Math 307: Practice Midterm 2

Date of exam: November 24, in class

Instructions: You have 50 minutes. Calculators and notes are not allowed. This practice exam is slightly longer than the actual midterm.

Problem 1. State precise definitions of (a) "linear independence", (b) "span" and (c) "basis".

Problem 2. Determine whether the following statements are true or false. You don't need to show your work or justify your answer.

(a) The matrix $\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$ is a rotation.

(b) Let A be an $n \times n$ matrix and $b \in \mathbb{R}^n$, then $AX = b$ has a unique solution X if $\ker(A) = \{0\}$.

(c) If A is a 4×6 matrix, then its nullspace is a plane passing through the origin.

(d) The set of functions $X = \{f : \mathbb{R} \to \mathbb{R} \mid f(.4) = 1\}$ forms a vector space.

(e) Any five vectors in $\mathbb{R}^4$ are linearly dependent.

(f) If A is an $m \times n$ matrix, then the dimension of the row space of A equals the dimension of the column space.

(g) If A, B are $n \times n$ matrices, then $\det(A + B) = \det(A) + \det(B)$.

(h) Let A be an $m \times n$ matrix, and let $T(X) = AX$ be a linear transformation. Then the domain of T is $\mathbb{R}^n$ and the range is $\mathbb{R}^m$.

(i) An $n \times n$ matrix has rank n if and only if its columns are linearly independent.

(j) Let A, B be $n \times n$ matrices. If A and B are invertible, then AB is also invertible.

(k) The set of 2×2 matrices of rank 1 is a subspace.

(l) If we know $T(v)$ for n different nonzero vectors in $\mathbb{R}^n$, then we know $T(v)$ for every vector $v \in \mathbb{R}^n$.

(m) If a square matrix A has independent columns, so does A^2.

(n) If $T : \mathbb{R}^3 \to \mathbb{R}^3$ is a linear transformation, then T^2 is also a linear transformation.

(o) Every subspace of $\mathbb{R}^4$ is the kernel of some matrix.

(p) The vectors $\begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$ form a basis for $\mathbb{R}^3$.

Problem 3. Let $A = \begin{bmatrix} 10 & -9 \\ 4 & -2 \end{bmatrix}$

(a) By what factor does the linear transformation of A scale area?

(b) Find the eigenvalues of A.

(c) For each eigenvalue of A, find a basis for the eigenspace.

Problem 4. Find the inverse of $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & -1 \\ 1 & 0 & 2 \end{bmatrix}$

Problem 5. Let

$$A = \begin{bmatrix} 1 & 0 & -1 & 3 \\ 2 & 1 & 4 & -1 \\ 4 & 1 & 2 & 5 \end{bmatrix}$$

Find bases for

(a) Find a basis for $RS(A)$

(b) Find a basis for $NS(A)$

(c) Find a basis for $CS(A)$

(d) What is the rank of A?

Problem 6. The planes given by the equations

$$x + 3y + 3z = 1 \quad \text{and} \quad x - y + z = 1$$

intersect to form a line. Find an equation of the line and write your answer in the form

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} + \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix} x_3$$

where x_3 is a free variable, and $c_1, c_2, c_3, d_1, d_2, d_3$ are scalars.

Problem 7. Let

$$A = \begin{bmatrix} 3 & -2 \\ 1 & 2 \end{bmatrix} \quad B = \begin{bmatrix} 1 & -4 \\ -2 & 3 \end{bmatrix}.$$

(a) Find the determinants of A and B.

(b) Find $\det(AB)$, $\det(A^{-1})$, and $\det(B^\top A^{-1})$.

(c) Show that $\det(A + B)$ is not the same as $\det(A) + \det(B)$.

Problem 8. Compute the determinant of

$$A = \begin{bmatrix} 6 & -5 & 1 & 3 \\ 3 & 1 & -2 & -1 \\ 0 & 10 & 0 & 0 \\ 3 & 3 & 0 & 3 \end{bmatrix}$$

Problem 9. Solve the system of equations using row reduction

$$\begin{aligned} 2x + 3y - 4z &= 3 \\ 4x + 6y - 4z &= 6 \\ 8x + 12y - 4z &= 14 \end{aligned}$$

Problem 10. Compute a basis for the nullspace of the following matrix:

$$\left[\begin{array}{ccccc|c} 1 & 0 & 2 & 3 & 4 & 0 \\ 0 & 1 & 5 & 6 & 7 & 0 \\ 0 & 1 & 5 & 6 & 7 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$