8:30 – 9:20, WAT 112
9:30 – 10:20, KEL 303

Section 2.4. Dimension; nullspace, rowspace and column space

Recall from last time:

If V is a vector space, a basis for V is a collection $v_1, \dots, v_n \in V$ such that:

- $v_1, \dots, v_n$ (span V) — means: any $v \in V$ can be written $v = c_1 v_1 + \dots + c_n v_n$ for some scalars c_i
- $v_1, \dots, v_n$ are (linearly independent) — means: if $c_1 v_1 + \dots + c_n v_n = 0$, then $c_1 = c_2 = \dots = c_n = 0$.

Examples:

- $\mathbb{R}^3$ has 'standard basis' $\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix}$ but there are many others e.g. $\begin{bmatrix}3\\3\\0\end{bmatrix}, \begin{bmatrix}4\\1\\6\end{bmatrix}, \begin{bmatrix}2\\3\\2\end{bmatrix}$ or $\begin{bmatrix}1\\0\\1\end{bmatrix}, \begin{bmatrix}1\\1\\1\end{bmatrix}, \begin{bmatrix}-1\\1\\1\end{bmatrix}$ from last lecture.

- $M_{m \times n}(\mathbb{R})$ (rows m, columns n) has basis $E_{11}, E_{12}, \dots, E_{1n}, E_{21}, \dots, E_{2n}, \dots, E_{m1}, \dots, E_{mn}$, where $E_{ij} = \begin{bmatrix} & & 0 & \\ & & \vdots & \\ 0 & \cdots & 1 & \\ & & & \end{bmatrix}$ (1 in row i, column j).
  Again, there are others, e.g. $\begin{bmatrix}1&0\\0&-1\end{bmatrix}, \begin{bmatrix}1&0\\0&1\end{bmatrix}, \begin{bmatrix}0&1\\1&0\end{bmatrix}, \begin{bmatrix}0&-1\\1&0\end{bmatrix}$ for $M_{2\times 2}(\mathbb{R})$

Theorem (see 2.9 in text – not obvious!)

If $v_1, \dots, v_n$ and $w_1, \dots, w_m$ are both bases for the same vector space V, then $m=n$.

Definition:

The dimension of a vector space is the number of elements in a basis. Notation for the dimension of V:

$$\boxed{\dim(V)}$$

Examples:

- $\dim(\mathbb{R}^3) = 3$, and more generally $\dim(\mathbb{R}^n)=n$
- $\dim(\mathbb{R}[x]_{\leq 2})=3$, and more generally $\dim(\mathbb{R}[x]_{\leq n}) = n+1$.

Comments:

- Some vector spaces have bases with infinitely many vectors
  (e.g. a basis for $\mathbb{R}[x]$ ← all polynomials is $1, x, x^2, x^3, \ldots$)
  In this case, we say the dimension of V is infinite
  Notation: $\dim(V) = \infty$.

  (In this course, you will mainly focus on finite dimensional vector spaces).

- If V is the 0 vector space, we write $\dim(V)=0$.

Some important properties of dimension (see 2.11 and 2.12):

Let V be a vector space with $\dim(V) = n$. Then:

1) If $v_1, \dots, v_k \in V$ are linearly independent, then $k \le n$ and there are $v_{k+1}, \dots, v_n \in V$ with $v_1, \dots, v_k, v_{k+1}, \dots, v_n$ a basis.
"lin. ind. collections cannot be too big"

2) If $v_1, \dots, v_k \in V$ span V, then $k \ge n$, and some collection of n vectors from $v_1, \dots, v_k$ is a basis.
"spanning collections cannot be too small"

Consequences:

- If $v_1, \dots, v_n$ span V, they are a basis.
- If $v_1, \dots, v_n$ are linearly independent, they are a basis.

