41 2025-12-05 | Week 15 | Lecture 41

What do solutions to systems of first order linear differential equations look like?

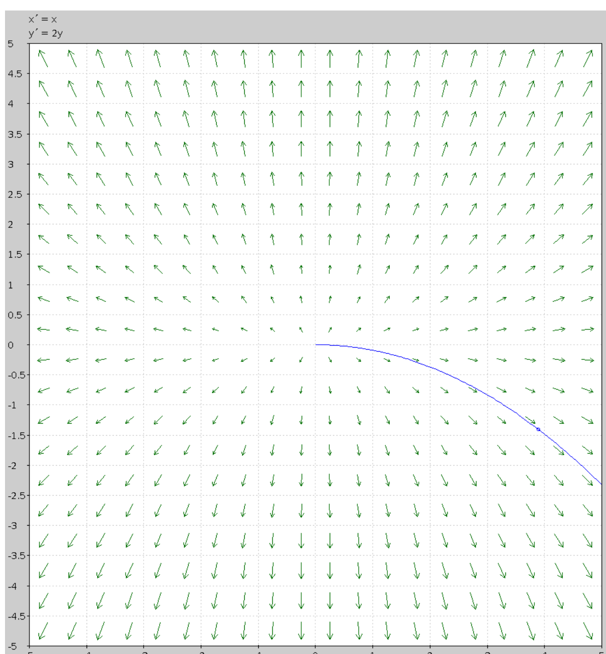
In the following examples, pay attention to two things:

- A ***nullcline*** is a region in space where either $\frac{dx}{dt} = 0$ or $\frac{dy}{dt} = 0$. When the derivatives take the form $ax + by$ (as they do for $Y' = AY$), for $a, b \neq 0$, these will always be lines.
- The eigenspaces $E_{\lambda_1}, E_{\lambda_2}$ of the matrix A. For real eigenvalues of 2×2 matrices, these are usualy lines. If the flow starts on one of these lines, it will never leave it. Depending on whether the eigenvalue is positive or negative, the flow along the line will be either:
  - attracted to the origin if $\lambda < 0$; or,
  - repelled from the origin if $\lambda > 0$.
- A point at the origin will always stay at the origin. The system $Y' = AY$ is "stable" if points starting near the origin either (1) get pulled into the origin, or at least (2) stay near the origin.

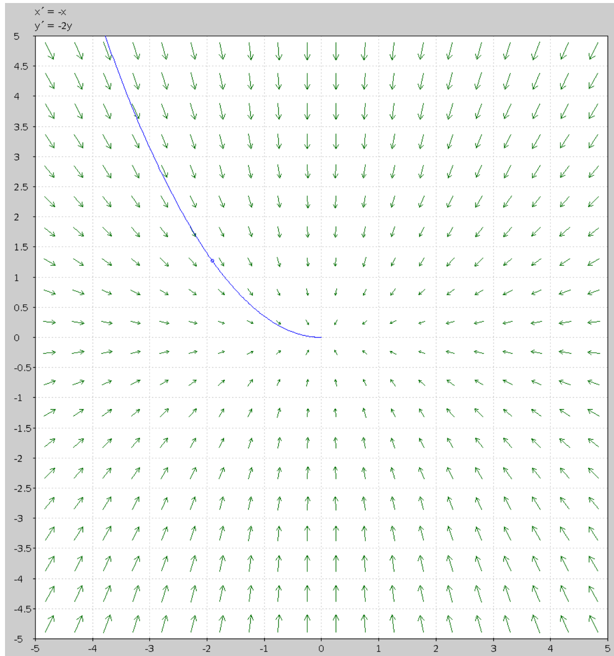
Example 179. Here are some slope field plots for then $Y' = AY$ for various values of A.

