9 2025-09-15 | Week 04 | Lecture 09

This lecture is based on sections 1.5 and 1.6 in the textbook. We are going to skip section 1.7

> ***The nexus question of this lecture:*** *How do we understand (and compute) the determinant, algebraically?*

9.1 Review of the "Key Theorem" of Linear Algebra

Theorem 28 (The Key Theorem of Linear Algebra (partial version))**.** *Let A be an $n \times n$ matrix. Then the following are equivalent:*

(i.) A^{-1} *exists (i.e., A is invertible)*

(ii.) $\det A \neq 0$

(iii.) The linear system $AX = B$ has a unique solution for each $B \in \mathbb{R}^n$.

(iv.) A is row equivalent to I

(v.) ...

Property (iv.) says we can row reduce A into I. The term for this is "row equivalence". Precisely, If A and B are matrices, we say that A is ***row equivalent*** to B if there is a sequence of elementary row operations which if applied to A will result in B.

9.2 Definition of the determinant

Consider a square matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix}.$$

Given an entry a_{ij}, the ***minor*** of a_{ij}, denoted M_{ij}, is the matrix obtained from A by deleting row i and column j of A. For example, if

$$A = \begin{bmatrix} a_{11}, a_{12}, a_{13} \\ a_{21}, a_{22}, a_{23} \\ a_{31}, a_{32}, a_{33} \end{bmatrix}$$

then some minors are

$$M_{11} = \begin{bmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{bmatrix}, \quad M_{12} = \begin{bmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{bmatrix}, \quad \text{and} \quad M_{32} = \begin{bmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{bmatrix}.$$

Definition 29 (Determinant)**.** For a 1×1 matrix, $A = [a_{11}]$, we define $\det(A) = 1$. If A is an $n \times n$ matrix with $n \geq 2$, we define the determinant recursively as

$$\det(A) = \sum_{j=1}^{n} (-1)^{j+1} a_{1j} \det(M_{1j}).$$

Note: Note that the determinant is defined only for square matrices.

Geometrically, the determinant is the (signed) volume scaling factor of the transformation of space, which is very useful to keep in mind. Definition 29 is also useful because it allows us to see how to actually compute determinants.

For a 2×2 matrix, Definition 29 simplifies to

$$\det\left(\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}\right) = a_{11}a_{22} - a_{12}a_{21}.$$

To simplify notation, we use vertical bars to denote determinant $|A| := \det(A)$, or something like this:

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} := \det\left(\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}\right).$$

Example 30 (Computing the determinant of a 3×3 matrix)**.**

$$\begin{aligned} \begin{vmatrix} 2 & 3 & -2 \\ -1 & 6 & 3 \\ 4 & -1 & 1 \end{vmatrix} &= 2\begin{vmatrix} 6 & 3 \\ -2 & 1 \end{vmatrix} - 3\begin{vmatrix} -1 & 3 \\ 4 & 1 \end{vmatrix} + (-2)\begin{vmatrix} -1 & 6 \\ 4 & -2 \end{vmatrix} \\ &= 2\left[6 \cdot 1 - 3(-2)\right] - 3\left[(-1)1 - 3 \cdot 4\right] - 2\left[(-1)(-2) - 6 \cdot 4\right] \\ &= 107. \end{aligned}$$

End of Example 30. □

9.3 Computing determinants using cofactor expansions

The formula for the determinant in Definition 29 is called a ***cofactor expansion*** (there are other formulas). A ***cofactor*** of an entry a_{ij} is the quantity

$$C_{ij} = (-1)^{i+j}\det(M_{ij})$$

so our formula was

$$\det(A) = \sum_{j=1}^{n} a_{1j}C_{1j}.$$

This is called the ***cofactor expansion about the first row.*** In fact, we could have picked any row or column and done a similar calculation to get the determinant:

Theorem 31 (Cofactor Expansion)**.** *If A is an $n \times n$ matrix with $n \geq 2$, then*

(i.) For any fixed $i = 1, 2, \dots, n$, we have

$$\det(A) = \sum_{j=1}^{n} a_{ij}C_{ij} \quad \textit{(cofactor expansion about the } i^{\text{th}} \textit{ row)}$$

(ii.) For any fixed $j = 1, 2, \dots, n$, we have

$$\det(A) = \sum_{i=1}^{n} a_{ij}C_{ij} \quad \textit{(cofactor expansion about the } j^{\text{th}} \textit{ column)}$$

This theorem is proved by induction on n in section 1.7, but the proof is technical, so we'll skip it. Two examples will illustrate this theorem.

