Additional practice problems for the final exam

Problem 1 (change of basis - similar to updated practice exam problem)**.** Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation defined geometrically by first **projecting** space onto the line $y = 2x$, and then reflecting across the line $y = -\frac{1}{2}x$.

(a) Find two linearly independent eigenvectors β_1 and β_2. Sketch them.
(Hint: the line $y = -\frac{1}{2}x$ is perpendicular to $y = 2x$)

(b) Find eigenvalues λ_1 and λ_2 corresponding to β_1 and β_2.

(c) Let $\beta = \{\beta_1, \beta_2\}$. Find $[T(\beta_1)]_\beta$ and $[T(\beta_2)]_\beta$.

(d) Find $[T]_\beta^\beta$.

(e) Let $\alpha = \{e_1, e_2\}$ where e_1, e_2 are the standard basis vectors. Find the change of basis matrix P from α to β, and also find P^{-1}.

(f) Use your answers to parts (d) and (e) to find $[T]_\alpha^\alpha$.

(g) Where does T send e_1 and e_2? In other words, find $[T(e_1)]_\alpha$ and $[T(e_2)]_\alpha$. (Hint: use your answer to part (d)).

(h) Verify $[T]_\alpha^\alpha$ and $[T]_\beta^\beta$ have the same characteristic polynomial.

Here are two more problems in this vein:

Problem 2 (change of basis)**.** Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the **projection** onto the line $y = 2x$.

(a) Find two linearly independent eigenvectors β_1, β_2 of T and sketch them.

(b) What are the eigenvalues λ_1 and λ_2 corresponding to β_1 and β_2?

(c) What is $[T]_\beta^\beta$?

(d) Let $\alpha = \{e_1, e_2\}$, and let P be the change-of-basis matrix from α to β. Find P and P^{-1}.

(e) Use your answer to the previous part to find $[T]_\alpha^\alpha$. Where does T send e_1 and e_2?

(f) Verify that $[T]_\alpha^\alpha$ and $[T]_\beta^\beta$ have the same characteristic polynomial.

Problem 3 (change of basis)**.** Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear tranformation defined by first **reflecting** across the line $y = x$ and then **dilating** space by a factor of 4.

(a) Find two linearly independent eigenvectors β_1, β_2 of T and sketch them.

(b) What are the eigenvalues λ_1 and λ_2 corresponding to β_1 and β_2?

(c) What is $[T]_\beta^\beta$?

(d) Let $\alpha = \{e_1, e_2\}$, and let P be the change-of-basis matrix from α to β. Find P and P^{-1}.

(e) Use your answer to the previous part to find $[T]_\alpha^\alpha$. Where does T send e_1 and e_2?

(f) Verify that $[T]_\alpha^\alpha$ and $[T]_\beta^\beta$ have the same characteristic polynomial.

Problem 4 (differential equation)**.** Let $A = \begin{bmatrix} 3 & 0 \\ 8 & -1 \end{bmatrix}$

(a) Let $Y(t) = \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix}$. Find the general solution of the differential equation $Y' = AY$.

(b) Solve the initial value problem

$$\begin{cases} Y' = AY \\ Y(0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \end{cases}$$

Problem 5 (differential equation)**.** Find the general solution to $Y' = AY$ when $A = \begin{bmatrix} 4 & 0 & 1 \\ -2 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}$.

Problem 6 (determinants)**.** Let

$$A = \begin{bmatrix} 3 & -2 \\ 1 & 2 \end{bmatrix} \quad B = \begin{bmatrix} 1 & -4 \\ -2 & 3 \end{bmatrix}.$$

(a) Find the determinants of A and B.

(b) Find $\det(AB)$, $\det(A^{-1})$, and $\det(B^\top A^{-1})$.

(c) Show that $\det(A + B)$ is not the same as $\det(A) + \det(B)$.

(d) Diagonalize B by writing it as $B = P\Lambda P^{-1}$, for some diagonal matrix Λ. (For full credit, you need to find P, P^{-1} and Λ.)

Problem 7 (composition)**.** Consider the two linear transformations $T : \mathbb{R}^3 \to \mathbb{R}^4$ and $S : \mathbb{R}^4 \to \mathbb{R}^2$ defined by

$$T \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_1 + x_2 \\ x_2 + x_3 \\ x_3 \end{bmatrix} \quad \text{and} \quad S \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 2x_1 - x_2 \\ x_3 - 2x_4 \end{bmatrix}$$

(a) Write the standard matrix for T

(b) Write the standard matrix for S

(c) Write the standard matrix for $S \circ T$

Problem 8 (system with variables)**.** Let $a, b, c, d \in \mathbb{R}$. Solve the following system of linear equations.

$$\begin{aligned} x + 2y - z &= a \\ x + y - 2z &= b \\ 2x + y - 3z &= c \end{aligned}$$

Your answer should be in terms of a, b, and c.

