

## Math 307: Homework 03

Due Wednesday, September 24 (at the beginning of class)

**Problem 1** (Important). Let  $A$  be a  $3 \times 3$  matrix. Suppose  $x_1, x_2$ , and  $x_3$  are column vectors such that

$$Ax_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad \text{and} \quad Ax_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \quad \text{and} \quad Ax_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

If the three solutions  $x_1, x_2$  and  $x_3$  are columns of a matrix  $X$ , what is  $AX$ ?

**Problem 2.** Given an  $m \times n$  matrix  $A = (a_{ij})$ , the **transpose** of  $A$ , denoted  $A^\top$ , is the  $n \times m$  matrix  $A^\top = (a_{ji})$ . For example,

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \quad \text{and} \quad A^\top = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$$

Prove the following theorem.

**Theorem.** (Properties of the transpose). Suppose  $A, B$  are matrices. Then whenever defined, the following properties hold:

- (i.)  $(A^\top)^\top = A$
- (ii.)  $(A + B)^\top = A^\top + B^\top$
- (iii.)  $(cA)^\top = cA^\top$
- (iv.)  $(AB)^\top = B^\top A^\top$  (this is sort of like the socks and shoes property)
- (v.)  $(A^\top)^{-1} = (A^{-1})^\top$

(This is Theorem 1.13 in the textbook. The textbook offers a proof of part (iv.), so if you understand that proof, you can use it in your answer. Hint for part (v.): by Theorem 22 (in Lecture 8), all you need to prove is  $(A^{-1})^\top A^\top = I$ .

**Problem 3.** If  $A = A^\top$ , then we say that  $A$  is a **symmetric matrix**. An example:

$$\begin{bmatrix} -5 & 1 & 2 \\ 1 & 4 & 3 \\ 2 & 3 & 6 \end{bmatrix}$$

Note that symmetric matrices are always square.

- (a) Let  $A$  be any matrix. Show that  $A^\top A$  and  $AA^\top$  are both symmetric matrices. *Hint: use property (iv.) from the Theorem in Problem 2.*
- (b) Let  $A$  be a symmetric matrix. Show that if  $A$  is invertible, then  $A^{-1}$  is also symmetric. *Hint: use property (v.) from the Theorem in Problem 2.*

**Problem 4.** An **involution** is a function  $f$  such that  $f(f(x)) = x$  for all  $x$ . In other words, an involution is a function which is its own inverse. By Part (i.) in the theorem from Problem 2, we know that matrix transposition is an involution, since if  $f(A) = A^\top$ , then

$$f(f(A)) = f(A^\top) = (A^\top)^\top = A.$$

Another example is matrix inversion, since  $(A^{-1})^{-1} = A$ . Give some other examples of involutions (from any area of math).

**Problem 5.** Let

$$A = \begin{bmatrix} 2 & -1 & 3 \\ 4 & 1 & -2 \\ -3 & 2 & 1 \end{bmatrix}$$

- (a) Find  $\det(A)$  by expanding about row 1
- (b) Find  $\det(A)$  by expanding about row 2
- (c) Find  $\det(A)$  by expanding about column 1
- (d) Find  $\det(A)$  by expanding about column 3

**Problem 6.** Find the inverse of the matrix

$$\begin{bmatrix} 0 & -2 & 1 \\ 2 & 4 & -1 \\ 2 & 1 & 2 \end{bmatrix}$$

**Problem 7.** Solve the following linear system:

$$\begin{aligned} 2x - 4y + 6z &= 2 \\ -3x + 6y - 9z &= 3 \end{aligned}$$

Interpret your results geometrically. Provide a sketch or an image (e.g., using Desmos) of the the solution.

**Problem 8.** The technical definition of “nonsingular” is the following:

*Definition.* An  $n \times n$  matrix  $A$  is said to be **nonsingular** if the only solution to the system of linear equations  $AX = \mathbf{0}$  is  $X = \mathbf{0}$ .

In other words,  $A$  produces the output  $\mathbf{0}$  only for the input  $\mathbf{0}$ . Note that in the above definition,

$$X = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \quad \text{and} \quad \mathbf{0} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}.$$

Prove that a matrix  $A$  is nonsingular if and only if  $\det A \neq 0$ . (*Hint: use the key theorem of linear algebra from lecture 9*).

**Problem 9** (More determinants).

- (a) Compute the determinant by doing a cofactor expansion across an appropriate row or column.

$$\begin{vmatrix} -3 & 0 & 4 \\ 2 & -1 & 3 \\ 4 & 0 & 5 \end{vmatrix}$$

- (b) Compute the determinant by doing a cofactor expansion across an appropriate row or column.

$$\begin{vmatrix} 6 & -5 & 1 & 3 \\ 3 & 1 & -2 & -1 \\ 0 & 7 & 0 & 0 \\ 3 & 3 & 0 & 9 \end{vmatrix}$$

*Hint: Don't try to brute force this calculation. Be clever. See Example 1 in section 1.5 of the textbook.*

**Problem 10** (Permutation Matrices). A square matrix called a *permutation matrix* if exactly one entry in each row and column is equal to 1 and all other entries are 0. Multiplication by such matrices affects a permutation of the rows or columns of the matrix multiplied. For example,

$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}$$

Left multiplication permutes the rows (as shown above). Right multiplication permutes the columns.

- (a) Consider the set  $\{1, 2, 3, 4, 5\}$ . One permutation of this set is  $(3, 2, 4, 1, 5)$ . Find the permutation matrix  $P$  such that

$$P \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ 4 \\ 1 \\ 5 \end{bmatrix}$$

- (b) Find a permutation matrix  $Q$  such that

$$Q \begin{bmatrix} 3 \\ 2 \\ 4 \\ 1 \\ 5 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{bmatrix}$$

- (c) What do you notice about the relationship between  $P$  and  $Q$ ?
- (d) Give a geometric argument for why the determinant of a permutation matrix is always  $+1$  or  $-1$ . (You don't need to give a proof, but try to be convincing.)

**Problem 11.** Solve the following linear system:

$$2x + 3y = 5$$

$$2x + y = 2$$

$$x - 2y = 1$$

Interpret your results geometrically. Provide a sketch or an image (e.g., using Desmos) of the the solution.