done in a low-regularity setting. In particular, in the low regularity setting we cannot use the same techniques as were used in the semilinear case. Thus the methods for proving LWP differ in these settings:

For the semilinear case, we use a **fixed point argument.** This may work even in a low regularity setting (depending on the PDE you are using). This technique may work for quasilinear equations, but only in the high-regularity setting. The form of Eq. (19) suggests a perturbative method. We can reformulate our problem as

$$u(t,x) = u_{\text{linear}} + DN(u) \tag{20}$$

Theorem 4 (Contraction mapping argument)**.** *Start with two Banach spaces $\mathcal{S}$ (not the Shwarz space) and $\mathcal{N}$. Suppose $D : \mathcal{N} \to \mathcal{S}$ is a linear operator such that there exists a positive constant $C_0 > 0$ such that*

$$\|DF\|_{\mathcal{S}} \le C_0 \|F\|_{\mathcal{N}}$$

for all $F \in \mathcal{N}$, and further suppose we have a nonlinear operator

$$N : \mathcal{S} \to \mathcal{N}$$

such that $N(0) = 0$ and such that N satisfies the Lipschitz bound

$$\|N(u) - N(v)\|_{\mathcal{N}} \le \frac{1}{2C_0} \|u - v\|_{\mathcal{S}}$$

where $u, v \in B_\epsilon = \{u \in \mathcal{S} : \|u\|_{\mathcal{S}} \le \epsilon\}$ for $\epsilon > 0$. Then for all $u_{\text{lin}} \in B_{\epsilon/2}$ there exists a unique solution

$$u \in B_\epsilon$$

which is a solution to Eq. (20), with the map

$$u_0 \mapsto u$$

is Lipschitz with constant at most 2. In particular, we have

$$\|u\|_{\mathcal{S}} \le 2 \|u_{\text{lin}}\|_{\mathcal{S}}$$

Example 6 (Nonlinear Schrodinger)**.** *Suppose we have the equation*

$$\left(i\partial_t + \frac{\Delta}{2}\right) u = |u|^{p-1}u$$

where $u : \mathbb{R}^{1+d} \to \mathbb{C}$, i.e. $u : [-T,T] * \times \mathbb{R}^d \to \mathbb{C}$. Then LWP in $e_t^0 H^s \to \mathcal{N}$ in L^2 [missed something here].

Consider the equation

$$\begin{cases} u_t + \frac{1}{2}\Delta u = \mu |u|^{p-1}p \\ u(t_0, x) = u_0(x) \in H_x^s(\mathbb{R}^d) \end{cases} \tag{21}$$

where we choose p to be prime (e.g. $p = 3$ is called the **cubic NLS**, $p = 5$ is the **quintic NLS**) as this often corresponds to something physcially relevant. If $\mu = 0$ there is no nonlinearity. If $\mu = +1$ or $\mu = -1$, these are called the **focusing** and **defocusing** cases respectively. The focusing case will not play a role for the rest of this lecture, but will come into play when proving global well-posedness. If you want to prove global well-posedness you have to show that (1) your solution in the nonlinear case looks like the linear case, or (2) your solution splits into two bits: a soliton and a solution to your linear probem. However, for LWP (the topic of this lecture), the sign of μ doesn't make much a difference (because we'll be taking absolute values).

The nonlinearity in Eq. (21) doesn't look too bad because it's a polynomial, but it's still a nonlinearity. We will study this problem on the real line, i.e. we are looking for a solution

$$u : [-T, T] \times R^d \to \mathbb{C}$$

and we want to show that such a solution exists. Then if we can make T large enough, we'll have a global solution.

We need to determine the regularity of the space we will be working on. Terry does a great job in Chapter 3 of discussing all the ways that you can treat this problem. You can treat it classically, or that you have solutions in a distributional sense, etc. There are multiple ways to think about this, so we need to be precise about what we mean by local well-posedness. This problem has many symmetries. To find scaling symmetry, we rescale our solutions

$$\begin{aligned} u(t, x) &\to \lambda^\alpha u(t\lambda^\beta, x\lambda^\gamma) \\ u_0(x) &\to \lambda^\alpha u_0(x\lambda^\gamma) \end{aligned}$$

where $\lambda > 0$. For our problem in particular, we will use

$$\begin{aligned} u(t, x) &\to \lambda^{-\frac{2}{p-1}} u\left(\frac{t}{\lambda^2}, \frac{x}{\lambda}\right) \\ u_0(x) &\to \lambda^{-\frac{2}{p-1}} u_0\left(\frac{x}{\lambda}\right) \end{aligned}$$

so that

$$\|u_0(x)\|_{\dot{H}^{s_c}} = \left\|\lambda^{-\frac{2}{p-1}} u_0\left(\frac{x}{\lambda}\right)\right\|_{\dot{H}^{s_c}}$$

where $s_c = \frac{d}{2} - \frac{2}{p-1}$. The critical value s_c is the threshold for the lowest regularity for which we can achieve a LWP result.

