18 2025-10-06 | Week 07 | Lecture 18

> ***The nexus question of this lecture:*** *What are the three fundamental linear subspaces associated with a matrix A?*

18.1 Three fundamental subspaces

Definition 81 (The fundamental subspaces of A)**.** Let $A \in M_{m\times n}(\mathbb{R})$. There are three important vector spaces associated with A :

- The ***column space***, which is the subspace of $M_{m\times 1}(\mathbb{R})$ spanned by the columns of A. Notation: $CS(A)$
- The ***row space***, which is the subspace of $M_{1\times n}(\mathbb{R})$ spanned by the rows of A. Notation $RS(A)$.
- The ***null space*** which is the subspace of R^n of vectors x such that $Ax = 0$. Notation $NS(A)$. The nullspace and the kernel are the same thing.

Remark 82 (Connection between column space and matrix multiplication)**.** The idea of column space is natural. If

$$A = \begin{bmatrix} | & | & & | \\ A_1 & A_2 & \dots & A_n \\ | & | & & | \end{bmatrix} \in M_{m\times n}(\mathbb{R})$$

then for any vector $x = (x_1, \dots, x_n)^\top \in \mathbb{R}^n$

$$Ax = A_1x_1 + A_2x_2 + \ldots + A_nx_n$$

This is linear combination of the columns of A, so the output Ax is always an element of the column space. Another word for column space is the ***image*** or ***range*** of the [linear transformation of the] matrix A.

Definition 83 (rank)**.** The ***rank*** of a matrix is the dimension of its column space (= dim of row space).

Theorem 84. *For $A \in M_{m\times n}(\mathbb{R})$, the dimensions satisfy*

$$\dim RS(A) = \dim CS(A)$$

and

$$\underbrace{\dim CS(A)}_{\text{‘rank’}} + \underbrace{\dim NS(A)}_{\text{‘nullity’}} = \underbrace{n}_{\#\text{ cols of } A}$$

The second part is called the rank-nullity theorem. Noting that $\operatorname{rank}(A) = \dim CS(A)$ and that $NS(A) = \ker(A)$, we have

18.2 Some examples of computing bases for the three fundamental subspaces

Example 85 (Row space, column space, null space)**.** Let

$$A = \begin{bmatrix} 1 & 0 & -1 & 3 \\ 2 & 1 & 4 & -1 \\ 4 & 1 & 2 & 5 \end{bmatrix} \in M_{3\times 4}(\mathbb{R}).$$

Find bases for

(a) $RS(A)$

(b) $NS(A)$

Solution

(a) Wrong:

$$\begin{bmatrix}1 & 0 & -1 & 3\end{bmatrix}, \begin{bmatrix}2 & 1 & 4 & -1\end{bmatrix}, \begin{bmatrix}4 & 1 & 2 & 5\end{bmatrix}$$

Better approach:

Idea: row reduction does not change row space, so row reduce until we get a linearly independent set.

The reduced row echelon form is

$$A_{\text{RREF}} = \begin{bmatrix}1 & 0 & -1 & 3\\ 0 & 1 & 6 & -7\\ 0 & 0 & 0 & 0\end{bmatrix} \tag{15}$$

A basis of A_{RREF} is

$$\begin{bmatrix}1 & 0 & -1 & 3\end{bmatrix},\ \begin{bmatrix}0 & 1 & 6 & 7\end{bmatrix}$$

(since these are linearly independent). Moreover, we note that since row reduction doesn't change the row space,

$$RS(A) = RS(A_{\text{RREF}}).$$

and hence

$$\begin{bmatrix}1 & 0 & -1 & 3\end{bmatrix},\ \begin{bmatrix}0 & 1 & 6 & 7\end{bmatrix}$$

are a basis for $RS(A)$ as well.

(b) We will use the fact that

$$NS(A) = NS(A_{\text{RREF}}).$$

So it suffices to find a basis for $NS(A_{\text{RREF}})$. Let's do a computation to sese what $NS(A_{\text{RREF}})$ looks like. Recall that $NS(A_{\text{RREF}})$ consists of the vectors x satisfying

$$A_{\text{RREF}}x = 0 \tag{16}$$

If $x = (x_1, \ldots, x_5)^\top$ satisfies Eq. (16), then we have

$$\begin{bmatrix}1 & 0 & -1 & 3\\ 0 & 1 & 6 & -7\\ 0 & 0 & 0 & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\\ x_4\end{bmatrix} = \begin{bmatrix}0\\ 0\\ 0\\ 0\end{bmatrix}.$$

Writing this as equations, we have

$$\begin{aligned} x_1 - x_3 + 3x_4 &= 0\\ x_2 + 6x_3 - 7x_4 &= 0\\ 0 &= 0. \end{aligned}$$

Therfore we have:

$$\begin{aligned} x_1 &= x_3 - 3x_4\\ x_2 &= -6x_3 + 7x_4 \end{aligned}$$

where x_3, x_4 are free variables.

Therefore if $x \in RS(A_{\text{RREF}})$ (equivlently, if x satisfies Eq. (16)), then it has the following form

$$\begin{bmatrix}x_1\\ x_2\\ x_3\\ x_4\end{bmatrix} = \begin{bmatrix}x_3 - 3x_4\\ -6x_3 + 7x_4\\ x_3\\ x_4\end{bmatrix} = x_3\begin{bmatrix}1\\ -6\\ 0\\ 1\end{bmatrix} + x_4\begin{bmatrix}-3\\ 7\\ 0\\ 1\end{bmatrix}.$$

This shows that the vectors

$$\begin{bmatrix} 1 \\ -6 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -3 \\ 7 \\ 0 \\ 1 \end{bmatrix}$$

span $NS(A)$. Moreover, they are also linearly independent (since two vectors are linearly dependent if and only if they are multiples of each other, which these are clearly not). Therefore

$$\begin{bmatrix} 1 \\ -6 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -3 \\ 7 \\ 0 \\ 1 \end{bmatrix}$$

form a basis for $NS(A_{\text{RREF}})$. Since $NS(A_{\text{RREF}}) = NS(A)$, they are a basis for $NS(A)$ as well. Since there are two vectors in the basis, $\dim NS(A) = 2$.

End of Example 85. $\square$