The first four examples have straight-line solutions $y = 0$ and $x = 0$, and eigenvectors $\begin{bmatrix}1\\0\end{bmatrix}$ and $\begin{bmatrix}0\\1\end{bmatrix}$

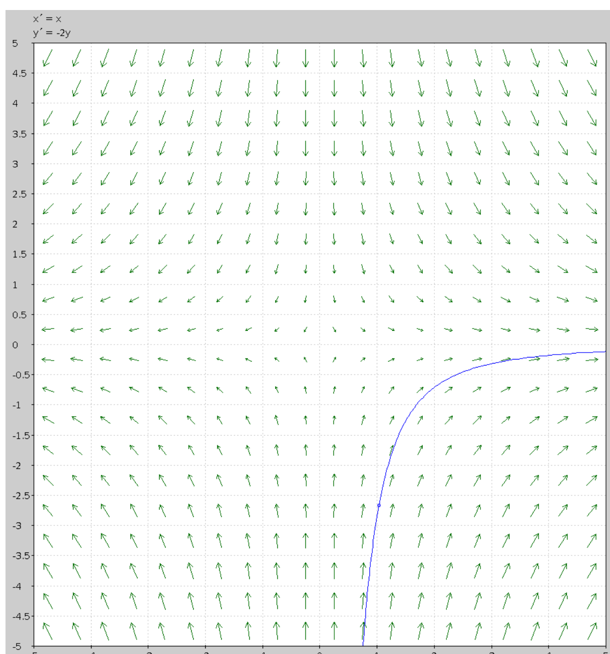
$$A = \begin{bmatrix}1 & 0\\0 & 2\end{bmatrix}$$



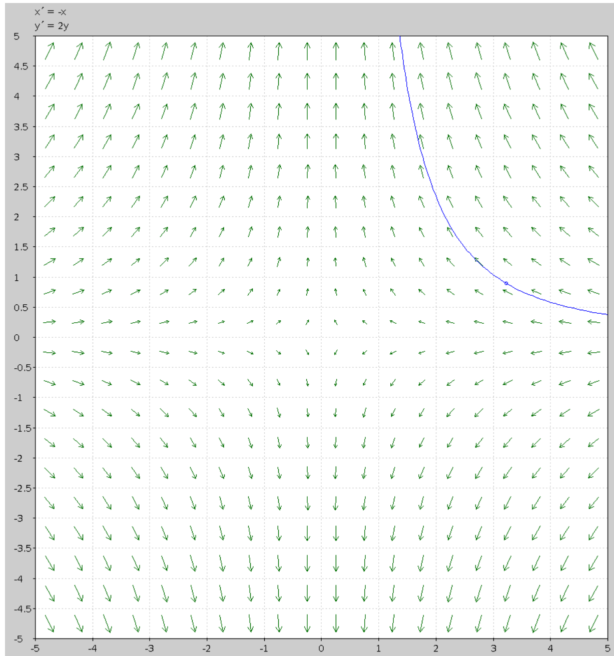
$$A = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix}$$



$$A = \begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix}$$



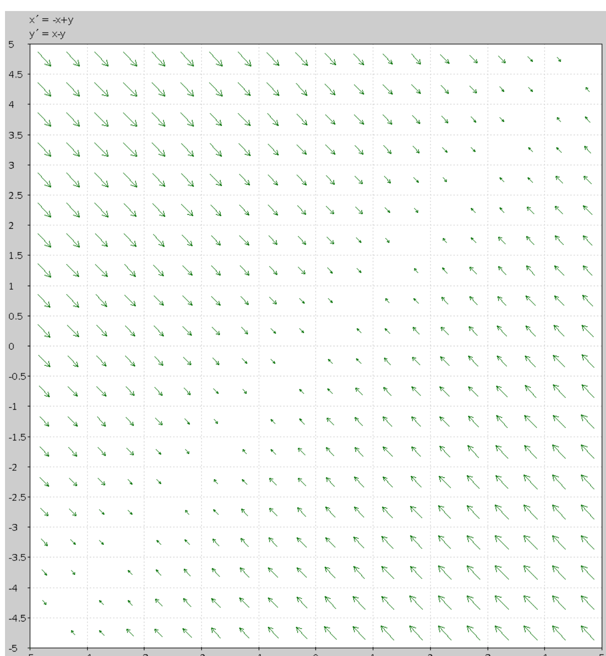
$$A = \begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix}$$



