3 2025-08-29 | Week 01 | Lecture 03

This lecture is based on section 1.1 in the textbook

> ***The nexus question of this lecture:*** *How can we solve a linear system without resorting to substitution?*

Recall in the boat example (Example 1), we had the system

$$\begin{cases} x + y = 12 \\ x - y = 8 \end{cases}$$

And we could solve this using subsitution. Another thing we could have done, would be to add the second equation to the first, giving us a new, simpler but equivalent system:

$$\begin{cases} 2x = 20 \\ x - y = 8 \end{cases}$$

Then divide the firs equation by two

$$\begin{cases} x = 10 \\ x - y = 8 \end{cases}$$

Then subtract the first equation from the second:

$$\begin{cases} x = 10 \\ -y = -2 \end{cases}$$

Then multiply the second equation by -1

$$\begin{cases} x = 10 \\ y = 2 \end{cases}$$

And tada! We have found our solution without doing substitution. But this example was very simple, so maybe it's special and we can't always do this sort of thing? Actually, we can. In the rest of the lecture, I'll try to formalize these sorts of steps we used here and apply them to a more complicated problem.

The reason I'm doing this is because, in the next lecture, I will begin to present ***Gauss-Jordan elimination*** (aka ***row reduction***), a general method which can be used to find the solutions of any system of linear equation which does not use substitution. For now, the we will work out an example which motivates the main ideas that will be used by Gauss-Jordan elimination.

3.1 Solving a linear system using via simplifying transformations

Example 9 (Solving a linear system with elementary operations)**.** Suppose we wish to solve the following system:

$$\begin{cases} x - y + z = 0 & (E_1) \\ 2x - 3y + 4z = -2 & (E_2) \\ -2x - y + z = 7 & (E_3) \end{cases} \tag{7}$$

This system has 3 equations, labeled E_1, E_2, E_3, and 3 variables x, y and z. Suppose that we know ahead of time that this system has a unique solution (we showed this graphically in Example 8). Then, in principle, we could solve this using substitution, but that would suck. Instead, I will illustrate an approach in which we iteratively transform this linear system into successively simpler systems until we get to a point where the solution is obvious.

To do this, we will play a game where there are three 'moves' available to us. The three moves are:

1. Interchange two equations in the system.
2. Multiply an equation by a nonzero number.

3. Replace an equation by itself plus a multiple of another equation.

These moves are called ***elementary operations***, and if we use them intelligently, they will allow us to transform the linear system into a simpler system.

Two systems of equations are said to be ***equivalent*** if they have the same solutions. Applying elementary operations always results in an equivalent system. Our goal will be to use some combination of elementary operations to produce a system of the form

$$\begin{cases} x = * \\ y = * \\ z = * \end{cases}$$

where each $*$ is a constant which we will have computed. This will be our solution to the linear system (7), because the two systems will be equivalent.

First, let's apply operation 3: specifically, by replacing E_2 with $E_2 - 2E_1$:

$$\begin{cases} x - y + z = 0 \\ -y + 2z = -2 \\ -2x - y + z = 7 \end{cases}$$

We have eliminated the x from the second equation, yielding a simpler system. Let's keep doing this. To eliminate x from equation 3, let's apply operation 3 again: This time, replace E_3 with $E_3 + 2E_1$:

$$\begin{cases} x - y + z = 0 \\ -y + 2z = -2 \\ -3y + 3z = 7 \end{cases}$$

Apply operation 3, replace E_1 with $E_1 - E_2$. This will allow us to eliminate y from E_1:

$$\begin{cases} x - z = 2 \\ -y + 2z = -2 \\ -3y + 3z = 7 \end{cases}$$

Apply operation 3, replace E_3 with $E_3 - 3E_2$. This will allow us to eliminate y from E_3:

$$\begin{cases} x - z = 2 \\ -y + 2z = -2 \\ -3z = 13 \end{cases}$$

Apply operation 2 twice: multiply both the first and second equations by 3:

$$\begin{cases} 3x - 3z = 6 \\ -3y + 6z = -6 \\ -3z = 13 \end{cases}$$

Apply operation 3, twice. First, replace E_1 with $E_1 - E_3$. Then replace E_2 with $E_2 + 2E_3$. Doing both of these, we get:

$$\begin{cases} 3x = -7 \\ -3y = 20 \\ -3z = 13 \end{cases}$$

Apply operation 2 by multiplying the first equation by $1/3$. Then multiply the second and third equations both by $-1/3$:

$$\begin{cases} x = -7/3 \\ y = -20/3 \\ z = -13/3 \end{cases}$$

This is the solution to the original equation. We have used elementary operatons to reduce our original linear system Eq. (7) to the above system, which equivalent to the original system.

While solving this system was still a lot of (tedious) work, it was still probably simpler than doing substitution.

End of Example 9. □

3.2 Representing a linear system as an augmented matrix

In the procedure presented in Example 9, we didn't really need to track the variables, only the *coefficients* and the *quantities on the right hand sides* of the equations. Instead of working with the equations directly, it will be simpler to work with the following matrix, called the ***augmented matrix*** corresponding go Eq. (7):

$$\left[\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 2 & -3 & 4 & -2 \\ -2 & -1 & 1 & 7 \end{array}\right].$$

Comparing this with system (7), it becomes clear that the augmented matrix was obtained essentially by just erasing the variables x, y, and z in (7), and then placing what remains into an array. We also drew a vertical line to the separate the left- and right-hand sides of the equations. Inside the augmented matrix, the 3×3 submatrix of coefficients

$$\begin{bmatrix} 1 & -1 & 1 \\ 2 & -3 & 4 \\ -2 & -1 & 1 \end{bmatrix}$$

is called the ***coefficient matrix*** of the system.

More precise definitions are as follows:

Definition 10 (Augmented Matrix)**.** Given a linear system of the form (2), the ***augmented matrix*** is

$$\left[\begin{array}{cccc|c} a_{11} & a_{12} & \dots & a_{1n} & b_1 \\ a_{21} & a_{22} & \dots & a_{2n} & b_2 \\ \vdots & \vdots & & \vdots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} & b_m \end{array}\right]$$

and the ***coefficient matrix*** is the matrix

$$\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}.$$

This is our first application of