

# Math 307: Homework 10

*Due Wednesday, December 10*

**Problem 1.** Let  $Y(t) = \begin{bmatrix} c_1 e^{2t} + c_2 e^{3t} \\ 2c_1 e^{2t} + c_2 e^{3t} \end{bmatrix}$ .

- (a) Show that  $Y(t)$  is a solution of  $Y' = \begin{bmatrix} 4 & -1 \\ 2 & 1 \end{bmatrix} Y$ . (To do this, compute the LHS and RHS independently and then observe that they are equal).
- (b) Use the webapp <https://homepages.bluffton.edu/~nesterd/apps/slopefields.html> (or the software PPLANE at <http://alun.math.ncsu.edu/pplane/>), to plot the slope field of the system. Clicking on a coordinate in the plot will show the trajectory of a particle starting at that point whose motion is governed by the differential equation. This can also be done precisely using the “initial points” tab. Do this to plot some trajectories.

**Problem 2.** Find the general solution to  $Y' = AY$  when  $A = \begin{bmatrix} 6 & -8 \\ 4 & -6 \end{bmatrix}$  and plot the slope field of the differential equation.

**Problem 3.** Find the general solution to  $Y' = AY$  when  $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$  and plot the slope field of the differential equation.

**Problem 4.** Let  $A = \begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix}$ .

- (a) Determine the general solution to  $Y' = AY$
- (b) Solve the initial value problem  $Y' = AY$  with  $Y(0) = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$
- (c) Plot the solution to the initial value problem using the software from the previous problems.

**Problem 5** (A non-diagonalizable example). Consider the system of differential equations

$$\begin{cases} x'(t) = y \\ y'(t) = -x - 2y \end{cases}$$

Letting  $Y(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$  and  $A = \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix}$ , we can write the above system of differential equations in matrix form as

$$Y' = AY. \tag{1}$$

- (a) Plot the slope field and some particle trajectories using the software from the previous problem.
- (b) Are there any straight-line trajectories?
- (c) Compute the eigenvalues and eigenvectors of  $A$ .
- (d) Show that  $A$  is not diagonalizable.
- (e) Since  $A$  is not diagonalizable, the techniques we've discussed so far for solving differential equations like this won't work. Show that for any scalars  $c_1$  and  $c_2$ , the function

$$\begin{aligned} Y(t) &= c_1 e^{-t} \begin{bmatrix} 1 \\ -1 \end{bmatrix} + c_2 e^{-t} \begin{bmatrix} 1 \\ 0 \end{bmatrix} + c_2 t e^{-t} \begin{bmatrix} 1 \\ -1 \end{bmatrix} \\ &= \begin{bmatrix} c_1 e^{-t} + c_2 e^{-t} + c_2 t e^{-t} \\ -c_1 e^{-t} - c_2 t e^{-t} \end{bmatrix} \end{aligned}$$

is a solution to the differential equation in Eq. (1). (Hint: computing the left-hand side and right-hand side independently and then observe that they are equal.)

(f) Using the general solution from part (e), solve the initial value problem

$$\begin{cases} Y' = AY \\ Y(0) = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \end{cases}$$

**Problem 6** (Continuation of Problem 5). Let  $A$  be the  $2 \times 2$  matrix from Problem 5.

- (a) Let  $B = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} \in \mathbb{R}^2$ . Using the fact that  $\frac{d}{dt} [e^{Mt}] = Me^{Mt}$  for any square matrix, show that  $Y(t) = e^{At}B$  is a solution to the initial value problem

$$\begin{cases} Y' = AY \\ Y(0) = B \end{cases}$$

(Hint: you don't need to calculate  $e^{At}$  for this part. The rest of this problem will guide you through one way to compute  $e^{At}$ .)

- (b) Assume that  $A = PJP^{-1}$  for some  $2 \times 2$  matrices  $J$  and  $P$ . Show that  $e^{At} = Pe^{Jt}P^{-1}$ .