Example:

Which (if any) of the following collections is a basis for $\mathbb{R}^3$?

a) $\begin{bmatrix}1\\2\\3\end{bmatrix}, \begin{bmatrix}4\\5\\6\end{bmatrix}$

b) $\begin{bmatrix}1\\2\\3\end{bmatrix}, \begin{bmatrix}4\\5\\6\end{bmatrix}, \begin{bmatrix}7\\8\\9\end{bmatrix}$

c) $\begin{bmatrix}1\\2\\3\end{bmatrix}, \begin{bmatrix}4\\5\\6\end{bmatrix}, \begin{bmatrix}7\\8\\9\end{bmatrix}, \begin{bmatrix}10\\11\\12\end{bmatrix}$

Solutions:

a) No: has too few vectors (so cannot span).

c) No: has too many vectors (so cannot be linearly independent).

not all of c_1, c_2, c_3 zero

b) Need to check if $c_1 \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + c_2 \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} + c_3 \begin{bmatrix} 7 \\ 8 \\ 9 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ has non-trivial solutions,

i.e. if $\begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ has non-trivial solutions.

Check linear independence by row reducing:

$$\begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 4 & 7 \\ 0 & -3 & -6 \\ 0 & -6 & -12 \end{bmatrix} \quad \begin{matrix} R2 - 2R1 \\ R3 - 3R1 \end{matrix}$$

$$\longrightarrow \begin{bmatrix} 1 & 4 & 7 \\ 0 & -3 & -6 \\ 0 & 0 & 0 \end{bmatrix} \quad R3 - 2R2$$

As we have a row of zeros, there are no conditions on c_3, and we see that there are infinitely many solutions

Conclude: the collection is not a basis (as not linearly independent)

Null space, row space, column space

Let $A \in M_{m \times n}(\mathbb{R})$

number rows, number columns

There are three important vector spaces associated with A:

- The column space, which is the subspace of $M_{m \times 1}(\mathbb{R})$ spanned by the columns of A.

  Notation: $\boxed{CS(A)}$

- The row space, which is the subspace of $M_{1 \times n}(\mathbb{R})$ spanned by the rows of A

  Notation: $\boxed{RS(A)}$

- The null space which is the subspace of $\mathbb{R}^n$ of vectors x such that $Ax = 0$.

  Notation: $\boxed{NS(A)}$

Example:

Let $A = \begin{bmatrix} 1 & 0 & -1 & 3 \\ 2 & 1 & 4 & -1 \\ 4 & 1 & 2 & 5 \end{bmatrix} \in M_{3\times 4}(\mathbb{R})$

find bases for:

a) $RS(A)$

b) $NS(A)$

Solution:

(a) WRONG: [1 0 -1 3], [2 1 4 -1], [4 1 2 5)

(Idea: row reduction does not change row space, so row reduce until we get a linearly independent set)

The reduced row echelon form is: $\begin{bmatrix} 1 & 0 & -1 & 3 \\ 0 & 1 & 6 & -7 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ (*)

So: a basis is $\boxed{[1\ 0\ -1\ 3],\ [0\ 1\ 6\ 7]}$

b) Can read off solution from (*):

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} x_3 - 3x_4 \\ -6x_3 + 7x_4 \\ x_3 \\ x_4 \end{bmatrix} = x_3 \begin{bmatrix} 1 \\ -6 \\ 0 \\ 1 \end{bmatrix} + x_4 \begin{bmatrix} -3 \\ 7 \\ 0 \\ 1 \end{bmatrix}$$

So: a basis is $\boxed{\begin{bmatrix} 1 \\ -6 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -3 \\ 7 \\ 0 \\ 1 \end{bmatrix}}$

Notes:

- To find $CS(A)$, take transpose, row reduce to find basis for $RS(A^T)$, then transpose back.
- For $A \in M_{m \times n}(A)$, the dimensions satisfy these relations:

$$\boxed{\dim(RS(A)) = \dim(CS(A))}$$

$$\boxed{\dim(CS(A)) + \dim(NS(A)) = n}$$

(see pages 102-3 in text).