Math 307: Homework 09

Due Wednesday, November 12. Material up to and including 5.4.

Problem 1. Let $A = \begin{bmatrix} 1 & -1 \\ 2 & 4 \end{bmatrix}$. The *trace* of A is the sum of its main diagonal entries. Find the eigenvalues of A, verify that trace equals the sum of the eigenvalues, and that the determinant equals their product.

Problem 2. Find the eigenvectors and eigenvalues of

$$A = \begin{bmatrix} 3 & 4 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 0 & 0 & 2 \\ 0 & 2 & 0 \\ 2 & 0 & 0 \end{bmatrix}$$

Check that $\lambda_1 + \lambda_2 + \lambda_3$ equals the trace and that $\lambda_1\lambda_2\lambda_3$ equals the determinant. (Here $\lambda_1, \lambda_2, \lambda_3$ are the eigenvalues of the matrix.)

Problem 3. Find the rank and nullspace of

$$A = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 0 & 0 & 1 & 2 \\ 0 & 0 & 1 & 2 \\ 1 & 1 & 1 & 0 \end{bmatrix}$$

Problem 4. Suppose all vectors x in the unit square $0 \leq x_1 \leq 1$, $0 \leq x_2 \leq 1$ are transformed to Ax. (Here A is a 2×2 matrix).

(a) What is the shape of the transformed region (all Ax)?

(b) For which matrices A is that region a square?

(c) For which A is it a line?

(d) For which A is the new area still 1?

Problem 5. Given an eigenvalue λ of matrix A, the *eigenspace* of λ is the subspace $NS(\lambda I - A)$. For the following matrices, find the eigenvalues, and find a basis for each eigenspace.

(a) $\begin{bmatrix} 3 & 0 \\ 8 & -1 \end{bmatrix}$

(b) $\begin{bmatrix} 4 & -4 \\ 1 & 0 \end{bmatrix}$

(c) $\begin{bmatrix} 4 & 0 & 1 \\ -2 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}$

Problem 6. Let A, B, C be square matrices. Prove the following statements:

(a) If B is similar to A, then A is similar to B. ("Similarity is reflexive")

(b) If A is similar to B, and B is similar to C, then A is similar to C. ("Similarity is transitive")

(c) If A, B are similar, then $\det(A) = \det(B)$.

(d) If A is invertible and A is similar to B, then B is invertible and B^{-1} is similar to A^{-1}.

Problem 7. *(Hint for this problem: refer to the examples in section 5.1 of the textbook.)* Let $T : \mathbb{R}^3 \to P_2$ be a linear transformation satisfying

$$T\begin{bmatrix}1\\1\\0\end{bmatrix} = x^2 + x, \quad T\begin{bmatrix}1\\-1\\1\end{bmatrix} = x^2 - x + 1, \quad T\begin{bmatrix}0\\1\\1\end{bmatrix} = x + 1.$$

(a) Find $T\begin{bmatrix}1\\0\\0\end{bmatrix}$.

(b) Find $T\begin{bmatrix}a\\b\\c\end{bmatrix}$.

Problem 8.

(a) Construct a matrix whose nullspace contains the vector $x = \begin{bmatrix}1\\1\\2\end{bmatrix}$

(b) Construct a matrix whose column space is spanned by $\begin{bmatrix}1\\1\\2\end{bmatrix}$ and whose row space is spanned by $\begin{bmatrix}1 & 5\end{bmatrix}$.

(c) How can you construct a matrix that transforms the coordinate vectors e_1, e_2, e_3 into three given vectors v_1, v_2, v_3. When will that matrix be invertible?

Problem 9. Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be given by

$$T\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} = \begin{bmatrix}17x_1 - 8x_2 - 12x_3\\16x_1 - 7x_2 - 12x_3\\16x_1 - 8x_2 - 11x_3\end{bmatrix}.$$

(a) Find $[T]_\alpha^\alpha$, where α is the standard basis for $\mathbb{R}^3$.

(b) Let $\beta = \left\{\begin{bmatrix}1\\1\\1\end{bmatrix}, \begin{bmatrix}1\\2\\0\end{bmatrix}, \begin{bmatrix}1\\-1\\2\end{bmatrix}\right\}$. Find the change of basis matrix from α to β.

(c) Find the change of basis matrix from β to α.

(d) Find $[T]_\beta^\beta$.

(e) Find $[v]_\beta$ for $v = \begin{bmatrix}2\\-1\\4\end{bmatrix}$.

(f) Find $[T(v)]_\beta$.

(g) Use the result of part (f) to find $T(v)$.

Problem 10. A linear transformation from V to W has an *inverse* from W to V when the range of is all of W and the kernel contains only $v = 0$. Why are these transformations not invertible?

(a) $T(v_1, v_2) = (v_2, v_2), \quad W = \mathbb{R}^2$

(b) $T(v_1, v_2) = (v_1, v_2, v_1 + v_2), \quad W = \mathbb{R}^3$

(c) $T(v_1, v_2) = v_1, \quad W = \mathbb{R}^1$

Problem 11. Suppose $T : V \to W$ is a linear transformation.

(a) Show that if $v \in V$ and $u \in \ker(T)$, then $T(u + v) = T(v)$.

(b) Show that if $u, v \in V$ such that $T(u) = T(v)$, then $u - v$ is in $\ker(T)$.

(c) Use part (b) to deduce that if $\ker(T) = \{0\}$, then T is injective. [Note: a function f is *injective* if $x_1 \neq x_2$ implies $f(x_1) \neq f(x_2)$].

Problem 12 (An application of eigenvalues)**.** If $f : \mathbb{R}^2 \to \mathbb{R}$ is a twice-differentiable function of two variables, then the *Hessian matrix* of f is the matrix

$$H(x, y) = \begin{bmatrix} f_{xx}(x, y) & f_{xy}(x, y) \\ f_{yx}(x, y) & f_{yy}(x, y) \end{bmatrix}.$$

In this problem, we'll use the following theorem:

The Second Derivative Test

Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a twice-differentiable function of two variables, and let (a, b) be a critical point of f. Let λ_1 and λ_2 be the eigenvalues of $H(a, b)$. Then:

- If λ_1, λ_2 are both positive, then f has a minimum at $(x, y) = (a, b)$.
- If λ_1, λ_2 are both negative, then f has a minimum at $(x, y) = (a, b)$.
- If one of λ_1, λ_2 is negative and the other positive, then f has a saddle point at $(x, y) = (a, b)$.
- If one or more eigenvalue is zero, then the test is inconclusive.

For parts (a) and (b), find the critical points of $f(x, y)$ and classify them using the second derivative test stated here.

(a) Let $f(x, y) = x^2 + 4xy + y^2$.

(b) Let $f(x, y) = x^3 - 3x + y^2 - 4y$

Problem 13. For the following matrices, find the eigenvalues, and find a basis for each eigenspace.

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & -2 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 4 & 0 & 0 \\ 1 & 4 & 0 \\ 0 & 1 & 4 \end{bmatrix}$$

Problem 14. Find a basis for the following subspace of $\mathbb{R}^4$:

(a) The vectors for which $x_1 = 2x_4$.

(b) The vectors for which $x_1 + x_2 + x_3 = 0$ and $x_3 + x_4 = 0$.

(c) The subspace spanned by $\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 4 \\ 5 \end{bmatrix}$

Problem 15. Suppose a linear transformation T transforms $(1, 1)$ to $(2, 2)$ and $(2, 0)$ to $(0, 0)$. Find $T(v)$ when

(a) $v = (2, 2)$

(b) $v = (3, 1)$

(c) $v = (-1, 1)$

(d) $v = (a, b)$