Problem 9 (system with variables)**.** Determine the conditions on $a, b, c \in \mathbb{R}$ such that the following linear system has at least one solution:

$$\begin{aligned} x + 2y - z &= a \\ x + y - 2z &= b \\ 2x + 2y - 4z &= c \end{aligned}$$

Problem 10 (diag onalization)**.** Let A be a 2×2 matrix such that

- $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is eigenvector with eigenvalue 2
- $\begin{bmatrix} 2 \\ 3 \end{bmatrix}$ is an eigenvector with eigenvalue 3

(a) Find $A^3 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $A^3 \begin{bmatrix} 2 \\ 3 \end{bmatrix}$ and $A^3 \begin{bmatrix} 3 \\ 4 \end{bmatrix}$

(b) Find A

(c) Find A^3

Problem 11 (diagonalization)**.** Consider the matrix $A = \begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix}$

(a) Diagonalize A. That is, find matrices P, D and P^{-1} such that $A = PDP^{-1}$.

(b) Use your diagonalization to find A^5.

Problem 12 (diagonalization)**.** Let $A = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 1 \\ 0 & 0 & 4 & 0 \end{bmatrix}$. This matrix is diagonalizable; that is, there exists a diagonal matrix D and an invertible matrix P such that $A = PAP^{-1}$. Find D and P. Do not compute P^{-1}.

Problem 13 (basis)**.** Let $\beta = \left\{ \begin{bmatrix} 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 3 \\ 4 \end{bmatrix} \right\}$ be a basis for $\mathbb{R}^2$. If $[v]_\beta = \begin{bmatrix} -3 \\ 2 \end{bmatrix}$, find v.

Problem 14 (Geometry)**.**

(a) Find the matrix of the linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ that rotates space by 90° counterclockwise.

(b) Find the matrix of the linear transformation defined by $S \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2x + y \\ 3y - x \\ x - 3y \end{bmatrix}$.

(c) Circle which transformation makes sense: $T \circ S \quad S \circ T$.

(d) Write a matrix for the transformation you circled.

Problem 15 (geometry)**.** The linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ can be achieved by rotating counterclockwise by 135° and then expanding vertically by a factor of 4. Find the standard matrix for T.

Problem 16 (computation)**.** Let

$$A = \begin{bmatrix} 0 & 0 & 1 & 2 \\ 0 & 0 & 1 & 2 \\ 1 & 1 & 1 & 0 \end{bmatrix}$$

(a) (2 points) Find a basis for the column space of A.

(b) (2 points) Find a basis for the row space of A.

(c) (2 points) Find a basis for the null space of A.

(d) (2 points) What is the rank of A?

(e) (1 point) Is A invertible?

Problem 17 (computation)**.** Let $A = \begin{bmatrix} 0 & -1 & 1 \\ 1 & 2 & 0 \\ 1 & 1 & 1 \end{bmatrix}$.

(a) Find the rank and nullity of A.

(b) Find a basis for the row space of A.

(c) Find a basis for the column space of A.

(d) Find a basis for the nullspace of A.

Problem 18. Solve the following system by reducing the augmented matrix to reduced row-echelon form:

$$\begin{aligned} x + 3y + 2z &= 2 \\ x \qquad\; - 4z &= -7 \\ -2x - 4y + 3z &= -1 \end{aligned}$$

Problem 19 (computation)**.** Let $A = \begin{bmatrix} 1 & 3 & -3 & -2 \\ 0 & -2 & 4 & 2 \\ 2 & 4 & -5 & -3 \end{bmatrix}$. Find a basis for the nullspace and column space of A.

Problem 20 (computation)**.** Suppose T is a linear transformation such that

$$T\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 7 \\ -1 \\ 5 \end{bmatrix}, \quad T\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 6 \\ -2 \\ 5 \end{bmatrix}, \quad T\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 6 \\ 2 \\ 5 \end{bmatrix}$$

(a) Find $T\begin{bmatrix} a \\ b \\ c \end{bmatrix}$

(b) Write down a 3×3 matrix that gives the transformation T.

Problem 21 (Subspace)**.**

(a) Give a precise definition of **linear subspace**.

(b) Show that $W = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \in \mathbb{R}^3 : x_1 + 2x_2 - 3x_3 = 0 \right\}$ a subspace of $\mathbb{R}^3$, and find a basis for it.

(c) Is $U = \text{Span}\left\{ \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix} \right\}$ a subspace of $\mathbb{R}^3$? If not, justify why not. If U is a subspace, find a basis for it.