25 2025-10-22 | Week 09 | Lecture 25

> ***The nexus question of this lecture:*** *The matrix we use to represent a linear transformation depends on a choice of bases. How does it change when whe choose different bases?*

This lecture is based on section 5.3 in the textbook

25.1 Introduction to change of basis

The following theorem summarizes the main ideas from the previous lecture.

Theorem 114 (Change of Basis I)**.** *Let $\mathbb{A}$ and $\mathbb{B}$ be vector spaces. If $T : \mathbb{A} \to \mathbb{B}$ is a linear map, $\alpha = \{\alpha_1, \ldots, \alpha_n\}$ is a basis of $\mathbb{A}$, and $\beta = \{\beta_1, \ldots, \beta_m\}$ is a basis of $\mathbb{B}$, then we can represent T by the matrix*

$$[T]_\alpha^\beta = \begin{bmatrix} | & | & & | \\ T(\alpha_1) & T(\alpha_2) & \ldots & T(\alpha_n) \\ | & | & & | \end{bmatrix}$$

where $T(\alpha_i) = A\alpha_i = a_{1i}\beta_1 + a_{2i}\beta_2 + \ldots + a_{mi}\beta_m$ for each $i = 1, \ldots, n$.
(Here, $\alpha_1, \ldots, \alpha_n$ are vectors in $\mathbb{A}$ and $\beta_1, \ldots, \beta_n$ are vectors in $\mathbb{B}$.)

Example 115 (Example 1 in Section 5.3)**.** Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear transformation

$$T \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 5x + z \\ 3x + 2y - 3z \\ 5x \end{bmatrix}$$

Let $\alpha = \{e_1, e_2, e_3\}$, where

$$e_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad e_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad e_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

Clearly α is a basis of $\mathbb{R}^3$.

(a) **Question:** Find the matrix of T with respect to the standard basis α, that is, find $[T]_\alpha^\alpha$.
Solution: Observe that

$$T(e_1) = T \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 5 \\ 3 \\ 5 \end{bmatrix} = 5e_1 + 3e_2 + 5e_3$$

Similarly,

$$T(e_2) = \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix} = 0e_1 + 2e_2 + 0e_3 \quad \text{and} \quad T(e_3) = \begin{bmatrix} 1 \\ -3 \\ 0 \end{bmatrix} = 1e_1 - 3e_2 + 0e_3$$

These becomes the columns of the matrix

$$[T]_\alpha^\alpha = \begin{bmatrix} 5 & 0 & 1 \\ 3 & 2 & -3 \\ 5 & 0 & 0 \end{bmatrix}$$

(b) **Question:** Let $\beta = \{\beta_1, \beta_2, \beta_3\}$, where

$$\beta_1 = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}, \quad \beta_2 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, \quad \beta_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

This is a basis of $\mathbb{R}^3$. Find $[T]_\beta^\beta$, the matrix of T with respect to the basis β.

(The point of this example is to show you how to find $[T]_\beta^\beta$ mechanistically; I'm not trying to illustrate why this particular choice of basis β is good or meaningful—it's not.)

Solution: The columns of $[T]_\beta^\beta$ are the vectors $T(\beta_1), T(\beta_2), T(\beta_3)$ expressed in terms of the vectors in β. By plugging $\beta_1, \beta_2, \beta_3$ into T, we see that

$$T(\beta_1) = \begin{bmatrix} 7 \\ -1 \\ 5 \end{bmatrix}, \quad T(\beta_2) = \begin{bmatrix} 6 \\ -2 \\ 5 \end{bmatrix}, \quad T(\beta_3) = \begin{bmatrix} 6 \\ 2 \\ 5 \end{bmatrix}$$

But in the above equations, the right hand sides are all expressed in terms of e_1, e_2, e_3. For example,

$$T(\beta_1) = \begin{bmatrix} 7 \\ -1 \\ 5 \end{bmatrix} = 7e_1 - e_2 + 5e_3.$$

This is no good: we need to express them using in basis β. Observe that

- $T(\beta_1) = \begin{bmatrix} 7 \\ -1 \\ 5 \end{bmatrix} \overset{*}{=} -2 \underbrace{\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}}_{\beta_1} +4 \underbrace{\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}}_{\beta_2} +5 \underbrace{\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}}_{\beta_3} = -2\beta_1 + 4\beta_2 + 5\beta_3 = \begin{bmatrix} -2 \\ 4 \\ 5 \end{bmatrix}_\beta$

  Similarly,
- $T(\beta_2) = \begin{bmatrix} 6 \\ -2 \\ 5 \end{bmatrix} \overset{*}{=} -1 \underbrace{\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}}_{\beta_1} +4 \underbrace{\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}}_{\beta_2} +3 \underbrace{\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}}_{\beta_3} = \begin{bmatrix} -1 \\ 4 \\ 3 \end{bmatrix}_\beta$
- $T(\beta_3) = \begin{bmatrix} 6 \\ 2 \\ 5 \end{bmatrix} \overset{*}{=} -1 \underbrace{\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}}_{\beta_1} +2 \underbrace{\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}}_{\beta_2} +5 \underbrace{\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}}_{\beta_3} = \begin{bmatrix} -1 \\ 2 \\ 5 \end{bmatrix}_\beta$