(Hint: use the definition  $e^M = \sum_{k=0}^{\infty} \frac{M^k}{k!} = I + M + \frac{M^2}{2!} + \frac{M^3}{3!} + \dots$ ).

- (c) Let  $P = \begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix}$  and  $J = \begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix}$ . Compute  $P^{-1}$  and verify that  $A = PJP^{-1}$ .

(Note:  $J$  is the Jordan normal form of  $A$ . Since  $A$  is not diagonalizable,  $J$  is the closest we can get to diagonalizing  $A$ .)

- (d) Compute  $(tJ)^k$  for  $k = 0, 1, 2, 3, 4, 5$ . Formulate a conjecture about what you think  $(tJ)^k$  is, for a general integer  $k$ .

- (e) Using part (d), one can show that  $e^{tJ} = \begin{bmatrix} e^{-t} & te^{-t} \\ 0 & e^{-t} \end{bmatrix}$  (you don't have to show this). Moreover, from part (b) we know that  $e^{At} = Pe^{Jt}P^{-1}$ . Using these two facts, compute  $e^{At}$ .

- (f) From part (a), we know that  $Y(t) = e^{At}B$  is a solution to the intial value problem. Find the solution when  $B = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$  and compare your answer to the answer to your answer in part (f) of Problem 5.

**Problem 7.** Let  $A = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 4 \end{bmatrix}$ .

- (a) Determine the general solution to  $Y' = AY$

- (b) Solve the initial value problem  $Y' = AY$  with  $Y(0) = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$

**Problem 8.** Suppose a rabbit population  $r$  and a wolf population  $w$  are governed by

$$\begin{cases} \frac{dr}{dt} = 4r - 2w \\ \frac{dw}{dt} = r + w \end{cases}$$

- (a) What is the solution to this differential equation? Interpret your answer in terms of rabbits and wolves.  
 (b) If  $r = 300$  and  $w = 200$  at time  $t = 0$ , what is the population of wolves and rabbits at time  $t > 0$ ?  
 (c) After a long time, what is the proportion of rabbits to wolves? Hint: compute  $\lim_{t \rightarrow \infty} \frac{r(t)}{w(t)}$ .

**Problem 9.** Find the general solution to  $Y' = AY$  when  $A = \begin{bmatrix} 6 & 0 & -8 \\ -4 & 2 & -4 \\ 4 & 0 & 6 \end{bmatrix}$  and plot the slope field of the differential equation.

**Problem 10.** Find the eigenvalues and eigenvectors, and the exponential  $e^{At}$  for

$$A = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}$$

Write the general solution to  $Y' = AY$ , and the specific solution that satisfies  $Y(0) = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$ . Plot the trajectory of your solution. What happens as  $t \rightarrow \infty$ ?

**Problem 11** (Kimura 2 parameter substitution model). This problem introduces a model of DNA mutation that is more sophisticated than Jukes Cantor, since it does not make the overly simplifying assumption that all nucleotides A, C, G, and T mutated between each other at equal rates. Instead, it assumes

- time  $t$  is measured in generations
- mutations from A between G happen at rate  $a > 0$  per generation
- mutations from C between T happen at rate  $a > 0$  per generation
- mutations between all other pairs of A, C, G, T happen at rate  $b > 0$  per generation

Let  $Y(t) = \begin{bmatrix} y_A(t) \\ y_C(t) \\ y_G(t) \\ y_T(t) \end{bmatrix}$ , where,  $y_A(t)$  is the proportion of the DNA with letter A at time  $t$ , and  $y_C, y_G, y_T$  are defined similarly. The differential equation describing the change in  $Y$  is

$$Y' = QY \tag{2}$$

where

$$Q = \begin{bmatrix} -a-2b & b & a & b \\ b & -a-2b & b & a \\ a & b & -a-2b & b \\ b & a & b & -a-2b \end{bmatrix}$$

(a) Show the following eigenvectors are linearly independent

$$\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \end{bmatrix}$$

(b) Show that the vectors from part (a) are eigenvectors of  $Q$ . What are the corresponding eigenvalues?

(c) What is the general solution to Eq. (2)?