32 2025-11-07 | Week 11 | Lecture 32

> ***The nexus question of this lecture:*** *How can we test whether a matrix is diagonalizable?*

Definition 142. Given a matrix A with eigenvalue λ, the ***eigenspace*** corresponding to λ is the subspace $NS(\lambda I - A)$. We denote this by E_λ.

Remark 143. E_λ is the subspace spanned by the eigenvectors corresponding to λ. It consists of all eigenvectors for λ plus the zero vector.

The following theorem gives a criterion for determining whether a matrix is diagonalizable.

Theorem 144 (Eigenspace dimension criterion)**.** *Suppose A is an $n \times n$ matrix with distinct eigenvalues $r_1, \ldots, r_k$. Let E_r be the eigenspace of r. Then A is diagonalizable if and only if*

$$\dim(E_{r_1}) + \dim(E_{r_2}) + \ldots + \dim(E_{r_k}) = n.$$

Remark 145. The core idea of Theorem 144 is this: in order for a matrix to be diagonalizable, it has to have enough eigenvectors.

The next two examples will illustrate the use of Theorem 144. In both examples, I'm going to skip most of the computational details.

Example 146 (Example 3 in section 5.4)**.** Let $A = \begin{bmatrix} 2 & -1 & 3 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ Then

$$A \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = 2 \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

and

$$A \begin{bmatrix} 1/3 \\ 1 \\ 0 \end{bmatrix} = (-1) \begin{bmatrix} 1/3 \\ 1 \\ 0 \end{bmatrix} \quad \text{and} \quad A \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} = (-1) \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

This shows that E_2 is spanned by $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ and E_{-1} is spanned by $\left\{ \begin{bmatrix} 1/3 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\}$.

Therefore by Theorem 144,

$$\dim(E_2) + \dim(E_{-1}) = 1 + 2 = 3$$

and hence A is diagonalizable.

We can say a bit more: we actually now have enough information to diabonalize A. In particular, we can take our 3 linearly independent eigenvectors and form a "prism" matrix P (technically, a change-of-basis matrix):

$$P = \begin{bmatrix} 1 & 1/3 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Since the three columns are linearly independent, the key theorem implies that P is invertible. Computing the inverse P^{-1} (not shown), it then follows that

$$P^{-1}AP = \begin{bmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}.$$

The right-hand side is a diagonal matrix, and so we have diagonalized A.

End of Example 146. $\square$

Example 147 (Example 4 in section 5.4)**.** The matrix $A = \begin{bmatrix} 1 & -2 & -6 \\ -2 & 2 & -5 \\ 2 & 1 & 8 \end{bmatrix}$ has characteristic polynomial

$$\det(\lambda I - A) = (\lambda - 3)^2(\lambda - 5).$$

Therefore A has two eigenvalues: $\lambda = 3$ and $\lambda = 5$. Computing E_3 and E_5, we obtain:

$$E_3 = \left\{ c \begin{bmatrix} -2 \\ -1 \\ 1 \end{bmatrix} : c \in \mathbb{R} \right\} \quad \text{and} \quad E_5 = \left\{ c \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} : c \in \mathbb{R} \right\}$$

Both E_3 and E_5 are one-dimensional subspaces of $\mathbb{R}^3$ (since they are lines passing through the origin). Therefore, since $\dim(E_3) + \dim(E_5) = 2 \neq 3$, Theorem 144 implies that A is not diagonalizable.

This makes sense: to diagonalize A, we need to form an invertible "prism" matrix P by taking 3 linearly independent eigenvectors of A. But there aren't three linearly independent eigenvectors, there are only two. So A isn't diagonalizable.

End of Example 147. □

Theorem 148 (The determinant is the product of eigenvalues)**.** *Let A be an $n \times n$ matrix, and let $\lambda_1, \ldots, \lambda_n$ be the eigenvalues of A (possibly with repetitions). Then*

$$\det(A) = \lambda_1 \lambda_2 \ldots \lambda_n.$$

Proof. Let A be an $n \times n$ matrix. Then $\det(A - \lambda I)$ is a polynomial of degree n in the variable λ. The eigenvalues $\lambda_1, \ldots, \lambda_n$ of A are the roots of this polynomial. Therefore by the fundamental theorem of algebra, we can write

$$\det(A - \lambda I) = (\lambda_1 - \lambda)(\lambda_2 - \lambda) \cdots (\lambda_n - \lambda)$$

Taking $\lambda = 0$ implies $\det(A) = \lambda_1 \lambda_2 \cdots \lambda_n$. □

The next lecture will address the topic of what can be done with matrices which are not diagonalizable.