Math 307: Homework 08

Due Friday, October 31. This homework draws on sections 5.1, 5,2 and 5.3 in the textbook.

Problem 1. Let $S, T : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformations

$$S\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2x - y \\ x + 2y \end{bmatrix} \quad \text{and} \quad T\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x + 3y \\ x - y \end{bmatrix}.$$

(a) Find matrices A, B such that T and S are expressed as the matrix transformations $T(X) = AX$ and $S(X) = BX$.

(b) Find the matrix C such that the composition $S \circ T$ is expressed in the form $(S \circ T)(X) = CX$. Then verify that $C = AB$.

(c) Find the matrix D such that the composition $T \circ S$ is expressed in the form $(T \circ S)(X) = DX$. Then verify that $D = BA$.

Problem 2. Let

$$A = \begin{bmatrix} 4 & -1 & 2 & 1 \\ 2 & 3 & -1 & -2 \\ 0 & 7 & -4 & -5 \\ 2 & -11 & 7 & 8 \end{bmatrix}$$

For the following parts, you must fully justify your work to recieve credit—this includes providing at least one or two complete sentences explaining why your calculations justify your answer.

(a) Find a basis for $NS(A)$

(b) Find a basis for $RS(A)$

(c) Find a basis for $CS(A)$

(d) Determine the rank of A.

(e) Determine whether the matrix is invertible

Problem 3. If S and T are linear transformations with $S(v) = T(v) = v$, then $S(T(v)) = v$ or v^2?

Problem 4. True or false, with counterexample if false:

(a) If the vectors $x_1, \ldots, x_m$ span a subspace S, then $\dim S = m$.

(b) The intersection of two subspaces of a vector space cannot be emptpy.

(c) If $Ax = Ay$, then $x = y$.

(d) The row space of A has a unique basis that can be computed by reducing A to reduced row-echelon form.

(e) If a square matrix A has independent columns, then so does A^2.

(f) Any two bases of a linear subspace have the same number vectors.

Problem 5. Find all solutions of

$$\begin{cases} x_1 + x_2 + x_3 + x_4 = 0 \\ x_1 + x_2 + x_3 - x_4 = 4 \\ x_1 + x_2 - x_3 + x_4 = -4 \\ x_1 - x_2 + x_3 + x_4 = 2 \end{cases}$$

Problem 6. A linear transformation must leave the zero vector fixed: $T(0) = 0$. Prove this from $T(u+v) = T(u) + T(v)$ by choosing $v =$ ____. Prove it also from the requirement $T(cv) = cT(v)$ by choosing $c =$ ____.

Problem 7. Every straight line remains straight after a linear transformation. If z is halfway between x and y, show that Az is halfway between Ax and Ay

Problem 8. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation

$$T\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 + x_2 \\ x_1 - x_2 \end{bmatrix}.$$

(a) Find $[T]_\alpha^\alpha$ where α is the standard basis for $\mathbb{R}^2$.

(b) Let β be the basis for $\mathbb{R}^2$ consisting of $\begin{bmatrix} 1 \\ -1 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \end{bmatrix}$. Find the change of basis matrix from α to β.

(c) Find the change of basis matrix from β to α.

(d) Find $[T]_\beta^\beta$

(e) Find $[v]_\beta$ for $v = \begin{bmatrix} -2 \\ 3 \end{bmatrix}$

(f) Find $[T(v)]_\beta$.

(g) Use the result of part (f) to find $T(v)$.

Problem 9. Let β be the basis of $\mathbb{R}^2$ consisting of the vectors $\begin{bmatrix} 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

(a) Write the vectors $\begin{bmatrix} 5 \\ -4 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \end{bmatrix}$, and $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ in terms of the basis β. Please write your final answers as column vectors of the form $\begin{bmatrix} x \\ y \end{bmatrix}_\beta$.

(b) Let T be the linear transformation $T = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 5x_1 + 3x_2 \\ -6x_1 - 4x_2 \end{bmatrix}$.

(c) Find $[T]_\alpha^\alpha$, where α is the standard basis of $\mathbb{R}^2$ consisting of the vectors $\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

(d) Find $[T]_\beta^\beta$.

Problem 10. Let c be a scalar, and let $T : V \to W$ be a linear transformation. Verify that cT is a linear transformation.

Problem 11. The matrix $A = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$ produces a *stretching* in the x-direction. Draw the circle $x^2 + y^2 = 1$. What happens to this circle when space is transformed by A? Illustrate your answer with a sketch.

Problem 12.

(a) What 2×2 matrix has the effect of rotating every vector counterclockwise $90°$ and then projecting the result onto the x-axis?

(b) What 2×2 matrix represents projection onto the x-axis followed by projection onto the y-axis?

Problem 13. The matrix $A = \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}$ yields a *shearing* transformation, which leaves the y-axis unchanged. Sketch its effect on the x-axis, by indicating what happens to the points $(x, y) = (1, 0), (2, 0)$, and $(-1, 0)$—and how the whole axis is transformed.

Problem 14. The 4-Hadamard matrix is

$$H = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}$$

Find H^{-1} and write $v = \begin{bmatrix} 7 \\ 5 \\ 3 \\ 1 \end{bmatrix}$ as a linear combination of the columns of H.