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Topics: section 2.4 - fundamental subspaces, rank nullity, rank, and dimension

19.1 An alternative characterization of linear dependence

The following theorem gives us another quite useful characterization of linear dependence:

Theorem 86. *Let V be a vector space, and let $v_1, \dots, v_n \in V$. Then the $v_1, \dots, v_n$ are linearly dependent if and only if one of the $v_1, \dots, v_n$ is a linear combination of the others.*

Taking $n = 2$ in the above theorem gives a useful consequence: two u, v, are linearly dependent if and only if they are scalar multiples of each other.

For example, the vectors $\begin{bmatrix}1\\2\\3\end{bmatrix}, \begin{bmatrix}3\\6\\9\end{bmatrix}$. are scalar multiples of each other, and hence are dependent. On the other hand, the vectors $\begin{bmatrix}1\\2\\3\end{bmatrix}, \begin{bmatrix}3\\6\\10\end{bmatrix}$. are not scalar multiples of each other, so they are linearly independent.

19.2 Rank-Nullity Theorem

Definition 87 (Rank)**.** The ***rank*** of a matrix A, denoted $\operatorname{rank}(A)$, is the dimension of the column space of A. That is,

$$\operatorname{rank}(A) = \dim CS(A) \quad (= \dim RS(A))$$

Theorem 88 (Rank-nullity)**.** *Let A be an $m \times n$ matrix. Then*

$$\underbrace{\dim CS(A)}_{\operatorname{rank}(A)} + \underbrace{\dim NS(A)}_{\text{nullity(A)}} = n$$

19.3 More examples of computing bases for the three fundamental subspaces

Example 89 (Example 85 continued)**.** Consider the matrix

$$A = \begin{bmatrix} 1 & 0 & -1 & 3 \\ 2 & 1 & 4 & -1 \\ 4 & 1 & 2 & 5 \end{bmatrix} \in M_{3\times 4}(\mathbb{R}).$$

Previously, we showed the following:

- The vectors $\begin{bmatrix}1 & 0 & -1 & 3\end{bmatrix}, \begin{bmatrix}0 & 1 & 6 & 7\end{bmatrix}$ form a basis for $RS(A)$. Hence, $\dim RS(A) = 2$.
- The vectors $\begin{bmatrix}1\\-6\\0\\1\end{bmatrix}, \begin{bmatrix}-3\\7\\0\\1\end{bmatrix}$ form a basis for $NS(A)$. Hence $\dim NS(A) = 2$.

Question: What is $\operatorname{rank}(A)$? In other words, what is $\dim CS(A)$?

Solution 1 (general method): *Idea: to find the column space, take the transpose, row-reduce to find a basis for $RS(A^\top)$, then transpose back.*

$$A^\top = \begin{bmatrix} 1 & 2 & 4 \\ 0 & 1 & 1 \\ -1 & 4 & 2 \\ 3 & -1 & 5 \end{bmatrix} \xrightarrow{\text{row reduction}} \begin{bmatrix} 1 & 2 & 4 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Thus, the vectors

$$\begin{bmatrix} 1 & 2 & 4 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 1 \end{bmatrix}$$

form a basis for $RS(A^\top)$. Transposing back, it follows that the vectors $\begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$ form a basis for $CS(A)$. Since there are two vectors in the basis, it follows that $\dim CS(A) = 2$.

Solution 2: By the Rank-Nullity Theorem (Theorem 88), we have

$$\dim CS(A) + \dim NS(A) = 4$$

Since $\dim NS(A) = 2$ (because 2 vectors in the basis), it follows that

$$\dim CS(A) = 2.$$

Solution 3: In the last lecture, we showed that $\dim RS(A) = 2$. Recall that $\text{rank}(A) := \dim CS(A) = \dim RS(A)$. Therefore $\dim CS(A) = 2$.

End of Example 89. □

Example 90. Find the column space of

$$A = \begin{bmatrix} 1 & 1 & 2 \\ -1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}.$$

Solution: We will again use the general method for finding the column space from the previous example. The three steps are: (1) take the transpose, (2) row-reduce to find a basis for $RS(A^\top)$, then (3) transpose back.

We row reduce $A^\top$ which gives

$$A^\top = \begin{bmatrix} 1 & -1 & 0 \\ 1 & 1 & 1 \\ 2 & 0 & 1 \end{bmatrix} \xrightarrow{\text{row reduction}} \begin{bmatrix} 1 & -1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

The nonzero rows are a basis for $R(A^\top)$. Transposing back, we have

$$\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}$$

This is a basis for the colum space of A.

End of Example 90. □

19.4 Connection between rank and invertibility

Theorem 91. *Let A be an $n \times n$ matrix. Then A is invertible if and only if rank$(A) = n$.*

One way to see why this is true involves thinking of A as a transformation of n-dimensional space $\mathbb{R}^n$. Observe that

- A is non-invertible exactly when the transformation collapses the dimension.
- The rank of A is the dimension of its range (since range = column space).
- So if $\text{rank}(A) < n$ then a dimension collapse occurs, in which case A is not invertible. But if $\text{rank}(A) = n$, then no dimension collapse occurs, so A is invertible in that case.

Thus, Theorem 91 gives another condition we can add to our key theorem.