Math 307: Practice Midterm 1

Date of exam: November 24, in class

Instructions: You have 50 minutes. Calculators and notes are not allowed. This practice exam is slightly longer than the actual midterm.

Problem 1. State precise definitions of the following terms:

(a) null space of A

(b) "eigenvector" and "eigenvalue" of a matrix A

Problem 2. Determine whether the following statements are true or false. You don't need to show your work or justify your answer.

(a) If A is an $n \times n$ matrix of rank $n-1$, then its nullspace is a line through the origin.

(b) The matrix $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ is a rotation

(c) The vectors $\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 5 \end{bmatrix}$ span a dimension 3 subspace of $\mathbb{R}^4$.

(d) If A is a 5×8 matrix with $\dim(CS(A)) = 3$, then $\dim(NS(A)) = 2$.

(e) Let A and B be $n \times n$ matrices. If $T(v) = Av$ and $S(v) = Bv$ for all $v \in \mathbb{R}^n$, then $T \circ S(v) = ABv$.

(f) The transformation $T(x) = x + 11$ is linear.

(g) The matrix $A^\top A$ is symmetric.

(h) The columns of an invertible $n \times n$ matrix form a basis of $\mathbb{R}^n$.

(i) If the columns of A are linearly independendent, then $Ax = b$ has exactly one solution for every b.

(j) If v is an eigenvector of A, then it is an eigenvector of A^2

(k) Let A be an $m \times n$ matrix, and let $T(X) = AX$ be a linear transformation. Then the domain of T is $\mathbb{R}^n$ and the range is $\mathbb{R}^m$.

(l) If A, B are $n \times n$ matrices, then $(A+B)^2 = A^2 + 2AB + B^2$.

(m) Suppose the only solution to $Ax = 0$ (m equations, n unknowns) is $x = 0$. Then A has rank n.

(n) Let A be an $n \times n$ matrix. If $Ax = 0$ for some nonzero vector x, then the equation $Ax = 0$ has infinitely many solutions.

(o) The eigenvalues of a projection matrix are always 0 or 1.

Problem 3. Let $A = \begin{bmatrix} 1 & 1 & 0 & 3 \\ 1 & 1 & 1 & -2 \\ 3 & 3 & 2 & -1 \end{bmatrix}$

(a) Find a basis for $CS(A)$.

(b) Find a basis for $RS(A)$.

(c) Find a basis for $NS(A)$.

(d) What is the rank of A?

(e) Is A invertible?

Problem 4. *[Note: there will be a problem like this on the exam.]* Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the projection onto the line $y = -x$. Let $\alpha = \{e_1, e_2\}$

(a) Find two linearly independent eigenvectors β_1, β_2 of T and sketch them.

(b) What is $[T]_\beta^\beta$?

(c) Let P be the change-of-basis matrix from α to β. Find P and P^{-1}.

(d) Use your answer to the previous part to find $[T]_\alpha^\alpha$.

(e) What are the eigenvalues of T?

(f) Verify that $[T]_\alpha^\alpha$ and $[T]_\beta^\beta$ have the same characteristic polynomial.

Problem 5. Let $A = \begin{bmatrix} 6 & -8 \\ 4 & -6 \end{bmatrix}$

(a) By what factor does the linear transformation of A scale area?

(b) Find the eigenvalues of A.

(c) For each eigenvalue of A, find a basis for the eigenspace.

Problem 6. Let $A = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$.

(a) Sketch the image of the unit square $\{(x, y) : 0 \leq x \leq 1 \text{ and } 0 \leq y \leq 1\}$ under the transformation of the matrix

(b) Compute the eigenvalues of A, and and eigenbasis for each eigenvalue.

(c) What is the inverse of A?

Problem 7. Show that $\begin{bmatrix} 1 \\ 3 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}$ form a basis for $\mathbb{R}^3$.

Problem 8. Solve the following linear system:

$$\begin{aligned} 2x + 3y &= 6 \\ 2x + y &= 2 \\ x - y &= -1 \end{aligned}$$

Provide a sketch and interpret your results geometrically.

Problem 9. Suppose T is a linear transformation such that $T \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 6 \\ 6 \end{bmatrix}$ and $T \begin{bmatrix} 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 4 \\ 0 \end{bmatrix}$.

(a) Find $T \begin{bmatrix} 2 \\ 2 \end{bmatrix}$

(b) Find $T \begin{bmatrix} 5 \\ 1 \end{bmatrix}$

(c) Find $T \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

(d) Find $T \begin{bmatrix} a \\ b \end{bmatrix}$ where a, b are arbitrary real numbers.

(e) Write down a 2×2 matrix that gives the transformation T.