# 8 2025-09-12 | Week 03 | Lecture 08

*This lecture is based on sections 1.3 in your textbook*

***The nexus question of this lecture:*** *What are the key properties of the matrix inverse?*

## 8.1 What is a matrix inverse?

**Definition 20** (Matrix Inverse)**.** The matrix $A$ is ***invertible*** if there is a matrix $A^{-1}$ such that

$$A^{-1}A = I \quad \text{and} \quad AA^{-1} = I$$

If so we call $A^{-1}$ the ***inverse*** of $A$.

**Note:**

- Invertible matrices are always square. Nonsquare matrices are not invertible.
- Whatever $A$ does, $A^{-1}$ undoes. If $A$ stretches spaces, $A^{-1}$ compresses it back. If $A$ flips space, $A^{-1}$ flips it back. If $A$ rotates spaces, $A^{-1}$ rotates it back, etc.

**Theorem 21** (The socks and shoes property)**.** *If $A$ and $B$ are invertible then so is $AB$, and*

$$(AB)^{-1} = B^{-1}A^{-1}$$

If you put on socks and then shoes, what is the inverse of that? Take off the shoes, and then the socks.

Note that Theorem 21 has two conclusions: first, that products of matrices are invertible, and second the formula $(AB)^{-1} = B^{-1}A^{-1}$.

*Proof of Theorem 21.* It is sufficient to show that $B^{-1}A^{-1}$ is the inverse of $AB$, which we can do by direct computation:

$$(AB)(B^{-1}A^{-1}) = A(BB^{-1})A^{-1} = A(I)A^{-1} = AA^{-1} = I.$$

and

$$(B^{-1}A^{-1})(AB) = B^{-1}(A^{-1}A)B = B^{-1}B = I$$

□

**Theorem 22.** *Suppose that $A$ and $B$ are square matrices such that either $AB = I$ or $BA = I$. Then $A$ is an invertible matrix and $A^{-1} = B$*

I'm going to omit the proof of this theorem, but I will introduce the general framework that that is used to prove it and results like it, because it is based on an important idea.

## 8.2 Row reduction is multiplication by elementary matrices

Section 1.3 in the textbook provides a framework, based on row reduction, for deducing properties of inverses. Recall that in row reduction, you have three basic operations:

- swapping two rows
- multiplying a row by a number
- adding a multiple of a row to another row.

These operations can all be done using matrix multiplication, using matrices called ***elementary matrices***. Here are some examples:

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} c & d \\ a & b \end{bmatrix} \qquad R_1 \leftrightarrow R_2$$

$$\begin{bmatrix} 5 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 5a & 5b \\ c & d \end{bmatrix} \qquad 5R_1 \to R_1$$

$$\begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} c & d \\ 3a + c & 3b + d \end{bmatrix} \qquad 3R_1 + R_2 \to R_2$$

**Fact:** Elementary matrices are always invertible, because one can always undo a row operation. Moreover, by Theorem 21 (which tells us that products of invertible matrices are invertible), we also know that products of elementary matrices are also invertible.

**What does it all mean?** If you can row reduce a matrix $A$ to $I$, that means there exists some sequence of elementary matrices $E_1, \ldots, E_m$ such that

$$E_1 E_2 E_3 \cdots E_m A = I.$$

Letting $M = E_1 E_2 E_3 \cdots E_m$, we have

$$MA = I.$$

By Theorem 22, this is enough to conclude that $A$ is invertible and

$$A^{-1} = M.$$

We've just proven part of the following theorem:

**Theorem 23.** *Let $A$ be a square matrix. The following are equivalent:*

*(i.) $A$ can be row-reduced to the identity matrix $I$.*

*(ii.) $A$ is invertible.*

In particular, we've shown that $(i.)$ implies $(ii.)$. The reverse direction (that $(ii.)$ implies $(i.)$) can also be proved using this framework of elementary matrices, but I'd rather spend the time showing an example of how to compute the inverse of a matrix using row reduction.

## 8.3 Computing matrix inverses using row-reduction

The next example will illustrate a general approach for computing a matrix inverse.

**Example 24** (Computing the inverse of a matrix)**.** In the previous lecture, we considered the matrix

$$A = \begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix}$$

which stretches space in some way, scaling area by a factor of 5. Suppose we wish to find $A^{-1}$.

Let

$$X = \begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{bmatrix}$$

We need $X$ to satisfy

$$AX = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Writing out $AX$, we have

$$AX = \begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{bmatrix} = \begin{bmatrix} 3x_{11} + x_{21} & 3x_{12} + x_{22} \\ x_{11} + 2x_{21} & x_{12} + 2x_{22} \end{bmatrix} \stackrel{\text{set}}{=} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

By setting $AX$ equal to $I$, we have a system of four linear equations in four variables $x_{11}, x_{12}, x_{21}, x_{22}$.

We have a nice way to solve linear systes: row reduction. Moreover, for finding inverses, there is a clever way to do it, as I now show.

**Step 1.** Set up an augmented matrix of the following form:

$$\left[\begin{array}{cc|cc} 3 & 1 & 1 & 0 \\ 1 & 2 & 0 & 1 \end{array}\right]$$

**Step 2.** Completely row reduce the matrix until the left part is the identity matrix. (I'm skipping these steps). After doing this for this example, we get:

$$\left[\begin{array}{cc|cc} 1 & 0 & 2/5 & -1/5 \\ 0 & 1 & -1/5 & 3/5 \end{array}\right]$$

**Step 3.** Draw the conclusion that $A^{-1}$ is the the matrix to the right of the bar:

$$A^{-1} = \begin{bmatrix} 2/5 & -1/5 \\ -1/5 & 3/5 \end{bmatrix}$$

End of Example 24. □

**Remark 25.** The approach in Example 24 works for larger matrices as well, see Example 1 in Section 1.3 of the textbook (p.31).

**Remark 26.** Recall that from the previous lecture, we saw that the matrix $A = \begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix}$ corresponds to a transformation of space that somehow stretched space in a way that scaled area by a factor of 5, because $\det(A) = 5$ (see Section 7.3). Since $A^{-1}$ undoes whatever $A$ did, so it must shrink space by a factor of 5. Indeed,

$$\det(A^{-1}) = \left(\frac{2}{5}\right)\left(\frac{3}{5}\right) - \left(-\frac{1}{5}\right)\left(-\frac{1}{5}\right) = \frac{6}{25} - \frac{1}{25} = \frac{1}{5}.$$

This illustrates the following theorem:

**Theorem 27.** *If $A^{-1}$ exists then*

$$\det(A^{-1}) = \frac{1}{\det(A)}.$$

Now that we know the determinant of a matrix measures how space is scaled by the transformation, the geometric reason why this theorem is true is obvious: the matrix $A$ transforms space in some way, and then $A^{-1}$ undoes that transformation. So we are left with the ***identity transformation***, which doesn't change space at all. So if $A$ scales space by a factor of $a$ then $A^{-1}$ must scale it by a factor of $\frac{1}{a}$. We'll probably give an actual proof later, but that's the idea.