16 2025-10-01: Week 06 | Lecture 16

> ***The nexus question of this lecture:*** *How can we build linear subspaces from a* ***minimal*** *set of vectors?*

16.1 Linear Indepdenence

Definition 66. Let V be a vector space and let $v_1, \ldots, v_n \in V$. We say that the set $\{v_1, \ldots, v_n\}$ is ***linearly dependent*** if there are scalars $c_1, c_2, \ldots, c_n$ *not all zero* such that

$$c_1 v_1 + c_2 v_2 + \ldots + c_n v_n = \vec{0}.$$

If the vectors $v_1, \ldots, v_n$ are not linearly dependent, we say that they are ***linearly independent.***

General method: To check any set of vectors $v_1, \ldots, v_n$ for independence, put them in the columns of A Then solve the system $Ac = \vec{0}$. The vectors are dependent if there is a solution other than $c = \vec{0}$

Example 67. Are the vectors

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} \right\}$$

linearly independent?

Solution: We need to check if the linear system

$$\begin{bmatrix} 1 & 1 & -1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

has any nontrivial solutions.

Using row reduction, we have

$$\left[\begin{array}{ccc|c} 1 & 1 & -1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \end{array} \right]$$

can be row reduced to

$$\left[\begin{array}{ccc|c} 1 & 1 & -1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 2 & 0 \end{array} \right]$$

We could go further, but this is sufficient. Converting back to a system of equations,

$$\begin{aligned} c_1 + c_2 - c_3 &= 0 \\ c_2 + c_3 &= 0 \\ 2c_3 &= 0 \end{aligned}$$

Back substitution gives $c_1 = c_2 = c_3 = 0$. Thus, the only solution to the linear system is the trival solution. Therefore the set of vectors is linearly independent.

End of Example 67. □

Example 68. Show that the columns of the following matrix are linearly independent:

$$\begin{bmatrix} 3 & 4 & 2 \\ 0 & 1 & 5 \\ 0 & 0 & 2 \end{bmatrix}$$

Look for a linear combination that makes zero:

$$c_1 \begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 4 \\ 1 \\ 0 \end{bmatrix} + c_3 \begin{bmatrix} 2 \\ 5 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

We need to show that c_1, c_2, c_3 are all forced to be zero.

Indeed, the third row forces c_3 to be zero. Then the second row forces c_2 to be zero. Then the first row forces c_1 to be zero. Hence, the columns are linearly independent

End of Example 68. □

16.2 Basis

Linear independence allows us to make precise the notion of what it means for a spaning set of vectors to be minimal. Such a set is called a basis:

Definition 69 (Basis)**.** Let V be a vector space and let $v_1, \ldots, v_n \in V$. We say that $v_1, \ldots, v_n$ are a ***basis*** for V if both the following conditions are satisfied:

(i.) $v_1, \ldots, v_n$ span V.

(ii.) $v_1, \ldots, v_n$ are linearly independent.

Condition (i.) says that you have enough vectors to generate the lienar space V, and condition (ii.) says that *you don't have too many vectors* (i.e., minimality). To answer the motivating question of the last few lectures, a basis is a minimal generating set for a linear subspace (which may include be whole space V).

This latter point about minimality is related to the following theorem

Theorem 70. *Let V be a vector space, and let $v_1, \ldots, v_n \in V$. Then the $v_1, \ldots, v_n$ are linearly dependent if and only if one of the $v_1, \ldots, v_n$ is a linear combination of the others.*

Example 71 (The standard basis vectors)**.** The vectors

$$e_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad e_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad e_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

are a basis for $\mathbb{R}^3$.

To check this, observe that

(i.) (linear independence) It is clear that equality holds in the equation

$$c_1 e_1 + c_2 e_2 + c_3 e_3 = 0$$

only if $c_1 = c_2 = c_3 = 0$. That is, there is no nontrivial linear combination of e_1, e_2, e_3 which equals the zero vector. So e_1, e_2, e_3 are linearly independent.

(ii.) (spanning) If $\begin{bmatrix} a \\ b \\ c \end{bmatrix}$ is an arbitrary vector in $\mathbb{R}^3$, then

$$ae_1 + be_2 + ce_3 = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

Therefore $\mathrm{Span}(e_1, e_2, e_3)$ includes every vector in $\mathbb{R}^3$.

End of Example 71. □

Example 72. Show that

$$\begin{bmatrix}1\\0\\1\end{bmatrix}, \quad \begin{bmatrix}1\\1\\1\end{bmatrix}, \quad \begin{bmatrix}-1\\1\\1\end{bmatrix}$$

form a basis for $\mathbb{R}^3$.

Solution: We already checked that these vectors are linearly independent in Example 67. To it suffices to show that we can write any vector $\begin{bmatrix}z\\y\\z\end{bmatrix}$ in $\mathbb{R}^3$ as a linear combination of the three vectors.

To see this, set up the system

$$\begin{bmatrix}1 & 1 & -1\\0 & 1 & 1\\1 & 1 & 1\end{bmatrix}\begin{bmatrix}c_1\\c_2\\c_3\end{bmatrix} = \begin{bmatrix}x\\y\\z\end{bmatrix}.$$

If this system always has a solution, then the vectors span $\mathbb{R}^3$. If there is some vector $\begin{bmatrix}x\\y\\z\end{bmatrix}$ such that the above equation has no solutions, then the vectors do not span $\mathbb{R}^3$. We can check it by (you guessed it) row reducing.

$$\left[\begin{array}{ccc|c}1 & 1 & -1 & x\\0 & 1 & 1 & y\\1 & 1 & 1 & z\end{array}\right] \xrightarrow{R_3-R_1\to R_3} \left[\begin{array}{ccc|c}1 & 1 & -1 & x\\0 & 1 & 1 & y\\0 & 0 & 2 & z-x\end{array}\right] \xrightarrow{R_1-R_2\to R_1} \left[\begin{array}{ccc|c}1 & 0 & -2 & x-y\\0 & 1 & 1 & y\\0 & 0 & 2 & z-x\end{array}\right] \tag{14}$$

$$\xrightarrow{R_1+R_3\to R_1} \left[\begin{array}{ccc|c}1 & 0 & 0 & z-y\\0 & 1 & 1 & y\\0 & 0 & 2 & z-x\end{array}\right] \xrightarrow{R_2-\frac{1}{2}R_3\to R_2} \left[\begin{array}{ccc|c}1 & 0 & 0 & z-y\\0 & 1 & 0 & y-\frac{1}{2}(z-x)\\0 & 0 & 2 & z-x\end{array}\right] \xrightarrow{\frac{1}{2}R_3\to R_3} \left[\begin{array}{ccc|c}1 & 0 & 0 & z-y\\0 & 1 & 0 & y-\frac{1}{2}(z-x)\\0 & 0 & 1 & \frac{1}{2}(z-x)\end{array}\right]$$

Therefore, the solution is $c_1 = z - y$, $c_2 = y + \frac{x}{2} - \frac{z}{2}$, $c_3 = \frac{z}{2} - \frac{x}{2}$. Since this solution exists for any x, y, z, we have shown that the vectors form a basis.

End of Example 72. $\square$