6 2025-09-08 | Week 03 | Lecture 06

> ***The nexus question of this lecture:*** *How do we encode a linear system using matrices? And once thus encoded, what can we say about the solutions to the linear system just by looking at the matrix?*

This lecture is based on sections 1.2 and 1.3 of the textbook.

6.1 Inverting matrices

We begin this lecture by introducing the idea of a matrix inverse, which we will use to help answer the main question of the lecture.

Recall that the identity matrix is a matrix of the form

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \text{et cetera}$$

and we usually denote an identity matrix by the letter I_n (or just I if the dimensions are clear from context). Such a matrix behaves like the number 1, in the sense that $AI = A$ and $IA = A$ for any matrix A.

Every nonzero number a has an inverse $a^{-1} = \frac{1}{a}$ such that

$$a \cdot a^{-1} = a^{-1}a = 1.$$

It would natural to conjecture that every nonzero matrix A also has an inverse A^{-1} such that

$$AA^{-1} = A^{-1}A = I$$

In fact, this conjecture is false: we cannot do this with every matrix, but sometimes we can.

To be precise, if two matrices satisfy the property that $AB = BA = I$, then they are said to be ***inverses***. In this case, we write $B = A^{-1}$ (which doesn't mean $1/A$). This is analogous to when we multiply two numbers like $3 \cdot \frac{1}{3} = \frac{1}{3} \cdot 3 = 1$.

Example 15 (Matrix inverses)**.** An example are the matrices

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 5 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} -5 & 2 \\ 3 & -1 \end{bmatrix}$$

Because

$$\begin{aligned} AB &= \begin{bmatrix} 1 & 2 \\ 3 & 5 \end{bmatrix} \begin{bmatrix} -5 & 2 \\ 3 & -1 \end{bmatrix} \\ &= \begin{bmatrix} 1(-5) + 2 \cdot 3 & 1 \cdot 2 + 2(-1) \\ 3(-5) + 5 \cdot 3 & 2 \cdot 2 + 5(-1)) \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\ &= I_2 \end{aligned}$$

And similarly, $BA = I_2$. Therefor $A^{-1} = B$.

End of Example 15. ☐

As noted, not every matrix has an inverse. A matrix that doesn't have an inverse is called ***singular*** or ***noninvertible***. A matrix that has an inverse, is called ***nonsingular*** or ***invertible***. Generally in mathematics, "singular" means "bad".

Moreover, note that it is possible for $AB = I$ but $BA \neq I$, as the following example shows.

Example 16. Let

$$A = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} \quad \text{and} \quad B = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{bmatrix}$$

Then

$$AB = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{but} \quad BA = \frac{1}{2} \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

End of Example 16. □

Incidentally this example illustrates one imporant way that matrix multiplication is not like multiplication of numbers.

6.2 An aside about commutatitivity

Consider two real numbers $a, b \in \mathbb{R}$. Then

$$ab = ba.$$

This property is called ***commutatitivity***. One way that matrix multiplication differs from multiplication of real numbers is that if we consider two matrices A, B, then it is usually not the case that $AB = BA$. Whenever it is the case that $AB = BA$, we say that the matrices A and B ***commute***, but again, this usually doesn't happen. Here are some examples:

Example 17 (Commutativity and noncommutatitivity of matrix multiplication)**.** Let

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & 3 \\ 0 & 2 \end{bmatrix} \quad \text{and} \quad C = \begin{bmatrix} 2 & 0 \\ 3 & 1 \end{bmatrix}.$$

Direct computation shows that

$$AB = BA = \begin{bmatrix} 2 & 5 \\ 0 & 2 \end{bmatrix}.$$

In this case, we say that the matrices A and B commute.

On the other hand, the matrices A and C do not commute because

$$AC = \begin{bmatrix} 5 & 1 \\ 3 & 1 \end{bmatrix} \quad \text{but} \quad CA = \begin{bmatrix} 2 & 2 \\ 3 & 4 \end{bmatrix}$$

so $AC \neq CA$.

End of Example 17. □

6.3 Encoding a linear system via matrix equations

In this section, we answer the first part of the main question of lecture. Consider the 2×2 linear system

$$\begin{aligned} 3x_1 + 7x_2 &= 5 \\ 2x_1 - 6x_2 &= 1 \end{aligned}$$

Using matrix multiplication, we can write this system as

$$\begin{bmatrix} 3 & 7 \\ 2 & -6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 5 \\ 1 \end{bmatrix}$$

In fact, we can do something similar with any linear system, as we now show:

Suppose we have any linear system, like this:

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{21}x_2 + \ldots + a_{2n}x_n = b_2 \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n = b_m \end{cases}$$

Define the three matrices

$$A = \begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \ldots & a_{mn} \end{bmatrix}, \quad X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

Then we can represent the linear system compactly as the matrix equation

$$AX = B.$$

This answers the question *"how can we encode a linear system using matrices?"* (It alsow provides us with the beginnings of an answer to the question *"why is matrix multiplication defined the way it is?"*, which is that matrix multiplication allows us to compactly represent linear systems, though a more compelling answer to this question will come later in the course.)

6.4 What does A tell us about the linear system $AX = B$?

For the remainder of the lecture, we consider the question, *"What can we tell about the solutions to the linear system $AX = B$, just by looking at the matrix A?"* It is not obvious that we should be able to tell anything at all about the solutions just by looking at A—after all, that's, like trying to says something about the solutions to a system of equations *by only looking at the left-hand sides of the equations.*

Observe that if we know A^{-1}, then we can multiply both sides of this equation on the left by

$$A^{-1}AX = A^{-1}B$$

which simplifies to

$$X = A^{-1}B. \tag{10}$$

Thus, X is the unique solution to our original system. This motivates the following theorem:

Theorem 18. *A linear system $AX = B$ with n equations and n variables has exactly one solution if and only if A is invertible.*

This theorem provides a partial answer to the main question of the lecture. What is remarkable is that it says we don't need to know anything at all about the vector B to know whether the system has a unique solution. We only need to know whether A is singular or nonsingular. This hints a deeper structure in linear algebra that we will explore more fully throughout this course.

As an aside, note that when doing computations, it is usually easier to solve a linear system $AX = B$ directly using row reduction, rather than (1) finding the inverse of A and then (2) multiplying both sides by A^{-1}. So the latter approach isn't recommended for solving a system of linear equations. Just use row reduction.

We can actually push our answer to the lecture question a little bit further, by drawing a connection with one of your homework problems (homework 1, problem 3). In that problem, you showed that for the matrix $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$, the linear system

$$AX = B$$

has a unique solution if and only if the determinant of A (defined as $a_{11}a_{22} - a_{12}a_{21}$) is nonzero. That is,

$$\text{determinant of } A \text{ is nonzero} \iff AX = B \text{ has a unique solution}$$

Connecting this with Theorem 18, we have:

$$\text{determinant of } A \text{ is nonzero} \iff AX = B \text{ has a unique solution} \iff A \text{ is invertible}$$

Putting it together, we have the following theorem, which gives us an answer to the main question of the lecture:

Theorem 19. *Let A be an $n \times n$ matrix. Then the following are equivalent:*

(i.) *A is invertible.*

(ii.) *The determinant of A is nonzero.*

(iii.) *For any vector $B \in \mathbb{R}^n$ the linear system $AX = B$ has exactly one solution.*

We haven't proved this theorem for all matrices; we've only shown that it holds for 2×2 matrices. In fact, it holds for all matries, as we'll see later.