

## 35 2025-11-14 | Week 12 | Lecture 35

*The nexus question of this lecture: What are the eigenvectors of a rotation?*

Suppose  $T(x) = Ax$  is a linear transformation, and  $v$  is an eigenvector of  $T$ . This means that there exists a scalar  $\lambda \in \mathbb{C}$  such that.

$$T(v) = \lambda v.$$

Geometrically, this equation says that an eigenvector is a direction that the transformation leaves intact: when  $T$  acts on  $v$ , it may stretch, shrink, or flip that vector, but doesn't rotate it into a new direction.

This naturally raises a striking question: what are the eigenvectors of a rotation? A rotation of the plane visibly changes *every* real direction in the plane. Every arrow in the plane is dragged away from where it once pointed. So if eigenvectors are “directions left intact”, then how can a rotation possibly have any?

The short answer to this question is that we get complex eigenvectors. The rest of the lecture will consist of an example of a linear transformation of  $\mathbb{R}^2$  involving a rotation which illustrates how this happens.

### 35.1 Eigenvectors and eigenvalues of a rotation

**Example 159** (Eigenvalues and eigenvectors of a rotation). Let

$$A = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}.$$

This is a counterclockwise rotation by  $45^\circ$  composed with a scaling by  $\sqrt{2}$ . That is,  $A = \sqrt{2}R_{45^\circ}$ .

**Question:** what are the eigenvalues and eigenvectors of  $A$ ?

**Solution:** The characteristic polynomial is

$$\det \begin{bmatrix} \lambda - 1 & 1 \\ -1 & \lambda - 1 \end{bmatrix} = (\lambda - 1)^2 + 1 = \lambda^2 - 2\lambda + 2 = (\lambda - (1+i))(\lambda - (1-i)).$$

This has solutions  $\lambda = 1+i$  and  $\lambda = 1-i$ . These two complex numbers are the eigenvalues of  $A$ .

- For  $\lambda = 1+i$ , we want to find the eigenspace  $E_{1+i} = NS((1+i)I - A)$ .

First observe that  $(1+i)I - A = \begin{bmatrix} 1+i & 0 \\ 0 & 1+i \end{bmatrix} - \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} i & 1 \\ -1 & i \end{bmatrix}$ . We wish to solve

$$\begin{bmatrix} i & 1 \\ -1 & i \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Next, set up and row reduce the augmented matrix:

$$\left[ \begin{array}{cc|c} i & 1 & 0 \\ -1 & i & 0 \end{array} \right] \xrightarrow{R_2 - iR_1 \rightarrow R_2} \left[ \begin{array}{cc|c} i & 1 & 0 \\ 0 & 0 & 0 \end{array} \right] \xrightarrow{(-i)R_1 \rightarrow R_1} \left[ \begin{array}{cc|c} 1 & -i & 0 \\ 0 & 0 & 0 \end{array} \right]$$

This corresponds to the system

$$\begin{cases} x - iy = 0 \\ 0 = 0 \end{cases}$$

Hence  $x = iy$  and  $y$  is a free variable. Therefore

$$E_{1+i} = \left\{ \begin{bmatrix} iy \\ y \end{bmatrix} : y \in \mathbb{C} \right\} = \left\{ y \begin{bmatrix} i \\ 1 \end{bmatrix} : y \in \mathbb{C} \right\}$$

In words, the eigenspace  $E_{1+i}$  is spanned by the vector  $\begin{bmatrix} i \\ 1 \end{bmatrix}$ .

- For  $\lambda = 1 - i$ , similar calculation shows that the eigenspace  $E_{1-i} = NS((1-i)I - A)$  is spanned by the eigenvector

$$\begin{bmatrix} -i \\ 1 \end{bmatrix}.$$

End of Example 159.  $\square$

Example 159 illustrates the following theorem:

**Theorem 160.** *Let  $A$  be an  $n \times n$  matrix with real entries. If  $\lambda = a + bi$  is an eigenvalue of  $A$  (here  $a, b \in \mathbb{R}$ ), then  $\bar{\lambda} = a - bi$  is also an eigenvalue.*

**Remark 161.** Every complex number  $z \in \mathbb{C}$  can be written as  $z = a + bi$  for some choice of real numbers  $a$  and  $b$ . The terminology used is that  $a + bi$  and  $a - bi$  are called **complex conjugates**.

## 35.2 Diagonalization of a rotation

We can diagonalize the rotation.

**Example 162.** Let us consider again the matrix

$$A = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}.$$

This is the rotation from Example 159; there, we show that this matrix has two eigenvectors:

$$\begin{bmatrix} i \\ 1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} -i \\ 1 \end{bmatrix}.$$

Theorem 144 implies that  $A$  is diagonalizable. The matrix  $P = \begin{bmatrix} i & -i \\ 1 & 1 \end{bmatrix}$  (whose columns are the two eigenvectors) is invertible (because the eigenvectors are linearly independent). This is a change-of-basis matrix (our “prism”), and we can use it to diagonalize  $A$  as follows:

$$P^{-1}AP = \begin{bmatrix} 1+i & 0 \\ 0 & 1-i \end{bmatrix} \tag{28}$$

(see below for details to justify this computation). The main diagonal entries on the right hand side are the eigenvalues of  $A$ . Somehow, in rotating the plane  $45^\circ$  (and scaling it by a factor of  $\sqrt{2}$ ), one of the complex eigenvectors gets multiplied by  $1+i$  and the other by  $1-i$ .

Here are the details which justify Eq. (28): first observe that we have

$$P = \begin{bmatrix} i & -i \\ 1 & 1 \end{bmatrix} \quad \text{and} \quad P^{-1} = \frac{1}{2i} \begin{bmatrix} 1 & i \\ -1 & i \end{bmatrix}.$$

Therefore

$$\begin{aligned} P^{-1}AP &= \left( \frac{1}{2i} \begin{bmatrix} 1 & i \\ -1 & i \end{bmatrix} \right) \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} i & -i \\ 1 & 1 \end{bmatrix} \\ &= \left( \frac{1}{2i} \begin{bmatrix} 1 & i \\ -1 & i \end{bmatrix} \right) \begin{bmatrix} i-1 & -i-1 \\ i+1 & -i+1 \end{bmatrix} \\ &= \frac{1}{2i} \begin{bmatrix} 2i-2 & 0 \\ 0 & 2i+2 \end{bmatrix} \\ &= \begin{bmatrix} 1-\frac{1}{i} & 0 \\ 0 & 1+\frac{1}{i} \end{bmatrix} \\ &= \begin{bmatrix} 1+i & 0 \\ 0 & 1-i \end{bmatrix} \quad \text{since } \frac{1}{i} = -i, \end{aligned}$$

and the right-hand side is  $B$ . In this calculation, we used the fact that  $\frac{1}{i} = -i$  (which holds because  $(-i)i = 1$ ).

End of Example 162.  $\square$