24 2025-10-20 | Week 09 | Lecture 24

> ***The nexus question of this lecture:*** *How do we find a matrix to represent a linear transformation?*

This lecture is based on section 5.2 and 5.3

24.1 The algebra of linear transformations

24.1.1 Algebraic operations

Let V and W be vector spaces. Consider two linear transformations $T : V \to W$ and $S : V \to W$. The function $T + S$ is defined as
$$(T+S)(v) := T(v) + S(v),\ v \in V$$
And for $c \in \mathbb{R}$, cT is defined as
$$(cT)(v) := cT(v),\ v \in V$$

Theorem 110. *If $T, S : V \to W$ are linear transformations, then so are $T + S$ and cT.*

In other words, the set of linear transformations $T : V \to W$ is closed under addition and scalar multiplication. This suggests that set of linear transformations from V to W forms a vector space. (It does. The rabbit hole goes deep...)

Example 111. Suppose

$$S\begin{bmatrix} x \\ y \end{bmatrix} := \begin{bmatrix} 2x - y \\ x + 2y \end{bmatrix} \quad \text{and} \quad T\begin{bmatrix} x \\ y \end{bmatrix} := \begin{bmatrix} x + y \\ x - y \end{bmatrix}$$

End of Example 111. □

Question 1: What is $S + 3T$?

$$\begin{aligned}
(S+3T)\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) &= S\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) + 3T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) \\
&= \begin{bmatrix} 2x - y \\ x + 2y \end{bmatrix} + 3\begin{bmatrix} x + y \\ x - y \end{bmatrix} \\
&= \begin{bmatrix} 2x - y \\ x + 2y \end{bmatrix} + \begin{bmatrix} 3x + 3y \\ 3x - 3y \end{bmatrix} \\
&= \begin{bmatrix} 5x + 2y \\ 4x - y \end{bmatrix}
\end{aligned}$$

Question 2: What is $S \circ T$?

$$\begin{aligned}
S \circ T\begin{bmatrix} x \\ y \end{bmatrix} &= S\left(T\begin{bmatrix} x \\ y \end{bmatrix}\right) \\
&= S\begin{bmatrix} x + y \\ x - y \end{bmatrix} \\
&= \begin{bmatrix} 2(x+y) - (x-y) \\ (x+y) + 2(x-y) \end{bmatrix} \\
&= \begin{bmatrix} x + 3y \\ 3x - y \end{bmatrix}
\end{aligned}$$

At this point we note that we can represent

$$S = \begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix}, \quad T = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}, \quad S \circ T = \begin{bmatrix} 1 & 3 \\ 3 & -1 \end{bmatrix}, \quad S + 3T = \begin{bmatrix} 5 & 2 \\ 4 & -1 \end{bmatrix}$$

Notice that

$$\underbrace{\begin{bmatrix}2 & -1\\1 & 2\end{bmatrix}}_{S}\underbrace{\begin{bmatrix}1 & 1\\1 & -1\end{bmatrix}}_{T}=\underbrace{\begin{bmatrix}1 & 3\\3 & -1\end{bmatrix}}_{S\circ T}\quad\text{and that}\quad\underbrace{\begin{bmatrix}2 & -1\\1 & 2\end{bmatrix}}_{S}+3\underbrace{\begin{bmatrix}1 & 1\\1 & -1\end{bmatrix}}_{T}=\underbrace{\begin{bmatrix}5 & 2\\4 & -1\end{bmatrix}}_{S+3T}$$

24.1.2 Algebraic rules

The standard algebraic rules work with linear transformations:

Theorem 112. *Let R, S, T be linear transformations and $c, d \in R$. Then*

- $S + T = T + S$
- $R + (S + T) = (R + S) + T$
- $c(dT) = (cd)T$
- $c(S + T) = cS + cT$
- $R(ST) = (RS)T$
- $R(S + T) = RS + RT$
- $(R + S)T = RT + ST$
- $c(ST) = (cS)T = S(cT)$

Observe the glaring lack of the rule that $ST = TS$, which doesn't hold in general. In fact, all of the above rules are are identical to the rules for matrix algebra (i.e., if S, T, and R are matrices).

This makes sense because, as we show in the next section, linear transformations can be encoded with matrices.

24.2 Representing a linear transformation with a matrix

Let $T : V \to W$ be a linear transformation.
Goal: Find a matrix to represent T.

We'll start by choosing a basis for V and W. Let $\alpha = \{v_1, \ldots, v_n\}$ be a basis for V and $\beta = \{w_1, \ldots, w_m\}$ be a basis for W.

Following the ideas of Example 100, it is enough to understand how T acts on the basis $v_1, \ldots, v_n$.

Observe that

- Every vector in W can be written as a unique linear combination of the basis vectors $w_1, \ldots, w_m$ (because these form a basis).
- The vectors $T(v_1), \ldots, T(v_n)$ are all vectors in W.

From these two observations, we can find scalars $a_{ij} \in \mathbb{R}$ such that

$$\begin{aligned}
T(v_1) &= a_{11}w_1 + a_{21}w_2 + \ldots + a_{m1}w_m\\
T(v_2) &= a_{12}w_1 + a_{22}w_2 + \ldots + a_{m2}w_m\\
&\vdots\\
T(v_n) &= a_{1n}w_1 + a_{2n}w_2 + \ldots + a_{mn}w_m.
\end{aligned}$$

We can change notation by writing thes as ***coordinate vectors***:

$$T(v_1) = \begin{bmatrix}a_{11}\\a_{21}\\\vdots\\a_{m1}\end{bmatrix}_\beta \qquad T(v_2) = \begin{bmatrix}a_{12}\\a_{22}\\\vdots\\a_{m2}\end{bmatrix}_\beta \qquad \cdots \qquad T(v_n) = \begin{bmatrix}a_{1n}\\a_{2n}\\\vdots\\a_{mn}\end{bmatrix}_\beta$$

Here, the subscript β indicates that these vectors represent linear combinations of the basis $\beta = \{w_1, \dots, w_m\}$. That is, we are using the notation

$$\begin{bmatrix} a_{1i} \\ a_{2i} \\ \vdots \\ a_{mi} \end{bmatrix}_\beta = a_{1i}w_1 + a_{2i}w_2 + \ldots + w_{mi}w_m, \quad i = 1, \dots, n.$$

We can then represent T simply as the $m \times n$ matrix

$$[T]_\alpha^\beta = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

whose columns are the above vectors. We call this matrix ***the matrix of T with respect to the bases α and β.*** It is denoted $[T]_\alpha^\beta$.

Example 113. Let $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x+y \\ x-y \end{bmatrix}$. Then we can write $v_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $v_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

Here, we take as our basis $\alpha = \beta = \{v_1, v_2\}$. Observe that

- $T(v_1) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$
- $T(v_2) = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

Then

$$[T]_\alpha^\beta = \begin{bmatrix} | & | \\ T(v_1) & T(v_2) \\ | & | \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

Which makes sense, because

$$T\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x+y \\ x-y \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}.$$

End of Example 113. $\square$