$$A = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}$$

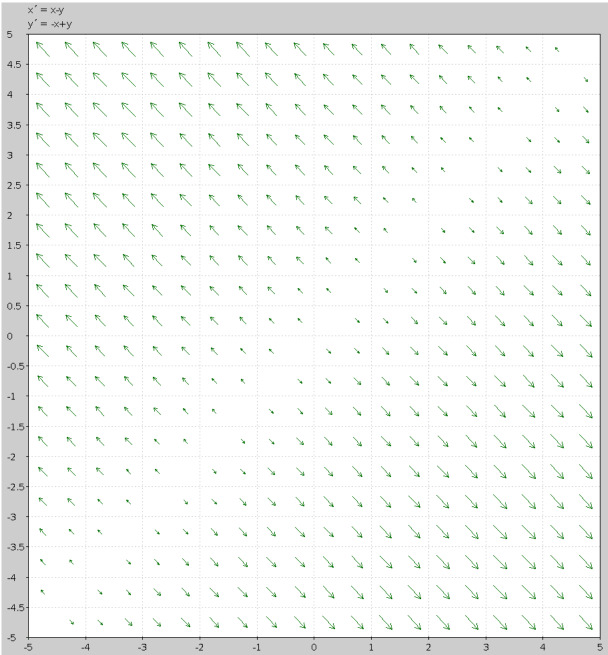
This is diffusion between two closed spaces. No matter the initial concentrations, they y_1, y_2 approach a steady state as $t \to \infty$.

If we start at $Y(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ The solution is $Y(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} \frac{1}{2} + \frac{1}{2}e^{-2t} \\ \frac{1}{2} - \frac{1}{2}e^{-2t} \end{bmatrix} \to \begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix}$



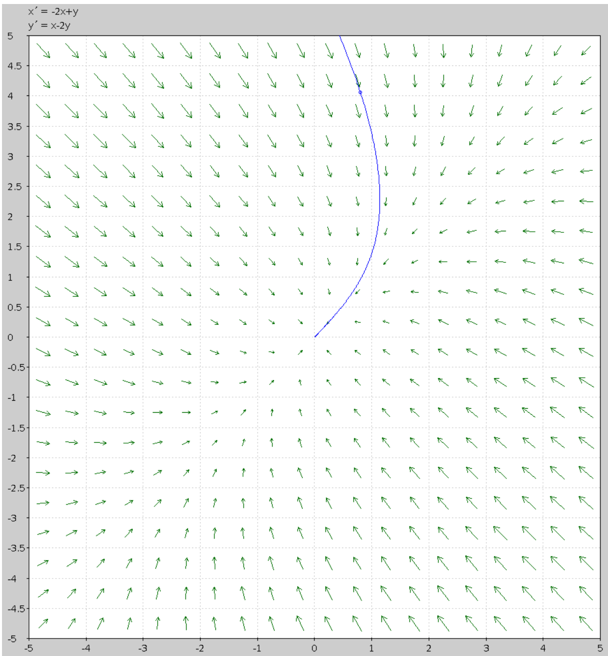
If we reverse time in the diffusion, the matrix is

$$A = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$



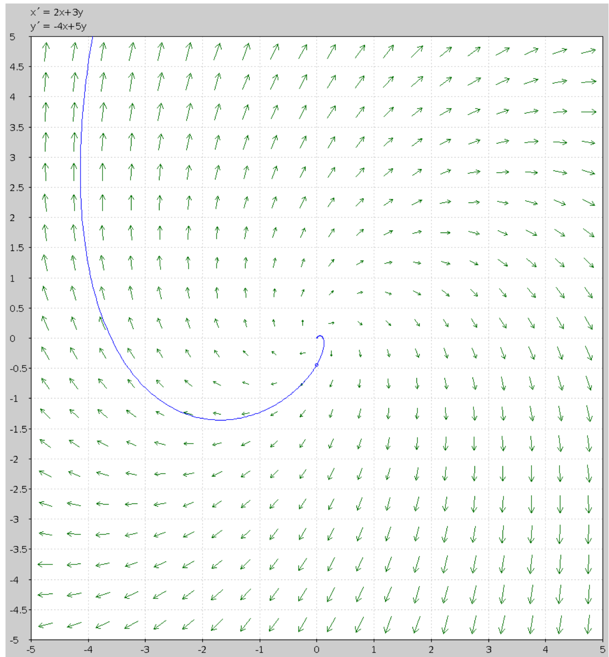
$$A = \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix}$$

