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39.1 Theorems and notation of section 6.1

Theorem 173 (Existence and uniqueness)**.** *If t_0 lies in an interval (a, b), and if $a_{ij}(t)$ and $g_i(t)$ are continuous on that interval for all i and j, then the initial value problem*

$$\begin{cases} Y' = AY + G \\ Y(t_0) = (b_1, \ldots, b_n)^\top \end{cases}$$

has a unique solution on the interval (a, b).

For a homogeneous system with no intial conditions, we may have many solutions. In fact

Theorem 174 (Solutions form a subspace)**.** *The solutions to a system of the form $Y' = AY$ (with n equations) form a vector space of dimension n.*

This means that if Y_1 and Y_2 are solutions to Eq. (34), then $\alpha Y_1 + \beta Y_2$ is also a solution as well, for any scalars α, β.

If $Y_1, \ldots, Y_n$ are a set of n linearly independent solutions to Eq. (34), we call them a ***fundamental set of solutions***, and the ***general solution*** is

$$Y_H = c_1 Y_1 + \ldots + c_n Y_n$$

where $c_1, \ldots, c_n$ are scalars. A matrix of the form

$$Y_H = \begin{bmatrix} | & & | \\ c_1 Y_1 & \cdots & c_n Y_n \\ | & & | \end{bmatrix}$$

is called a ***matrix of fundamental solutions***.

Theorem 175. *Suppose $Y_1, \ldots, Y_n$ form a fundamental set of solutions to $Y' = AY$, and that Y_P is a solution to*

$$Y' = AY + G. \tag{36}$$

Then every solution to Eq. (36) has the form

$$Y = Y_H + Y_P$$

where $Y_H = c_1 Y_1 + \ldots + c_n Y_n$.

The function Y_P is called a ***particular solution*** to the inhomogeneous system.

39.2 Examples (see section 6.2)

Example 176. Consider the homogeneous differential equation

$$\begin{bmatrix} y_1' \\ y_2' \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \tag{37}$$

This has form $Y' = AY$.

Eq. (37) has two pure exponential solutions:

$$Y_1(t) = e^{-t} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \text{and} \quad Y_2(t) = e^{3t} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

These are a fundamental set of solutions. The general solution is

$$\begin{aligned} Y &= c_1 Y_1 + c_2 Y_2 \\ &= c_1 \begin{bmatrix} e^{-t} \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ e^{3t} \end{bmatrix} \end{aligned}$$

A matrix of fundamental solutions is

$$\begin{bmatrix} c_1 e^{-t} & 0 \\ 0 & c_2 e^{3t} \end{bmatrix}$$

End of Example 176. □

Example 177. Let $A = \begin{bmatrix} 1 & -3 \\ -2 & 2 \end{bmatrix}$. Find the general solution of $Y' = AY$.

Solution: A has two eigenvalues, 4 and -1. The corresponding eigenvectors are $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 3 \\ 2 \end{bmatrix}$. Therefore there are two pure exponential solutions:

$$Y_1(t) = e^{4t} \begin{bmatrix} -1 \\ 1 \end{bmatrix} \quad \text{and} \quad Y_2(t) = e^{-t} \begin{bmatrix} 3 \\ 2 \end{bmatrix}.$$

The general solution is

$$Y(t) = c_1 e^{4t} \begin{bmatrix} -1 \\ 1 \end{bmatrix} + c_2 e^{-t} \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

End of Example 177. □

Example 178. Let $A = \begin{bmatrix} 2 & -3 & -3 \\ 2 & -2 & -2 \\ -2 & 1 & 1 \end{bmatrix}$.

(a) Find the general solution of $Y' = AY$

(b) Solve the initial value problem

$$\begin{cases} Y' = AY \\ Y(0) = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \end{cases}$$

Solution to (a): The matrix A has three eigenvalues:

- $\lambda_1 = -1$ with eigenvector $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$
- $\lambda_2 = 0$ with eigenvector $\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$
- $\lambda_3 = 2$ with eigenvector $\begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$

Therefore there are three pure exponenetial solutions:

$$Y_1(t) = e^{-t} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \quad Y_2(t) = e^{0t} \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}, \quad \text{and} \quad Y_3(t) = e^{2t} \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}.$$

Therefore the general solution is

$$Y(t) = c_1 e^{-t} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} + c_3 e^{2t} \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$$

Solution to (b): Plugging $t = 0$ into the general solution from part (a), we get

$$Y(0) = c_1 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} + c_3 \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}.$$

Moreover, using the initial condition $Y(0) = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, we have

$$c_1 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} + c_3 \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

Solving this system gives $c_1 = 0$, $c_2 = c_3 = -1$. Plugging these values into the general solution from part (a) gives

$$Y(t) = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} + e^{2t} \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} e^{2t} \\ -1 + e^{2t} \\ 1 - e^{2t}. \end{bmatrix}$$

This is the solution to the initial value problem.

End of Example 178. □