Math 307: Practice final exam

Instructions: You have 120 minutes. Calculators and notes are not allowed. There are a total of 70 points on the exam.

Problem 1 (Definitions)**.** Your answer must be precise to recieve full credit.

(a) Let $v_1, \dots, v_n$ be vectors in a vector space V. Define $\operatorname{Span}\{v_1, \dots, v_n\}$.

(b) Define **linear independence**.

(c) Let $v_1, \dots, v_n$ be vectors in a vector space V. The vectors $v_1, \dots, v_n$ are a **basis** of V if...

Problem 2 (dimension)**.** Fill in the blanks.

(a) (2 points) If the dimension of the row space of a 3×5 matrix is 2, then the nullspace of A is a linear subspace of ____ with dimension ____.

(b) (4 points) Suppose $Ax = 0$ is a system of linear equations with 10 equations in 5 variables, and the only solution is $x = 0$. The rank is ____. The columns of A are linearly ________. The rows of A are linearly ________. The rank of A is _____.

(c) Let A is a 9×5 matrix. If $Ax = 0$ has 3 free variables in its solution set, which one of the following describes the column space of A? Circle one:

- $CS(A)$ is a 2 dimensional subspace of $\mathbb{R}^9$
- $CS(A)$ is a 2 dimensional subspace of $\mathbb{R}^5$
- $CS(A)$ is a 7 dimensional subspace of $\mathbb{R}^9$
- $CS(A)$ is a 7 dimensional subspace of $\mathbb{R}^5$

Problem 3 (change of basis)**.** Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the **reflection** across the line $y = 2x$.

(a) Find two linearly independent eigenvectors β_1 and β_2. Sketch them.
(Hint: the line $y = -\frac{1}{2}x$ is perpendicular to $y = 2x$)

(b) Find eigenvalues λ_1 and λ_2 corresponding to β_1 and β_2.

(c) Let $\beta = \{\beta_1, \beta_2\}$. Find $[T(\beta_1)]_\beta$ and $[T(\beta_2)]_\beta$.

(d) Find $[T]_\beta^\beta$.

(e) Let $\alpha = \{e_1, e_2\}$ where e_1, e_2 are the standard basis vectors. Find the change of basis matrix P from α to β, and also find P^{-1}.

(f) Use your answers to parts (d) and (e) to find $[T]_\alpha^\alpha$.

(g) Find $[T(e_1)]_\alpha, [T(e_2)]_\alpha$. (Hint: use your answer to part (f)).

(h) Verify $[T]_\alpha^\alpha$ and $[T]_\beta^\beta$ have the same characteristic polynomial.

Problem 4 (differential equation)**.** Let

$$A = \begin{bmatrix} 6 & -8 \\ 4 & -6 \end{bmatrix}.$$

Solve the initial value problem

$$\begin{cases} Y' = AY \\ Y(0) = \begin{bmatrix} 11 \\ 6 \end{bmatrix} \end{cases}$$

Find the general solution to $Y' = AY$ when

Problem 5 (True-false).

(a) **TRUE | FALSE** If v is an eigenvector A, then $5v$ is also an eigenvector.

(b) **TRUE | FALSE** An 4×5 matrix A is surjective if $\text{Rank}(A) = 4$.

(c) **TRUE | FALSE** If $T : \mathbb{R}^n \to \mathbb{R}^m$ is a surjective linear transformation, then it must be the case that $m \leq n$.

(d) **TRUE | FALSE** The intersection of two subspaces of a vector space is never empty.

(e) **TRUE | FALSE** If $v_1, \ldots, v_n$ is a linearly independent set of vectors, then $\lambda v_1, \lambda v_2, \ldots, \lambda v_n$ are also linearly independent.

(f) **TRUE | FALSE** If $v_1, \ldots, v_n$ are linearly dependent, then at least 2 of the vectors are scalar multiples of each other.

(g) **TRUE | FALSE** If A is an $n \times n$ diagonalizable matrix, then A has n distinct eigenvalues.

(h) **TRUE | FALSE** If A is an $m \times n$ matrix, then $\ker(A)$ is a linear subspace of $\mathbb{R}^n$.

(i) **TRUE | FALSE** If $\lambda = 0$ is an eigenvalue of A, then for any b, the equation $Ax = b$ has infinitely many solutions.

