Math 307: Homework 01

Due Wednesday, September 10 (at the beginning of class)

Problem 1. Consider the function $f(x, y, z) = x^2 + yz$, which is defined for all $x, y, z \in \mathbb{R}$.

(a) Use the method of Lagrange multipliers to find the critical points of f subject to the constraint

$$xy + xz - 2 = 0.$$

You don't have to classify the critical points, just find them. If you don't recall the method of Lagrange multipliers, a good resources is `https://math.libretexts.org/Bookshelves/Calculus/Calculus_(OpenStax)/14%3A_Differentiation_of_Functions_of_Several_Variables/14.08%3A_Lagrange_Multipliers`.

(b) Your solution to part (a) involved solving a system of linear equations. Identify the system. How many equations does it have? How many variables?

Problem 2. In Section 2.1 of the lecture notes (lecture 2), we introduced a trichotomy for solution sets of a system of linear equations. For each of **Cases 1**, **Case 2**, and **Case 3** from that section, give an example of a linear system of equations with two equations and two variables x and y. Plot and label the lines.

Problem 3. Consider a 2×2 matrix $\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$, where $a_{11}, a_{12}, a_{21}, a_{22} \in \mathbb{R}$ and assume that a_{12} and a_{22} are both nonzero. Show that the following are equivalent:

(i) The determinant[1] of the matrix is nonzero.

(ii) The linear system $\begin{cases} a_{11}x + a_{12}y = b_1 \\ a_{21}x + a_{22}y = b_2 \end{cases}$ has exactly one solution.

Problem 4 (Row reduction)**.** Solve the system of equations using row reduction

$$\begin{aligned} x + y - z &= 0 \\ 2x + 3y - 2z &= 6 \\ x + 2y + 2z &= 10 \end{aligned}$$

Interpret your result geometrically. Provide a sketch or an image (e.g., using Desmos) of the the solution.

Problem 5 (Row reduction)**.** Solve the system of equations using row reduction

$$\begin{aligned} 2x + y - 2z &= 0 \\ 2x - y - 2z &= 0 \\ x + 2y - 4z &= 0 \end{aligned}$$

Interpret your result geometrically. Provide a sketch or an image (e.g., using Desmos) of the the solution.

Problem 6 (Row reduction)**.** Solve the system of equations using row reduction

$$\begin{aligned} 2x + 3y - 4z &= 3 \\ 2x + 3y - 2z &= 3 \\ 4x + 6y - 2z &= 7 \end{aligned}$$

Interpret your result geometrically. Provide a sketch or an image (e.g., using Desmos) of the the solution.

[1] The *determinant* of the 2×2 matrix $\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ is the quantity $a_{11}a_{22} - a_{12}a_{21}$.

Problem 7. Define the following matrices

$$A = \begin{bmatrix} 1 & 2 \\ 3 & -1 \\ 2 & -1 \end{bmatrix} \quad B = \begin{bmatrix} 2 & -1 \\ -3 & -2 \\ 0 & 4 \end{bmatrix} \quad C = \begin{bmatrix} 2 & -1 \\ 1 & 5 \end{bmatrix} \quad D = \begin{bmatrix} 0 & 1 \\ 3 & -1 \end{bmatrix}$$

$$E = \begin{bmatrix} 1 & -3 & 5 \\ 2 & 1 & -1 \\ 1 & 1 & 0 \end{bmatrix} \quad F = \begin{bmatrix} 1 & -1 & 4 \\ 2 & -3 & 6 \\ 1 & 0 & 1 \end{bmatrix}.$$

Either perform the indicated operation or state the the expression is undefined (e.g., because it asks you to add/multiply matrices whose dimenions don't allow for addition/multiplication):

(a) $D - 2C$

(b) $A + 2E$

(c) CD

(d) DC

(e) EF

(f) FE

(g) AE

(h) EA

(i) $B(C + D)$

(j) A^2