

## 36 2025-11-17 | Week 13 | Lecture 36

We begin Section 6.1. This lecture is based on Section 5.4 in Gilbert Strang's *Linear Algebra and its Applications*)

### 36.1 Examples of systems of first-order linear differential equations

**Example 163.** An ant is moving in the  $xy$ -plane with velocity vector

$$v = \begin{bmatrix} 2x - 5y \\ x - 2y \end{bmatrix}$$

Suppose the ant starts at  $(1, 1)$ .

**Question:** Find the position of the ant at time  $t > 0$

**Solution:** First recognize that

$$v = \begin{bmatrix} \frac{dx}{dt} \\ \frac{dy}{dt} \end{bmatrix}$$

and therefore we have the following system of equations:

$$\begin{cases} \frac{dx}{dt} = 2x - 5y \\ \frac{dy}{dt} = x - 2y \end{cases} \quad (29)$$

with initial conditions  $x(0) = 1$  and  $y(0) = 1$ .

A solution to this system would consist of functions of the form

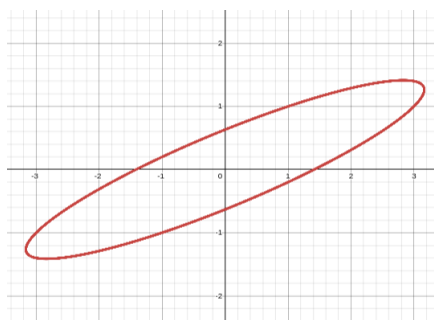
$$x = x(t) \quad \text{and} \quad y = y(t)$$

which satisfy the system of equations and initial conditions.

Without showing how we got it, here is one possible solution to the differential equation given in Eq. (29):

$$\begin{cases} x(t) = \cos(t) - 3\sin(t) \\ y(t) = \cos(t) - \sin(t) \end{cases}$$

These two equations parameterize the following curve:



To verify that this is indeed a solution, we can check first that  $x(0) = 1$  and  $y(0) = 1$ , and that the right hand sides of Eq. (29) actually equal  $x'(t)$  and  $y'(t)$ . To simplify, let  $c = \cos(t)$  and  $s = \sin(t)$ :

$$2x - 5y = 2(c - 3s) - 5(c - s) = -s - 3c = x'(t)$$

and

$$x - 2y = (c - 3s) - 2(c - s) = -c - s = y'(t).$$

End of Example 163.  $\square$

**Example 164** (Strang's example). Consider the differential equation:

$$\begin{bmatrix} y_1' \\ y_2' \end{bmatrix} \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}. \quad (30)$$

Here,  $y_1, y_2$  are both functions of  $t$  ("time"). This differential equation has the form

$$Y' = AY$$

where  $A$  is an  $n \times n$  matrix and  $Y = Y(t) = \begin{bmatrix} y_1(t) \\ \vdots \\ y_n(t) \end{bmatrix}$ .

The only thing we need is the following:

**Theorem 165** (Pure exponential solutions). *Let  $A$  be an  $n \times n$  matrix, let  $v$  be an eigenvector of  $A$  with eigenvalue  $\lambda$ , and let  $c$  be any scalar. Then*

$$Y(t) = ce^{\lambda t}v$$

*is a solution of the system  $Y' = AY$ .*

*Proof.*

$$\begin{aligned} Y'(t) &= \frac{d}{dt} [ce^{\lambda t}v] \\ &= ce^{\lambda t}\lambda v \\ &= ce^{\lambda t}Av \\ &= A(ce^{\lambda t}v) \\ &= AY(t) \end{aligned}$$

So  $Y' = AY$ , meaning  $Y(t) = ce^{\lambda t}v$  is a solution of the differential equation. □

In our problem Eq. (30), the first step is to find the eigenvalues and eigenvectors:

$$A \begin{bmatrix} 1 \\ 1 \end{bmatrix} = (-1) \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad \text{and} \quad A \begin{bmatrix} 1 \\ -1 \end{bmatrix} = (-3) \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

By Theorem 165, we have two solution:

$$Y_1(t) = c_1 e^{-t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad \text{and} \quad Y_2(t) = c_2 e^{-3t} \begin{bmatrix} 1 \\ -1 \end{bmatrix},$$

where  $c_1, c_2$  are arbitrary constants. Combining these, we get the general solution

$$Y(t) = c_1 e^{-t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 e^{-3t} \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

If we further know that  $Y(0) = B = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$ , we can solve for  $c_1, c_2$  by observing that plugging  $t = 0$  into the above formula gives

$$\begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = c_1 e^{-t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 e^{-3t} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

We can write this as

$$\begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

which implies that

$$\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}^{-1} \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

(Note that the matrix  $\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$  is invertible since its columns are linearly independent eigenvectors of  $A$ .)

End of Example 165. □