7 2025-09-10 | Week 03 | Lecture 07

This lecture isn't really based on any textbook section, but sections 1.5 and 1.6 cover determinants. Please read sections 1.3 and 1.4 for Friday.

> ***The nexus question of this lecture:*** *What is a determinant, geometrically?*

7.1 The determinant of 2×2 matrix

In your homework, you had a problem in which you computed the determinant of a 2×2 matrix. Recall that the determinant of a 2×2 matrix is defined as the quantity

$$\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc.$$

Later we'll see how to define determinants for $n \times n$ matrices, but for this lecture I'm going to focus on the 2×2 case in order to hopefully demonstrate why you should even care about determinants at all.

7.2 A matrix is a transformation of space

Consider the 2×2 matrix

$$A = \begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix}.$$

Preliminaries: Some elementary properties of A

Let us consider what this matrix does to the standard basis vectors $e_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $e_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$:

$$Ae_1 = \begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \end{bmatrix} \quad \text{and} \quad Ae_2 = \begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

In other words, multiplication by A sends the vector e_1 to the vector $\begin{bmatrix} 3 \\ 1 \end{bmatrix}$. It sends the vector e_2 to $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

Let's give these vectors names and compute some values that will be useful later:

$$u = \begin{bmatrix} 3 \\ 1 \end{bmatrix} \quad \text{and} \quad v = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

- The magnitudes of u and v are

$$|u| = \sqrt{3^2 + 1^2} = \sqrt{10} \quad \text{and} \quad |v| = \sqrt{2^2 + 1^2} = \sqrt{5}$$

- The cosine of angle θ between u and v can be computed using the dot product, by the formula

$$u \cdot v = |u||v|\cos(\theta).$$

  Doing the computation, we get

$$\cos(\theta) = \frac{1}{\sqrt{2}}. \tag{11}$$

The key idea

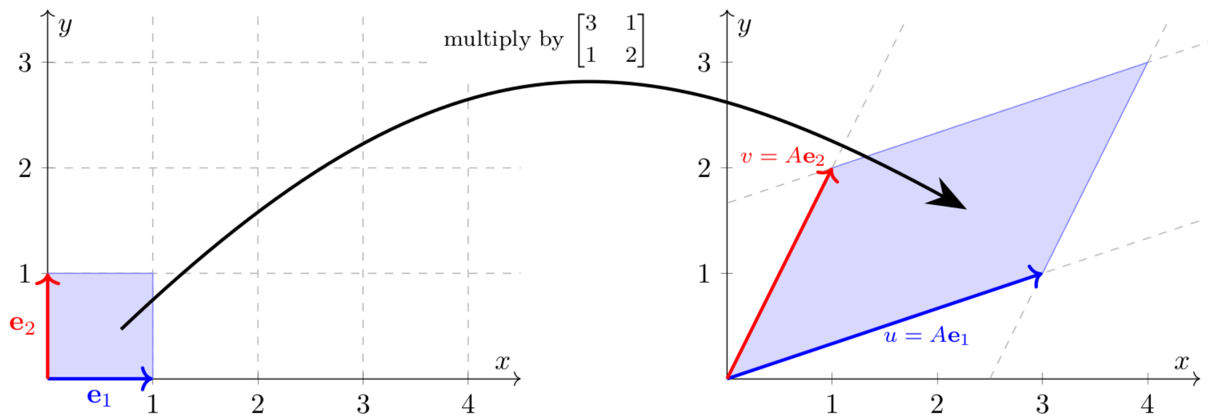
Suppose we decided to multiply *every* vector in the plane by the matrix $\begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix}$. In that case, we could conceive of the matrix transforming space (i.e., the plane) in some way. For a general vector in the plane $\begin{bmatrix} x \\ y \end{bmatrix}$, the action of A is the following:

$$\begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3x + y \\ x + 2y \end{bmatrix}$$

If we use "$\mapsto$" to mean "gets sent to", then we have

$$\begin{bmatrix} x \\ y \end{bmatrix} \mapsto \begin{bmatrix} 3x + y \\ x + 2y \end{bmatrix}$$

The following plot shows the effect of multiplying *all* vectors in the unit square by A:

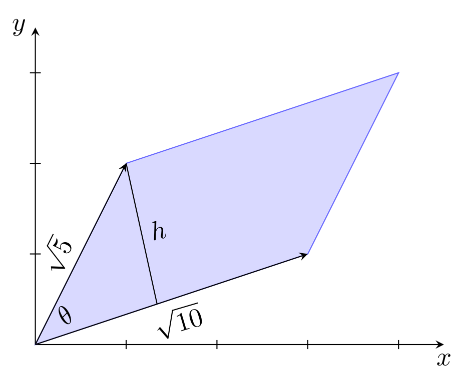


The effect of applying A is to transform space (i.e., the plane) by stretching it in some way. For this particular matrix, the unit square get mapped to the shown parallelogram. Squares adjacent to the unit square get sent to adjacent parallelograms.

