Theorem 127. *Let A be an $n \times n$ matrix. Then $\lambda \in \mathbb{C}$ is an eigenvalue of A if and only if* $\det(\lambda I - A) = 0$.

Definition 128. The ***characteristic equation of*** A is

$$\det(\lambda I - A) = 0.$$

When A is an $n \times n$ matrix, the left hand side of the characteristic equation is a polynomial in the variable λ of degree n, and is called the ***characteristic polynomial*** of A.

Example 129. Let

$$A = \begin{bmatrix} 1 & -3 \\ -2 & 2 \end{bmatrix}.$$

Then the characteristic polynomial of A is

$$\begin{aligned} \det(\lambda I - A) &= \det \begin{bmatrix} \lambda - 1 & 3 \\ 2 & \lambda - 2 \end{bmatrix} \\ &= (\lambda - 1)(\lambda - 2) - 6 \\ &= \lambda^2 - 3\lambda - 4 \\ &(\lambda - 4)(\lambda + 1) \end{aligned}$$

This is equal to zero if and only if $\lambda = 4$ or $\lambda = -1$. Therefore the eigenvalues of A are $\lambda = 4$ and $\lambda = -1$.

End of Example 129. □

28.3 How do we find eigenvectors?

Idea: First find the eigenvalues λ. Then for each eigenvalue λ, the eigenvectors are the nontrivial solutions of the homogeneous ssytem

$$(\lambda I - A)X = 0.$$

(This is a linear system which we can solve using row reduction.)

In other words, the eigenvectors are the nonzero vectors in the linear subspace

$$NS(\lambda I - A).$$

So we just need to compute a basis of this nullspace, which is called the ***eigenspace***. When we ask to find the eigenvalues, it is always enough to just compute the basis of the eigenspace.

Example 130. Find the eigenvectors of the matrix

$$A = \begin{bmatrix} 1 & -3 \\ -2 & 2 \end{bmatrix}$$

The equations we need to solve are

- **When $\lambda = 4$:** $4I - A = 0$ or

  $$\begin{bmatrix} 3 & 3 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

  Reducing find the nullspace is

  $$NS(4I - A) = \left\{ y \begin{bmatrix} -1 \\ 1 \end{bmatrix}, y \in \mathbb{R} \right\}$$

  Technically, all vectors in $NS(4I - A)$ are eigenvectors for $\lambda = 4$. To give a concrete example, we have eigenvector $v = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$.

End of Example 130. □

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Examples of computing eigenvectors and eigenvalues

29.1 Recall definitions

Definition 131 (Eigenvalue,eigenvector)**.** If A is an $n \times n$ matrix, an ***eigenvector*** of A is a nonzero column vector v such that

$$Av = \lambda v$$

for some scalar $\lambda \in \mathbb{C}$. The scalar λ is called an ***eigenvalue***.

Definition 132. The ***characteristic polynomial of*** A is

$$\det(\lambda I - A).$$

Theorem 133. *Let A be an $n \times n$ matrix. Then $\lambda \in \mathbb{C}$ is an eigenvalue of A if and only if* $\det(\lambda I - A) = 0$.

29.2 Examples of eigenvector/eigenvalue computations

Idea: Find the eigenvalues before finding the eigenvectors. Then for each eigenvalue λ, find the nullspace $NS(\lambda I - A)$. The vectors in the nullspace are the eigenvectors corresponding to λ. (Usually we just pick out a basis.)

Example 134. Find the eigenvectors of the matrix

$$A = \begin{bmatrix} 1 & -3 \\ -2 & 2 \end{bmatrix}$$

Last class, we showed that A has eigenvalues $\lambda = 4$ and $\lambda - 1$ The equations we need to solve are

- **When $\lambda = 4$:** we computed the nullsapce of the matrix $4I - A$, which gave

  $$NS(4I - A) = \left\{ y \begin{bmatrix} -1 \\ 1 \end{bmatrix}, y \in \mathbb{R} \right\}$$

  So $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$ is an eigenvector of A corresponding to $\lambda = 4$

- **When $\lambda = -1$:** $-I - A = 0$ Here we get a NS generated by a single basis element $\begin{bmatrix} 3 \\ 2 \end{bmatrix}$ so this is an eigenvector as well.

End of Example 134. □

Example 135. Find the eigenvaluess and eigenvectors of

$$A = \begin{bmatrix} 2 & -1 & 3 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

First we compute the characteristic polynomial:

$$\begin{aligned} \det(\lambda I - A) &= \det \begin{bmatrix} \lambda - 2 & 1 & -3 \\ 0 & \lambda + 1 & 0 \\ 0 & 0 & \lambda + 1 \end{bmatrix} \\ &= (\lambda - 2)(\lambda + 1)^2 \end{aligned}$$

This equals zero iff $\lambda = 2$ or $\lambda = -1$. These are the eigvenvalues.

To find eigenvectors, we need to find a basis for the nullspaces $NS(2I - A)$ and $NS(-I - A)$.

- $\lambda = 2$. Need to find $NS(2I - A)$.

  Row reducing the augmented matrix

$$\left[\begin{array}{ccc|c} 0 & 1 & -3 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 3 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

  This corresponds to the system with $x_2 = 0$, $x_3 = 0$, and x_1 a free variable. That is,

$$NS(2I - A) = \left\{ \begin{bmatrix} x_1 \\ 0 \\ 0 \end{bmatrix} : x_1 \in \mathbb{R} \right\} = \left\{ x_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} : x_1 \in \mathbb{R} \right\}$$

  So $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ is a basis for the nullspace. It is an eigenvector. Indeed, $A \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = 2 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$.

- $\lambda = -1$. Need to find $NS(-I - A)$.

End of Example 135. □