

Additional practice problems for the final exam

Problem 1 (differential equation). Let $A = \begin{bmatrix} 3 & 0 \\ 8 & -1 \end{bmatrix}$

(a) Let $Y(t) = \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix}$. Find the general solution of the differential equation $Y' = AY$.

(b) Solve the initial value problem

$$\begin{cases} Y' = AY \\ Y(0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \end{cases}$$

Problem 2 (differential equation). Find the general solution to $Y' = AY$ when $A = \begin{bmatrix} 4 & 0 & 1 \\ -2 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}$.

Problem 3 (determinants). Let

$$A = \begin{bmatrix} 3 & -2 \\ 1 & 2 \end{bmatrix} \quad B = \begin{bmatrix} 1 & -4 \\ -2 & 3 \end{bmatrix}.$$

- (a) Find the determinants of A and B .
- (b) Find $\det(AB)$, $\det(A^{-1})$, and $\det(B^\top A^{-1})$.
- (c) Show that $\det(A + B)$ is not the same as $\det(A) + \det(B)$.
- (d) Diagonalize B by writing it as $B = P\Lambda P^{-1}$, for some diagonal matrix Λ . (For full credit, you need to find P , P^{-1} and Λ .)

Problem 4 (composition). Consider the two linear transformations $T : \mathbb{R}^3 \rightarrow \mathbb{R}^4$ and $S : \mathbb{R}^4 \rightarrow \mathbb{R}^2$ defined by

$$T \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_1 + x_2 \\ x_2 + x_3 \\ x_3 \end{bmatrix} \quad \text{and} \quad S \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} x_1 \\ 2x_1 - x_2 \\ x_3 - 2x_4 \end{bmatrix}$$

- (a) Write the standard matrix for T
- (b) Write the standard matrix for S
- (c) Write the standard matrix for $S \circ T$

Problem 5 (system with variables). Let $a, b, c \in \mathbb{R}$. Solve the following system of linear equations.

$$\begin{aligned} x + 2y - z &= a \\ x + y - 2z &= b \\ 2x + y - 3z &= c \end{aligned}$$

Your answer should be in terms of a, b , and c .

Problem 6 (system with variables). Determine the conditions on $a, b, c \in \mathbb{R}$ such that the following linear system has at least one solution:

$$\begin{aligned} x + 2y - z &= a \\ x + y - 2z &= b \\ 2x + 2y - 4z &= c \end{aligned}$$

Problem 7 (diagonalization). Let A be a 2×2 matrix such that

- $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is eigenvector with eigenvalue 2
- $\begin{bmatrix} 2 \\ 3 \end{bmatrix}$ is an eigenvector with eigenvalue 3

(a) Find $A^3 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $A^3 \begin{bmatrix} 2 \\ 3 \end{bmatrix}$ and $A^3 \begin{bmatrix} 3 \\ 4 \end{bmatrix}$

(b) Find A

(c) Find A^3

Problem 8 (diagonalization). Consider the matrix $A = \begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix}$

(a) Diagonalize A . That is, find matrices P, D and P^{-1} such that $A = PDP^{-1}$.

(b) Use your diagonalization to find A^5 .

$$\begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 1 \\ 0 & 0 & 4 & 0 \end{bmatrix}$$

Problem 9 (diagonalization). Let $A = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 1 \\ 0 & 0 & 4 & 0 \end{bmatrix}$. This matrix is diagonalizable; that is, there exists a diagonal matrix D and an invertible matrix P such that $A = PAP^{-1}$. Find D and P . Do not compute P^{-1} .

Problem 10 (basis). Let $\beta = \left\{ \begin{bmatrix} 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 3 \\ 4 \end{bmatrix} \right\}$ be a basis for \mathbb{R}^2 . If $[v]_\beta = \begin{bmatrix} -3 \\ 2 \end{bmatrix}$, find v .

Problem 11 (Geometry).

(a) Find the matrix of the linear transformation $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ that rotates space by 90° counterclockwise.

(b) Find the matrix of the linear transformation defined by $S \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2x + y \\ 3y - x \\ x - 3y \end{bmatrix}$.

(c) Circle which transformation makes sense: $T \circ S$ $S \circ T$.

(d) Write a matrix for the transformation you circled.

Problem 12 (geometry). The linear transformation $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ can be achieved by rotating counterclockwise by 135° and then expanding vertically by a factor of 4. Find the standard matrix for T .

Problem 13 (computation). Let

$$A = \begin{bmatrix} 0 & 0 & 1 & 2 \\ 0 & 0 & 1 & 2 \\ 1 & 1 & 1 & 0 \end{bmatrix}$$

- (a) (2 points) Find a basis for the column space of A .
- (b) (2 points) Find a basis for the row space of A .
- (c) (2 points) Find a basis for the null space of A .
- (d) (2 points) What is the rank of A ?
- (e) (1 point) Is A invertible?

Problem 14 (computation). Let $A = \begin{bmatrix} 0 & -1 & 1 \\ 1 & 2 & 0 \\ 1 & 1 & 1 \end{bmatrix}$.

- (a) Find the rank and nullity of A .
- (b) Find a basis for the row space of A .
- (c) Find a basis for the column space of A .
- (d) Find a basis for the nullspace of A .

Problem 15. Solve the following system by reducing the augmented matrix to reduced row-echelon form:

$$\begin{aligned} x + 3y + 2z &= 2 \\ x &\quad - 4z = -7 \\ -2x - 4y + 3z &= -1 \end{aligned}$$

Problem 16 (computation). Let $A = \begin{bmatrix} 1 & 3 & -3 & -2 \\ 0 & -2 & 4 & 2 \\ 2 & 4 & -5 & -3 \end{bmatrix}$. Find a basis for the nullspace and column space of A .

Problem 17 (computation). Suppose T is a linear transformation such that

$$T \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 7 \\ -1 \\ 5 \end{bmatrix}, \quad T \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 6 \\ -2 \\ 5 \end{bmatrix}, \quad T \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 6 \\ 2 \\ 5 \end{bmatrix}$$

(a) Find $T \begin{bmatrix} a \\ b \\ c \end{bmatrix}$

- (b) Write down a 3×3 matrix that gives the transformation T .

Problem 18 (Subspace).

- (a) Give a precise definition of **linear subspace**.

(b) Show that $W = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \in \mathbb{R}^3 : x_1 + 2x_2 - 3x_3 = 0 \right\}$ a subspace of \mathbb{R}^3 , and find a basis for it.

(c) Is $U = \text{Span} \left\{ \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix} \right\}$ a subspace of \mathbb{R}^3 ? If not, justify why not. If U is a subspace, find a basis for it.