37 2025-11-19 | Week 13 | Lecture 37

topic: continuation of introduction to systems of linear differential equations

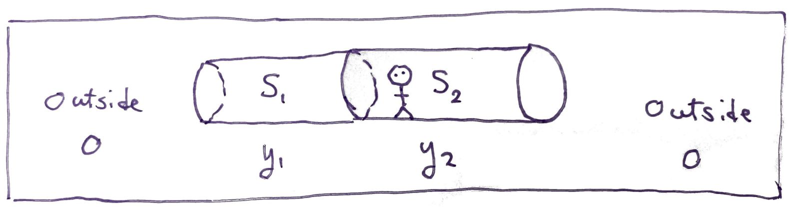
Theorem 166 (Pure exponential solutions)**.** *Let A be an $n \times n$ matrix, let v be an eigenvector of A with eigenvalue λ, and let c be any scalar. Then*

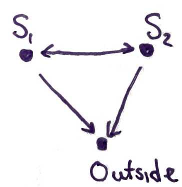
$$Y(t) = ce^{\lambda t}v$$

is a solution of the system $Y' = AY$.

37.1 Diffusion example

Example 167 (Diffusion)**.** A person is standing in a large pipe drops a flask of chemicals, releasing a chemical gas. The tube has 2 chambers, S_1 and S_2, as shown:





Let

$$\begin{aligned} y_1(t) &= \text{ concentration of chemical in chamber } S_1 \text{ at time } t \\ y_2(t) &= \text{ concentration of chemical in chamber } S_2 \text{ at time } t \end{aligned}$$

and assume that the concentration outside is always zero (the gas gets immediately blown away by the wind).

We make the following assumption:

At each time t, the diffusion rate between adjacent areas is the difference in concentrations.

Based on this assumption, y_1 (the concentration in S_1) is changing due to two factors:

- the diffusion to the outside, which is $(0 - y_1)$
- the diffusion into or out of S_2, which is $(y_2 - y_1)$.

Hence

$$\frac{dy_1}{dx} = (y_2 - y_1) + (0 - y_1) = -2y_1 + y_2.$$

Similar reasoning implies that

$$\frac{dy_2}{dx} = (y_1 - y_2) + (0 - y_2) = y_1 - 2y_2.$$

This gives us the following system of differential equations:

$$\begin{aligned} \frac{dy_1}{dt} &= -2y_1 + y_2 \\ \frac{dy_2}{dt} &= y_1 - 2y_2 \end{aligned}$$

Writing this in matrix form gives the matrix equation

$$\begin{bmatrix} y_1'(t) \\ y_2'(t) \end{bmatrix} \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} \tag{31}$$

This is a differential equation of the form

$$Y' = AY.$$

Observe that A has eigenvalue -1 with eigenvector $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and eigenvalue -3 with eigenvector $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$. Therefore by Theorem 166, our system of differential equations has solutions two pure exponential solutions:

$$Y_1(t) = c_1 e^{-t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad \text{and} \quad Y_2(t) = c_2 e^{-3t} \begin{bmatrix} 1 \\ -1 \end{bmatrix},$$

where c_1, c_2 are arbitrary constants. By superposition, the general solution is

$$Y(t) = c_1 e^{-t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 e^{-3t} \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

We can write this as

$$Y(t) = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} e^{-t} & 0 \\ 0 & e^{-3t} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}. \tag{32}$$

Further, suppose we are given the initial condition

$$Y(0) = B = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}.$$

(For our problem, a reasonable initial condition would be $Y(0) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, since the chemical gas gets released in section 2 of the pipe at time $t = 0$, but it hasn't reached S_1 yet.)

Plugging $t = 0$ into Eq. (32) gives

$$B = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

or equivalently

$$\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}^{-1} B$$

Plugging this back into Eq. (31) yeilds

$$Y(t) = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} e^{-t} & 0 \\ 0 & e^{-3t} \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}^{-1} B$$

Writing

$$P = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}, \quad \Lambda = \begin{bmatrix} -1 & 0 \\ 0 & -3 \end{bmatrix}, \quad \text{and} \quad e^{\Lambda t} = \begin{bmatrix} e^{-t} & 0 \\ 0 & e^{-3t} \end{bmatrix},$$

we see that our solution takes the form

$$Y(t) = P e^{\Lambda t} P^{-1} B. \tag{33}$$

If we take our initial condition to be $Y(0) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, as noted earlier, then we have $b_1 = 0$ and $b_2 = 1$. In that case, the matrix multiplication gives

$$\begin{aligned} Y(t) &= P e^{\Lambda t} P^{-1} B \\ &= \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & -3 \end{bmatrix} \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\ &= \frac{1}{2} \begin{bmatrix} e^{-t} + e^{-3t} & e^{-t} - e^{-3t} \\ e^{-t} - e^{-3t} & e^{-t} + e^{-3t} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} e^{-t} - e^{-3t} \\ e^{-t} + e^{-3t} \end{bmatrix}. \end{aligned}$$

In other words, at time t, the concentration of gas in S_1 is $y_1(t) = e^{-t} - e^{-3t}$ and the concentration of gas in S_2 is $y_2(t) = e^{-t} + e^{-3t}$. One can check that this satisfies

$$\begin{cases} Y'(t) = \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix} Y(t) \\ Y(0) = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \end{cases}$$

(check this!), so it is a solution to the differential equation. Phsycially, it also makes since, since it indicates that both concentrations are positive and tend to 0 as $t \to \infty$ as the gas diffuses. From the equations we can also see that diffusion causes the gas concentration to decay exponentially in time.

End of Example 167. □

37.2 General solution to $Y' = AY$

In Example 167 we obtained the fundamental solution shown in Eq. (33). The form that it took illustrate the following theorem:

Theorem 168. *If A can be diagonalized $A = P\Lambda P^{-1}$, then*

$$Y = Pe^{\Lambda t}P^{-1}B$$

is a solution to the differential equation

$$\begin{cases} Y' = Ay \\ Y(0) = B \end{cases}$$

Remark 169. In the above theorem, the columns of P are the eigenvectors $v_1, \ldots, v_n$ of A. So the formula is for the solution can be written as

$$Y(t) = \begin{bmatrix} | & & | \\ v_1 & \cdots & v_n \\ | & & | \end{bmatrix} \begin{bmatrix} e^{\lambda_1 t} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & e^{\lambda_n t} \end{bmatrix} P^{-1}Y(0),$$

Another way to write the solution: Let $c = (c_1, \ldots, c_n)^\top = P^{-1}Y(0)$, in which case we get

$$Y(t) = c_1 e^{\lambda_1 t} v_1 + \ldots + c_n e^{\lambda_n t} v_n.$$

Remark 170. When A is diagonalizable (as is assumed in Theorem 174), we have

$$Pe^{\Lambda t}P^{-1} = e^{At}$$

so the formula for the solution can be written in the wonderfully simple form

$$Y(t) = e^{At}Y(0).$$

In fact, this formula holds even if A isn't diagonalizable, but to understand that case, we'll need to first understand what e^{At} even means.