9 2025-09-15 | Week 04 | Lecture 09

This lecture is based on sections 1.5 and 1.6 in the textbook. We are going to skip section 1.7

> ***The nexus question of this lecture:*** *How do we understand (and compute) the determinant, algebraically?*

9.1 Review of the "Key Theorem" of Linear Algebra

Theorem 28 (The Key Theorem of Linear Algebra (partial version))**.** *Let A be an $n \times n$ matrix. Then the following are equivalent:*

(i.) A^{-1} *exists (i.e., A is invertible)*

(ii.) $\det A \neq 0$

(iii.) The linear system $AX = B$ has a unique solution for each $B \in \mathbb{R}^n$.

(iv.) A is row equivalent to I

(v.) ...

Property (iv.) says we can row reduce A into I. The term for this is "row equivalence". Precisely, If A and B are matrices, we say that A is ***row equivalent*** to B if there is a sequence of elementary row operations which if applied to A will result in B.

9.2 Definition of the determinant

Consider a square matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix}.$$

Given an entry a_{ij}, the ***minor*** of a_{ij}, denoted M_{ij}, is the matrix obtained from A by deleting row i and column j of A. For example, if

$$A = \begin{bmatrix} a_{11}, a_{12}, a_{13} \\ a_{21}, a_{22}, a_{23} \\ a_{31}, a_{32}, a_{33} \end{bmatrix}$$

then some minors are

$$M_{11} = \begin{bmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{bmatrix}, \quad M_{12} = \begin{bmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{bmatrix}, \quad \text{and} \quad M_{32} = \begin{bmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{bmatrix}.$$

Definition 29 (Determinant)**.** For a 1×1 matrix, $A = [a_{11}]$, we define $\det(A) = 1$. If A is an $n \times n$ matrix with $n \geq 2$, we define the determinant recursively as

$$\det(A) = \sum_{j=1}^{n} (-1)^{j+1} a_{1j} \det(M_{1j}). \tag{13}$$

Note: Note that the determinant is defined only for square matrices.

Geometrically, the determinant is the (signed) volume scaling factor of the transformation of space, which is very useful to keep in mind. Definition 29 is also useful because it allows us to see how to actually compute determinants.

For a 2×2 matrix, Definition 29 simplifies to

$$\det\left(\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}\right) = a_{11}a_{22} - a_{12}a_{21}.$$

To simplify notation, we use vertical bars to denote determinant $|A| := \det(A)$, or something like this:

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} := \det\left(\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}\right).$$

Example 30 (Computing the determinant of a 3×3 matrix)**.**

$$\begin{aligned} \begin{vmatrix} 2 & 3 & -2 \\ -1 & 6 & 3 \\ 4 & -2 & 1 \end{vmatrix} &= 2\begin{vmatrix} 6 & 3 \\ -2 & 1 \end{vmatrix} - 3\begin{vmatrix} -1 & 3 \\ 4 & 1 \end{vmatrix} + (-2)\begin{vmatrix} -1 & 6 \\ 4 & -2 \end{vmatrix} \\ &= 2\left[6 \cdot 1 - 3(-2)\right] - 3\left[(-1)1 - 3 \cdot 4\right] - 2\left[(-1)(-2) - 6 \cdot 4\right] \\ &= 107. \end{aligned}$$

End of Example 30. $\square$

9.3 Computing determinants using cofactor expansions

The formula for the determinant in Definition 29 is called a ***cofactor expansion*** (there are other formulas). A ***cofactor*** of an entry a_{ij} is the quantity

$$C_{ij} = (-1)^{i+j} \det(M_{ij})$$

so our formula in Eq. (13) can be written as

$$\det(A) = \sum_{j=1}^{n} a_{1j} C_{1j}.$$

This is called the ***cofactor expansion about the first row.*** The next theorem tell us that, in fact, we could have picked *any* row or column and done a similar calculation to get the determinant:

Theorem 31 (Cofactor Expansion)**.** *If A is an $n \times n$ matrix with $n \geq 2$, then*

