

# Math 307: Practice Midterm 1

*Date of exam: November 24, in class*

*Instructions: You have 50 minutes. Calculators and notes are not allowed. This practice exam is slightly longer than the actual midterm.*

**Problem 1.** State precise definitions of the following terms:

- (a) null space of  $A$
- (b) “eigenvector” and “eigenvalue” of a matrix  $A$

**Problem 2.** Determine whether the following statements are true or false. You don’t need to show your work or justify your answer.

- (a) If  $A$  is an  $n \times n$  matrix of rank  $n - 1$ , then its nullspace is a line through the origin.
- (b) The matrix  $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$  is a rotation
- (c) The vectors  $\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$ ,  $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ ,  $\begin{bmatrix} 2 \\ 3 \\ 5 \end{bmatrix}$  span a dimension 3 subspace of  $\mathbb{R}^4$ .
- (d) If  $A$  is a  $5 \times 8$  matrix with  $\dim(CS(A)) = 3$ , then  $\dim(NS(A)) = 2$ .
- (e) Let  $A$  and  $B$  be  $n \times n$  matrices. If  $T(v) = Av$  and  $S(v) = Bv$  for all  $v \in \mathbb{R}^n$ , then  $T \circ S(v) = ABv$ .
- (f) The transformation  $T(x) = x + 11$  is linear.
- (g) The matrix  $A^\top A$  is symmetric.
- (h) The columns of an invertible  $n \times n$  matrix form a basis of  $\mathbb{R}^n$ .
- (i) If the columns of  $A$  are linearly independent, then  $Ax = b$  has exactly one solution for every  $b$ .
- (j) If  $v$  is an eigenvector of  $A$ , then it is an eigenvector of  $A^2$
- (k) Let  $A$  be an  $m \times n$  matrix, and let  $T(X) = AX$  be a linear transformation. Then the domain of  $T$  is  $\mathbb{R}^n$  and the range is  $\mathbb{R}^m$ .
- (l) If  $A, B$  are  $n \times n$  matrices, then  $(A + B)^2 = A^2 + 2AB + B^2$ .
- (m) Suppose the only solution to  $Ax = 0$  ( $m$  equations,  $n$  unknowns) is  $x = 0$ . Then  $A$  has rank  $n$ .
- (n) Let  $A$  be an  $n \times n$  matrix. If  $Ax = 0$  for some nonzero vector  $x$ , then the equation  $Ax = 0$  has infinitely many solutions.
- (o) The eigenvalues of a projection matrix are always 0 or 1.

**Problem 3.** Let  $A = \begin{bmatrix} 1 & 1 & 0 & 3 \\ 1 & 1 & 1 & -2 \\ 3 & 3 & 2 & -1 \end{bmatrix}$

- (a) Find a basis for  $CS(A)$ .
- (b) Find a basis for  $RS(A)$ .
- (c) Find a basis for  $NS(A)$ .
- (d) What is the rank of  $A$ ?

(e) Is  $A$  invertible?

**Problem 4.** [Note: there will be a problem like this on the exam.] Let  $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  be the projection onto the line  $y = -x$ . Let  $\alpha = \{e_1, e_2\}$

(a) Find two linearly independent eigenvectors  $\beta_1, \beta_2$  of  $T$  and sketch them.

(b) What is  $[T]_{\beta}^{\beta}$ ?

(c) Let  $P$  be the change-of-basis matrix from  $\alpha$  to  $\beta$ . Find  $P$  and  $P^{-1}$ .

(d) Use your answer to the previous part to find  $[T]_{\alpha}^{\alpha}$ .

(e) What are the eigenvalues of  $T$ ?

(f) Verify that  $[T]_{\alpha}^{\alpha}$  and  $[T]_{\beta}^{\beta}$  have the same characteristic polynomial.

**Problem 5.** Let  $A = \begin{bmatrix} 6 & -8 \\ 4 & -6 \end{bmatrix}$

(a) By what factor does the linear transformation of  $A$  scale area?

(b) Find the eigenvalues of  $A$ .

(c) For each eigenvalue of  $A$ , find a basis for the eigenspace.

**Problem 6.** Let  $A = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$ .

(a) Sketch the image of the unit square  $\{(x, y) : 0 \leq x \leq 1 \text{ and } 0 \leq y \leq 1\}$  under the transformation of the matrix

(b) Compute the eigenvalues of  $A$ , and an eigenbasis for each eigenvalue.

(c) What is the inverse of  $A$ ?

**Problem 7.** Show that  $\begin{bmatrix} 1 \\ 3 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}$  form a basis for  $\mathbb{R}^3$ .

**Problem 8.** Solve the following linear system:

$$2x + 3y = 6$$

$$2x + y = 2$$

$$x - y = -1$$

Provide a sketch and interpret your results geometrically.

**Problem 9.** Suppose  $T$  is a linear transformation such that  $T \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 6 \\ 6 \end{bmatrix}$  and  $T \begin{bmatrix} 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 4 \\ 0 \end{bmatrix}$ .

(a) Find  $T \begin{bmatrix} 2 \\ 2 \end{bmatrix}$

(b) Find  $T \begin{bmatrix} 5 \\ 1 \end{bmatrix}$

(c) Find  $T \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

(d) Find  $T \begin{bmatrix} a \\ b \end{bmatrix}$  where  $a, b$  are arbitrary real numbers.

(e) Write down a  $2 \times 2$  matrix that gives the transformation  $T$ .