

18 2025-10-06 | Week 07 | Lecture 18

The nexus question of this lecture: What are the three fundamental linear subspaces associated with a matrix A ?

18.1 Three fundamental subspaces

Definition 81 (The fundamental subspaces of A). Let $A \in M_{m \times n}(\mathbb{R})$. There are three important vector spaces associated with A :

- The **column space**, which is the subspace of $M_{m \times 1}(\mathbb{R})$ spanned by the columns of A . Notation: $CS(A)$
- The **row space**, which is the subspace of $M_{1 \times n}(\mathbb{R})$ spanned by the rows of A . Notation $RS(A)$.
- The **null space** which is the subspace of \mathbb{R}^n of vectors x such that $Ax = 0$. Notation $NS(A)$. The nullspace and the kernel are the same thing.

Remark 82 (Connection between column space and matrix multiplication). The idea of column space is natural. If

$$A = \begin{bmatrix} & & & \\ A_1 & A_2 & \dots & A_n \\ & & & \end{bmatrix} \in M_{m \times n}(\mathbb{R})$$

then for any vector $x = (x_1, \dots, x_n)^\top \in \mathbb{R}^n$

$$Ax = A_1x_1 + A_2x_2 + \dots + A_nx_n$$

This is linear combination of the columns of A , so the output Ax is always an element of the column space. Another word for column space is the **image** or **range** of the [linear transformation of the] matrix A .

Definition 83 (rank). The **rank** of a matrix is the dimension of its column space ($= \dim$ of row space).

Theorem 84. For $A \in M_{m \times n}(\mathbb{R})$, the dimensions satisfy

$$\dim RS(A) = \dim CS(A)$$

and

$$\underbrace{\dim CS(A)}_{\text{'rank'}} + \underbrace{\dim NS(A)}_{\text{'nullity'}} = \underbrace{n}_{\# \text{ cols of } A}$$

The second part is called the rank-nullity theorem. Noting that $\text{rank}(A) = \dim CS(A)$ and that $NS(A) = \ker(A)$, we have

18.2 Some examples of computing bases for the three fundamental subspaces

Example 85 (Row space, column space, null space). Let

$$A = \begin{bmatrix} 1 & 0 & -1 & 3 \\ 2 & 1 & 4 & -1 \\ 4 & 1 & 2 & 5 \end{bmatrix} \in M_{3 \times 4}(\mathbb{R}).$$

Find bases for

- (a) $RS(A)$
- (b) $NS(A)$

Solution

(a) Wrong:

$$[1 \ 0 \ -1 \ 3], [2 \ 1 \ 4 \ -1], [4 \ 1 \ 2 \ 5]$$

Better approach:

Idea: row reduction does not change row space, so row reduce until we get a linearly independent set.

The reduced row echelon form is

$$A_{\text{RREF}} = \begin{bmatrix} 1 & 0 & -1 & 3 \\ 0 & 1 & 6 & -7 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad (15)$$

A basis of A_{RREF} is

$$[1 \ 0 \ -1 \ 3], [0 \ 1 \ 6 \ 7]$$

(since these are linearly independent). Moreover, we note that since row reduction doesn't change the row space,

$$RS(A) = RS(A_{\text{RREF}}).$$

and hence

$$[1 \ 0 \ -1 \ 3], [0 \ 1 \ 6 \ 7]$$

are a basis for $RS(A)$ as well.

(b) We will use the fact that

$$NS(A) = NS(A_{\text{RREF}}).$$

So it suffices to find a basis for $NS(A_{\text{RREF}})$. Let's do a computation to see what $NS(A_{\text{RREF}})$ looks like. Recall that $NS(A_{\text{RREF}})$ consists of the vectors x satisfying

$$A_{\text{RREF}}x = 0 \quad (16)$$

If $x = (x_1, \dots, x_5)^\top$ satisfies Eq. (16), then we have

$$\begin{bmatrix} 1 & 0 & -1 & 3 \\ 0 & 1 & 6 & -7 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

Writing this as equations, we have

$$\begin{aligned} x_1 - x_3 + 3x_4 &= 0 \\ x_2 + 6x_3 - 7x_4 &= 0 \\ 0 &= 0. \end{aligned}$$

Therefore we have:

$$\begin{aligned} x_1 &= x_3 - 3x_4 \\ x_2 &= -6x_3 + 7x_4 \end{aligned}$$

where x_3, x_4 are free variables.

Therefore if $x \in RS(A_{\text{RREF}})$ (equivalently, if x satisfies Eq. (16)), then it has the following form

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} x_3 - 3x_4 \\ -6x_3 + 7x_4 \\ x_3 \\ x_4 \end{bmatrix} = x_3 \begin{bmatrix} 1 \\ -6 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -3 \\ 7 \\ 0 \\ 1 \end{bmatrix}.$$

This shows that the vectors

$$\begin{bmatrix} 1 \\ -6 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 7 \\ 0 \\ 1 \end{bmatrix}$$

span $NS(A)$. Moreover, they are also linearly independent (since two vectors are linearly dependent if and only if they are multiples of each other, which these are clearly not). Therefore

$$\begin{bmatrix} 1 \\ -6 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 7 \\ 0 \\ 1 \end{bmatrix}$$

form a basis for $NS(A_{\text{RREF}})$. Since $NS(A_{\text{RREF}}) = NS(A)$, they are a basis for $NS(A)$ as well. Since there are two vectors in the basis, $\dim NS(A) = 2$.

End of Example 85. \square