Math 307: Homework 07

Due Wednesday, October 22. This homework is mostly based on section 2.4 and 5.1 in the textbook.

Instructions for problems 1-4:

(a) Find a basis for the nullspace of the matrix

(b) Find a basis for the row space of the matrix

(c) Find a basis for the column space of the matrix

(d) Determine the rank of the matrix

(e) Determine whether the matrix is invertible

Note that parts (a)-(c) do not have unique answers.

Problem 1. $\begin{bmatrix} -2 & 4 \\ 1 & -2 \end{bmatrix}$

Problem 2. $\begin{bmatrix} 2 & -1 & 3 & 4 \\ 1 & 0 & -1 & 3 \end{bmatrix}$

Problem 3. $\begin{bmatrix} 1 & -1 & 2 & 1 \\ 2 & 1 & -1 & 0 \\ 1 & 2 & -3 & -1 \\ 0 & 3 & -5 & -2 \\ 4 & -1 & 3 & 2 \end{bmatrix}$

Problem 4. $\begin{bmatrix} 1 & -1 & -1 & 2 & 0 \\ -2 & 1 & 1 & -1 & 0 \\ 1 & 1 & -2 & 1 & 1 \end{bmatrix}$

Problem 5. Find a basis for each of the following subspaces:

(a) Span $\left\{ \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 5 \\ -6 \\ -7 \end{bmatrix} \right\}$

(b) Span $\left\{ \begin{bmatrix} 0 \\ -2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 4 \\ -2 \\ -1 \end{bmatrix} \right\}$

(c) Span $\left\{ \begin{bmatrix} -2 \\ 1 \\ 3 \\ -4 \end{bmatrix}, \begin{bmatrix} 1 \\ -2 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} -1 \\ -4 \\ 9 \\ -11 \end{bmatrix} \right\}$

Problem 6. Decide the dependence or independence of

(a) The vectors $\begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}$, and $\begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$

(b) The vectors $\begin{bmatrix} 1 \\ -3 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ -3 \end{bmatrix}$, and $\begin{bmatrix} -3 \\ 2 \\ 1 \end{bmatrix}$

Problem 7. Show that

$$\begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad \text{and} \quad \begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix}$$

form a basis for $M_{2\times 2}(\mathbb{R})$.

Problem 8.

(a) Find the largest possible number of independent vectors among

$$v_1 = \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix}, \quad v_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix}, \quad v_4 = \begin{bmatrix} 0 \\ 1 \\ -1 \\ 0 \end{bmatrix}, \quad v_5 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \end{bmatrix}, \quad v_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \end{bmatrix}.$$

(b) This number is the _________ of the space spanned by the v's.

Problem 9. Let $S : \mathbb{R}^2 \to \mathbb{R}^2$ and $T : \mathbb{R}^2 \to \mathbb{R}^2$ be linear transformations defined by

$$S \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2x - y \\ x + 2y \end{bmatrix} \quad \text{and} \quad T \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x + 2y \\ x - y \end{bmatrix}.$$

(a) Find $(S+T)\begin{bmatrix} x \\ y \end{bmatrix}$

(b) Find $(S-4T)\begin{bmatrix} x \\ y \end{bmatrix}$

(c) Find $ST\begin{bmatrix} x \\ y \end{bmatrix}$

(d) Find $TS\begin{bmatrix} x \\ y \end{bmatrix}$

(e) What are the matrices that corresponds to the linear transformations S and T?

Problem 10 (Extra credit)**.** Recall that for any real number x, the *exponential function* e^x is defined by the power series

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}. \tag{1}$$

(a) Remarkably, the exponential function is defined even if x is a matrix. In particular, letting A be any square matrix, we can define the *matrix exponential* as

$$e^A = \sum_{k=0}^{\infty} \frac{A^k}{k!},$$

(this is not so easy to compute). Note that e^A is a square matrix of the same dimensions as A. Compute e^A when $A = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$.

(b) Let b and t be real numbers. By differentiating the series in Eq. (1) term-by-term (with $x = bt$), show that

$$\frac{d}{dt}\left[e^{bt}\right] = be^{bt}.$$

(This shows that $y(t) = e^{bt}$ is a solution to the differential equation $y'(t) - by(t) = 0$.)

(c) Let B be any square matrix, and let t be a real number. Following your approach in part (b), show that

$$\frac{d}{dt}\left[e^{Bt}\right] = Be^{Bt}.$$

(This shows that $y(t) = e^{Bt}$ is a solution to the system of differential equations $y'(t) - By(t) = 0$.)