1 2025-08-25 | Week 01 | Lecture 01

This lecture is based on textbook section 1.1. Introduction to Systems of Linear Equations

> ***The nexus question of this lecture:*** *What is a system of linear equations, and what does it mean to 'solve' a system of linear equations?*

1.1 A first example of a system of linear equations

We begin with something concrete.

Example 1 (A first example of a *system of linear equations*)**.** Consider the following word problem:

> *A boat travels between two ports on a river 48 miles apart. When traveling downstream (i.e., with the current), the trip takes 4 hours, but when traveling upstream (i.e., fighting the current), the trip takes 6 hours.*
>
> *Assume that the boat and the current are both moving at a constant speed. What is the speed of the boat in still water, and what is the speed of the current?*

This problem is hard to reason through without writing something down, but becomes much simpler when we formalize it mathematically with equations. The unknowns are (1) **the speed of the boat in still water** and (2) **the speed of the current**. So let

$$x := (\text{the speed of the boat in still water})$$
$$y := (\text{the speed of the current}).$$

The speed of the boat going downstream is $x + y$. Therefore, since (speed) $\times$ (time) = (distance travelled), we have

$$4(x + y) = 48, \quad \text{or equivalently} \quad x + y = 12.$$

Similarly, the sped of the boat going upstream is $x - y$, so

$$6(x - y) = 48, \quad \text{or equivalently} \quad x - y = 8$$

Thus, we have the following *system of linear equations:*

$$\begin{cases} x + y = 12 \\ x - y = 8. \end{cases} \tag{1}$$

This system has **two equations** and **two variables** (x and y). You have encountered systems of equations like this many times. With the help of the technology of algebra, solving this problem (namely, solving System (1)) is much easier than solving the original word problem.

- In this case, the problem can be easily solved **algebraically** using a substitution (e.g., plug $x = 8 + y$ into the first equation and solve for y, then solve for x after finding y). This gives the solution $(x, y) = (10, 2)$. The speed of the boat in still water is 10mph. The speed of the river current is 2mph.
- We can conceive of another type of solution, which uses a **geometric**, rather than algebraic perspective: observe that each equation $x + y = 12$ and $x - y = 8$ represents a line on the xy-plane. Plot the lines. Their intersection is the point $(10, 2)$, which is the solution.
- However, solving systems of equations like in (1) becomes more cumbersome when there are lots of variables and equations. Doing substitutions and algebraic manipulations will still work, but will be tedius and difficult if you have many equations and variables.

  Later, we will introduce a general algorithm which can solve any such system. This algorithm is called ***Gauss-Jordan elimination***, and it will be one of the core techniques that we will use to solve many types of problems in this class.

End of Example 1. □

1.2 Key definitions: linear systems and their solutions

In this section, we formalize the mathematical objects we are studying.

Definition 2 (Linear equation)**.** A ***linear equation*** in the variables $x_1, \ldots, x_n$ is an equation that can be written in the form

$$a_1x_1 + a_2x_2 + \ldots + a_nx_n = b,$$

where $a_1, \ldots, a_n$ and b are constants (e.g., fixed real numbers). The numbers $a_1, \ldots, a_n$ are called ***coefficients***.

Note that the variables $x_1, \ldots, x_n$ are not raised to any powers. That's what makes the equation *linear*. If we had squares or cubes of some of the x_i's, or products like x_1x_3, then the equation would be quadratic or cubic, or something else, but not linear.

Example 3 (Examples of linear equations)**.**

- The equation
$$2x - 3y = 1$$
is a linear equation in the variables x and y. Its graph is a line on the xy-plane.

- The equation
$$3x - y + 2z = 9$$
is a linear equation in the variables x, y and z. Its graph is a plane in 3-dimensional space (denoted $\mathbb{R}^3$).

- The equation
$$-x_1 + 5x_2 + \pi^2 x_3 + \sqrt{2}x_4 = e^2$$
is a linear equation in the variables x_1, x_2, x_3, and x_4. The coefficients are
$$a_1 = -1, \quad a_2 = 5, \quad a_3 = \pi, \quad \text{and} \quad a_4 = \sqrt{2}.$$
The graph of this linear system is a 3-dimensional hyperplane in 4d-space (i.e., $\mathbb{R}^4$).

  Observation: There is a simple relationship between the number of variables and the dimension of the graph:
$$\text{dimension of graph} = (\#\text{ of variables}) - 1.$$

End of Example 3. □

Definition 4 (Linear system, solution of a linear system)**.** When considered together, a collection of m linear equations

$$\left\{ \begin{array}{c} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{21}x_2 + \ldots + a_{2n}x_n = b_2 \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n = b_m \end{array} \right. \tag{2}$$

is called a ***system of linear equations***, or ***linear system*** for short. A ***solution*** to a system of linear equations is a set of values for $x_1, \ldots, x_n$ which satisfy all equations in system (2).

Example 5 (A system of linear equations)**.** An example of a system of linear equations is

$$\left\{ \begin{array}{rcrcrcr} x & - & y & + & z & = & 0 \\ 2x & - & 3y & + & 4z & = & -2 \\ -2x & - & y & + & z & = & 7 \end{array} \right.$$

When a linear system like this walks in the door, we always first ask two basic questions: (1) 'how many equations does it have?' and (2) 'how many variables does it have?'. In this case, we have $m = 3$ equations and $n = 3$ variables.

End of Example 5. □

1.3 How to understand solutions of linear systems geometrically

Here is a very useful geometric perspective. In system (2), we have a system of m equations expressed in n variables $x_1, \ldots, x_n$. Each of the m equations is the equation of some hyperplane[1] which lives in n-dimensional space ($\mathbb{R}^n$). *The solution to the linear system is the intersection of these hyperplanes.*

For example, in Example 5, the 'hyperplanes' were lines, and their intersection was the point $(x, y) = (10, 2)$.

We will spend a lot of time understanding what hyperplanes look like, and what intersections of hyperplanes look like.

[1]Note: Hyperplanes will be defined more formally later, but for now can be thought of as generalized lines or planes, since a 1-dimensional hyperplanes is a *line* and a 2-dimensional hyperplane is a *plane*.