

## Math 307: Homework 09

*Due Wednesday, November 12. Material up to and including 5.4.*

**Problem 1.** Let  $A = \begin{bmatrix} 1 & -1 \\ 2 & 4 \end{bmatrix}$ . The *trace* of  $A$  is the sum of its main diagonal entries. Find the eigenvalues of  $A$ , verify that trace equals the sum of the eigenvalues, and that the determinant equals their product.

**Problem 2.** Find the eigenvectors and eigenvalues of

$$A = \begin{bmatrix} 3 & 4 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 0 & 0 & 2 \\ 0 & 2 & 0 \\ 2 & 0 & 0 \end{bmatrix}$$

Check that  $\lambda_1 + \lambda_2 + \lambda_3$  equals the trace and that  $\lambda_1\lambda_2\lambda_3$  equals the determinant. (Here  $\lambda_1, \lambda_2, \lambda_3$  are the eigenvalues of the matrix.)

**Problem 3.** Find the rank and nullspace of

$$A = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 0 & 0 & 1 & 2 \\ 0 & 0 & 1 & 2 \\ 1 & 1 & 1 & 0 \end{bmatrix}$$

**Problem 4.** Suppose all vectors  $x$  in the unit square  $0 \leq x_1 \leq 1, 0 \leq x_2 \leq 1$  are transformed to  $Ax$ . (Here  $A$  is a  $2 \times 2$  matrix).

- (a) What is the shape of the transformed region (all  $Ax$ )?
- (b) For which matrices  $A$  is that region a square?
- (c) For which  $A$  is it a line?
- (d) For which  $A$  is the new area still 1?

**Problem 5.** Given an eigenvalue  $\lambda$  of matrix  $A$ , the *eigenspace* of  $\lambda$  is the subspace  $NS(\lambda I - A)$ . For the following matrices, find the eigenvalues, and find a basis for each eigenspace.

(a)  $\begin{bmatrix} 3 & 0 \\ 8 & -1 \end{bmatrix}$

(b)  $\begin{bmatrix} 4 & -4 \\ 1 & 0 \end{bmatrix}$

(c)  $\begin{bmatrix} 4 & 0 & 1 \\ -2 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}$

**Problem 6.** Let  $A, B, C$  be square matrices. Prove the following statements:

- (a) If  $B$  is similar to  $A$ , then  $A$  is similar to  $B$ . (“Similarity is reflexive”)
- (b) If  $A$  is similar to  $B$ , and  $B$  is similar to  $C$ , then  $A$  is similar to  $C$ . (“Similarity is transitive”)
- (c) If  $A, B$  are similar, then  $\det(A) = \det(B)$ .
- (d) If  $A$  is invertible, then  $B$  is invertible and  $B^{-1}$  is similar to  $A^{-1}$ .

**Problem 7.** (*Hint for this problem: refer to the examples in section 5.1 of the textbook.*) Let  $T : \mathbb{R}^3 \rightarrow P_2$  be a linear transformation satisfying

$$T \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = x^2 + x, \quad T \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} = x^2 - x + 1, \quad T \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = x + 1.$$

(a) Find  $T \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ .

(b) Find  $T \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ .

**Problem 8.**

(a) Construct a matrix whose nullspace contains the vector  $x = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$

(b) Construct a matrix whose column space is spanned by  $\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$  and whose row space is spanned by  $\begin{bmatrix} 1 & 5 \end{bmatrix}$ .

(c) How can you construct a matrix that transforms the coordinate vectors  $e_1, e_2, e_3$  into three given vectors  $v_1, v_2, v_3$ . When will that matrix be invertible?

**Problem 9.** Let  $T : \mathbb{R}^3 \rightarrow \mathbb{R}^3$  be given by

$$T \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 17x_1 - 8x_2 - 12x_3 \\ 16x_1 - 7x_2 - 12x_3 \\ 16x_1 - 8x_2 - 11x_3 \end{bmatrix}.$$

(a) Find  $[T]_{\alpha}^{\alpha}$ , where  $\alpha$  is the standard basis for  $\mathbb{R}^3$ .

(b) Let  $\beta = \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} \right\}$ . Find the change of basis matrix from  $\alpha$  to  $\beta$ .