(j) **TRUE | FALSE** If v is a vector in $NS(B)$, then it is in $NS(AB)$ as well.

(k) **TRUE | FALSE** If v is a vector in $NS(B)$, then it is in $NS(BA)$ as well.

(l) **TRUE | FALSE** Let A, B be $n \times n$ matrices. If v is in the column space of A, then it is also in the column space of AB.

(m) **TRUE | FALSE** A linear transformation $T : \mathbb{R}^n \to \mathbb{R}^n$ is invertible if and only if it is surjective.

(n) **TRUE | FALSE** A linear transformation $T : \mathbb{R}^n \to \mathbb{R}^n$ is invertible if and only if it is injective.

(o) **TRUE | FALSE** An $n \times n$ matrix is diagonalizable if and only if it has n linearly independent eigenvectors.

Problem 6 (Subspace problem)**.** Determine whether the following sets are subspaces or not. If the set is not a subspace, give a reason why not. If the set is a subspace, find a basis for it.

(a) The set $\left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \in \mathbb{R}^4 : x_1 + 2x_2 + 7x_3 - x_4 = 0 \right\}$.

(b) The set $\left\{ \begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^2 : x + y = 0 \text{ or } x - y = 0 \right\}$

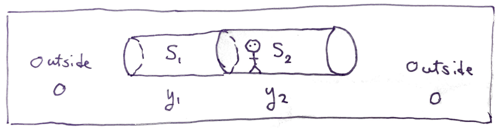
(c) The set $\left\{ \begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^2 : xy = 0 \right\}$ is a linear subspace of $\mathbb{R}^2$.

(d) The vectors in $\mathbb{R}^4$ for which $x_1 = 2x_4$.

(e) The vectors in $\mathbb{R}^4$ for which $x_1 + x_2 + x_3 = 0$ and $x_3 + x_4 = 0$.

(f) The subspace spanned by $\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 4 \\ 5 \end{bmatrix}$

Problem 7 (differential equation)**.** A person is standing in a large pipe drops a flask of chemicals, releasing a chemical gas. The tube has 2 sections, S_1 and S_2, as shown:



Let $Y(t) = \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix}$, where

$$\begin{aligned} y_1(t) &= \text{ concentration of chemical in section } S_1 \text{ at time } t \\ y_2(t) &= \text{ concentration of chemical in section } S_2 \text{ at time } t. \end{aligned}$$

Further, assume that the ends of the pipe are open, so the chemical can leak out the ends of the tube. (The concentration outside is always zero, since the gas gets immediately blown away by the wind). Assume that at each time t, the diffusion rate between adjacent areas is the difference in concentrations.

(a) Set up a system of differential equations $Y' = AY$ describing the diffusion of the gas.

(b) Solve your system, assuming that the initial concentration $Y(0) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. Interpret your result in words.

Problem 8 (Geometry)**.** Let $M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Find values for a, b, c, d such that the column space of M is the line $y = x$ and the null space is the line $y = -3x$.

Problem 9 (computation, diagonalizability)**.** Let $A = \begin{bmatrix} 2 & 0 & 0 \\ -2 & -3 & 2 \\ 4 & -2 & 1 \end{bmatrix}$.

(a) Find the eigenvalues of A.

(b) For each eigenvalue λ, find the eigenspace E_λ.

(c) Is A diagonalizable?

Problem 10 (system with variables)**.** Consider the system of equations

$$\begin{cases} x_1 + x_2 + 3x_3 = a \\ 2x_1 + x_2 + 4x_3 = b \\ 3x_1 + x_2 + 5x_3 = c \end{cases}$$

where $a, b, c \in \mathbb{R}$.

(a) For conditions on a, b, c such that the system has at least one solution.

(b) When the conditions in part (a) are satisfied, find all solutions $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ of the system.

(c) Describe the geometric shape of your answer to part (b). What is its dimension?

Problem 11 (Geometry)**.** The linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ can be achieved by reflecting across the y-axis, and then rotating counter-clockwise by $45°$. Find the standard matrix for T and use it to find $T\begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

Problem 12 (computation)**.** Let $A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 6 \end{bmatrix}$.

(a) Find a basis for $\ker(A)$. What is the dimension of the kernel?

(b) Find a basis for the column space of A. What is its dimension?