30 2025-11-03 | Week 11 | Lecture 30

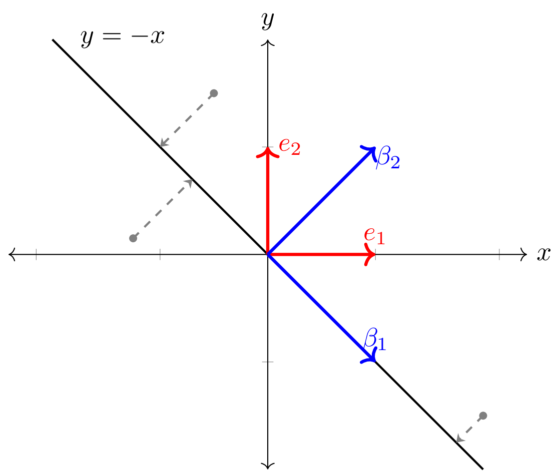
> ***The nexus question of this lecture:*** *What can we say about two matrices that represent the same linear transformation?*

This lecture draws from G. Strang's textbook "Linear Algebra and its Applications"

30.1 Similarity

Example 136 (Two matrices that represent the same projection)**.** Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the projection onto the line $y = -x$. We don't need a basis to describe this transformation. Even without a matrix, we can observe geometric properties of T:

- the dimension of its range is 1, so it has rank 1
- the dimension of its kernel (the line $y = x$) is 1, so the nullity is 1
- it collapses space, so has determinant 0
- it has two eigenvalues: 1 (with eigenvector β_1) and 0 (with eigenvector β_2)



Recall we can always represent a linear transformation $T : V \to V$ in the form of a matrix, but this representation depends on a choice of basis for V (see Section 24.2). We'll compare two choices of basis, $\alpha = \{e_1, e_2\}$ and $\beta = \{\beta_1, \beta_1\}$ (shown in the picture above).

- **The matrix $[T]_\beta^\beta$.** Note that β_1 and β_2 are eigenvectors of T. They are linearly independent and therefore form a basis (this doesn't always happen, but when it does, eigenvectors are the best basis). Using the notation $\begin{bmatrix} a \\ b \end{bmatrix}_\beta = a\beta_1 + b\beta_2$, observe from the figure that

  $$T(\beta_1) = \beta_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}_\beta \quad \text{and} \quad T(\beta_2) = 0 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}_\beta .$$

  Hence

  $$[T]_\beta^\beta = \begin{bmatrix} | & | \\ T(\beta_1) & T(\beta_2) \\ | & | \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}.$$

  This is the matrix of T with respect to the basis β. The first column come from the first basis vector (projected onto itself). The second colunn comes from the basis vector that is projected to zero.

- **The matrix $[T]_\alpha^\alpha$.** We'll compute this matrix using a change-of-basis matrix.

  Let P be the change-of-basis matrix from α to β. To compute P, write β_1, β_2 in terms of α_1, α_2:
$$\beta_1 = e_1 - e_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}_\alpha \quad \text{and} \quad \beta_2 = e_1 + e_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}_\alpha$$
  Therefore the change of basis matrix from α to β is
$$P = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$$
  This matrix has inverse
$$P^{-1} = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}.$$
  Now we compute $[T]_\alpha^\alpha$. By Theorem 116,
$$[T]_\beta^\beta = P^{-1}[T]_\alpha^\alpha P, \tag{25}$$
  which implies
$$[T]_\alpha^\alpha = P[T]_\alpha^\alpha P^{-1}.$$
  Therefore
$$\begin{aligned} [T]_\alpha^\alpha &= \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} \\ &= \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix}. \end{aligned}$$

Conclusion: the matrices
$$[T]_\alpha^\alpha = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix} \quad \text{and} \quad [T]_\beta^\beta = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$
both represent the same linear transformation T (a projection), but with different choice of bases. The linear transformation T has many geometric properties which don't depend on the choice of basis:

- the dimension of its range (the 'rank')
- the dimension of its kernel (the 'nullity')
- the determinant
- its eigenvalues

These properties don't hinge on how we choose to represent T as a matrix. Therefore although the matrices $[T]_\alpha^\alpha$ and $[T]_\beta^\beta$ are different, they share the same rank, determinant, nullity, etc.

End of Example 136. □

The two matrices $[T]_\alpha^\alpha, [T]_\beta^\beta$ from Example 136 are an example of *similar matrices*, which we define next.

Definition 137 (Similar)**.** Let $A, B \in \mathcal{M}_{n\times n}$. We say that B is ***similar*** to A if there exists an invertible matrix P such that
$$B = P^{-1}AP. \tag{26}$$

Question What does it *mean* for two matrices to be similar?

Answer: It means that they represent the same linear transformation with respect to different bases. This is because the formula Eq. (26) is the change-of-basis formula from Theorem 116 (like we used in Eq. (25)).

To be precise, if $A, B \in \mathcal{M}_{n\times n}(\mathbb{R})$ are similar, then there is a linear transformation $T : \mathbb{R}^n \to \mathbb{R}^n$ such that
$$A = [T]_\alpha^\alpha \quad \text{and} \quad B = [T]_\beta^\beta$$
for some choice of bases α and β of $\mathbb{R}^n$. Since quantities like rank, nullity, determinant, trace, and eigenvalues all represent geometric properties inherent to the linear transformation T, and since A and B are merely different representations of this transformation, the following theorem holds:

Theorem 138. *If A and B are similar, then they have the same rank, nullity, determinant, trace, eigenvalues, and characteristic polynomial.*