28 2025-10-29 | Week 10 | Lecture 28

The nexus question of this lecture: What are eigenvalues and eigenvectors and how do we find them?

One way to understand what a linear linear transformation does is to undertand which parts of space are invariant — that is, which parts of space don't change.

28.1 Eigenvalues

Definition 124 (Eigenvalue). If A is an $n \times n$ matrix, an **eigenvector** of A is a nonzero column vector $v \in \mathbb{R}^n$ such that

$$Av = \lambda v$$

for some scalar $\lambda \in \mathbb{C}$. The scalar λ is called the eigenvalue.

Example 125. Let

$$A = \frac{1}{2} \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$$

In Example 121, we say that this transformation corresponds to a dilation by a factor of 2 in the direction of 45° .

By geometric considerations, we can see that the line y = x is invariant under this transformation (vectors along this line get scaled by 2 but don't jump off the line). Also we see that the line y = -x is invariant (vectors along this line don't change at all).

The vector $v = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ lies on the line y = x. We see that

$$Av = 2v$$

so that $v = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is an eigenvector with eigenvalue 2.

Similarly, the following vector lies on the line y = -x:

$$v = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

It satisfies Av = 1v. Hence it is an eigenvector with eigenvalue 1.

End of Example 125. \square

Example 126. Let

$$A = \begin{bmatrix} 1 & -3 \\ -2 & 2 \end{bmatrix}$$

has eigenvewctor

$$v = \begin{bmatrix} -1\\1 \end{bmatrix}$$

with eigenvalue 4.

End of Example 126. \square

28.2 How do we find eigenvalues?

Idea: look to the system $AX = \lambda X$. This is equivalent to

$$(\lambda I - A)X = 0.$$

Since $X \neq 0$ (since by definition eigenvectors must be nonzero), we conclude that the matrix $\lambda I - A$ is singular, and hence by the key theorem, $\det(\lambda I - A) = 0$.

Theorem 127. Let A be an $n \times n$ matrix. Then $\lambda \in \mathbb{R}$ is an eigenvalue of A if and only if $\det(\lambda I - A) = 0$.

Definition 128. The characteristic equation of A is

$$\det(\lambda I - A) = 0.$$

When A is an $n \times n$ matrix, the left hand side of the characteristic equation is a polynomial in the variable λ of degree n, and is called the **characteristic polynomial** of A.

Example 129. Let

$$A = \begin{bmatrix} 1 & -3 \\ -2 & 2 \end{bmatrix}.$$

Then the characteristic polynomial of A is

$$\det(\lambda I - A) = \det\begin{bmatrix} \lambda - 1 & 3\\ 2 & \lambda - 2 \end{bmatrix}$$
$$= (\lambda - 1)(\lambda - 2) - 6$$
$$= \lambda^2 - 3\lambda - 4$$
$$(\lambda - 4)(\lambda + 1)$$

This is equal to zero if and only if $\lambda=4$ or $\lambda=-1$. Therefore the eigenvalues of A are $\lambda=4$ and $\lambda=-1$.

End of Example 129. \square

28.3 How do we find eigenvectors?

Idea: First find the eigenvalues λ . Then for each eigenvalue λ , the eigenvectors are the nontrivial solutions of the homogeneous system

$$(\lambda I - A)X = 0.$$

(This is a linear system which we can solve using row reduction.)

In other words, the eigenvectors are the nonzero vectors in the linear subspace

$$NS(\lambda I - A)$$
.

So we just need to compute a basis of this nullspace, which is called the *eigenspace*. When we ask to find the eigenvalues, it is always enough to just compute the basis of the eigenspace.

Example 130. Find the eigenvectors of the matrix

$$A = \begin{bmatrix} 1 & -3 \\ -2 & 2 \end{bmatrix}$$

The equations we need to solve are

• When $\lambda = 4$: 4I - A = 0 or

$$\begin{bmatrix} 3 & 3 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Reducing find the nullspace is

$$NS(4I - A) = \left\{ y \begin{bmatrix} -1\\1 \end{bmatrix}, y \in \mathbb{R} \right\}$$

Technically, all vectors in NS(4I-A) are eigenvectors for $\lambda=4$. To give a concrete example, we have eigenvector $v=\begin{bmatrix} -1\\1 \end{bmatrix}$.

End of Example 130. \square