36 2025-11-17 | Week 13 | Lecture 36

We begin Section 6.1. This lecture is based on Section 5.4 in Gilbert Strang's Linear Algebra and its Applications)

36.1 Examples of systems of first-order linear differential equations

Example 163. An ant is moving in the xy-plane with velocity vector

$$v = \begin{bmatrix} 2x - 5y \\ x - 2y \end{bmatrix}$$

Suppose the ant starts at $(1, 1)$.

Question: Find the position of the ant at time $t > 0$
Solution: First recognize that

$$v = \begin{bmatrix} \frac{dx}{dt} \\ \frac{dy}{dt} \end{bmatrix}$$

and therefore we have the following system of equations:

$$\begin{cases} \frac{dx}{dt} = 2x - 5y \\ \frac{dy}{dt} = x - 2y \end{cases} \tag{29}$$

with initial conditions $x(0) = 1$ and $y(0) = 1$.

A solution to this system would consist of functions of the form

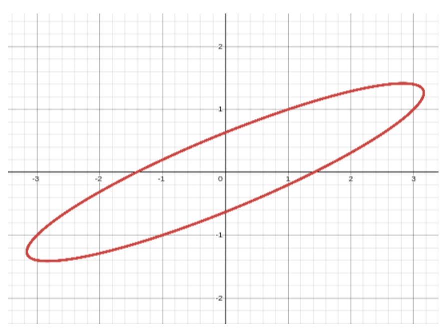
$$x = x(t) \quad \text{and} \quad y = y(t)$$

which satisfy the system of equations and initial conditions.

Without showing how we got it, here is one possible solution to the differential equation given in Eq. (29):

$$\begin{cases} x(t) = \cos(t) - 3\sin(t) \\ y(t) = \cos(t) - \sin(t) \end{cases}$$

These two equations parameterize the following curve:



To verify that this is indeed a solution, we can check first that $x(0) = 1$ and $y(0) = 1$, and that the right hand sides of Eq. (29) actually equal $x'(t)$ and $y'(t)$. To simplify, let $c = \cos(t)$ and $s = \sin(t)$:

$$2x - 5y = 2(c - 3s) - 5(c - s) = -s - 3c = x'(t)$$

and

$$x - 2y = (c - 3s) - 2(c - s) = -c - s = y'(t).$$

End of Example 163. □

Example 164 (Strang's example)**.** Consider the differential equation:

$$\begin{bmatrix} y_1' \\ y_2' \end{bmatrix} \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}. \tag{30}$$

Here, y_1, y_2 are both functions of t ("time"). This differential equation has the form

$$Y' = AY$$

where A is an $n \times n$ matrix and $Y = Y(t) = \begin{bmatrix} y_1(t) \\ \vdots \\ y_n(t) \end{bmatrix}$.

The only thing we need is the following:

Theorem 165 (Pure exponential solutions)**.** *Let A be an $n \times n$ matrix, let v be an eigenvector of A with eigenvalue λ, and let c be any scalar. Then*

$$Y(t) = ce^{\lambda t}v$$

is a solution of the system $Y' = AY$.

Proof.

$$\begin{aligned} Y'(t) &= \frac{d}{dt}\left[ce^{\lambda t}v\right] \\ &= ce^{\lambda t}\lambda v \\ &= ce^{\lambda t}Av \\ &= A\left(ce^{\lambda t}v\right) \\ &= AY(t) \end{aligned}$$

So $Y' = AY$, meaning $Y(t) = ce^{\lambda t}v$ is a solution of the differential equation. $\square$

In our problem Eq. (30), the first step is to find the eigenvalues and eigenvectors:

$$A\begin{bmatrix} 1 \\ 1 \end{bmatrix} = (-1)\begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad \text{and} \quad A\begin{bmatrix} 1 \\ -1 \end{bmatrix} = (-3)\begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

By Theorem 165, we have two solution:

$$Y_1(t) = c_1e^{-t}\begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad \text{and} \quad Y_2(t) = c_2e^{-3t}\begin{bmatrix} 1 \\ -1 \end{bmatrix},$$

where c_1, c_2 are arbitrary constants. Combining these, we get the general solution

$$Y(t) = c_1e^{-t}\begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2e^{-3t}\begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

If we further know that $Y(0) = B = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$, we can solve for c_1, c_2 by observing that plugging $t = 0$ into the above formula gives

$$\begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = c_1e^{-t}\begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2e^{-3t}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

We can write this as

$$\begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

which implies that

$$\begin{bmatrix} c_1 \\ c_2 = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}^{-1} \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} \end{bmatrix}.$$

(Note that the matrix $\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$ is invertible since its columns are linearly independent eigenvectors of A.)

End of Example 165. □