(c) Find the change of basis matrix from  $\beta$  to  $\alpha$ .

(d) Find  $[T]_{\beta}^{\beta}$ .

(e) Find  $[v]_{\beta}$  for  $v = \begin{bmatrix} 2 \\ -1 \\ 4 \end{bmatrix}$ .

(f) Find  $[T(v)]_{\beta}$ .

(g) Use the result of part (f) to find  $T(v)$ .

**Problem 10.** A linear transformation from  $V$  to  $W$  has an *inverse* from  $W$  to  $V$  when the range of is all of  $W$  and the kernel contains only  $v = 0$ . Why are these transformations not invertible?

(a)  $T(v_1, v_2) = (v_2, v_2), \quad W = \mathbb{R}^2$

(b)  $T(v_1, v_2) = (v_1, v_2, v_1 + v_2), \quad W = \mathbb{R}^3$

(c)  $T(v_1, v_2) = v_1, \quad W = \mathbb{R}^1$

**Problem 11.** Suppose  $T : V \rightarrow W$  is a linear transformation.

- (a) Show that if  $v \in V$  and  $u \in \ker(T)$ , then  $T(u + v) = T(v)$ .
- (b) Show that if  $u, v \in V$  such that  $T(u) = T(v)$ , then  $u - v$  is in  $\ker(T)$ .
- (c) Use part (b) to deduce that if  $\ker(T) = \{0\}$ , then  $T$  is injective. [Note: a function  $f$  is *injective* if  $x_1 \neq x_2$  implies  $f(x_1) \neq f(x_2)$ ].

**Problem 12** (An application of eigenvalues). If  $f : \mathbb{R}^2 \rightarrow \mathbb{R}$  is a twice-differentiable function of two variables, then the *Hessian matrix* of  $f$  is the matrix

$$H(x, y) = \begin{bmatrix} f_{xx}(x, y) & f_{xy}(x, y) \\ f_{yx}(x, y) & f_{yy}(x, y) \end{bmatrix}.$$

In this problem, we'll use the following theorem:

**The Second Derivative Test**

Let  $f : \mathbb{R}^2 \rightarrow \mathbb{R}$  be a twice-differentiable function of two variables, and let  $(a, b)$  be a critical point of  $f$ . Let  $\lambda_1$  and  $\lambda_2$  be the eigenvalues of  $H(a, b)$ . Then:

- If  $\lambda_1, \lambda_2$  are both positive, then  $f$  has a minimum at  $(x, y) = (a, b)$ .
- If  $\lambda_1, \lambda_2$  are both negative, then  $f$  has a maximum at  $(x, y) = (a, b)$ .
- If one of  $\lambda_1, \lambda_2$  is negative and the other positive, then  $f$  has a saddle point at  $(x, y) = (a, b)$ .
- If one or more eigenvalue is zero, then the test is inconclusive.

For parts (a) and (b), find the critical points of  $f(x, y)$  and classify them using the second derivative test stated here.

- (a) Let  $f(x, y) = x^2 + 4xy + y^2$ .
- (b) Let  $f(x, y) = x^3 - 3x + y^2 - 4y$

**Problem 13.** For the following matrices, find the eigenvalues, and find a basis for each eigenspace.

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & -2 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 4 & 0 & 0 \\ 1 & 4 & 0 \\ 0 & 1 & 4 \end{bmatrix}$$

**Problem 14.** Find a basis for the following subspace of  $\mathbb{R}^4$ :

- (a) The vectors for which  $x_1 = 2x_4$ .
- (b) The vectors for which  $x_1 + x_2 + x_3 = 0$  and  $x_3 + x_4 = 0$ .

- (c) The subspace spanned by  $\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 4 \\ 5 \end{bmatrix}$

**Problem 15.** Suppose a linear transformation  $T$  transforms  $(1, 1)$  to  $(2, 2)$  and  $(2, 0)$  to  $(0, 0)$ . Find  $T(v)$  when

- (a)  $v = (2, 2)$
- (b)  $v = (3, 1)$
- (c)  $v = (-1, 1)$
- (d)  $v = (a, b)$