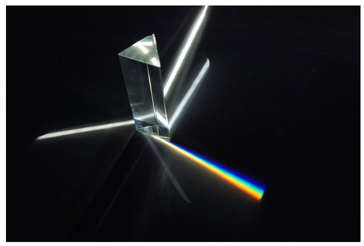
31 2025-11-05 | Week 11 | Lecture 31

The nexus question of this lecture: *When is a matrix similar to a diagonal matrix?*

Recall that two square matrices A, B are ***similar*** if $B = P^{-1}AP$ for some invertible matrix P.

31.1 Prism Analogy

The following analogy goes back to David Hlbert (early 1900s) and Wilhelm Wirtinger (1897). White light consists of a mixture of wavelengths. These can be seen clearly by passing the light through a prism that separates those wavelengths into a *spectrum of colors*:



Source: `https://en.wikipedia.org/wiki/Prism_(optics)`

The same can be done with matrices. For a matrix, the ***spectrum*** is the list of eigenvalues.

Example 139 (First diagonalization)**.** Consider the diagonal matrix

$$D = \begin{bmatrix} 4 & 0 \\ 0 & -1 \end{bmatrix}$$

This matrix has two pure colors: 4 and -1, which are the eigenvalues. The linear transformation does exactly two things: it stretches space by a factor of 4 in the x-direction (since it sends e_1 to $4e_1$) and it reflects space across the y-axis (since it sends e_2 to $-e_2$).

Now consider the matrix

$$A = \begin{bmatrix} 1 & -3 \\ -2 & 2 \end{bmatrix}$$

It's harder to see what the linear transformation of A does; it's like complicated white light consisting of several wavelengths that are hard to separate. We need a prism, and for that we'll choose $P = \begin{bmatrix} -1 & \frac{3}{2} \\ 1 & 1 \end{bmatrix}$. Now, one can check that

$$\underbrace{\begin{bmatrix} -\frac{2}{5} & \frac{3}{5} \\ \frac{2}{5} & \frac{2}{5} \end{bmatrix}}_{P^{-1}} \underbrace{\begin{bmatrix} 1 & -3 \\ -2 & 2 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} -1 & \frac{3}{2} \\ 1 & 1 \end{bmatrix}}_{P} = \underbrace{\begin{bmatrix} 4 & 0 \\ 0 & -1 \end{bmatrix}}_{D}.$$

We have passed the matrix A through a prism. Immediately, we can see from this that the eigenvalues of A are 4 and -1 (since A and D are similar). We now see the spectrum of A.

Where did P come from? The columns of P are eigenvectors of A:

$$V_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \quad \text{and} \quad V_2 = \begin{bmatrix} \frac{3}{2} \\ 1 \end{bmatrix}.$$

Interpreting A through the prism. In addition to knowing the eigenvalues of A, we can now more easily see how it transforms space. Let T be the linear transformation of A, and let $\beta = \{V_1, V_2\}$. P is the change-of-basis matrix from the standard basis $\{e_1, e_2\}$ to β. Since $D = P^{-1}AP$, we have shown that

$$[T]_\beta^\beta = \begin{bmatrix} 4 & 0 \\ 0 & -1 \end{bmatrix}.$$

We can interpret this as follows: T stretches space in the direction of V_1 by a factor of 4 and flips space so that V_2 points in the opposite direction.

End of Example 139. □

31.2 Diagonalizability

Definition 140 (Diagonalizable)**.** A square matrix is ***diagonalizable*** if it is similar to a diagonal matrix.

In Example 145„ we were able to form an invertible matrix P whose columns consisted of eigenvectors of A. The eigenvectors formed an ***eigenbasis***, or a basis consisting of eigenvectors. When this occurs, something special happens: $P^{-1}AP$ becomes a diagonal matrix.

Theorem 141. *A matrix $A \in \mathcal{M}_{n\times n}$ is diagonalizable if and only if there is a basis for $\mathbb{R}^n$ consisting of eigenvectors of A.*

This theorem answers the question of the lecture. (In fact, we could say a lot more. For example, every symmetric matrix $A \in \mathcal{M}_{n\times n}(\mathbb{R})$ is diagonalizable.)