Both this example and the last have eigenvectors $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$. Note the straight-line solutions $y = x$ and $y = -x$



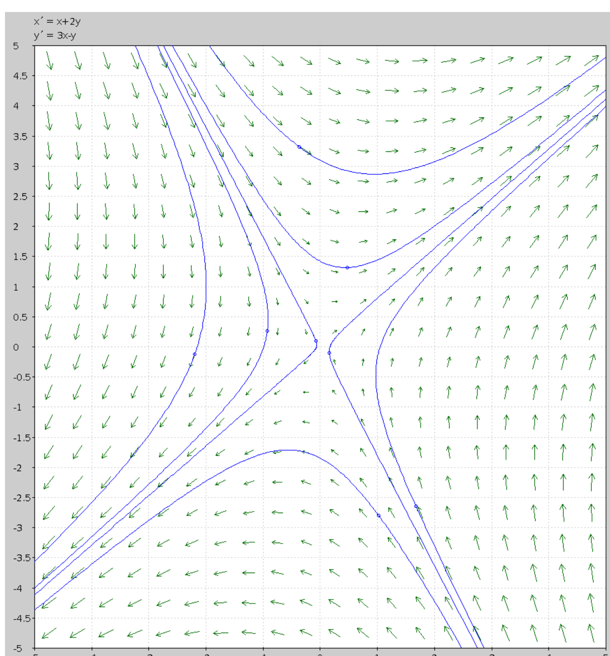
$$A = \begin{bmatrix} 2 & 3 \\ -4 & 5 \end{bmatrix}$$

This has complex eigenvalues.



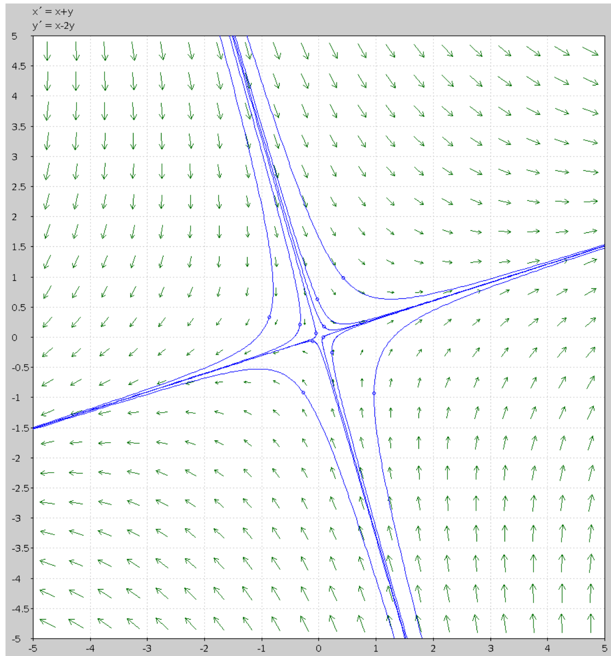
$$A = \begin{bmatrix} 1 & 2 \\ 3 & -1 \end{bmatrix}$$

This has eigenvectors $\begin{bmatrix} \pm(\sqrt{7}+1)/2 \\ 1 \end{bmatrix}$ and the straight line solutions are the respective spans of these two vectors, which are the approximately lines $y = .82x$ and $y = -.82x$.



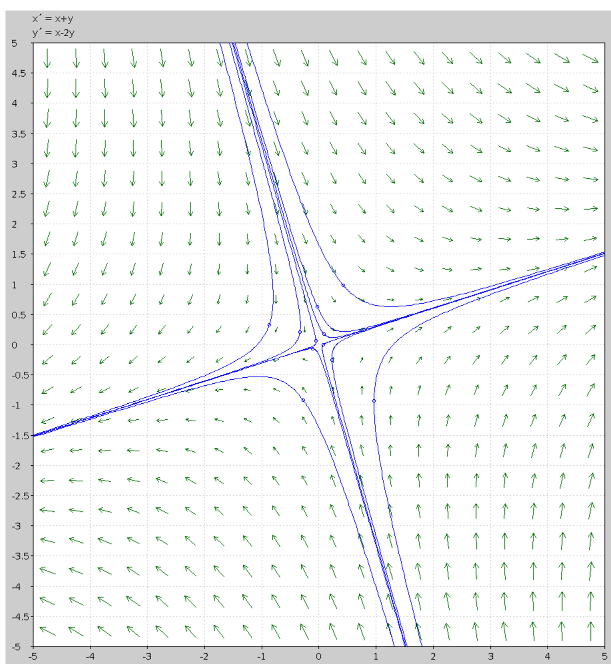
$$A = \begin{bmatrix} 1 & 1 \\ 1 & -2 \end{bmatrix}$$