Therefore by Theorem 114

$$[T]_\beta^\beta = \begin{bmatrix} | & | & | \\ T(\beta_1) & T(\beta_2) & T(\beta_3) \\ | & | & | \end{bmatrix} = \begin{bmatrix} -2 & -1 & -1 \\ 4 & 4 & 2 \\ 5 & 3 & 5 \end{bmatrix}$$

This matrix represents the same linear transformation, T, but now we are using β as the basis for $\mathbb{R}^3$ rather than α as in part (a). For some linear transformations, the is a "best" basis to use (and often it is not the standard basis!)

There is also a

The starred equalities above

Note that the equalities marked with a $(*)$ required solving a linear system. In the $T(\beta_1)$ case, for example, we needed to solve the linear system

$$\begin{bmatrix} 7 \\ -1 \\ 5 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} + c_3 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix},$$

which has solution $c_1 = -2$, $c_2 = 4$, $c_3 = 5$.

End of Example 115. □

25.2 The change of basis matrix

Suppose $\alpha = \{\alpha_1, \ldots, \alpha_n\}$ and $\beta = \{\beta_1, \ldots, \beta_n\}$ are bases for the same vector space V.

We can write the vectors of basis β in terms of the vectors of basis α:

$$\begin{aligned}
\beta_1 &= p_{11}\alpha_1 + p_{21}\alpha_2 + \ldots + p_{n1}\alpha_n \\
\beta_2 &= p_{12}\alpha_1 + p_{22}\alpha_2 + \ldots + p_{n2}\alpha_n \\
&\vdots \\
\beta_n &= p_{1n}\alpha_1 + p_{2n}\alpha_2 + \ldots + p_{nn}\alpha_n.
\end{aligned}$$

where the p's are all scalars. In other notation, we can write the above equations as

$$\beta_1 = \begin{bmatrix} p_{11} \\ p_{21} \\ \vdots \\ p_{n1} \end{bmatrix}_\alpha, \quad \beta_2 = \begin{bmatrix} p_{12} \\ p_{22} \\ \vdots \\ p_{n2} \end{bmatrix}_\alpha, \quad \cdots \beta_n = \begin{bmatrix} p_{1n} \\ p_{2n} \\ \vdots \\ p_{nn} \end{bmatrix}_\alpha.$$

Concatenating these column vectors, we obtain the matrix

$$P = \begin{bmatrix} p_{11} & p_{12} & \cdots & p_{1n} \\ p_{21} & p_{22} & \cdots & p_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ p_{n1} & p_{n2} & \cdots & p_{nn} \end{bmatrix}$$

This matrix is called ***the change of basis matrix from α to β***.

The following theorem is Corollary 5.13 in the textbook.

Theorem 116 (Change of basis II)**.** *If $T : V \to V$ is a linear transformation, α and β are bases of V, and P is the change of basis matrix from α to β, then $[T]_\beta^\beta = P^{-1}[T]_\alpha^\alpha P$.*

Example 117 (Using a change of basis matrix)**.** In Example 115, we had

$$[T]_\alpha^\alpha = \begin{bmatrix} 5 & 0 & 1 \\ 3 & 2 & -3 \\ 5 & 0 & 0 \end{bmatrix} \quad \text{and} \quad [T]_\beta^\beta = \begin{bmatrix} -2 & -1 & -1 \\ 4 & 4 & 2 \\ 5 & 3 & 5 \end{bmatrix}$$

One can compute the change of basis matrix P and its inverse P^{-1} for α and β. (For details, see Example 2 pages 260-261.) These are

$$P = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 2 & 1 & 1 \end{bmatrix} \quad \text{and} \quad P^{-1} = \begin{bmatrix} -1 & 0 & 1 \\ 1/2 & -1/2 & 0 \\ 3/2 & 1/2 & -1 \end{bmatrix}.$$

Thus we could solve part (b) using the formula in Theorem 116:

$$\begin{aligned}
[T]_\beta^\beta &= P^{-1}[T]_\alpha^\alpha P \\
&= \begin{bmatrix} -1 & 0 & 1 \\ 1/2 & -1/2 & 0 \\ 3/2 & 1/2 & -1 \end{bmatrix} \begin{bmatrix} 5 & 0 & 1 \\ 3 & 2 & -3 \\ 5 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 2 & 1 & 1 \end{bmatrix} \\
&= \begin{bmatrix} -2 & -1 & -1 \\ 4 & 4 & 2 \\ 5 & 3 & 5 \end{bmatrix}.
\end{aligned}$$

This computation agrees with the answer for $[T]_\beta^\beta$ that we obtained in Example 115.

End of Example 117. □