38 2025-11-26 | Week 14 | Lecture 38

Please read chapters 6.1 and 6.2

38.1 Recap of first order linear differential equations

Recall from the previous two lectures, we considered differential equations of the form

$$Y' = AY \tag{34}$$

where $Y = \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix}$ is a function of time t and A is an $n \times n$ matrix.

Eq. (34) is a system of n equations, each taking the form

$$\frac{dy_i}{dt} = a_{i1}y_1(t) + a_{i2}y_2(t) + \ldots + a_{in}y_n(t), \quad (i = 1, 2, \ldots, n).$$

Differential equations of the form given in Eq. (34) are called ***homogeneous systems of first order linear differential equations.***

A system is ***inhomogeneous*** if takes the form

$$Y' = AY + G \tag{35}$$

for some nonzero vector $G = G(t) = \begin{bmatrix} g_1(t) \\ \vdots \\ g_n(t) \end{bmatrix}$ is not the zero vector. So far we have focused on the homogenous case.

Remark 171. *The textbook uses the variable name x rather than t, but I like to think of it as time, so I'll use t in my lectures. One can also allow the matrix A to be a function of t, and this is done in the textbook, but I'd rather avoid this more general setting for now.*

End of Remark 171. □

Applications: We saw two physical interpretations of Eq. (34):

- as the position of a moving particle (or ant)
- as the concentration of a chemical diffusing between several discrete locations

We also saw that, given the ***initial condition*** $Y(0) = B$, the solution to the differential equation in Eq. (34) takes the form

$$Y(t) = e^{At}B.$$

This describes either the position of the particle at time t, or the concentrations of the chemical in the n compartements at time t.

When A is diagonalizable (that is, when $A = P\Lambda P^{-1}$), the above solution can be written as

$$Y(t) = Pe^{\Lambda t}P^{-1}B.$$

where $\Lambda = \mathrm{diag}(\lambda_1, \ldots, \lambda_n)$ and $e^{\Lambda t} = \mathrm{diag}(e^{\lambda_1 t}, \ldots, e^{\lambda_n t})$. This was the content of Theorem 168.

Next we discuss what e^{At} really means.

38.2 The matrix exponential

In Example 164, we wrote down $e^{\Lambda t}$ but Λt is a matrix. So we need to define e^M where M is a matrix.

If x is a real number, then the power series of e^x is

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots = \sum_{k=0}^{\infty} \frac{x^k}{k!}$$

If M is a square matrix, then we define the ***exponential of a matrix*** e^M as

$$e^M = I + M + \frac{M^2}{2!} + \frac{M^3}{3!} + \ldots = \sum_{k=0}^{\infty} \frac{M^k}{k!}$$

kk Using this definition we have two cases:

- If $M = \Lambda$ is diagonal $\begin{bmatrix} \lambda_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \lambda_n \end{bmatrix}$ then it's easy:

$$e^{\Lambda t} = \begin{bmatrix} e^{\lambda_1 t} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & e^{\lambda_n t} \end{bmatrix}$$

- If M is not a diagonal matrix, computing the exponential is harder. For example, if $M = At$, we have

$$e^{At} = I + At + \frac{(At)^2}{2!} + \frac{(At)^3}{3!} + \ldots$$

This may be hard to compute in general. Nonetheless, this series always converges, and satisfies:

  * $e^{At}e^{As} = e^{A(t+s)}$
  * $e^{At}e^{-tA} = I$ (so e^{At} is always invertible)
  * $\frac{d}{dt}\left[e^{At}\right] = Ae^{At}$

The next proposition says that for diagonalizable matrices, it is possible to compute the matrix exponential without too much difficulty (need to compute some eigenvalues and eigenvectors, and a matrix inverse).

Proposition 172. *If A is diagonalizable, then $e^{At} = Pe^{\Lambda t}P^{-1}$, where $\Lambda = \operatorname{diag}(\lambda_1, \ldots, \lambda_n)$ and $\lambda_1, \ldots, \lambda_n$ is the list of eigvenvalues of A.*

Proof. Since A is diagonal, we can write $A = P\Lambda P^{-1}$, where P is a matrix whose columns are eigenvectors of A. Using this, we can write

$$A^k = \left(P\Lambda P^{-1}\right)^k = P\Lambda^k P^{-1}.$$

(see Section 34.2 for details). Then

$$\begin{aligned} e^{At} &= \sum_{k=0}^{\infty} \frac{(At)^k}{k!} \\ &= \sum_{k=0}^{\infty} \frac{\left(P\Lambda^k P^{-1}\right) t^k}{k!} \\ &= P\left(\sum_{k=0}^{\infty} \frac{\Lambda^k t^k}{k!}\right) P^{-1} \\ &= P\left(\sum_{k=0}^{\infty} \frac{(\Lambda t)^k}{k!}\right) P^{-1} \\ &= Pe^{\Lambda t}P^{-1}. \end{aligned}$$

□