17 2025-10-03 | Week 06 | Lecture 17

Sub lecture by R. Willett.

Lecture on section 2.4

If V is a vector space, a ***basis*** for V is a collection $v_1, \ldots, v_n \in V$ such that

- $v_1, \ldots, v_n$ span V
- $v_1, \ldots, v_n$ are linearly independent

Example 73 (Bases)**.** $\mathbb{R}^3$ has 'standard basis'

$$\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix},$$

but there may be others, e.g.,

$$\begin{bmatrix}3\\0\\0\end{bmatrix}, \begin{bmatrix}4\\1\\0\end{bmatrix}, \begin{bmatrix}2\\5\\2\end{bmatrix},$$

or

$$\begin{bmatrix}1\\0\\1\end{bmatrix}, \begin{bmatrix}1\\1\\1\end{bmatrix}, \begin{bmatrix}-1\\1\\1\end{bmatrix},$$

End of Example 73. □

Example 74. $M_{m\times n}(\mathbb{R})$ has basis $E_{11}, E_{12}, \ldots, E_{1n}, E_{21}, \ldots, E_{2n}, \ldots, E_{m1}, \ldots E_{mn}$ where E_{ij} is an $m \times n$ matrix with 1 at position (i, j) and zero else.

Again, there are others, e.g.

$$\begin{bmatrix}1 & 0\\0 & -1\end{bmatrix}, \begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}, \begin{bmatrix}0 & 1\\1 & 0\end{bmatrix}, \begin{bmatrix}0 & -1\\1 & 0\end{bmatrix}$$

End of Example 74. □

Theorem 75 (Theorem 2.9 in textbook – not obvious!)**.** *If $v_1, \ldots, v_n$ and $w_1, \ldots, w_m$ are both bases for the same vector space, then $m = n$.*

Definition 76 (Dimension)**.** The ***dimension*** of a vector space is the number of elements in a basis. Notation for the dimension of V:

$$\dim(V).$$

Example 77 (Dimension)**.**

- $\dim(\mathbb{R}^3) = 3$ and more generally $\dim(\mathbb{R}^n) = n$.
- $\dim(\mathbb{R}[x]_{\leq 2}) = 3$ and more generally $\dim(\mathbb{R}[x]_{\leq n}) = n + 1$

Comments:

- Some vector spaces have bases with infinitely many vectors (e.g. a basis for $\mathbb{R}[x]$ is $1, x, x^2, x^3, \ldots$). In this case, the dimension of V is infinite. Notation: $\dim(V) = \infty$.

  (In this course, you will mainly focus on finite dimensional vector spaces.)
- If V is the 0 vector space, we write $\dim(V) = 0$.

End of Example 77. □

Theorem 78 (Some important properties of dimension (see 2.11 and 2.12))**.** *Let V be a vector space with $\dim(V) = n$. Then* *(a) If $v_1, \ldots, v_k \in V$ are linearly independent, then $k \leq n$ and there are $v_{k+1}, v_{k+2}, \ldots, v_n$ with $v_1, \ldots, v_k, v_{k+1}, \ldots, v_n$ a basis ("linear independent collections cannot be too big")*

(b) If $v_1, \ldots, v_k \in V$ span V, then $k \geq n$, and some collection of n vectors from $v_1, \ldots, v_k$ is a basis '("spanning collections collections cannot be too small")

Theorem 79 (Dimension-basis)**.** *Suppose $v_1, \ldots, v_n \in V$, where $\dim(V) = n$. Then*

- *If $v_1, \ldots v_n$ span V, then they are a basis.*
- *If $v_1, \ldots, v_n$ are linearly independent, they are a basis.*

Example 80. Which (if any) of the following collections is a basis for $\mathbb{R}^3$?

(a) $\begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix}, \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$

(b) $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}, \begin{bmatrix} 7 \\ 8 \\ 9 \end{bmatrix}$

(c) $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}, \begin{bmatrix} 7 \\ 8 \\ 9 \end{bmatrix}, \begin{bmatrix} 10 \\ 11 \\ 12 \end{bmatrix}$

Solution:

(a) No: has too few vectors (so cannot span)

(c) No: has too many vectors (so cannot be linearly independeent)

(b) To check linear independence, we need to check whether

$$c_1 \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + c_2 \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} + c_3 \begin{bmatrix} 7 \\ 8 \\ 9 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

has nontrivial solutions.

Check linear independence by row reducing:

$$\begin{aligned} \begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{bmatrix} &\longrightarrow \begin{bmatrix} 1 & 4 & 7 \\ 0 & -3 & -6 \\ 0 & -6 & -12 \end{bmatrix} && R_2 - 2R_1 \text{ and } R_3 - 3R_1 \\ &\longrightarrow \begin{bmatrix} 1 & 4 & 7 \\ 0 & -3 & -6 \\ 0 & 0 & 0 \end{bmatrix} && R_3 - 2R_2 \end{aligned}$$

As we have a row of zeroes, there are no conditions on c_3, and we see that there are infinitely many solutions.

We conclude that the collection is not a basis (since it's not linearly independent.)

End of Example 80. □

17.1 Null space, row space, column space

Let $A \in M_{m\times n}(\mathbb{R})$. There are three important vector spaces associated with A :

- The ***column space***, which is the subspace of $M_{m\times 1}(\mathbb{R})$ spanned by the columns of A. Notation: $CS(A)$
- The ***row space***, which is the subspace of $M_{1\times n}(\mathbb{R})$ spanned by the rows of A. Notation $RS(A)$.
- The ***null space*** (aka: kernel) which is the subspace of R^n of vectors x such that $Ax = 0$. Notation $NS(A)$ or $\ker(A)$.