

## 37 2025-11-19 | Week 13 | Lecture 37

topic: continuation of introduction to systems of linear differential equations

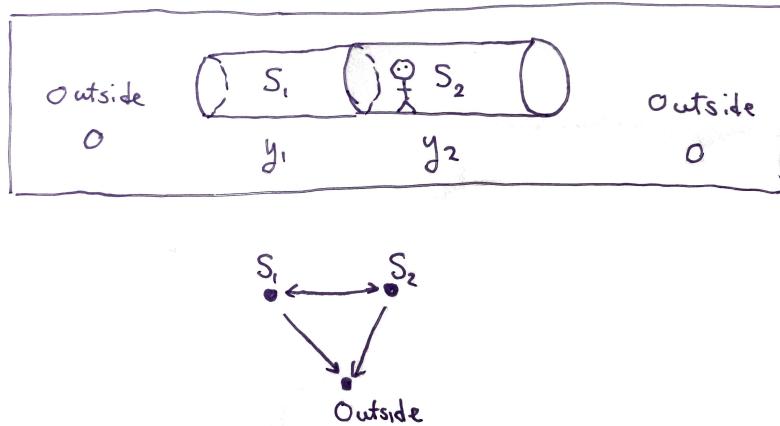
**Theorem 166** (Pure exponential solutions). Let  $A$  be an  $n \times n$  matrix, let  $v$  be an eigenvector of  $A$  with eigenvalue  $\lambda$ , and let  $c$  be any scalar. Then

$$Y(t) = ce^{\lambda t}v$$

is a solution of the system  $Y' = AY$ .

### 37.1 Diffusion example

**Example 167** (Diffusion). A person is standing in a large pipe drops a flask of chemicals, releasing a chemical gas. The tube has 2 sections,  $S_1$  and  $S_2$ , as shown:



Let

$$\begin{aligned} y_1(t) &= \text{concentration of chemical in section } S_1 \text{ at time } t \\ y_2(t) &= \text{concentration of chemical in section } S_2 \text{ at time } t \end{aligned}$$

and assume that the concentration outside is always zero (the gas gets immediately blown away by the wind).

We make the following assumption:

**At each time  $t$ , the diffusion rate between adjacent areas is the difference in concentrations.**

Based on this assumption,  $y_1$  (the concentration in  $S_1$ ) is changing due to two factors:

- the diffusion to the outside, which is  $(0 - y_1)$
- the diffusion into or out of  $S_2$ , which is  $(y_2 - y_1)$ .

Hence

$$\frac{dy_1}{dx} = (y_2 - y_1) + (0 - y_1) = -2y_1 + y_2.$$

Similar reasoning implies that

$$\frac{dy_2}{dx} = (y_1 - y_2) + (0 - y_2) = y_1 - 2y_2.$$

This gives us the following system of differential equations:

$$\begin{aligned} \frac{dy_1}{dt} &= -2y_1 + y_2 \\ \frac{dy_2}{dt} &= y_1 - 2y_2 \end{aligned}$$

Writing this in matrix form gives the matrix equation

$$\begin{bmatrix} y'_1(t) \\ y'_2(t) \end{bmatrix} \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} \quad (31)$$

This is a differential equation of the form

$$Y' = AY.$$

Observe that  $A$  has eigenvalue  $-1$  with eigenvector  $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$  and eigenvalue  $-3$  with eigenvector  $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ . Therefore by Theorem 166, our system of differential equations has solutions two pure exponential solutions:

$$Y_1(t) = c_1 e^{-t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad \text{and} \quad Y_2(t) = c_2 e^{-3t} \begin{bmatrix} 1 \\ -1 \end{bmatrix},$$

where  $c_1, c_2$  are arbitrary constants. By superposition, the general solution is

$$Y(t) = c_1 e^{-t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 e^{-3t} \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

We can write this as

$$Y(t) = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} e^{-t} & 0 \\ 0 & e^{-3t} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}. \quad (32)$$

Further, suppose we are given the initial condition

$$Y(0) = B = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}.$$

(For our problem, a reasonable initial condition would be  $Y(0) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ , since the chemical gas gets released in section 2 of the pipe at time  $t = 0$ , but it hasn't reached  $S_1$  yet.)

