Math 307: Homework 10

Due Wednesday, December 10

Problem 1. Let $Y(t) = \begin{bmatrix} c_1 e^{2t} + c_2 e^{3t} \\ 2c_1 e^{2t} + c_2 e^{3t} \end{bmatrix}$.

(a) Show that $Y(t)$ is a solution of $Y' = \begin{bmatrix} 4 & -1 \\ 2 & 1 \end{bmatrix} Y$. *(To do this, compute the LHS and RHS independently and then observe that they are equal).*

(b) Use the webapp `https://homepages.bluffton.edu/~nesterd/apps/slopefields.html` (or the sofrware `PPLANE` at `http://alun.math.ncsu.edu/pplane/`), to plot the slope field of the system. Clicking on a coordinate in the plot will show the trajectory of a particle starting at that point whose motion is governed by the differential equation. This can also be done precisely using the "initial points" tab. Do this to plot some trajectories.

Problem 2. Find the general solution to $Y' = AY$ when $A = \begin{bmatrix} 6 & -8 \\ 4 & -6 \end{bmatrix}$ and plot the slope field of the differential equation.

Problem 3. Find the general solution to $Y' = AY$ when $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ and plot the slope field of the differential equation.

Problem 4. Let $A = \begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix}$.

(a) Determine the general solution to $Y' = AY$

(b) Solve the initial value problem $Y' = AY$ with $Y(0) = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

(c) Plot the solution to the initial value problem using the software from the previous problems.

Problem 5 (A non-diagonalizable example)**.** Consider the system of differential equations

$$\begin{cases} x'(t) = y \\ y'(t) = -x - 2y \end{cases}$$

Letting $Y(t) = \begin{bmatrix} x(t) \\ y(y) \end{bmatrix}$ and $A = \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix}$, we can write the above system of differential equations in matrix form as

$$Y' = AY. \tag{1}$$

(a) Plot the slope field and some particle trajectories using the software from the previous problem.

(b) Are there any straight-line trajectories?

(c) Compute the eigenvalues and eigenvectors of A.

(d) Show that A is not diagonalizable.

(e) Since A is not diagonalizable, the techniques we've discussed so far for solving differential equations like this won't work. Show that for any scalars c_1 and c_2, the function

$$\begin{aligned} Y(t) &= c_1 e^{-t} \begin{bmatrix} 1 \\ -1 \end{bmatrix} + c_2 e^{-t} \begin{bmatrix} 1 \\ 0 \end{bmatrix} + c_2 t e^{-t} \begin{bmatrix} 1 \\ -1 \end{bmatrix} \\ &= \begin{bmatrix} c_1 e^{-t} + c_2 e^{-t} + c_2 t e^{-t} \\ -c_1 e^{-t} - c_2 t e^{-t} \end{bmatrix} \end{aligned}$$

is a solution to the differential equation in Eq. (1). (Hint: computing the left-hand side and right-hand side independently and then observe that they are equal.)

(f) Using the general solution from part (e), solve the initial value problem

$$\begin{cases} Y' = AY \\ Y(0) = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \end{cases}$$

Problem 6 (Continuation of Problem 5)**.** Let A be the 2×2 matrix from Problem 5.

(a) Let $B = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} \in \mathbb{R}^2$. Using the fact that $\frac{d}{dt}\left[e^{Mt}\right] = Me^{Mt}$ for any square matrix, show that $Y(t) = e^{At}B$ is a solution to the initial value problem

$$\begin{cases} Y' = AY \\ Y(0) = B \end{cases}$$

(Hint: you don't need to calculate e^{At} for this part. The rest of this problem will guide you through one way to compute e^{At}.)

(b) Assume that $A = PJP^{-1}$ for some 2×2 matrices J and P. Show that $e^{At} = Pe^{Jt}P^{-1}$.

(Hint: use the definition $e^M = \sum_{k=0}^{\infty} \frac{M^k}{k!} = I + M + \frac{M^2}{2!} + \frac{M^3}{3!} + \ldots$).

