11 2025-09-19 | Week 04 | Lecture 11

This lecture is based on sections 2.1 and 2.2 in the textbook.

> ***The nexus question of this lecture:*** *What are the essential properties of vectors?*

So far we have defined a "vector" as any $n \times 1$ column matrix

$$x = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$$

whose entries are real numbers. We can think of the vector x in several ways

- as a data structure consisting of an ordered list of entries
- as an arrow in space, possessing both magnitude and direction (e.g., a velocity)
- as a "point" in the n-dimensional space $\mathbb{R}^n$.

In some sense, these different notions are just superficially different ways of thinking about the same fundamental underlying mathematical object.

So what are the *essential* properties of vectors?

11.1 The criteria for vectorhood

Thinking abstractly, I propose that any definition of vectors should capture three key properties:

1. **Vector addition:** we have to be able to add vectors together, which always gives us another vector (as opposed to some other sort of mathematical object)

$$\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} = \begin{bmatrix} x_1 + y_1 \\ \vdots \\ x_n + y_n \end{bmatrix}$$

2. **Scalar multiplication:** We can *scale* vectors, by multiplying by a scalar $c \in \mathbb{R}$

$$c \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} cx_1 \\ \vdots \\ cx_n \end{bmatrix}$$

   and the act of scaling a vector always returns a vector.

3. **The algebra:** When doing algebra with vectors, the algebra should behave mostly how we'd expect (e.g., if u, v are vectors then $u + v = v + u$, and if c is a scalar then $c(u + v) = cu + cv$, etc.). The exception is that you don't need to be able to multiply or divide vectors.

We'll offer a more precise definition later, but in some sense, these three things are really all it should take to define a "vector". But understanding vectors in this way opens up a lot of room for things that don't necessarily "look like" the vectors in $\mathbb{R}^n$, but nonetheless behave precisely in the ways that vectors should behave.

11.2 Some new examples of vectors

Example 41 (The infinite-dimensional vector space of sequences)**.**

$$\mathbb{R}^{\mathbb{N}} := \{(a_1, a_2, a_3, \ldots) : a_1, a_2, \ldots \in \mathbb{R}\}$$

Here, the "vectors" are actually infinite sequences, like

$$(1, 1/2, 1/3, 1/4, \ldots) \quad \text{or} \quad (0, 1, 0, 1, 0, 1, \ldots) \quad \text{or} \quad (0, 0, 0, 0, 0, \ldots) \quad \text{or} \quad (1, 2, 4, 8, 16, 32, \ldots)$$

Of course we can add two sequences

$$(a_1, a_2, a_3, \ldots) + (b_1, b_2, b_3, \ldots) = (a_1 + b_1, a_2 + b_2, a_3 + b_3, \ldots)$$

and we can scale them

$$c(a_1, a_2, \ldots) = (ca_1, ca_2, \ldots)$$

and there's no reason to think that arithmetic would behave differently with infinite sequences than with finite column vectors.

While $\mathbb{R}^n$ is n-dimensional space with vectors of length n, $\mathbb{R}^{\mathbb{N}}$ is infinite-dimensional with vectors which are infinitely long.

End of Example 41. □

Polynomials seem to fit the same criteria for vectorhood as well:

Example 42 (Polynomials of degree at most 2)**.**

$$\mathbb{R}[x]_{\leq 2} = \left\{a_0 + a_1 x + a_2 x^2 : a_0, \ldots, a_2 \in \mathbb{R} \text{ and } x \text{ is an symbolic variable}\right\}$$

Here, the "vectors" are polynomials like

$$1 + x + 3x^2 \quad \text{or} \quad x - x^2 \quad \text{or} \quad -1 + x.$$

Here, we can add the "vectors" of this set in the usual way. If

$$u = a_0 + a_1 x + a_2 x^2 \quad \text{and} \quad v = b_0 + b_1 x + b_2 x^2,$$

Then the vector $u + v$ is

$$(a_0 + a_0) + (a_1 + a_1)x + (a_2 + a_2)x^2$$

which is again a polynomial.

We also have a notion of scalar multiplication. If $c \in \mathbb{R}$, then

$$\begin{aligned} cu &= c\left(a_0 + a_1 x + a_2 x^2\right) \\ &= ca_0 + ca_1 x + ca_2 x^2 \end{aligned}$$

which is still a polynomial of degree at most 2.

Question: What is the dimension of the "space" $\mathbb{R}[x]_{\leq 2}$?

To specify an element of $\mathbb{R}[x]_{\leq 2}$, which takes the form

$$a_0 + a_1 x + a_2 x^2,$$

we need to know three numbers: a_0, a_1, a_2. So there are three independent variables, meaning the dimension is 3. We can see this by "rewriting" the elements of $\mathbb{R}[x]_{\leq 2}$ as

$$a_0 + a_1 x + a_2 x^2 \quad \leftrightarrow \quad \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix}$$

in which it becomes clear that the dimension really is 3.

We can define $\mathbb{R}[x]_{\leq n}$ similarly to $\mathbb{R}[x]_{\leq 2}$, as

$$\mathbb{R}[x]_{\leq n} = \left\{a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n : a_0, a_1, \ldots, a_n \in \mathbb{R} \text{ and } x \text{ is a symbolic variable}\right\}$$

in which case the dimension is $n + 1$ (because you need to specify $n + 1$ coefficients).

End of Example 42. □

But why stop at polynomials? We can regard functions as "vectors" too!

Example 43 (Real-valued continuous functions on the closed unit interval)**.** Let

$$C[0,1] = \{f : [0,1] \to \mathbb{R} \mid f \text{ is continuous}\}$$

For example, this includes functions like

$$\sin(x), \quad e^x, \quad 4x^2 - 1, \quad 4$$

Here "vector addition" is defined in the usual way of adding functions (i.e., $f+g$ is the function $f(x)+g(x)$), and "scalar multiplication" as well (i.e., af is the function $af(x)$).
Question: What is the dimension of $C[0,1]$?

Consider that $\mathbb{R}[x]_{\leq 2} \subseteq C[0,1]$ (since polynomials are continuous). Therefore, $C[0,1]$ contains a set of dimension 2, so its dimension is at least 2.

Similarly, for every positive integer n (no matter how large), we have $\mathbb{R}[x]_{\leq n} \subseteq C[0,1]$. Hence, $C[0,1]$ contains a set of dimension at least $n+1$, for every $n \geq 1$. So $C[0,1]$ must be infinite-dimensional.

End of Example 43. □

Example 44 (Possible velocities of a particle)**.** Consider the set of possible velocities of an electron in space. This is a vector space. Clearly this is 3-dimensional (assuming space itself is 3-dimensional).

But it's not immediately clear how to represent these velocity vectors in the form

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

since in real life, space doesn't have coordinate axis.

End of Example 44. □

Although these examples don't necessarily "look like" sets of vectors (at least superficially), they all share the common essential structure embodied in properties 1,2, and 3.

We can finally define our subject matter:

Linear algebra is the study of finite dimensional "spaces" of vectors.