

35 2025-11-14 | Week 12 | Lecture 35

The nexus question of this lecture: What are the eigenvectors of a rotation?

Suppose $T(x) = Ax$ is a linear transformation, and v is an eigenvector of T . This means that there exists a scalar $\lambda \in \mathbb{C}$ such that.

$$T(v) = \lambda v.$$

Geometrically, this equation says that an eigenvector is a direction that the transformation leaves intact: when T acts on v , it may stretch, shrink, or flip that vector, but doesn't rotate it into a new direction.

This naturally raises a striking question: what are the eigenvectors of a rotation? A rotation of the plane visibly changes *every* real direction in the plane. Every arrow in the plane is dragged away from where it once pointed. So if eigenvectors are "directions left intact", then how can a rotation possibly have any?

The short answer to this question is that we get complex eigenvectors. The rest of the lecture will consist of an example of a linear transformation of \mathbb{R}^2 involving a rotation which illustrates how this happens.

35.1 Eigenvectors and eigenvalues of a rotation

Example 159 (Eigenvalues and eigenvectors of a rotation). Let

$$A = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}.$$

This is a counterclockwise rotation by 45° composed with a scaling by $\sqrt{2}$. That is, $A = \sqrt{2}R_{45^\circ}$.

Question: what are the eigenvalues and eigenvectors of A ?

Solution: The characteristic polynomial is

$$\det \begin{bmatrix} \lambda - 1 & 1 \\ -1 & \lambda - 1 \end{bmatrix} = (\lambda - 1)^2 + 1 = \lambda^2 - 2\lambda + 2 = (\lambda - (1 + i))(\lambda - (1 - i)).$$

This has solutions $\lambda = 1 + i$ and $\lambda = 1 - i$. These two complex numbers are the eigenvalues of A .

- For $\lambda = 1 + i$, we want to find the eigenspace $E_{1+i} = NS((1 + i)I - A)$.

First observe that $(1 + i)I - A = \begin{bmatrix} 1 + i & 0 \\ 0 & 1 + i \end{bmatrix} - \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} i & 1 \\ -1 & i \end{bmatrix}$. We wish to solve

$$\begin{bmatrix} i & 1 \\ -1 & i \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Next, set up and row reduce the augmented matrix:

$$\left[\begin{array}{cc|c} i & 1 & 0 \\ -1 & i & 0 \end{array} \right] \xrightarrow{R_2 - iR_1 \rightarrow R_2} \left[\begin{array}{cc|c} i & 1 & 0 \\ 0 & 0 & 0 \end{array} \right] \xrightarrow{(-i)R_1 \rightarrow R_1} \left[\begin{array}{cc|c} 1 & -i & 0 \\ 0 & 0 & 0 \end{array} \right]$$

This corresponds to the system

$$\begin{cases} x - iy = 0 \\ 0 = 0 \end{cases}$$

Hence $x = iy$ and y is a free variable. Therefore

$$E_{1+i} = \left\{ \begin{bmatrix} iy \\ y \end{bmatrix} : y \in \mathbb{C} \right\} = \left\{ y \begin{bmatrix} i \\ 1 \end{bmatrix} : y \in \mathbb{C} \right\}$$

In words, the eigenspace E_{1+i} is spanned by the vector $\begin{bmatrix} i \\ 1 \end{bmatrix}$.

- For $\lambda = 1 - i$, similar calculation shows that the eigenspace $E_{1-i} = \text{NS}((1 - i)I - A)$ is spanned by the eigenvector

$$\begin{bmatrix} -i \\ 1 \end{bmatrix}.$$

End of Example 159. \square

Example 159 illustrates the following theorem:

Theorem 160. *Let A be an $n \times n$ matrix with real entries. If $\lambda = a + bi$ is an eigenvalue of A (here $a, b \in \mathbb{R}$), then $\bar{\lambda} = a - bi$ is also an eigenvalue.*

Remark 161. Every complex number $z \in \mathbb{C}$ can be written as $z = a + bi$ for some choice of real numbers a and b . The terminology used is that $a + bi$ and $a - bi$ are called **complex conjugates**.

35.2 Diagonalization of a rotation

We can diagonalize the rotation.

Example 162. Let us consider again the matrix

$$A = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}.$$

This is the rotation from Example 159; there, we shows that this matrix has two eigenvectors:

$$\begin{bmatrix} i \\ 1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} -i \\ 1 \end{bmatrix}.$$

Theorem 144 implies that A is diagonalizable. The matrix $P = \begin{bmatrix} i & -i \\ 1 & 1 \end{bmatrix}$ (whose columns are the two eigenvectors) is invertible (because the eigenvectors are linearly independent). This is a change-of-basis matrix (our “prism”), and we can use it to diagonalize A as follows:

$$P^{-1}AP = \begin{bmatrix} 1+i & 0 \\ 0 & 1-i \end{bmatrix} \tag{28}$$

(see below for details to justify this computation). The main diagonal entries on the right hand side are the eigenvalues of A . Somehow, in rotating the plane 45° (and scaling it by a factor of $\sqrt{2}$), one of the complex eigenvectors gets multiplied by $1 + i$ and the other by $1 - i$.

Here are the details which justify Eq. (28): first observe that we have

$$P = \begin{bmatrix} i & -i \\ 1 & 1 \end{bmatrix} \quad \text{and} \quad P^{-1} = \frac{1}{2i} \begin{bmatrix} 1 & i \\ -1 & i \end{bmatrix}.$$

Therefore

$$\begin{aligned} P^{-1}AP &= \left(\frac{1}{2i} \begin{bmatrix} 1 & i \\ -1 & i \end{bmatrix} \right) \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} i & -i \\ 1 & 1 \end{bmatrix} \\ &= \left(\frac{1}{2i} \begin{bmatrix} 1 & i \\ -1 & i \end{bmatrix} \right) \begin{bmatrix} i-1 & -i-1 \\ i+1 & -i+1 \end{bmatrix} \\ &= \frac{1}{2i} \begin{bmatrix} 2i-2 & 0 \\ 0 & 2i+2 \end{bmatrix} \\ &= \begin{bmatrix} 1-\frac{1}{i} & 0 \\ 0 & 1+\frac{1}{i} \end{bmatrix} \\ &= \begin{bmatrix} 1+i & 0 \\ 0 & 1-i \end{bmatrix} \end{aligned} \quad \text{since } \frac{1}{i} = -i,$$

and the right-hand side is B . In this calculation, we used the fact that $\frac{1}{i} = -i$ (which holds because $(-i)i = 1$).

End of Example 162. \square