Guess what the spans of each of the eigenvectors are?



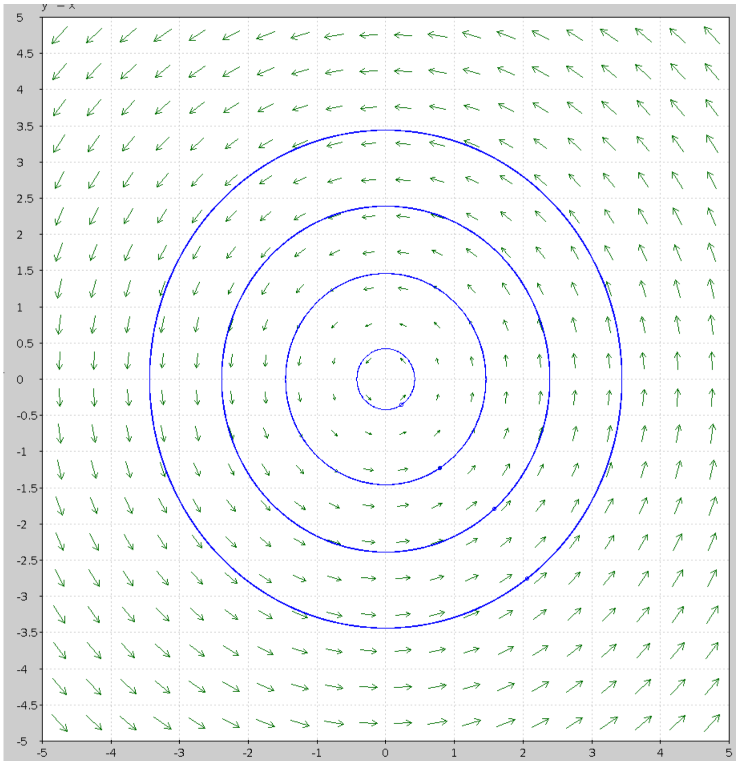
$$A = \begin{bmatrix} -1 & -1 \\ -1 & -1 \end{bmatrix}$$

Guess what the spans of each of the eigenvectors are?



End of Example 179. □

Example 180. Let $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ then $Y' = AY$ has slope field:



Solution: The eigenvalues are $\lambda = i$ and $-i$, with eigenvectors $\begin{bmatrix} 1 \\ -i \end{bmatrix}, \begin{bmatrix} 1 \\ i \end{bmatrix}$. The general solution is

$$Y(t) = c_1 \begin{bmatrix} 1 \\ -i \end{bmatrix} e^{it} + c_2 \begin{bmatrix} 1 \\ i \end{bmatrix} e^{-it}$$

Suppose the initial condition is $Y(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. Plugging $t = 0$ into the above equation gives $c_1 = c_2 = \frac{1}{2}$. Therefore

$$Y(t) = \frac{1}{2} \begin{bmatrix} 1 \\ -i \end{bmatrix} e^{it} + \frac{1}{2} \begin{bmatrix} 1 \\ i \end{bmatrix} e^{-it}$$

To obtain a real solution, we use the formulas

$$e^{it} = \cos(t) + i\sin(t) \quad \text{and} \quad e^{-it} = \cos(-t) + i\sin(-t) = \cos(t) - i\sin(t).$$

This gives

$$Y(t) = \begin{bmatrix} \cos(t) \\ \sin(t) \end{bmatrix}$$

which parametrizes the unit circle.

End of Example 180. □