Math 307: Homework 02

Due Wednesday, September 17 (at the beginning of class)

Problem 1. The *standard unit vectors* in $\mathbb{R}^3$ are the vectors

$$e_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad e_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad e_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Let $A = (a_{ij}) \in \mathcal{M}_{3\times 3}(\mathbb{R})$. Compute the following:

$$Ae_1, \quad Ae_2, \quad Ae_3.$$

What do you notice about these products?

Problem 2. Suppose that $A, B \in M_{n\times n}(\mathbb{R})$.

(a) Show that $(A+B)^2 = A^2 + AB + BA + B^2$

(b) Explain why $(A+B)^2$ is not equal to $A^2 + 2AB + B^2$ in general.

(c) Compute $A^2 + AB + BA + B^2$ when $A = \begin{bmatrix} 1 & 3 \\ -1 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & -2 \\ 1 & -3 \end{bmatrix}$

Problem 3 (Row reduction)**.** Solve the system of linear equations using row reduction

$$\begin{aligned} 3x + y - 2z &= 3 \\ x - 8y - 14z &= -14 \\ x + 2y + z &= 2 \end{aligned}$$

Interpret your result geometrically. Provide a sketch or an image (e.g., using Desmos) of the the solution.

Problem 4 (Row reduction)**.** Solve the system of linear equations using row reduction

$$\begin{aligned} x + 3z &= 0 \\ 2x + y - z &= 0 \\ 4x + y + 5z &= 0 \end{aligned}$$

Interpret your result geometrically. Provide a sketch or an image (e.g., using Desmos) of the the solution.

Problem 5. Solve the system of linear equations

$$\begin{aligned} x + 2y + z &= -2 \\ 2x + 2y - 2z &= 3 \end{aligned}$$

Problem 6. Determine conditions on the numbers a, b, and c so that the following system of linear equations has at least one solution:

$$\begin{aligned} 2x - y + 3z &= a \\ x - 3y + 2z &= b \\ x + 2y + z &= c \end{aligned}$$

Hint: use row reduction.

Problem 7. Write the linear system in matrix form $AX = B$:

$$\begin{aligned} 2x - y + 4z &= 1 \\ x + y - z &= 4 \\ y + 3x &= 5 \\ x + y &= 2 \end{aligned}$$

Problem 8. Let $f(x, y) = \frac{1}{2}(x^2 + y^2)$. Suppose that $f(x, y)$ represents the temperature of the point (x, y) on the plane, in Kelvin. For example $f(1, 3) = 5$, so the temperature of the point point $(1, 3)$ is 5 Kelvin.

(a) What is the coldest point on the plane? What is its temperature?

(b) Let $g(x, y) = 2x - y - 5$. Use the methods of Lagrange multipliers to find the minimum of $f(x, y)$ subject to the constraint $g(x, y) = 0$. (In other words, we are finding the coldest point on the line $y = 2x - 5$.)

(c) In part (b), you had to solve a system of equations. Identify the system. How many equations and how many variables does it have? Is it a linear?

Problem 9. Let

$$A = \begin{bmatrix} 2 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

(a) Show that A can be row-reduced to the matrix

$$U = \begin{bmatrix} 2 & 1 & 0 & 0 \\ 0 & 3/2 & 1 & 0 \\ 0 & 0 & 4/3 & 1 \\ 0 & 0 & 0 & 5/4 \end{bmatrix}$$

(b) Find a matrix L of the form

$$L = \begin{bmatrix} x_{11} & 0 & 0 & 0 \\ x_{21} & x_{22} & 0 & 0 \\ x_{31} & x_{32} & x_{33} & 0 \\ x_{41} & x_{42} & x_{43} & x_{44} \end{bmatrix}$$

such that

$$A = LU$$

Hint: first compute the matrix product LU symbolically and set the result equal to A. This will give a bunch of linear equations. Use the linear equations to deduce the values of the x_{ij}*'s.*

Problem 10. A 2×2 matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is said to be *positive semi-definite* if

$$\begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \geq 0$$

for all real numbers x, y. Show that the matrix

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

is positive semi-definite.

Problem 11. Let

$$A = \begin{bmatrix} 1 & -2 & 3 \\ 2 & -1 & 4 \\ 1 & 1 & 1 \end{bmatrix}$$

(a) Compute the determinant of A.

(b) Is A invertible? Justify your answer.

Problem 12. Compute the inverse of the matrix using row reduction:

$$A = \begin{bmatrix} 2 & -1 & 3 \\ 1 & 1 & -2 \\ 1 & 1 & 5 \end{bmatrix}$$