26 2025-10-24 | Week 09 | Lecture 26

> ***The nexus question of this lecture:*** *What do linear transformations of 2-dimensional space look like?*

Example 118 (Horizontal and Vertical Dilations)**.** Scale space by $a > 0$ in the x direction and $b > 0$ in the y-direction

$$\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$$

This can be undone by

$$\begin{bmatrix} \frac{1}{a} & 0 \\ 0 & \frac{1}{b} \end{bmatrix}$$

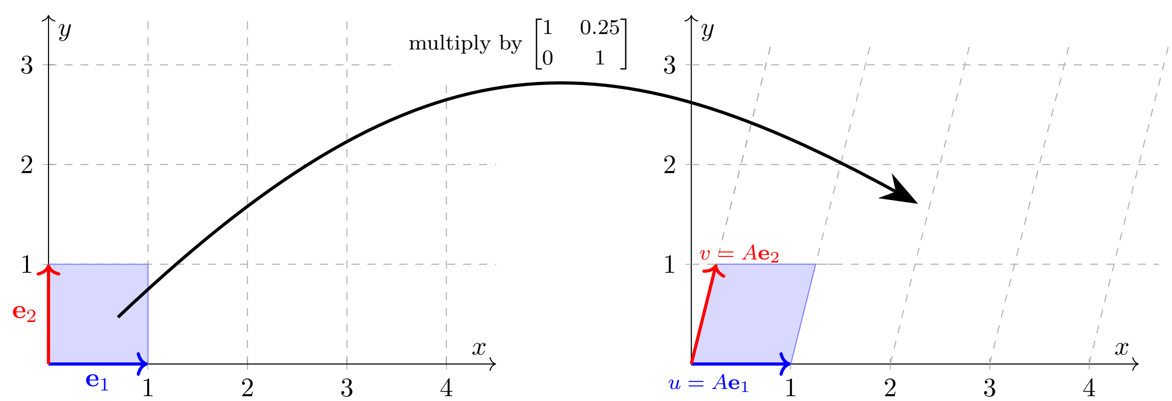
End of Example 118. □

Example 119 (Shear)**.**

$$A = \begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & 0 \\ b & 1 \end{bmatrix}$$

These have inverses

$$A^{-1} = \begin{bmatrix} 1 & -a \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad B^{-1} = \begin{bmatrix} 1 & 0 \\ -b & 1 \end{bmatrix}$$



This transformation is a ***horizontal shear***. It does not change the height of any point. Points above the x-axis get shifted right (because $a = 0.25$ is positive) and points below the x-axis get shifted left. The further away from the x-axis, the greater the horizontal shift. The x-axis is not changed at all by this transformation—it is ***invariant*** under the linear transformation. This transformation can be undone by $\begin{bmatrix} 1 & -0.25 \\ 0 & 1 \end{bmatrix}$, which is the inverse.

The transformation

$$\begin{bmatrix} 1 & 0 \\ b & 1 \end{bmatrix}$$

is similar, but effectuates a ***vertical shear***.

End of Example 119. □

A square matrix A is said to be an ***orthogonal matrix*** is A and $A^\top$ are inverses.

The determinant of an orthogonal matrix is alwasy ± 1 because

$$1 = \det(I) = \det(A^\top A) = \det(A^\top)\det(A) = (\det(A))^2\,.$$

Orthogonal matrices has the property that they *preserve distances*, i.e., that

$$\operatorname{dist}(x, y) = \operatorname{dist}(Ax, Ay).$$

In words, if you choose any two points, their distance doesn't change under the linear transformation—the points may get sent to new coordinates, but the distance between them doesn't change.

If A is an orthogonal matrix and $\det(A) = 1$, then we say that A is a ***rotation***.

Example 120 (Rotation)**.** Let $R_\theta : \mathbb{R}^2 \to \mathbb{R}^2$ be a rotation of the plane about origin by θ radians counter-clockwise. This linear transformation is represented by the matrix

$$R_\theta = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$$

Using the trig identities

$$\cos(a+b) = \cos(a)\cos(b) - \sin(a)\sin(b)$$
$$\sin(a+b) = \sin(a)\cos(b) + \cos(a)\sin(b)$$

we can show that

$$R_a R_b = R_{a+b}$$

That says that, eg if you rotate by say 10° and then by 24°, the result is a rotation by 34°.

Observe that $R_\theta^\top = R_{-\theta}$. In other words,

$$R_\theta^\top R_\theta = R_{-\theta} R_\theta = R_0 = I$$

Therefore R and $R^\top$ are inverses.

End of Example 120. □