Math 307: Homework 03

Due Wednesday, September 24 (at the beginning of class)

Problem 1 (Important)**.** Let A be a 3×3 matrix. Suppose x_1, x_2, and x_3 are column vectors such that

$$Ax_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad \text{and} \quad Ax_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \quad \text{and} \quad Ax_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

If the three solutions x_1, x_2 and x_3 are columns of a matrix X, what is AX?

Problem 2. Given an $m \times n$ matrix $A = (a_{ij})$, the ***transpose*** of A, denoted $A^\top$, is the $n \times m$ matrix $A^\top = (a_{ji})$. For example,

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \quad \text{and} \quad A^\top = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$$

Prove the following theorem.

> **Theorem.** (Properties of the transpose). Suppose A, B are matrices. Then whenever defined, the following properties hold:
>
> (i.) $\left(A^\top\right)^\top = A$
>
> (ii.) $(A + B)^\top = A^\top + B^\top$
>
> (iii.) $(cA)^\top = cA^\top$
>
> (iv.) $(AB)^\top = B^\top A^\top$ *(this is sort of like the socks and shoes property)*
>
> (v.) $\left(A^\top\right)^{-1} = \left(A^{-1}\right)^\top$

(This is Theorem 1.13 in the textbook. The textbook offers a proof of part (iv.), so if you understand that proof, you can use it in your answer. Hint for part (v.): by Theorem 22 (in Lecture 8), all you need to prove is $(A^{-1})^\top A^\top = I$.

Problem 3. If $A = A^\top$, then we say that A is a ***symmetric matrix***. An example:

$$\begin{bmatrix} -5 & 1 & 2 \\ 1 & 4 & 3 \\ 2 & 3 & 6 \end{bmatrix}$$

Note that symmetric matrices are always square.

(a) Let A be any matrix. Show that $A^\top A$ and $AA^\top$ are both symmetric matrices. *Hint: use property (iv.) from the Theorem in Problem 2.*

(b) Let A be a symmetric matrix. Show that if A is invertible, then A^{-1} is also symmetric. *Hint: use property (v.) from the Theorem in Problem 2.*

Problem 4. An ***involution*** is a function f such that $f(f(x)) = x$ for all x. In other words, an involution is a function which is its own inverse. By Part (i.) in the theorem from Problem 2, we know that matrix transposition is an involution, since if $f(A) = A^\top$, then

$$f(f(A)) = f(A^\top) = \left(A^\top\right)^\top = A.$$

Another example is matrix inversion, since $\left(A^{-1}\right)^{-1} = A$. Give some other examples of involutions (from any area of math).

Problem 5. Let

$$A = \begin{bmatrix} 2 & -1 & 3 \\ 4 & 1 & -2 \\ -3 & 2 & 1 \end{bmatrix}$$

(a) Find $\det(A)$ by expanding about row 1

(b) Find $\det(A)$ by expanding about row 2

(c) Find $\det(A)$ by expanding about column 1

(d) Find $\det(A)$ by expanding about column 3

Problem 6. Find the inverse of the matrix

$$\begin{bmatrix} 0 & -2 & 1 \\ 2 & 4 & -1 \\ 2 & 1 & 2 \end{bmatrix}$$

Problem 7. Solve the following linear system:

$$\begin{aligned} 2x - 4y + 6z &= 2 \\ -3x + 6y - 9z &= 3 \end{aligned}$$

Interpret your results geometrically. Provide a sketch or an image (e.g., using Desmos) of the the solution.

Problem 8. The technical definition of "nonsingular" is the following:

> *Definition.* An $n \times n$ matrix A is said to be ***nonsingular*** if the only solution to the system of linear equations $AX = \mathbf{0}$ is $X = \mathbf{0}$.

In other words, A produces the output $\mathbf{0}$ only for the input $\mathbf{0}$. Note that in the above definition,

$$X = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \quad \text{and} \quad \mathbf{0} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}.$$

Prove that a matrix A is nonsingular if and only if $\det A \neq 0$. *(Hint: use the key theorem of linear algebra from lecture 9).*

Problem 9 (More determinants)**.**

(a) Compute the determinant by doing a cofactor expansion across an approriate row or column.

$$\begin{vmatrix} -3 & 0 & 4 \\ 2 & -1 & 3 \\ 4 & 0 & 5 \end{vmatrix}$$

(b) Compute the determinant by doing a cofactor expansion across an approriate row or column.

$$\begin{vmatrix} 6 & -5 & 1 & 3 \\ 3 & 1 & -2 & -1 \\ 0 & 10 & 0 & 0 \\ 3 & 3 & 0 & 3 \end{vmatrix}$$

Hint: Don't try to brute force this calculation. Be clever. See Example 1 in section 1.5 of the textbook.

Problem 10 (Permutation Matrices)**.** A square matrix called a ***permutation matrix*** if exactly one entry in each row and column is equal to 1 and all other entries are 0. Multiplication by such matrices permutes the rows or columns of the matrix multiplied. For example,

$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}$$

Left multiplication permutes the rows (as shown above). Right multiplication permutes the columns.

(a) Consider the set $\{1, 2, 3, 4, 5\}$. One permutation of this set is $(3, 2, 4, 1, 5)$. Find the permutation matrix P such that

$$P \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ 4 \\ 1 \\ 5 \end{bmatrix}$$

(b) Find a permtuation matrix Q such that

$$Q \begin{bmatrix} 3 \\ 2 \\ 4 \\ 1 \\ 5 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{bmatrix}$$

(c) What do you notice about the relationship between P and Q?

(d) Give a geometric argument for why the determinant of a permutation matrix is always $+1$ or -1. (You don't need to give a proof, but try to be convincing.)

Problem 11. Solve the following linear system:

$$\begin{aligned} 2x + 3y &= 5 \\ 2x + y &= 2 \\ x - 2y &= 1 \end{aligned}$$

Interpret your results geometrically. Provide a sketch or an image (e.g., using Desmos) of the the solution.