

Math 307: Homework 09

Due Wednesday, November 12. Material up to and including 5.4.

Problem 1. Let $A = \begin{bmatrix} 1 & -1 \\ 2 & 4 \end{bmatrix}$. The *trace* of A is the sum of its main diagonal entries. Find the eigenvalues of A , verify that trace equals the sum of the eigenvalues, and that the determinant equals their product.

Problem 2. Find the eigenvectors and eigenvalues of

$$A = \begin{bmatrix} 3 & 4 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 0 & 0 & 2 \\ 0 & 2 & 0 \\ 2 & 0 & 0 \end{bmatrix}$$

Check that $\lambda_1 + \lambda_2 + \lambda_3$ equals the trace and that $\lambda_1\lambda_2\lambda_3$ equals the determinant. (Here $\lambda_1, \lambda_2, \lambda_3$ are the eigenvalues of the matrix.)

Problem 3. Find the rank and nullspace of

$$A = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 0 & 0 & 1 & 2 \\ 0 & 0 & 1 & 2 \\ 1 & 1 & 1 & 0 \end{bmatrix}$$

Problem 4. Suppose all vectors x in the unit square $0 \leq x_1 \leq 1, 0 \leq x_2 \leq 1$ are transformed to Ax . (Here A is a 2×2 matrix).

- (a) What is the shape of the transformed region (all Ax)?
- (b) For which matrices A is that region a square?
- (c) For which A is it a line?
- (d) For which A is the new area still 1?

Problem 5. Given an eigenvalue λ of matrix A , the *eigenspace* of λ is the subspace $NS(\lambda I - A)$. For the following matrices, find the eigenvalues, and find a basis for each eigenspace.

(a) $\begin{bmatrix} 3 & 0 \\ 8 & -1 \end{bmatrix}$

(b) $\begin{bmatrix} 4 & -4 \\ 1 & 0 \end{bmatrix}$

(c) $\begin{bmatrix} 4 & 0 & 1 \\ -2 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}$

Problem 6. Let A, B, C be square matrices. Prove the following statements:

- (a) If B is similar to A , then A is similar to B . (“Similarity is reflexive”)
- (b) If A is similar to B , and B is similar to C , then A is similar to C . (“Similarity is transitive”)
- (c) If A, B are similar, then $\det(A) = \det(B)$.
- (d) If A is invertible, then B is invertible and B^{-1} is similar to A^{-1} .

Problem 7. (*Hint for this problem: refer to the examples in section 5.1 of the textbook.*) Let $T : \mathbb{R}^3 \rightarrow P_2$ be a linear transformation satisfying

$$T \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = x^2 + x, \quad T \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} = x^2 - x + 1, \quad T \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = x + 1.$$

(a) Find $T \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$.

(b) Find $T \begin{bmatrix} a \\ b \\ c \end{bmatrix}$.

Problem 8.

(a) Construct a matrix whose nullspace contains the vector $x = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$

(b) Construct a matrix whose column space is spanned by $\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$ and whose row space is spanned by $[1 \ 5]$.

(c) How can you construct a matrix that transforms the coordinate vectors e_1, e_2, e_3 into three given vectors v_1, v_2, v_3 . When will that matrix be invertible?

Problem 9. Let $T : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ be given by

$$T \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 17x_1 - 8x_2 - 12x_3 \\ 16x_1 - 7x_2 - 12x_3 \\ 16x_1 - 8x_2 - 11x_3 \end{bmatrix}.$$

(a) Find $[T]_{\alpha}^{\alpha}$, where α is the standard basis for \mathbb{R}^3 .

(b) Let $\beta = \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} \right\}$. Find the change of basis matrix from α to β .

(c) Find the change of basis matrix from β to α .

(d) Find $[T]_{\beta}^{\beta}$.

(e) Find $[v]_{\beta}$ for $v = \begin{bmatrix} 2 \\ -1 \\ 4 \end{bmatrix}$.

(f) Find $[T(v)]_{\beta}$.

(g) Use the result of part (f) to find $T(v)$.

Problem 10. A linear transformation from V to W has an *inverse* from W to V when the range of is all of W and the kernel contains only $v = 0$. Why are these transformations not invertible?

(a) $T(v_1, v_2) = (v_2, v_2)$, $W = \mathbb{R}^2$

(b) $T(v_1, v_2) = (v_1, v_2, v_1 + v_2)$, $W = \mathbb{R}^3$

(c) $T(v_1, v_2) = v_1$, $W = \mathbb{R}^1$

Problem 11. Suppose $T : V \rightarrow W$ is a linear transformation.

- (a) Show that if $v \in V$ and $u \in \ker(T)$, then $T(u + v) = T(v)$.
- (b) Show that if $u, v \in V$ such that $T(u) = T(v)$, then $u - v$ is in $\ker(T)$.
- (c) Use part (b) to deduce that if $\ker(T) = \{0\}$, then T is injective. [Note: a function f is *injective* if $x_1 \neq x_2$ implies $f(x_1) \neq f(x_2)$].

Problem 12 (An application of eigenvalues). If $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ is a twice-differentiable function of two variables, then the *Hessian matrix* of f is the matrix

$$H(x, y) = \begin{bmatrix} f_{xx}(x, y) & f_{xy}(x, y) \\ f_{yx}(x, y) & f_{yy}(x, y) \end{bmatrix}.$$

In this problem, we'll use the following theorem:

The Second Derivative Test

Let $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ be a twice-differentiable function of two variables, and let (a, b) be a critical point of f . Let λ_1 and λ_2 be the eigenvalues of $H(a, b)$. Then:

- If λ_1, λ_2 are both positive, then f has a minimum at $(x, y) = (a, b)$.
- If λ_1, λ_2 are both negative, then f has a maximum at $(x, y) = (a, b)$.
- If one of λ_1, λ_2 is negative and the other positive, then f has a saddle point at $(x, y) = (a, b)$.
- If one or more eigenvalue is zero, then the test is inconclusive.

For parts (a) and (b), find the critical points of $f(x, y)$ and classify them using the second derivative test stated here.

(a) Let $f(x, y) = x^2 + 4xy + y^2$.

(b) Let $f(x, y) = x^3 - 3x + y^2 - 4y$

Problem 13. For the following matrices, find the eigenvalues, and find a basis for each eigenspace.

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & -2 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 4 & 0 & 0 \\ 1 & 4 & 0 \\ 0 & 1 & 4 \end{bmatrix}$$

Problem 14. Find a basis for the following subspace of \mathbb{R}^4 :

- (a) The vectors for which $x_1 = 2x_4$.
- (b) The vectors for which $x_1 + x_2 + x_3 = 0$ and $x_3 + x_4 = 0$.

(c) The subspace spanned by $\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 4 \\ 5 \end{bmatrix}$

Problem 15. Suppose a linear transformation T transforms $(1, 1)$ to $(2, 2)$ and $(2, 0)$ to $(0, 0)$. Find $T(v)$ when

- (a) $v = (2, 2)$
- (b) $v = (3, 1)$
- (c) $v = (-1, 1)$
- (d) $v = (a, b)$