14 2025-09-26 | Week 05 | Lecture 14

This lecture is based on section 2.2 in the text.

> ***The nexus question of this lecture:*** *How can we build linear subspaces from vectors?*

14.1 Linear Span

We begin by showing how to construct a linear subspace from a collection of vectors.

Definition 58 (Linear combination)**.** Let V be a vector space and let $v_1, \ldots, v_n \in V$. An expression of the form

$$c_1 v_1 + c_2 v_2 + \ldots + c_n v_n \quad (c_1, \ldots, c_n \in \mathbb{R})$$

is called a ***linear combination*** of $v_1, \ldots, v_n$. The linear combination with $c_1 = c_2 = \ldots = c_n = 0$ is called the ***trivial linear combination***. If at leat one of the c_i's is nonzero we say that the linear combination is ***nontrivial***.

Definition 59 (Span)**.** Given a set of vectors $S = \{v_1, \ldots, v_n\}$, the set of all their linear combinations is called the ***span of*** S, and is denoted $\mathrm{Span}(S)$. In set notation,

$$\begin{aligned} \mathrm{Span}(S) &= \mathrm{Span}\{v_1, \ldots, v_n\} \\ &= \{c_1 v_1 + \ldots + c_n v_n : c_1, \ldots, c_n \in \mathbb{R}\} \subseteq V \end{aligned}$$

Note that $\mathrm{Span}(S)$ always contains the point $\vec{0}$, which is achieved by the trivial linear combination.

If $\mathrm{Span}(S) = V$ then we say that S ***spans*** V. This means that every vector in V can be written as a linear combination of vectors in S.

Example 60. Is the vector

$$v = \begin{bmatrix} 2 \\ -5 \\ 1 \\ 10 \end{bmatrix} \quad \text{in} \quad \mathrm{Span}\left\{ \begin{bmatrix} 1 \\ -1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ -2 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \\ 3 \end{bmatrix} \right\}?$$

To check this, we need to determine if there exists constants $c_1, c_2, c_3 \in \mathbb{R}$ such that

$$c_1 \begin{bmatrix} 1 \\ -1 \\ 2 \\ 3 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ -2 \\ -1 \\ 2 \end{bmatrix} + c_3 \begin{bmatrix} -1 \\ 0 \\ 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 2 \\ -5 \\ 1 \\ 10 \end{bmatrix}.$$

If there exists at least one choice of c_1, c_2, c_3 such that the above holds, then v is in the span.

Converting this to a system of equations, we have

$$\begin{aligned} c_1 + c_2 - c_3 &= 2 \\ -c_1 - 2c_2 &= -5 \\ 2c_1 - c_2 + c_3 &= 1 \\ 3c_1 + 2c_2 + 3c_3 &= 10. \end{aligned}$$

We can solve this by setting up an augmented matrix and row-reducing. Doing this we get

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

which tells us that there is a solution: $(c_1, c_2, c_3) = (1, 2, 1)$. Hence v is in the span.

End of Example 60. □

Theorem 61. *If V is a vector space and $v_1, \ldots, v_n \in V$, then* $\operatorname{Span}(v_1, \ldots, v_n)$ *is a subspace of V.*

Proof. Follows from an application of Theorem 53. (You should check this – hw problem, probably). $\square$

Example 62. Let $S = \left\{ \begin{bmatrix} 1 \\ -2 \end{bmatrix}, \begin{bmatrix} 2 \\ -4 \end{bmatrix} \right\}$. Does S span $\mathbb{R}^2$? In other words, we are asking whether we write every vector $\begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^2$ as a linear combination of the form

$$c_1 \begin{bmatrix} 1 \\ -2 \end{bmatrix} + c_2 \begin{bmatrix} 2 \\ -4 \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$$

for some values of c_1 and c_2?

Converting this linear system into an augmented matrix, we have

$$\left[\begin{array}{cc|c} 1 & 2 & x \\ -2 & -2 & y \end{array} \right] \xrightarrow{R_2 + 2R_1 \to R_2} \left[\begin{array}{cc|c} 1 & 2 & x \\ 0 & 0 & y + 2x \end{array} \right]$$

From this form, we see that there is a solution if and only if $y = -2x$. So for example, if we pick the vector $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ then there is no solution, meaning we cannot find c_1, c_2 such that

$$c_1 \begin{bmatrix} 1 \\ -2 \end{bmatrix} + c_2 \begin{bmatrix} 2 \\ -4 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

Therefore $\operatorname{Span}(S) \neq \mathbb{R}^2$. (This answers the question, but we can push a little bit further.)

Looking at the reduced form of the augmented matrix, we see that there is a solution for any vector $\begin{bmatrix} x \\ y \end{bmatrix}$ with $y = -2x$ (that is, for any vector of the form $\begin{bmatrix} x \\ -2x \end{bmatrix}$). In particular, we can take $c_1 = x$ and $c_2 = 0$. Therefore such vectors are in $\operatorname{Span}(S)$. This tells us that

$$\operatorname{Span}(S) = \left\{ \begin{bmatrix} x \\ -2x \end{bmatrix} : x \in \mathbb{R} \right\}$$

This is the line $y = -2x$. Note that this line passes through the origin, which we know from the last lecture is one of the possible forms a subspace can take.

End of Example 62. $\square$

We've shown how to build a subspace using a set of vectors: namely, take the span of the vectors. This is nice, but insufficient. It is of interest to know what is the *minimal* number of vectors needed to build a given subspace? That is, how can we build a subspace with as few vectors as possible? And how can we know that we can't use fewer vectors? To answer these question, we need the notion of *linear independence*, which will be next lecture.