Plugging  $t = 0$  into Eq. (32) gives

$$B = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

or equivalently

$$\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}^{-1} B$$

Plugging this back into Eq. (31) yeilds

$$Y(t) = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} e^{-t} & 0 \\ 0 & e^{-3t} \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}^{-1} B$$

Writing

$$P = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}, \quad \Lambda = \begin{bmatrix} -1 & 0 \\ 0 & -3 \end{bmatrix}, \quad \text{and} \quad e^{\Lambda t} = \begin{bmatrix} e^{-t} & 0 \\ 0 & e^{-3t} \end{bmatrix},$$

we see that our solution takes the form

$$Y(t) = P e^{\Lambda t} P^{-1} B. \quad (33)$$

If we take our initial condition to be  $Y(0) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ , as noted earlier, then we have  $b_1 = 0$  and  $b_2 = 1$ . In that case, the matrix multiplication gives

$$\begin{aligned} Y(t) &= P e^{\Lambda t} P^{-1} B \\ &= \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & -3 \end{bmatrix} \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\ &= \frac{1}{2} \begin{bmatrix} e^{-t} + e^{-3t} & e^{-t} - e^{-3t} \\ e^{-t} - e^{-3t} & e^{-t} + e^{-3t} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} e^{-t} - e^{-3t} \\ e^{-t} + e^{-3t} \end{bmatrix}. \end{aligned}$$

In other words, at time  $t$ , the concentration of gas in  $S_1$  is  $y_1(t) = e^{-t} - e^{-3t}$  and the concentration of gas in  $S_2$  is  $y_2(t) = e^{-t} + e^{-3t}$ . One can check that this satisfies

$$\begin{cases} Y'(t) = \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix} Y(t) \\ Y(0) = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \end{cases}$$

(check this!), so it is a solution to the differential equation. Physically, it also makes sense, since it indicates that both concentrations are positive and tend to 0 as  $t \rightarrow \infty$  as the gas diffuses. From the equations we can also see that diffusion causes the gas concentration to decay exponentially in time.

End of Example 167.  $\square$

### 37.2 General solution to $Y' = AY$

In Example 167 we obtained the fundamental solution shown in Eq. (33). The form that it took illustrate the following theorem:

**Theorem 168.** *If  $A$  can be diagonalized  $A = P\Lambda P^{-1}$ , then*

$$Y = Pe^{\Lambda t}P^{-1}B$$

*is a solution to the differential equation*

$$\begin{cases} Y' = Ay \\ Y(0) = B \end{cases}$$

**Remark 169.** *In the above theorem, the columns of  $P$  are the eigenvectors  $v_1, \dots, v_n$  of  $A$ . So the formula is for the solution can be written as*

$$Y(t) = \begin{bmatrix} | & & | \\ v_1 & \cdots & v_n \\ | & & | \end{bmatrix} \begin{bmatrix} e^{\lambda_1 t} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & e^{\lambda_n t} \end{bmatrix} P^{-1}Y(0),$$

*Another way to write the solution: Let  $c = (c_1, \dots, c_n)^\top = P^{-1}Y(0)$ , in which case we get*

$$Y(t) = c_1 e^{\lambda_1 t} v_1 + \dots + c_n e^{\lambda_n t} v_n.$$

End of Remark 169.  $\square$

**Remark 170.** *When  $A$  is diagonalizable (as is assumed in Theorem 175), we have*

$$Pe^{\Lambda t}P^{-1} = e^{At}$$

*so the formula for the solution can be written in the wonderfully simple form*

$$Y(t) = e^{At}Y(0).$$

*In fact, this formula holds even if  $A$  isn't diagonalizable, but to understand that case, we'll need to first understand what  $e^{At}$  even means.*

End of Remark 170.  $\square$