## 29 2025-10-31 | Week 10 | Lecture 29

*Examples of computing eigenvectors and eigenvalues*

### 29.1 Recall definitions

**Definition 131** (Eigenvalue,eigenvector)**.** If $A$ is an $n \times n$ matrix, an ***eigenvector*** of $A$ is a nonzero column vector $v$ such that

$$Av = \lambda v$$

for some scalar $\lambda \in \mathbb{C}$. The scalar $\lambda$ is called an ***eigenvalue***.

**Definition 132.** The ***characteristic polynomial of*** $A$ is

$$\det(\lambda I - A).$$

**Theorem 133.** *Let $A$ be an $n \times n$ matrix. Then $\lambda \in \mathbb{C}$ is an eigenvalue of $A$ if and only if* $\det(\lambda I - A) = 0$.

### 29.2 Examples of eigenvector/eigenvalue computations

**Idea:** Find the eigenvalues before finding the eigenvectors. Then for each eigenvalue $\lambda$, find the nullspace $NS(\lambda I - A)$. The vectors in the nullspace are the eigenvectors corresponding to $\lambda$. (Usually we just pick out a basis.)

**Example 134.** Find the eigenvectors of the matrix

$$A = \begin{bmatrix} 1 & -3 \\ -2 & 2 \end{bmatrix}$$

Last class, we showed that $A$ has eigenvalues $\lambda = 4$ and $\lambda - 1$ The equations we need to solve are

- **When $\lambda = 4$:** we computed the nullsapce of the matrix $4I - A$, which gave

  $$NS(4I - A) = \left\{ y \begin{bmatrix} -1 \\ 1 \end{bmatrix}, y \in \mathbb{R} \right\}$$

  So $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$ is an eigenvector of $A$ corresponding to $\lambda = 4$

- **When $\lambda = -1$:** $-I - A = 0$ Here we get a NS generated by a single basis element $\begin{bmatrix} 3 \\ 2 \end{bmatrix}$ so this is an eigenvector as well.

End of Example 134. □

**Example 135.** Find the eigenvaluess and eigenvectors of

$$A = \begin{bmatrix} 2 & -1 & 3 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

First we compute the characteristic polynomial:

$$\begin{aligned} \det(\lambda I - A) &= \det \begin{bmatrix} \lambda - 2 & 1 & -3 \\ 0 & \lambda + 1 & 0 \\ 0 & 0 & \lambda + 1 \end{bmatrix} \\ &= (\lambda - 2)(\lambda + 1)^2 \end{aligned}$$

This equals zero iff $\lambda = 2$ or $\lambda = -1$. These are the eigvenvalues.

To find eigenvectors, we need to find a basis for the nullspaces $NS(2I - A)$ and $NS(-I - A)$.

- $\lambda = 2$. Need to find $NS(2I - A)$.

  Row reducing the augmented matrix

$$\left[\begin{array}{ccc|c} 0 & 1 & -3 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 3 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

  This corresponds to the system with $x_2 = 0$, $x_3 = 0$, and $x_1$ a free variable. That is,

$$NS(2I - A) = \left\{ \begin{bmatrix} x_1 \\ 0 \\ 0 \end{bmatrix} : x_1 \in \mathbb{R} \right\} = \left\{ x_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} : x_1 \in \mathbb{R} \right\}$$

  So $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ is a basis for the nullspace. It is an eigenvector. Indeed, $A \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = 2 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$.

- $\lambda = -1$. Need to find $NS(-I - A)$.

End of Example 135. □