Math 307: Homework 04

Due Wednesday, October 1 (at the beginning of class)

Problem 1. Determine which of the following sets of vectors are subspaces of $\mathbb{R}^2$.

(a) All vectors of the form $\begin{bmatrix} 0 \\ y \end{bmatrix}$

(b) All vectors of the form $\begin{bmatrix} x \\ 3x \end{bmatrix}$

(c) All vectors of the form $\begin{bmatrix} x \\ 2-5x \end{bmatrix}$

(d) All vectors $\begin{bmatrix} x \\ y \end{bmatrix}$ where $x + y = 0$.

Problem 2. A linear system is said to be *homogeneous* if the right hand sides are all zero, like this:

$$\begin{aligned} x - y + z &= 0 \\ 2x + y + 2z &= 0 \\ 3x - 5y + 3z &= 0 \end{aligned}$$

Such systems always have at least one solution, namely the "trivial solution" in which all the variables are zero (i.e., $x = y = z = 0$ in this case). Using row reduction, find all solutions to the above linear system, and include a sketch or plot of the solutions.

Problem 3. Determine whether the following matrices are invertible

(a) $\begin{bmatrix} 6 & -3 \\ -4 & 2 \end{bmatrix}$

(b) $\begin{bmatrix} 5 & -1 \\ 3 & 4 \end{bmatrix}$

(c) $\begin{bmatrix} -1 & 0 & 2 \\ 1 & 1 & -1 \\ 3 & 0 & 1 \end{bmatrix}$

Problem 4. Let

$$A = \begin{bmatrix} 3 & -2 \\ 1 & 4 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 2 \\ -2 & 3 \end{bmatrix}.$$

(a) Find the determinants of A and B.

(b) Find $\det(AB)$, $\det(A^{-1})$, and $\det(B^\top A^{-1})$ without finding AB, A^{-1}, or $B^\top A^{-1}$

(c) Show that $\det(A+B)$ is not the same as $\det(A) + \det(B)$.

Problem 5. Recall from lecture 10 the vector space

$$C[0,1] = \{f : [0,1] \to \mathbb{R} : f \text{ is a continuous function}\}.$$

Suppose $p \in [0,1]$ is any fixed number in the closed unit interval, and define the set A_p as

$$A_p = \{f : [0,1] \to \mathbb{R} : f \text{ is a continuous function with } f(p) = 0\}.$$

Show that A_p is a subspace of $C[0,1]$.

Problem 6. The set of complex numbers $\mathbb{C}$ is defined aso $\mathbb{C} = \{x + yi : x, y \in \mathbb{R}\}$, where $i^2 = -1$. Does the set of complex numbers under addition and scalar multiplciation

$$(a + bi) + (c + di) = (a + c) + (b + d)i$$
$$c(a + bi) = ca + cbi$$

(where a, b, c and d are real numbers) form a vector space? If not, why not?

Problem 7. Determine whether the vectors $\begin{bmatrix}1\\1\\0\end{bmatrix}, \begin{bmatrix}1\\-1\\-1\end{bmatrix}, \begin{bmatrix}1\\3\\1\end{bmatrix}$ span $\mathbb{R}^3$.

Problem 8. Determine the conditions on $a, b, c, d \in \mathbb{R}$ such that the following system has solutions:

$$\begin{aligned} x + 2y - z &= a \\ x + y - 2z &= b \\ 2x + y - 3z &= c \end{aligned}$$

Problem 9. Let V be the vector space of real-valued sequences:

$$V = \{(a_1, a_2, \ldots) : a_1, a_2, \ldots \in \mathbb{R}\}.$$

In this vector space, vector addition and scalar multiplication are defined as follows:

- If $a = (a_1, a_2, a_3, \ldots) \in V$ and $b = (b_1, b_2, b_3, \ldots) \in V$, define "vector addition" as

$$a + b = (a_1 + b_1,\ a_2 + b_2,\ a_3 + b_3,\ \ldots).$$

- For $a = (a_1, a_2, a_3, \ldots) \in V$ and $c \in \mathbb{R}$, define "scalar multiplication" by

$$ca = (ca_1,\ ca_2,\ ca_3,\ \ldots).$$

Let F be the following subset of V:

$$F = \{(a_1, a_2, a_3, \ldots) \in V : a_n = a_{n-1} + a_{n-2} \text{ for each } n = 3, 4, 5, \ldots\}.$$

For example, F contains the Fibonacci sequence

$$(a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8, a_9, a_{10}, a_{11}, ...) = (0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, \ldots)$$

as well as other "Fibonacci-like" sequences.

Verify that F is a linear subspace of V. (You may assume that V is a vector spaces.)