(i.) For any fixed $i = 1, 2, \ldots, n$, we have

$$\det(A) = \sum_{j=1}^{n} a_{ij} C_{ij} \quad \textit{(cofactor expansion about the } i^{\text{th}} \textit{ row)}$$

(ii.) For any fixed $j = 1, 2, \ldots, n$, we have

$$\det(A) = \sum_{i=1}^{n} a_{ij} C_{ij} \quad \textit{(cofactor expansion about the } j^{\text{th}} \textit{ column)}$$

This theorem is proved by induction on n in section 1.7, but the proof is technical, so we'll skip it. Two examples will illustrate this theorem.

Example 32 (Alternative cofactor expansions)**.** Let's compute the determinant of the matrix from Example 30 in two different ways, using Theorem 31:

The cofactor expansion about the third row:

$$\begin{aligned}
\begin{vmatrix} 2 & 3 & -2 \\ -1 & 6 & 3 \\ 4 & -2 & 1 \end{vmatrix} &= a_{31}C_{31} + a_{32}C_{32} + a_{33}C_{33} \\
&= 4\begin{vmatrix} 3 & -2 \\ 6 & 3 \end{vmatrix} - (-2)\begin{vmatrix} 2 & -2 \\ -1 & 3 \end{vmatrix} + 1\begin{vmatrix} 2 & 3 \\ -1 & 6 \end{vmatrix} \\
&= 107.
\end{aligned}$$

The cofactor expansion about the second column:

$$\begin{aligned}
\begin{vmatrix} 2 & 3 & -2 \\ -1 & 6 & 3 \\ 4 & -2 & 1 \end{vmatrix} &= a_{21}C_{21} + a_{22}C_{22} + a_{32}C_{32} \\
&= -3\begin{vmatrix} -1 & 3 \\ 4 & 1 \end{vmatrix} + 6\begin{vmatrix} 2 & -2 \\ 4 & 1 \end{vmatrix} - (-2)\begin{vmatrix} 2 & -2 \\ -1 & 3 \end{vmatrix} \\
&= 107.
\end{aligned}$$

The -3 at the beginning of this is not a typo. It's -3 rather than 3 because of the -1 introduced by the cofactor C_{21}, which is $C_{21} = (-1)^{2+1}\begin{vmatrix} -1 & 3 \\ 4 & 1 \end{vmatrix} = -\begin{vmatrix} -1 & 3 \\ 4 & 1 \end{vmatrix}$.

End of Example 32. □

9.4 Key properties of the determinant

Theorem 33. *If I is an $n \times n$ identity matrix, then $\det(I) = 1$. If $\mathbf{O}$ is an $n \times n$ zero matrix, then $\det(\mathbf{O}) = 0$.*

I claim that when this theorem is considered geometrically, its truth becomes obvious. Why is it obvious?

- Because the identity matrix I corresponds to the transformation of space that *doesn't change anything*. This is called the ***identity transformation***. This doesn't stretch (or reflect) space at all, and hence areas/volumes are not changed at all. So the determinant, being the volume scaling factor of the transformation, is 1.
- And because the zero matrix $\mathbf{O}$ maps every vector to the vector $\vec{0} = (0, 0, \ldots, 0)$. This means $\mathbf{O}$ collapses the entirety of n-dimensional space into a single point (i.e., the origin), which has dimension zero. The dimensional collapse means that area/volume gets destroyed, and hence $\det(\mathbf{O}) = 0$.

The next theorem says that the determinant "preserves multiplication".

Theorem 34 (The determinant preserves multiplication)**.** *If A and B are $n \times n$ matrices, then*

$$\det(AB) = \det(A)\det(B).$$

The textbook provides a nice algebraic proof of this in terms of the row reduciton framework presented in Section 8.2 [namely, the proof of Theorem 1.24 in section 1.6 (p.52-53), which I encourage you to read]. But I claim that this theorem is *obvious* when its geometric meaning is considered (i.e., in terms of matrices as transformations of space). We'll start with this idea next lecture.