# Math 307: Practice Midterm 1

*Date of exam: November 24, in class*

*Instructions: You have 50 minutes. Calculators and notes are not allowed. This practice exam is slightly longer than the actual midterm.*

**Problem 1.** State precise definitions of the following terms:

(a) null space of $A$

(b) "eigenvector" and "eigenvalue" of a matrix $A$

**Problem 2.** Determine whether the following statements are true or false. You don't need to show your work or justify your answer.

(a) If $A$ is an $n \times n$ matrix of rank $n - 1$, then its nullspace is a line through the origin.

(b) The matrix $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ is a rotation

(c) The vectors $\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$, $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$, $\begin{bmatrix} 2 \\ 3 \\ 5 \end{bmatrix}$ span a dimension 3 subspace of $\mathbb{R}^4$.

(d) If $A$ is a $5 \times 8$ matrix with $\dim(CS(A)) = 3$, then $\dim(NS(A)) = 2$.

(e) Let $A$ and $B$ be $n \times n$ matrices. If $T(v) = Av$ and $S(v) = Bv$ for all $v \in \mathbb{R}^n$, then $T \circ S(v) = ABv$.

(f) The transformation $T(x) = x + 11$ is linear.

(g) The matrix $A^\top A$ is symmetric.

(h) The columns of an invertible $n \times n$ matrix form a basis of $\mathbb{R}^n$.

(i) If the columns of $A$ are linearly independendent, then $Ax = b$ has exactly one solution for every $b$.

(j) If $v$ is an eigenvector of $A$, then it is an eigenvector of $A^2$

(k) Let $A$ be an $m \times n$ matrix, and let $T(X) = AX$ be a linear transformation. Then the domain of $T$ is $\mathbb{R}^n$ and the range is $\mathbb{R}^m$.

(l) If $A, B$ are $n \times n$ matrices, then $(A + B)^2 = A^2 + 2AB + B^2$.

(m) Suppose the only solution to $Ax = 0$ ($m$ equations, $n$ unknowns) is $x = 0$. Then $A$ has rank $n$.

(n) Let $A$ be an $n \times n$ matrix. If $Ax = 0$ for some nonzero vector $x$, then the equation $Ax = 0$ has infinitely many solutions.

(o) The eigenvalues of a projection matrix are always 0 or 1.

**Problem 3.** Let $A = \begin{bmatrix} 1 & 1 & 0 & 3 \\ 1 & 1 & 1 & -2 \\ 3 & 3 & 2 & -1 \end{bmatrix}$

(a) Find a basis for $CS(A)$.

(b) Find a basis for $RS(A)$.

(c) Find a basis for $NS(A)$.

(d) What is the rank of $A$?

(e) Is $A$ invertible?

**Problem 4.** *[Note: there will be a problem like this on the exam.]* Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the projection onto the line $y = -x$. Let $\alpha = \{e_1, e_2\}$

   (a) Find two linearly independent eigenvectors $\beta_1, \beta_2$ of $T$ and sketch them.

   (b) What is $[T]_\beta^\beta$?

   (c) Let $P$ be the change-of-basis matrix from $\alpha$ to $\beta$. Find $P$ and $P^{-1}$.

   (d) Use your answer to the previous part to find $[T]_\alpha^\alpha$.

   (e) What are the eigenvalues of $T$?

   (f) Verify that $[T]_\alpha^\alpha$ and $[T]_\beta^\beta$ have the same characteristic polynomial.

**Problem 5.** Let $A = \begin{bmatrix} 6 & -8 \\ 4 & -6 \end{bmatrix}$

   (a) By what factor does the linear transformation of $A$ scale area?

   (b) Find the eigenvalues of $A$.

   (c) For each eigenvalue of $A$, find a basis for the eigenspace.

**Problem 6.** Let $A = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$.

   (a) Sketch the image of the unit square $\{(x, y) : 0 \le x \le 1 \text{ and } 0 \le y \le 1\}$ under the transformation of the matrix

   (b) Compute the eigenvalues of $A$, and and eigenbasis for each eigenvalue.

   (c) What is the inverse of $A$?

**Problem 7.** Show that $\begin{bmatrix} 1 \\ 3 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}$ form a basis for $\mathbb{R}^3$.

**Problem 8.** Solve the following linear system:

$$2x + 3y = 6$$
$$2x + y = 2$$
$$x - y = -1$$

Provide a sketch and interpret your results geometrically.

**Problem 9.** Suppose $T$ is a linear transformation such that $T \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 6 \\ 6 \end{bmatrix}$ and $T \begin{bmatrix} 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 4 \\ 0 \end{bmatrix}$.

   (a) Find $T \begin{bmatrix} 2 \\ 2 \end{bmatrix}$

   (b) Find $T \begin{bmatrix} 5 \\ 1 \end{bmatrix}$

   (c) Find $T \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

   (d) Find $T \begin{bmatrix} a \\ b \end{bmatrix}$ where $a, b$ are arbitrary real numbers.

   (e) Write down a $2 \times 2$ matrix that gives the transformation $T$.