15 2025-09-29 | Week 06 | Lecture 15

> ***The nexus question of this lecture:*** *How do we know if a linear system is minimal—i.e., that it doesn't have any redundant equations?*

(Investigating this question will help set us up to answer the question of how to build a linear subspace with a **minimal** set of vectors.)

15.1 Some review

Theorem 63 (The Key Theorem of Linear Algebra (partial version))**.** *Let A be an $n \times n$ matrix. Then the following are equivalent:*

(i.) *A is invertible (i.e., A^{-1} exists)*

(ii.) $\det A \neq 0$

(iii.) *The linear system $AX = B$ has a unique solution for each $B \in \mathbb{R}^n$.*

(iv.) *A is row equivalent to I*

(v.) *The only solution to $AX = \vec{0}$ is $X = \vec{0}$ (i.e., A is nonsingular)*

(vi.) $\ker(A) = \left\{\vec{0}\right\}$

(vii.) *???*

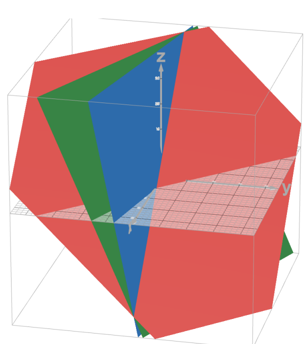
Review definition of kernel, too.

15.2 Dependent linear systems

Consider the linear system

$$\begin{cases} E_1: & x + y + z = 5 \\ E_2: & x + 5y + z = 6 \\ E_3: & 3x + 7y + 3z = 16 \end{cases}$$

The set of solutions to this system is the the intersection of three planes in 3d space, one for each equation. In principle, each equation imposes some restriction on what the solution set can be. For examplep, E_1 and E_2 intersect to form a line, so solutions must lie on that line. E_3 is the equation of another plane, and while it is not parallel to either E_1 or E_2, when we plot all three planes, we see that the intersection is the same line we get by just intersecting E_1, E_2:



Hence, the solutions to the system of equations form a line. The third plane in our system failed to to cut the line of intersection down to a single point. The equation E_3 didn't impose any additional restrictions on the solution set that weren't already imposed by E_1 and E_2. This is because $E_3 = 2E_1 + E_2$. In some sense, E_3 is just E_1 and E_2 in disguise.

Indeed, if we row reduce, we get a row of zeros:

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 5 \\ 1 & 5 & 1 & 6 \\ 3 & 7 & 3 & 16 \end{array}\right] \xrightarrow{R_3 - R_2 \to R_3} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 5 \\ 1 & 5 & 1 & 6 \\ 2 & 2 & 1 & 10 \end{array}\right] \xrightarrow{R_3 - 2R_1 \to R_3} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 5 \\ 1 & 5 & 1 & 6 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

This implies that the solutions of the system can be expressed as the intersection of only the first two planes.

How can we predict exactly when this will happen? To understand exactly when a linear system yields exactly one solution vs. infinitely many solutions, we need the concept of "linear independence".

15.3 Linear independence

Definition 64. Let V be a vector space and let $v_1, \ldots, v_n \in V$. We say that the set $\{v_1, \ldots, v_n\}$ is ***linearly dependent*** if there are scalars $c_1, c_2, \ldots, c_n$ *not all zero* such that

$$c_1 v_1 + c_2 v_2 + \ldots + c_n v_n = \vec{0}.$$

If the vectors $v_1, \ldots, v_n$ are not linearly dependent, we say that they are ***linearly independent.***

Geometrically, we can visualize linear dependence in the following way. For vectors in 3-dimensional space, two vectors are linearly dependent if they lie on the same line. *Three vectors are linearly dependent if they lie in the same plane.*

Example 65. Are the vectors

$$\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \\ 5 \end{bmatrix}$$

linearly dependent or linearly independent?

To determine the answer we need to solve the linear system, to see if there are any solutions other than $c_1 = c_2 = c_3 = 0$.

$$\begin{bmatrix} 1 & 3 & -1 \\ 2 & 2 & 2 \\ 3 & 1 & 5 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Row reducing, we get

$$\begin{bmatrix} 1 & 3 & -1 & 0 \\ 2 & 2 & 2 & 0 \\ 3 & 1 & 5 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Convert back to equations

$$\begin{aligned} c_1 + 2c_3 &= 0 \\ c_2 - c_3 &= 0 \\ 0 &= 0 \end{aligned}$$

We can write this as

$$\begin{aligned} c_1 &= -2c_3 \\ c_2 &= c_3 \end{aligned}$$

with free variable $c_3 \in \mathbb{R}$. Therefore, there are infinitely many solutions. For example, if $c_3 = 1$, we get the solution

$$\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$$

We've shown that there is a nontrivial linear combination of the vectors which equals zero. Therefore they are linearly dependent.

End of Example 65. □

This example illustrates a general method: To check any set of vectors $v_1, \ldots, v_n$ for independence, put them in the columns of A Then solve the system $Ac = \vec{0}$. The vectors are dependent if there is a solution other than $c = \vec{0}$

To relate today's discussion of linear independence back to our nexus question, we utilize our key theorem. Let A be an $n \times n$ matrix, and let $B \in \mathbb{R}^n$ be an arbitrary vector. We wish to know whether the linear system $AX = B$ has a unique solution or not. (After all, if it has infinitely many solutions, that means we have at least one redundant equation.)

Then the following statements are equivalent:

$$\begin{gathered}
\text{The columns of } A \text{ are linearly independent.} \\
\Updownarrow \\
\text{The only solution to } AX = \vec{0} \text{ is } X = \vec{0}. \\
\Updownarrow \\
A \text{ is nonsingular} \\
\Updownarrow \\
\text{The equation } AX = B \text{ has exactly one solution for any } B \in \mathbb{R}^n.
\end{gathered}$$