22 2025-10-15 | Week 08 | Lecture 22

> ***The nexus question of this lecture:*** *Why do the row space and column space always have the same dimension?*

This lecture is based on sections 5.1

22.1 The dimension of the column space equals the dimension of the row space

Consider the 3×2 matrix

$$\begin{bmatrix} 1 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

The row space is the vector space $\left\{ \begin{bmatrix} x & x \end{bmatrix} : x \in \mathbb{R} \right\}$. The column space is the vector space $\left\{ \begin{bmatrix} x \\ 0 \\ 0 \end{bmatrix} : x \in \mathbb{R} \right\}$.

These are not the same. But they have the same dimesnion: 1.

The fact that the dimensions are equal is not a coincidence.

Theorem 102. *Given any $m \times n$ matrix A, the row space and column space always have the same dimesion. That is,* $\dim RS(A) = \dim CS(A)$.

22.2 Preliminaries

We begin with the following theorem.

Theorem 103. *Let X, Y be vector spaces. If $T : X \to Y$ is a linear transformation, then* $\ker(T)$ *is a subspace of X and* $\operatorname{range}(T)$ *is a subspace of Y.*

Proof. The fact that $\ker(T)$ is a subspace follows similarly to the proof of Theorem 56 (the "kernels are subspaces" theorem from lecture 13).

To show that $\operatorname{range}(T)$ is a subspace of Y, we need only verify two facts:

(i) $y + y' \in \operatorname{range}(T)$ whenever $y, y' \in \operatorname{range}(T)$

(ii) $cy \in \operatorname{range}(T)$ whenever $y \in Y$ and $c \in \mathbb{R}$

- **Proof of (i).** Let $y, y' \in Y$. Then there exist x, x' such that $T(x) = y$ and $T(x') = y'$. Since X is closed under vector addition, $x + x' \in X$. Moreover,

  $$T(x + x') = T(x) + T(x') = y + y'$$

  Therefore $y + y' \in \operatorname{range}(T)$.

- **Proof of (ii).** Let $y \in \operatorname{range}(T)$. Then there exists $x \in X$ such that $T(x) = y$. Therefore

  $$\begin{aligned} cy &= cT(x) \\ &= T(cx) \end{aligned}$$

  Since X is closed under scalar multiplication $cx \in X$. Therefore $cy \in \operatorname{range}(T)$.

□

Because $\ker(T), \operatorname{range}(T)$ are subspaces by Theorem 103, they each have bases, a fact which we will need to use to prove the following theorem.

22.3 Rank-nullity, again

Theorem 104 (Rank Nullity Theorem – Theorem 5.4 in textbook)**.** *If $T : V \to W$ is a linear transformation where V is a finite dimensional vector space, then*

$$\dim\ker(T) + \dim\operatorname{range}(T) = \dim V$$

If T is an $m \times n$ matrix A, then $A : \mathbb{R}^n \to R^m$, and we can restate this as

$$\dim NS(A) + \dim CS(A) = n \tag{19}$$

(recall that the column space is the range, and that nullspace and kernel are synonyms).

Proof sketch. Suppose $\dim\ker(T) = k$ and $\dim V = n$ (note: $k \leq n$). Then with some work (omitted, see p242), one can find vectors $v_1, \ldots, v_n \in V$ which form a basis for V with the property that

- $v_1, \ldots, v_k$ is a basis for $\ker(T)$; and,
- $\underbrace{T(v_{k+1}), T(v_{k+1}), \ldots, T(v_n)}_{n-k \text{ vectors}}$ is a basis for range(T).

This shows that $\dim\operatorname{range}(T) = n - k$. Therefore we have:

- $\dim\ker(T) = k$
- $\dim\operatorname{range}(T) = n - k$
- $\dim V = n$

Putting these together implies

$$\dim\ker(T) + \dim\operatorname{range}(T) = \dim(V).$$

□

22.4 Dimension of the nullspace

Theorem 105. *Let A be an $n \times m$ matrix. Then*

$$\dim NS(A) = n - r,$$

where $r = \dim RS(A)$.

An example will illustrate why Theorem 105 is true. Suppose we have a 4×5 matrix A and we want to solve the linear system $AX = 0$. Suppose that row-reducing the augmented matrix gives

$$\left[\begin{array}{ccccc|c} 1 & 0 & 2 & 3 & 4 & 0 \\ 0 & 1 & 5 & 6 & 7 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

This has two nonzero zero rows after row-reduction, so $r = 2$. Theorem 105 says that

$$\dim NS(A) = n - r = 5 - 2 = 3.$$

We can check this. The augmented matrix above corresponds to the linear system

$$x_1 + 2x_3 + 3x_4 + 4x_5 = 0$$
$$x_2 + 5x_3 + 6x_4 + 7x_5 = 0$$

or

$$x_1 = -2x_3 - 3x_4 - 4x_5$$
$$x_2 = -5x_3 - 6x_4 - 7x_5$$

and this has three free variables x_3, x_4, x_5. Therefore the solutions to $AX = 0$ are of the form

$$\begin{bmatrix} -2x_3 - 3x_4 - 4x_5 \\ -5x_3 - 6x_4 - 7x_5 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = x_3 \begin{bmatrix} -2 \\ -5 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -3 \\ -6 \\ 0 \\ 1 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} -4 \\ -7 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

Therefore the nullspace has dimension 3 because

$$NS(A) = \text{Span} \left\{ \begin{bmatrix} -2 \\ -5 \\ 1 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} -3 \\ -6 \\ 0 \\ 1 \\ 0 \end{bmatrix} \begin{bmatrix} -4 \\ -7 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

22.5 Proof that $\dim CS(A) = \dim RS(A)$

We can now prove the following theorem, which says that the row space and column space of a matrix are the same thing.

Theorem 106. *If A is any $m \times n$ matrix, then*

$$\dim RS(A) = \dim NS(A)$$

Proof. By Theorem 104,
$$\dim NS(A) + \dim CS(A) = n. \tag{20}$$
By Theorem 105,
$$\dim NS(A) = n - \dim RS(A) \tag{21}$$
Plugging Eq. (21) into Eq. (20) gives

$$n - \dim RS(A) + \dim CS(A) = n$$

wich simplifies to
$$\dim RS(A) = \dim CS(A).$$

□

Theorem 106 says that we are justified in writing

$$\text{rank}(A) = \dim NS(A) = \dim CS(A)$$

and hence that
$$\text{rank}(A) + \text{nullity}(A) = n$$
for any $m \times n$ matrix A.