2 2025-08-27 | Week 01 | Lecture 02

The nexus question of this lecture: *What do solutions to systems of equations look like?*

2.1 How to understand solutions of linear systems geometrically

Here is a very useful geometric perspective. In system (2), we have a system of m equations expressed in n variables $x_1, \ldots, x_n$. Each of the m equations is the equation of some hyperplane[2] which lives in n-dimensional space ($\mathbb{R}^n$). *The solution to the linear system is the intersection of these hyperplanes.*

The clearest example of this can be seen in the linear system:

Example 6 (The case with two variables)**.**

$$\begin{cases} a_{11}x + a_{12}y = b_1 \\ a_{21}x + a_{22}y = b_2 \end{cases} \tag{3}$$

where $a_{12}, a_{22} \neq 0$. (In this case, the "hyperplanes" are simply lines.) Here, the solutions to the first equation are the points on the line

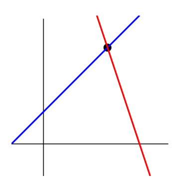
$$y = -\frac{a_{11}}{a_{12}}x + \frac{b_1}{a_{12}} \tag{4}$$

Similarly, the solutions to the second equation are the points on the line

$$y = -\frac{a_{21}}{a_{22}}x + \frac{b_2}{a_{22}}. \tag{5}$$

There are three possible things that can happen when we intersect the two lines in Eqs. (4) and (5):

- **Case 1.** The two line equations Eqs. (4) and (5) represent distinct lines and are not parallel. In this case, their intersection consists of a unique point, like this:



  In this case, the system (3) has **exactly one solution**—namely, the intersection of the two lines, just like we saw in the boat example.

- **Case 2.** The two line equations Eqs. (4) and (5) represent two parallel but different lines. In this case, the two lines never intersect each other (i.e., there is no point that lies on both lines), so the system (3) has **no solutions**.

- **Case 3.** The two equations of lines are the same, so they represent the same line. Therefore the intersection of the two lines is the entire line. Therefore, there are **infinitely many solutions** to the linear system (3). Namely, any point (x, y) on the line is a solution to the linear system.

End of Example 6. □

These three cases desribed in Example 6 constitute the following trichotomy:

Theorem 7. *A system of linear equations either has (1) exactly one solution, (2) no solution, or (3) infinitely many solutions.*

We haven't proven this fact, only illustrated it for systems of linear equations like (3) that have 2 equations and 2 variables. In fact, as we shall see, this fact always holds for all linear systems of the form given in (2), no matter how many equations and variables.

[2] Note: Hyperplanes will be defined more formally later, but for now can be thought of as generalized lines or planes, since a 1-dimensional hyperplanes is a *line* and a 2-dimensional hyperplane is a *plane*.

2.2 The planar case

Recall that, geometrically, a line is determined by two features:

1. A slope m which determines the direction of hte line
2. A point (x_0, y_0) which the line passes through, as this determines where the line lives on the xy-plane

It is easy to see that these two things determine everything about a line because the equation of a line can be expressed as

$$y - y_0 = m(x - x_0)$$

and to write this down, all we need are m and (x_0, y_0).

Just like a line, a plane is determined by two things:

1. A normal vector $n = \langle A, B, C \rangle$ which determines the tilt of the plane. (Here, A, B, and C are fixed constants)
2. A point (x_0, y_0, z_0) which the plane passes through, as this determines where in 3-d space ($\mathbb{R}^3$) the plane lives.

Indeed, a plane $\mathbb{P}$ consists of the set of points (x, y, z) satisfying the following equation:

$$A(x - x_0) + B(y - y_0) + C(z - z_0) = 0 \tag{6}$$

This is the standard form equation of a plane, and we can write it down if we know both $n = \langle A, B, C \rangle$ and (x_0, y_0, z_0). So if we know those two things, then we know the equation of the plane, meaning we know everything about it.

By a little bit of algebra, we can rewrite Eq. (6) as

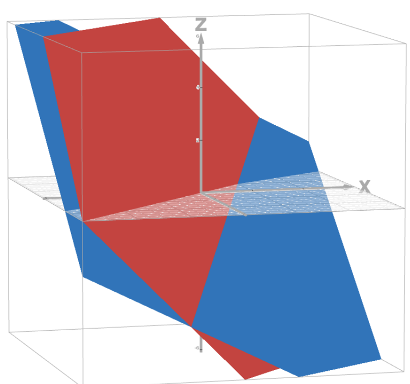
$$Ax + By + Cz = D$$

where $D = Ax_0 + By_0 + Cz_0$. This is a linear equation. The solutions to a linear equation with 3 variables is a plane.

Example 8 (A system with three variables)**.** Suppose we wish to solve the linear system

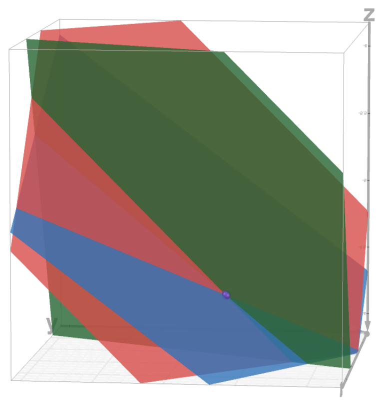
$$\begin{cases} x \;-\; y + \;\; z = \;\;\, 0 \\ 2x \;-\; 3y + \; 4z = -2 \\ -2x \;-\; y + \;\; z = \;\;\, 7 \end{cases}$$

In this case, each equation is the equation of a plane. The planes for the first two equations are the following:



The plane for the first equation is in red. The plane for the second equation is blue. Any point on the red plane is a solution to the first equation $x - y + z = 0$. Any point on the blue plane is a solution to the second equation $2x - 3y + 4z = -2$. The two planes interesect in a line. If I pick any point on this line, then it satisfies both equations.

But our system has three equations, so we have a third plane, and the intersection of all three planes is a point, as shown:

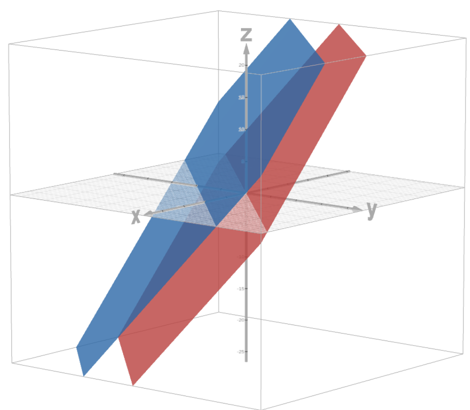


In this case, the system of equations has a unique solution, which is the unique point of intersection of hte planes.

Here's the desmos link to see the plots of these three planes, if you want to play around with it.

`https://www.desmos.com/3d/gpgtw2rjaf`

Of course ther are other ways that three planes could have intersected. For example, two of the planes might be parallel, like the following picture, in which case the system will have no solutions:



There are other ways that three planes could intersect as well, but the trichotomy stated earlier always holds: their intersection either constists of (1) exactly one point, (2) infinitely many points, or (3) zero points.

End of Example 8. □

We've seen in this lecture that for systems of linear equations with two variables, the solutions are the intersection of lines. For systems of linear equations with three variables, the solutions are the intersections of planes. ... And for systems with $n > 3$ variables, the solutions are the intersections of hyperplanes.