10 2025-09-17 | Week 04 | Lecture 10

This lecture is based on sections 1.5 and 1.6 in the textbook.

> ***The nexus question of this lecture:*** *What are some useful connections between the geometric and algebraic interpretations of the the determinant?*

Recall that for $A \in M_{n\times n}$, we have

$$\det(A) = \begin{pmatrix}\text{the (signed) volume scaling} \\ \text{factor of the transformation}\end{pmatrix} = \sum_{j=1}^{n} a_{1j}(-1)^{1+j}\det(M_{1j})$$

These are the geometric interpretation (left) and algebraic interpretation (right) of the determinant.

10.1 Determinants preserve multiplication

Theorem 35 (The determinant preserves multiplication)**.** *If A and B are $n \times n$ matrices, then*

$$\det(AB) = \det(A)\det(B).$$

Key background - Matrices as tranformations: The geometric idea of understanding matrices as transformations of space makes this theorem obvious. Let $P = AB$. The transformation of space given by P is

- first, do the transformation of B
- then, do the transformation of A.

Why is it in this order? To see why, let $P = AB$, and consider how P acts on a vector X:

$$PX = (AB)X = A(BX)$$

The placement of the parentheses means we first transform X with B. Then, whatever we get from that, we transform with A. In other words, the product P acts on X by first applying B and then applying A.

In other words, matrix multiplication can be understood as *function composition* of the transformations of A and B.

Explanation of why Theorem 35 is true: When two transformations are composed (by multiplying the matrices), the total scaling is the product of the scalings of each transformation. If B scales volume by a factor of 2, and A scales it again by a factor of 5, then the final scaling induced by $P = AB$ will be 10. In symbols

$$\det(AB) = 10 = 5 \cdot 2 = \det(A)\det(B).$$

The idea is similar when the negative determinants (i.e., corresponding to transformations which include some sort of reflection) are used.

One consequence of Theorem 35 is the following relationship:

Theorem 36. *If A is invertible, then*

$$\det(A^{-1}) = \frac{1}{\det(A)}$$

Proof.

$$\begin{aligned}
\det(A)\det(A^{-1}) &= \det(AA^{-1}) && \text{(by Theorem 35)} \\
&= \det(I) && \text{(since } AA^{-1} = I\text{)} \\
&= 1 && \text{(since the determinant of the identity matrix is always 1).}
\end{aligned}$$

Dividing both sides by $\det(A^{-1})$ gives the result. □

10.2 Determinants and Dimension Collapse

The next theorem describes a class of matrices which have dimension zero because they collapse space:

Theorem 37 (Zero row/column)**.** *Let A be an $n \times n$ matrix. If A has a row of zeros (or a column of zeroes), then the determinant is zero.*

Proof. Suppose that the k^{th} row of A has only zero entries. That means

$$a_{k1} = 0, \quad a_{k2} = 0, \quad a_{k3} = 0, \quad \ldots \quad \text{and} \quad a_{kn} = 0$$

By Theorem 31(i.),

$$\det(A) = \sum_{j=1}^{n} a_{ij} C_{ij}$$

for any choice of $j \in \{1, 2, \ldots, n\}$. Picking $i = k$, we get

$$\begin{aligned} \det(A) &= \sum_{j=1}^{n} a_{kj} C_{kj} \\ &= \sum_{j=1}^{n} (0) C_{kj} \\ &= 0. \end{aligned}$$

The proof for the case where A has a column of zero entries is similar. $\square$

To see why Theorem 37 is true geometrically, it suffices to consider an example

Example 38 (A row of zeroes implies zero determinant)**.**

$$A = \begin{bmatrix} 4 & 5 & -1 \\ 6 & 2 & 3 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 4 & 5 & -1 \\ 6 & 2 & 3 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4x + 5y - z \\ 6x + 2y + 3z \\ 0 \end{bmatrix}$$

As a transformation of space, this matrix sends every point $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ to another point of the form $\begin{bmatrix} * \\ * \\ 0 \end{bmatrix}$. Therefore every point gets mapped to a point in the set

$$\left\{(x, y, z) \in \mathbb{R}^3 : z = 0\right\}$$

This is the plane $z = 0$. So the matrix A effectuates a dimension collapse, from 3 dimensonal space into to a 2-dimensional plane. This destroys volume, so $\det(A) = 0$.

End of Example 38. $\square$

Example 39 (Remark on Theorem 37)**.** Of course, not every matrix that has determinant zero has a row or column of zeroes. For example, one can check that

$$\begin{vmatrix} 0 & 1 & 2 \\ 3 & 1 & 2 \\ 5 & 2 & 4 \end{vmatrix} = 0.$$

Yet this matrix doesn't have a row or column of zeroes. Notice that the second and third columns are colinear: one is a scalar multiple of the other. This has something to do with it.

End of Example 39. $\square$

10.3 Determinants and Axis-Aligned Stretching

A special case of matrices are those which ***triangular***. A matrix is said to be ***upper triangular***, if all entries below the main diagonal are equal to zero. Like this:

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 5 & 6 & 7 \\ 0 & 0 & 8 & 9 \\ 0 & 0 & 0 & 10 \end{bmatrix}$$

Similarly, a matrix is said to be ***lower triangular*** if all the entries above the main diagonal are zero.

Triangular matrices always correspond to some combination of the following two transformations:

- stretch space in the direction of one or more the coordinate axes (e.g., in the direction of the x-axis or y-axis or z-axis, etc.)
- possibly also one or more "shear transforms"

Shear transformations don't stretch space at all, so the determinant of a triangular matrix is determined only by how much it stretches space in the directions of the coordinate axes x, y, and z directions. And it turns out, these stretchings are easy to see just by looking at the matrix.

Here's an example. The matrix

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{bmatrix}$$

stretches space in the x direction by 1, stretches spaces in the y direction by 4, and stretches space in the z direction by 6. The overall stretching of volume is therefore $24 = 1 \cdot 4 \cdot 6$. So

$$\det(A) = 1 \cdot 4 \cdot 6 = 24.$$

This is the idea behind the following theorem:

Theorem 40 (Determinants of triangular matrices)**.** *Let $T = (t_{ij}) \in M_{n\times n}(\mathbb{R})$ be a triangular matrix . Then*

$$\det(T) = t_{11}t_{22}\cdots t_{nn}.$$

In words, the determinant equals the product of the diagonal entries.

This theorem can be proved by induction on n.