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34.1 Multiplicity

Given a polynomial $p(x)$ with root r, the ***multiplicity*** of r in $p(x)$ is the highest power of $(x-r)$ which divides $p(x)$. For example, if

$$p(x) = (x-3)(x-7)^4(x+1)^2(x-1/2)^{10}.$$

then $p(x)$ four roots, $x=3$, $x=7$, $x=-1$, and $x=1/2$. The multiplicities of these roots are 1, 4, 2, and 10 respectively.

Definition 149 (Algebraic and geometric multiplicity)**.** Let A be an $n \times n$ matrix, and let λ be an eigenvalue of A. The ***algebraic multiplicity (AM)*** of λ is its multiplicity in the characteristic polynomial $\det(\lambda I - A)$. The ***geometric multiplicity (GM)*** of λ is the dimension of the eigenspace $E_\lambda = NS(\lambda I - A)$.

Theorem 150. *Given an eigenvalue λ of a matrix, we have*

$$1 \leq GM \leq AM$$

Theorem 151. *Let A be an $n \times n$ matrix. The following are equivalent:*

- *For every eigenvalue of A, the algebraic multiplicity equals the geometric multiplicity.*
- *A is diagonalizable.*

The next example shows that the second inequality can be strict.

Example 152 (A nondiagonalizable linear transformation)**.** Consider the linear transformation given by the matrix

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

This is a shear transform. We will use Theorem 151 to show that it is not diagonalizable.

Obser that A has only one eigenvalue, namely $\lambda = 1$. We can see this by computing the characteristic polynomial:

$$\det(A - \lambda I) = \det \begin{bmatrix} 1-\lambda & 1 \\ 0 & 1-\lambda \end{bmatrix} = (1-\lambda)^2$$

The algebraic multiplicity of $\lambda = 1$ is 2. The geometric multiplicity is the dimension of the nullspace of

$$B = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

This matrix has rank 1. By the rank nullity theorem, we have

$$\operatorname{rank}(B) + \dim NS(B) = \text{number of columns} = 2$$

Therefore

$$\dim NS(B) = 1.$$

So the geometric multiplicy of the eigenvalue $\lambda = 1$ is 1.

In this case, the geometric multiplicy (2) is less than the algebraic multiplicity (1). So A is not diagonalizable.

End of Example 152. □

34.2 An application of diagonalizability

Example 153. Let $A = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix}$. Suppose that we want to compute A^4 or A^{100}, but we don't want to do that much matrix multiplication.

First observe that

$$A = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

(You should check this). This equation is of the form

$$A = PDP^{-1}$$

where $D = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ is a diagonal matrix.

Fact: If $D = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$ then $D^k = \begin{bmatrix} a^k & 0 \\ 0 & b^k \end{bmatrix}$. And a similar equation holds for $n \times n$ diagonal matrices.

By this fact, we have $D^4 = D$. (Check this.)

Since $A = PDP^{-1}$, we have

$$\begin{aligned}
A^4 &= (PDP^{-1})^4 \\
&= (PDP^{-1})(PDP^{-1})(PDP^{-1})(PDP^{-1}) \\
&= PD(P^{-1}P)D(P^{-1}P)D(P^{-1}P)DP^{-1} && \text{by associativity} \\
&= PD^4P^{-1} && \text{this is the key step I wanted to show you} \\
&= PDP^{-1} && \text{since } D^4 = D \\
&= A.
\end{aligned}$$

That's neat: $A^4 = A$.

End of Example 153. □