In this context, the terminology is as follows. The case in which $s > s_c$ is called **subcritical**. Subcritical LWP theory is not that bad. The case when $s = s_c$ is called **critical**. The case when $s < s_c$ is called **supercritical**. Supercritical LWP theory is harder.

In general, showing LWP means showing

1. Existence of a solution (e.g. by a contraction mapping argument for semilinear problems; different story for quasilinear problem) in $C_t^0[-T, T]H_x^s$.

2. Uniqueness of solutions in the same regularity class (easy for both semi- and quasi-linear).

3. Continuous dependence on the initial data (we call this continuous dependence, but when dealing with semilinear PDEs what we really need is Lipschitz continuity). That is, we want the map

   $$u_0 \to u$$

   to be continuous.

Recall the Benjamin-Ono equation $u_t + H\partial_x^2 u = uu_x$, which is semilinear, but behaves more like a quasilinear problem in low regularity.

Definition 10 (Well-Posedness)**.** *We say the NLS is* ***locally well-posed (LWP)*** *in $H^s_x(R^d)$ if for any $u_0^* \in H^s_x(\mathbb{R}^d)$, there exists a time $T > 0$ and an open ball $B \subset H^s_x$ containing u_0^*, and $X \subset C^0_t H^s_x([-T,T]\times\mathbb{R}^d)$ such that for each $u_0 \in B$ there exists a strong unique solution $u \in X$ to the integral equation associated to hte NLS via Duhamel formulation, and furthermore, the map $u_0 \to u$ is continuous from B to X.*

- *If, in addition $X = C^0_t H^s_x\left([-T,T]\times\mathbb{R}^d\right)$ from this H^s_x-well posedness is* ***unconditional****. [This isn't correct?]*
- *If T can be arbitrarily large, then we have* ***global well-posedness****. If T only one H^s_x of the u_0 then we say we hañve WP in the* ***subcritical sense****. [fix this]*
- *If $u_0 \to u$ is uniformly continuous from B to X then we call it* ***uniform well-posedness****.*

9 Lecture Notes: 2022-10-20

Recall

$$\begin{cases} iu_t + \frac{\Delta}{2}u = |u|^{p-1}u \\ u(0,x) = u_0(x) \end{cases} \tag{22}$$

has solution

$$u(t,x) = e^{i(t-t_0)}\frac{\Delta}{2}u(t_0) - i\mu\int_{t_0}^{t} e^{i(t-s)\frac{\Delta}{2}}F(s)ds$$

where $F(s) = |u(s,x)|^{p-1}u(s,x)$.

We will discuss classical solutions (LWP) on $[-T,T]$ or $[0,T]$. This means our solutions are going to be continuous in time and space. Our initial data is in $H^s_x(R^d)$, $s \in \mathbb{R}$ where $s > d/2$. Sobolev embedding will give continuity. The critcal threshold is $s_c = \frac{d}{2} - \frac{2}{p-1}$.

Remark 13. *Critical case means the time interval on which the solution is defined depends on the size of your initial data. If your data is small (i.e. is in a certain ball around 0, then you have a unique solution.)*

The next theorem will use the fact that $H^s_x(R^d)$ is an algebra for certain values of s, depending on d. This follows from the inequality

$$\|fg\|_{H^s_x} \lesssim \|f\|_{H^s_x}\|g\|_{H^s_x} \tag{23}$$

In the subcritical case, the inequality Eq. (23) will not hold. We will use Strichartz estimates instead.

Theorem 5 (Classical NLS solutions)**.** *Let $p > 1$ be a prime number, $s \in \mathbb{R}$ such that $s > d/2$, and $\mu = \pm 1$. Then Eq. (22) is unconditionally well-posed in $H^s_x(\mathbb{R}^d)$ in the subcritical sense: for all $R > 0$ there exists $T = T(s,d,p,R) > 0$ such that for all $u_0 \in B_R = \left\{u_0 \in H^s_x(\mathbb{R}^d), \|u_0\|_{H^s_x(\mathbb{R}^d)} < R\right\}$ there exists a unique solution $u \in C^0_t H^s_x\left([-T,T]\times\mathbb{R}^d\right)$ to Eq. (22). Furthermore, the map*

$$B_R \to C^0_t H^s_x \quad \textit{given by} \quad u_0 \mapsto u$$

is Lipshitz continuous.