(c) Let $P = \begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix}$ and $J = \begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix}$. Compute P^{-1} and verify that $A = PJP^{-1}$.

(Note: J is the Jordan normal form of A. Since A is not diagonalizable, J is the closest we can get to diagonalizing A.)

(d) Compute $(tJ)^k$ for $k = 0, 1, 2, 3, 4, 5$. Formulate a conjecture about what you think $(tJ)^k$ is, for a general integer k.

(e) Using part (d), one can show that $e^{tJ} = \begin{bmatrix} e^{-t} & te^{-t} \\ 0 & e^{-t} \end{bmatrix}$ (you don't have to show this). Moreover, from part (b) we know that $e^{At} = Pe^{Jt}P^{-1}$. Using these two facts, compute e^{At}.

(f) From part (a), we know that $Y(t) = e^{At}B$ is a solution to the intial value problem. Find the solution when $B = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ and compare your answer to the answer to your answer in part (f) of Problem 5.

Problem 7. Let $A = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 4 \end{bmatrix}$.

(a) Determine the general solution to $Y' = AY$

(b) Solve the initial value problem $Y' = AY$ with $Y(0) = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$

Problem 8. Suppose a rabbit population r and a wolf population w are governed by

$$\begin{cases} \frac{dr}{dt} = 4r - 2w \\ \frac{dw}{dt} = r + w \end{cases}$$

(a) What is the solution to this differential equation? Interpret your answer in terms of rabbits and wolves.

(b) If $r = 300$ and $w = 200$ at time $t = 0$, what is the population of wolves and rabbits at time $t > 0$?

(c) After a long time, what is the proportion of rabbits to wolves? Hint: compute $\lim_{t\to\infty} \frac{r(t)}{w(t)}$.

Problem 9. Find the general solution to $Y' = AY$ when $A = \begin{bmatrix} 6 & 0 & -8 \\ -4 & 2 & -4 \\ 4 & 0 & 6 \end{bmatrix}$ and plot the slope field of the differential equation.

Problem 10. Find the eigenvalues and eigenvectors, and the exponential e^{At} for

$$A = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}$$

Write the general solution to $Y' = AY$, and the specific solution that satisfies $Y(0) = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$. Plot the trajectory of your solution. What happens as $t \to \infty$?

Problem 11 (Kimura 2 parameter substitution model)**.** This problem introduce a model of DNA mutation that is more sophisticaed than Jukes Cantor, since it does not make the overly simplifying assumption that all nucleotides `A`, `C`, `G`, and `T` mutated between each other at equal rates. Instead, it assumes

- time t is measured in generations
- mutations from `A` between `G` happen at rate $a > 0$ per generation
- mutations from `C` between `T` happen at rate $a > 0$ per generation
- mutations between all other pairs of `A`, `C`, `G`, `T` happen at rate $b > 0$ per generation

Let $Y(t) = \begin{bmatrix} y_{\mathtt{A}}(t) \\ y_{\mathtt{C}}(t) \\ y_{\mathtt{G}}(t) \\ y_{\mathtt{T}}(t) \end{bmatrix}$, where, $y_{\mathtt{A}}(t)$ is the proportion of the DNA with letter `A` at time t, and $y_{\mathtt{C}}, y_{\mathtt{G}} y_{\mathtt{T}}$ are defined similarly. The differential equation describing the change in Y is

$$Y' = QY \tag{2}$$

where

$$Q = \begin{bmatrix} -a-2b & b & a & b \\ b & -a-2b & b & a \\ a & b & -a-2b & b \\ b & a & b & -a-2b \end{bmatrix}$$

(a) Show the following eigenvectors are linearly independent

$$\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \end{bmatrix}$$

(b) Show that the vectors from part (a) are eigenvectors of Q. What are the corresponding eigenvalues?

(c) What is the general solution to Eq. (2)?