7.3 The "volume scaling factor" of a transformation

Question: By what factor does the matrix $\begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix}$ scale the area of a region in space?

Answer: The unit square has area 1. What about the parallelogram?



The parallelogram has area

$$\text{(area of parallelogram)} = \text{(base)} \times \text{(height)} \tag{12}$$

In this case, the base of the parallelogram is $|u|= \sqrt{10}$. To find the height h, use the definition of $\sin(\theta)$:

$$\sin(\theta) = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{h}{\sqrt{5}}.$$

which implies

$$h = \sqrt{5}\sin(\theta)$$

We already know that $\cos(\theta) = \frac{1}{\sqrt{2}}$ by Eq. (11). Therefore

$$\begin{aligned} \sin^2(\theta) &= 1 - \cos^2(\theta) && (\text{since } \sin^2(\theta) + \cos^2(\theta) = 1) \\ &= 1 - \frac{1}{2} \\ &= \frac{1}{2}. \end{aligned}$$

Taking square roots, we get $\sin(\theta) = \frac{1}{\sqrt{2}}$. Hence $h = \frac{\sqrt{5}}{\sqrt{2}}$. Therefore, by Eq. (12),

$$(\text{area of parallelogram}) = \left(\sqrt{10}\right)\left(\frac{\sqrt{5}}{\sqrt{2}}\right) = 5.$$

To conclude, the linear transformation of space obtained by multiplying every vector by the matrix $A = \begin{bmatrix} 1 & 3 \\ 2 & 1 \end{bmatrix}$ took a region of area 1 (the unit square) and mapped it to a parallelogram of area 5. The scaling is uniform throughout the plane, so every region of area 1 gets mapped to a region of area 5. In other words, the transformation scales the volume by a factor 5.

Now let's look at the determinant of the matrix:

$$\det \begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix} = 3 \cdot 2 - 1 \cdot 1 = 5$$

The determinant is also 5. This is not a coincidence. **The determinant tells us how much space gets scaled by the linear transformation induced by the matrix.**

7.4 How do we interpret the determinant when it's zero or negative?

The determinant measures how much the linear transformation scales volume. But then what does it mean geometrically for a matrix to have determinant zero or negative?

- If the determinant is zero, that means regions with positive area get mapped to regions of zero area. **This always occurs as a result of dimension collapse.** For example, the following matrix has determinant zero:

  $$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

  Note that

  $$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ 0 \end{bmatrix}$$

  Geometrically, this matrix projects every point in the plane to its location on the x-axis. This means that every point in the plane (a 2-dimensional surface) gets mapped to the x-axis (a 1-dimensional line with zero area). The dimension collapse here is the reduction in dimension from 2D to 1D.

  Another example would be a transformation that projects every point in 3D space onto a specified plane; this is because mapping a 3D region onto a 2D plane destroys volume. An example of such a transformation is obtained from the matrix

  $$P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

which projects points in 3D space onto the xy-plane. We haven't discussed how to define the determinant for 3×3 matrices, but based on our geometric intuition, we'd expect P to have determinant zero (it does).

- If the determinant is negative, that means the transformation reverses the orientation of space in the same way a mirror changes left and right hands. In 2D, this occurs when the transformation reflects the plane across a line. For example, the matrix

  $$R = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

  reflects the xy-plane across the line $y = x$, but doesn't stretch space at all. The determinant is -1. Under this transformation, regions don't stretch or shrink, but they do get flipped.

  Another example can be obtained if we decided to *combine* two transformations:

  - **First**, flip the xy-plane using R.
  - **Then**, transform space using the matrix A.

  To combine these two tranformations, we multiply the matrices like this:

  $$AR = \underbrace{\begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix}}_{2^{\text{nd}}} \underbrace{\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}}_{1^{\text{st}}} = \begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix}$$

  This new matrix has determinant -5. It is similar to the matrix A we considered earlier, but the columns are swapped. Geometrically, this matrix transforms space by first reflecting the plane across the line $y = x$ and then doing the same stretchy thing done by the previous matrix.

  The picture is almost the same as the previous one, but note how the red and blue vectors swapped compared to the first picture. The orientation has changed, which is why the determinant is negative.