

## 38 2025-11-26 | Week 14 | Lecture 38

Please read chapters 6.1 and 6.2

### 38.1 Recap of first order linear differential equations

Recall from the previous two lectures, we considered differential equations of the form

$$Y' = AY \quad (34)$$

where  $Y = \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix}$  is a function of time  $t$  and  $A$  is an  $n \times n$  matrix.

Eq. (34) is a system of  $n$  equations, each taking the form

$$\frac{dy_i}{dt} = a_{i1}y_1(t) + a_{i2}y_2(t) + \dots + a_{in}y_n(t), \quad (i = 1, 2, \dots, n).$$

Differential equations of the form given in Eq. (34) are called **homogeneous systems of first order linear differential equations**.

A system is **inhomogeneous** if takes the form

$$Y' = AY + G \quad (35)$$

for some nonzero vector  $G = G(t) = \begin{bmatrix} g_1(t) \\ \vdots \\ g_n(t) \end{bmatrix}$  is not the zero vector. So far we have focused on the homogenous case.

**Remark 171.** The textbook uses the variable name  $x$  rather than  $t$ , but I like to think of it as time, so I'll use  $t$  in my lectures. One can also allow the matrix  $A$  to be a function of  $t$ , and this is done in the textbook, but I'd rather avoid this more general setting for now.

End of Remark 171.  $\square$

**Applications:** We saw two physical interpretations of Eq. (34):

- as the position of a moving particle (or ant)
- as the concentration of a chemical diffusing between several discrete locations

We also saw that, given the **initial condition**  $Y(0) = B$ , the solution to the differential equation in Eq. (34) takes the form

$$Y(t) = e^{At}B.$$

This describes either the position of the particle at time  $t$ , or the concentrations of the chemical in the  $n$  compartments at time  $t$ .

When  $A$  is diagonalizable (that is, when  $A = P\Lambda P^{-1}$ ), the above solution can be written as

$$Y(t) = Pe^{\Lambda t}P^{-1}B.$$

where  $\Lambda = \text{diag}(\lambda_1, \dots, \lambda_n)$  and  $e^{\Lambda t} = \text{diag}(e^{\lambda_1 t}, \dots, e^{\lambda_n t})$ . This was the content of Theorem 168.

Next we discuss what  $e^{At}$  really means.

### 38.2 The matrix exponential

In Example 164, we wrote down  $e^{\Lambda t}$  but  $\Lambda t$  is a matrix. So we need to define  $e^M$  where  $M$  is a matrix.

If  $x$  is a real number, then the power series of  $e^x$  is

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots = \sum_{k=0}^{\infty} \frac{x^k}{k!}$$

If  $M$  is a square matrix, then we define the **exponential of a matrix**  $e^M$  as

$$e^M = I + M + \frac{M^2}{2!} + \frac{M^3}{3!} + \dots = \sum_{k=0}^{\infty} \frac{M^k}{k!}$$

Using this definition we have two cases:

- If  $M = \Lambda$  is diagonal  $\begin{bmatrix} \lambda_1 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & \lambda_n \end{bmatrix}$  then it's easy:

$$e^{\Lambda t} = \begin{bmatrix} e^{\lambda_1 t} & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & e^{\lambda_n t} \end{bmatrix}$$

- If  $M$  is not a diagonal matrix, computing the exponential is harder. For example, if  $M = At$ , we have

$$e^{At} = I + At + \frac{(At)^2}{2!} + \frac{(At)^3}{3!} + \dots$$

This may be hard to compute in general. Nonetheless, this series always converges, and satisfies:

- \*  $e^{At} = e^{As} = e^{A(t+s)}$
- \*  $e^{At}e^{-tA} = I$  (so  $e^{At}$  is always invertible)
- \*  $\frac{d}{dt}[e^{At}] = Ae^{At}$

The next proposition says that for diagonalizable matrices, it is possible to compute the matrix exponential without too much difficulty (need to compute some eigenvalues and eigenvectors, and a matrix inverse).

**Proposition 172.** *If  $A$  is diagonalizable, then  $e^{At} = Pe^{\Lambda t}P^{-1}$ , where  $\Lambda = \text{diag}(\lambda_1, \dots, \lambda_n)$  and  $\lambda_1, \dots, \lambda_n$  is the list of eigenvalues of  $A$ .*

*Proof.* Since  $A$  is diagonal, we can write  $A = P\Lambda P^{-1}$ , where  $P$  is a matrix whose columns are eigenvectors of  $A$ . Using this, we can write

$$A^k = (P\Lambda P^{-1})^k = P\Lambda^k P^{-1}.$$

(see Section 34.2 for details). Then

$$\begin{aligned} e^{At} &= \sum_{k=0}^{\infty} \frac{(At)^k}{k!} \\ &= \sum_{k=0}^{\infty} \frac{(P\Lambda^k P^{-1})t^k}{k!} \\ &= P \left( \sum_{k=0}^{\infty} \frac{\Lambda^k t^k}{k!} \right) P^{-1} \\ &= P \left( \sum_{k=0}^{\infty} \frac{(\Lambda t)^k}{k!} \right) P^{-1} \\ &= Pe^{\Lambda t}P^{-1}. \end{aligned}$$

□