Proof. Fix $R > 0$. We will decide what $T > 0$ is later. Now we will apply the contraction mapping argument and will ultimately obtain unconditional well-posedness. Let

$$\mathcal{S} = \mathcal{N} = C^0_t H^s_x$$

and consider the map $D : \mathcal{N} \to \mathcal{S}$ given by

$$DF(t,x) = i\mu\int_0^t e^{i(t-s)\Delta/2}F(s,x)ds$$

and let $N : \mathcal{S} \to \mathcal{N}$ be given by

$$N(u(t,x)) = |u|^{p-1}u.$$

We will need to show that there exists a constant C_0 such that

$$\|DF\|_{\mathcal{S}} \leq C_0 \|F\|_{\mathcal{N}} \tag{24}$$

and

$$\|Nu - Nv\|_{\mathcal{N}} \leq \frac{1}{2C_0} \|u - v\|_{\mathcal{S}} \tag{25}$$

(We are hiding the fact that we need $u_{\text{lin}} \in B_{R/2}$. This won't screw anything up). First we will prove Eq. (24):

$$\begin{aligned}
\|DF\|_{\mathcal{S}} &= \left\| \int_0^t e^{i(t-s)\Delta/2} F(s) ds \right\|_{C_t^0 H_x^s([-T,T]\times\mathbb{R}^d)} \\
&\leq C_0(t,d) \|F\|_{H_x^s}.
\end{aligned}$$

Next, to prove Eq. (25),

$$\begin{aligned}
\|N(u) - N(v)\|_{C_t^0 H_x^s} &= \left\| |u|^{p-1}u - |v|^{p-1}v \right\|_{C_t^0 H_x^s} \\
&\leq \left\| \left(|u|^{p-1} + |v|^{p-1}\right)(u - v) \right\|_{C_t^0 H_x^s} \\
&\leq \tilde{C}_0(p,s,d,R) \|u - v\|_{C_t^0 H_x^s}.
\end{aligned}$$

Now I want to apply the contraction mapping which means we have to have T sufficiently small (so that $\tilde{C}_0 < \frac{1}{2C_0}$). Then contraction mapping implies that if if $u_{\text{lin}} \in C_t^0 H_x^s$, $\|u_{\text{lin}}\|_{C_t^s H_x^s} \lesssim R$, there exists a unique solution $u \in C_t^\infty H_x^s$ with $\|u\|_{C_t^0 H_x^s} < R$. And it also implies that the map $H_x^s \to C_t^0 H_x^s$ given by $u \to u_0$ is Lipschitz on the ball in on radius of order $O(R)$. [This is proposition 1.38 in Terry Tao's notes]. This looks like **conditional** well-posedness. Next we explain why we also get unconditional well-posedness using a bootstrap argument.

We get uniqueness as long as $C_t^0 H_x^s$ of the solution is $O(R)$. But $\|u_0\|_{H_x^s}$ at most R at time $t = 0$. We prove unconditional uniqueness as follows. Let $u \in C_t^0 H_x^s$ be what constructed earlier. So $\|u\|_{\mathcal{S}} \leq C_1 R$. Let $u^* \in C_t^0 H_x^s$ is another solution. We'll show that u^* is also in the ball. Let $H(t)$ be the proposition

$$H(t) : \|u^*\|_{C_t^0 H_x^s([-t,t]\times R^d)} \leq 2C_1 R.$$

This is our bootstrap assumption. We will show that this implies that u^* actually lives in a smaller ball:

$$C(t) : \|u^*\|_{C_t^0 H_x^s([-t,t]\times R^d)} \leq C_1 R,$$

which means we are in a smaller ball! It's like we want to be in the bathroom, and we are assuming that we are in the house (a decent assumption) and then finding out that we are actually in bathroom. The above bootstrap argument works for every t up to some fixed maximal value, which is T. This implies unconditional LWP. $\square$

For the next theorem, we will need to introduce some new notation. If $s = 0$ then $H^s = L^2$.

$$s_c < 0$$

and since $s_c = \frac{d}{2} - \frac{2}{p-1}$. Therefore

$$1 < p < 1 + \frac{4}{d}$$

For the case with $s \leq d/2$ then sour solutions are in H_x^s but this does not give classical solutions because there is no algebra property for H^s when $s \leq d/2$.

Theorem 6 (Subcritical L^2 Solutions to NLS)**.**