Example 32 (Alternative cofactor expansions)**.** Let's compute the determinant of the matrix from Example 30 in two different ways, using Theorem 31:

The cofactor expansion about the third row:

$$\begin{aligned} \begin{vmatrix} 2 & 3 & -2 \\ -1 & 6 & 3 \\ 4 & -1 & 1 \end{vmatrix} &= a_{31}C_{31} + a_{32}C_{32} + a_{33}C_{33} \\ &= 4\begin{vmatrix} 3 & -2 \\ 6 & 3 \end{vmatrix} - (-2)\begin{vmatrix} 2 & -2 \\ -1 & 3 \end{vmatrix} + 1\begin{vmatrix} 2 & 3 \\ -1 & 6 \end{vmatrix} \\ &= 107. \end{aligned}$$

The cofactor expansion about the second column:

$$\begin{aligned}\begin{vmatrix} 2 & 3 & -2 \\ -1 & 6 & 3 \\ 4 & -1 & 1 \end{vmatrix} &= a_{21}C_{21} + a_{22}C_{22} + a_{32}C_{32} \\ &= -3\begin{vmatrix} -1 & 3 \\ 4 & 1 \end{vmatrix} + 6\begin{vmatrix} 2 & -2 \\ 4 & 1 \end{vmatrix} - (-2)\begin{vmatrix} 2 & -2 \\ -1 & 3 \end{vmatrix} \\ &= 107. \end{aligned}$$

End of Example 32. □

9.4 Key properties of the determinant

The following theorem is obvious geometrically

Theorem 33. *If I is an $n \times n$ identity matrix, then* $\det(I) = 1$*. If* $\mathbf{O}$ *is an $n \times n$ zero matrix, then* $\det(\mathbf{O}) = 0$.

Theorem 34 (Zero row/column)**.** *Let A be an $n \times n$ matrix. If A has a row of zeros (or a column of zeroes), then the determinant is zero.*

Proof. Suppose that the k^{th} row of A has only zero entries. That means

$$a_{k1} = 0, \quad a_{k2} = 0, \quad a_{k3} = 0, \quad \ldots \quad \text{and} \quad a_{kn} = 0$$

By Theorem 31(i.),

$$\det(A) = \sum_{j=1}^{n} a_{ij}C_{ij}$$

for any choice of $j \in \{1, 2, \ldots, n\}$. Picking $i = k$, we get

$$\begin{aligned}\det(A) &= \sum_{j=1}^{n} a_{kj}C_{kj} \\ &= \sum_{j=1}^{n} (0)C_{kj} \\ &= 0.\end{aligned}$$

The proof for the case where A has a column of zero entries is similar. □

To see why Theorem 34 is true geometrically, it suffices to consider an example

Example 35 (A row of zeroes implies zero determinant)**.**

$$A = \begin{bmatrix} 4 & 5 & -1 \\ 6 & 2 & 3 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 4 & 5 & -1 \\ 6 & 2 & 3 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4x + 5y - z \\ 6x + 2y + 3z \\ 0 \end{bmatrix}$$

As a transformation of space, this matrix sends every point $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ to another point of the form $\begin{bmatrix} * \\ * \\ 0 \end{bmatrix}$. Therefore every point gets mapped to a point in the set

$$\left\{(x, y, z) \in \mathbb{R}^3 : z = 0\right\}$$

This is the plane $z = 0$. So the matrix A effectuates a dimension collapse, from 3 dimensonal space into to a 2-dimensional plane. This destroys volume, so $\det(A) = 0$.

Of course, not every matrix that has determinant zero has a row or column of zeroes. For example, one can check that

$$\begin{vmatrix} 0 & 1 & 2 \\ 3 & 1 & 2 \\ 5 & 2 & 4 \end{vmatrix} = 0.$$

End of Example 35. □

Theorem 36 (The determinant preserves multiplication)**.** *If A and B are $n \times n$ matrices, then*

$$\det(AB) = \det(A)\det(B).$$

The geometric idea of understanding matrices as transformations of space makes this theorem obvious. The transformation of space given by AB is

- first, do the transformation of B
- then, do the transformation of A.

If B scales volume by a factor of 2, and A scales it again by a factor of 5, then the final scaling induced by AB will be 10. In symbols

$$\det(AB) = 10 = 5 \cdot 2 = \det(A)\det(B).$$

The idea is similar when the negative determinants (i.e., corresponding to transformations which include some sort of reflection) are used.