Math 307: Homework 08

Due Wednesday, October 29. This homework is mostly based on section 5.1 in the textbook.

Problem 1. Find all solutions of

$$\begin{cases} x_1 + x_2 + x_3 + x_4 = 0 \\ x_1 + x_2 + x_3 - x_4 = 4 \\ x_1 + x_2 - x_3 + x_4 = -4 \\ x_1 - x_2 + x_3 + x_4 = 2 \end{cases}$$

Problem 2. Let

$$A = \begin{bmatrix} 4 & -1 & 2 & 1 \\ 2 & 3 & -1 & -2 \\ 0 & 7 & -4 & -5 \\ 2 & -11 & 7 & 8 \end{bmatrix}$$

For the following parts, you must fully justify your work to recieve credit—this includes providing at least one or two complete sentences explaining why your calculations justify your answer.

(a) Find a basis for $NS(A)$

(b) Find a basis for $RS(A)$

(c) Find a basis for $CS(A)$

(d) Determine the rank of A.

(e) Determine whether the matrix is invertible

Problem 3. The matrix $A = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$ produces a *stretching* in the x-direction. Draw the circle $x^2 + y^2 = 1$. What happens to this circle when space is transformed by A? Illustrate your answer with a sketch.

Problem 4. The matrix $A = \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}$ yields a *shearing* transformation, which leaves the y-axis unchanged. Sketch its effect on the x-axis, by indicating what happens to the points $(x, y) = (1, 0)$, $(2, 0)$, and $(-1, 0)$—and how the whole axis is transformed.

Problem 5.

(a) What 2×2 matrix has the effect of rotating every vector counterclockwise $90°$ and then projecting the result onto the x-axis?

(b) What 2×2 matrix represents projection onto the x-axis followed by projection onto the y-axis?

Problem 6. Let c be a scalar, and let $T : V \to W$ be a linear transformation. Verify that cT is a linear transformation.

Problem 7. Let $S, T : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformations

$$S \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2x - y \\ x + 2y \end{bmatrix} \quad \text{and} \quad T \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x + 3y \\ x - y \end{bmatrix}.$$

(a) Find matrices A, B such that T and S are expressed as the matrix transformations $T(X) = AX$ and $S(X) = BX$.

(b) Find the matrix C such that the composition $S \circ T$ is expressed in the form $(S \circ T)(X) = CX$. Then verify that $C = AB$.

(c) Find the matrix D such that the composition $T \circ S$ is expressed in the form $(T \circ S)(X) = DX$. Then verify that $D = BA$.

Problem 8. The 4 Hadamard matrix is

$$H = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}$$

Find H^{-1} and write $v = \begin{bmatrix} 7 \\ 5 \\ 3 \\ 1 \end{bmatrix}$ as a linear combination of the columns of H.

Problem 9. Find the rank and nullsapce of

$$A = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 0 & 0 & 1 & 2 \\ 0 & 0 & 1 & 2 \\ 1 & 1 & 1 & 0 \end{bmatrix}$$

Problem 10. If S and T are linear transformations with $S(v) = T(v) = v$, then $S(T(v)) = v$ or v^2?

Problem 11. A linear transformation must leave the zero vector fixed: $T(0) = 0$. Prove this from $T(u+v) = T(u) + T(v)$ by choosing $v =$ ____. Prove it also from the requirement $T(cv) = cT(v)$ by choosing $c =$ ____.

Problem 12. Every straight line remains straight after a linear transformation. If z is halfway between x and y, show that Az is halfway between Ax and Ay

Problem 13. True or false, with counterexample if false:

(a) If the vectors $x_1, \ldots, x_m$ span a subspace S, then $\dim S = m$.

(b) The intersection of two subspaces of a vector space cannot be emptpy.

(c) If $Ax = Ay$, then $x = y$.

(d) The row space of A has a unique basis that can be computed by reducing A to reduced row-echelon form.

(e) If a square matrix A has independent columns, then so does A^2.