13 2025-09-24 | Week 05 | Lecture 13

This lecture is based on section 2.2 in the text.

> ***The nexus question of this lecture:*** *What do linear subspaces look like?*

13.1 Subspaces

Recall that given a vector space V, a nonempty subset W is a ***linear subspace*** if W is a vector space (under the same vector addition and scalar multiplication as V.)

Recall also the following theorem from last time:

Theorem 53. *Let W be a nonempty subset of a vector space V. Then W is a linear subspace iff the following conditions are satisfied:*

(i.) $u + v \in W$ *whenever* $u, v \in V$.

(ii.) $cu \in W$ *whenever* $c \in \mathbb{R}$ *and* $u \in W$.

Example 54. Do the vectors of the form

$$W = \left\{ \begin{bmatrix} x \\ y \\ x - 2y \end{bmatrix} : x, y, z \in \mathbb{R} \right\} \subseteq \mathbb{R}^3$$

form a subspace of $\mathbb{R}^3$?

Yes. To check this, we will apply Theorem 53.

- **Condition (i):** Let $u, v \in W$. We need to show that $u + v \in W$. First, write

$$u = \begin{bmatrix} x_1 \\ y_1 \\ x_1 + 2y_1 \end{bmatrix} \quad \text{and} \quad v = \begin{bmatrix} x_2 \\ y_2 \\ x_2 + 2y_2. \end{bmatrix}$$

  Then

$$u + v = \begin{bmatrix} x_1 + x_2 \\ y_1 + y_2 \\ (x_1 + x_2) + 2(y_1 + y_2) \end{bmatrix}$$

  And we observe that this vector is in W. So $u + v \in W$, as desired.

- **Condition (ii):** Let $u \in W$ and $c \in \mathbb{R}$. We need to show that $cu \in W$.

$$cu = c \begin{bmatrix} x_1 \\ y_1 \\ x_1 + 2y_1 \end{bmatrix} = \begin{bmatrix} cx_1 \\ cy_1 \\ cx_1 + 2cy_1 \end{bmatrix}$$

  Observe that this vector is in W (with $x = cx_1$ and $y = cy_1$). So $cu \in W$, as desired.

End of Example 54. □

Example 55. Let $\mathbb{R}[x]$ be the set of all polynomials in the variable x. And let $\mathbb{R}[x]_{\leq n}$ be the set of all polynomials of degree $\leq n$. That is,

$$\mathbb{R}[x]_{\leq n} = \left\{ a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n : a_0, \ldots, a_n \in \mathbb{R} \right\}$$

Then $\mathbb{R}[x]_{\leq n}$ is a subspace of $\mathbb{R}[x]$. To see why, we just need to check the conditions of Theorem 53.

- closure under addition
- closure under scalar multiplication

See Example 42.

End of Example 55. □

13.2 The kernel of a matrix

Given an $m \times n$ matrix A, the ***kernel*** of A, denoted $\ker(A)$, is the set

$$\ker(A) = \left\{ X = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \in \mathbb{R}^n : AX = \vec{0} \right\}$$

Geometrically, the kernel is the set of of points in space that get A sent to the origin under the transormation of A. Moreover, since the kernel is the solution set to the linear system $AX = \vec{0}$, we know that it must either

- consist of a unique point (i.e., the linear system has exactly one solution)
- consist infinitely many points. (i.e., the system as infinitely many solutions)
- consist of zero points (i.e., the system has no solutions)

But note that no matter what the matrix A is, the linear system $AX = \vec{0}$ always has at least one solution, namely $X = \vec{0}$. Thus, $\ker(A)$ either consists of the single point $X = \vec{0}$, or it contains infinitely many points.

In fact, we can say more than that: the kernel of a matrix is always a linear subspace.

Theorem 56 (Kernels are subspaces)**.** *Let A be an $m \times n$ matrix. The solutions to the linear system*

$$AX = \vec{0}$$

is a subspace in $\mathbb{R}^n$. In other words, $\ker(A)$ is a linear subspace of $\mathbb{R}^n$.

Proof. To prove this theorem, we will apply Theorem 53. It will suffice to show that (i) if $u, v \in \ker(A)$ then $u + v \in \ker(A)$, and (ii) if $c \in \mathbb{R}$ then $cu \in \ker(A)$.

- **Proof of (i):** Let $u, v \in \ker(A)$. Then

  $$Au = \vec{0} \quad \text{and} \quad Av = \vec{0}.$$

  Then

  $$A(u + v) = Au + Av = \vec{0} + \vec{0} = \vec{0}$$

  Therefore $u + v \in \ker(A)$.

- **Proof of (ii):** Let $c \in \mathbb{R}$ and $u \in \ker(A)$. Then

  $$A(cu) = cAu = c(\vec{0}) = \vec{0}.$$

  This shows that $cu \in \ker(A)$.

□

Theorem 56 is only half the story; the converse is also true

Theorem 57 (Every subspace is a kernel)**.** *Every subspace of $\mathbb{R}^n$ is the kernel of some matrix.*

We don't yet have the technical machinery to express why this is true, but we'll encounter it in the coming weeks (linear spans, linear independence, and bases).

If we accept Theorem 57 on face for now, it tells us some useful geometric information about linear subspaces of $\mathbb{R}^n$. Namely, a linear subspaces of $\mathbb{R}^n$ is always one of the following:

- the point $\left\{\vec{0}\right\}$
- a line passing through the origin
- a plane passing through the origin

- an n-dimensional hyperplane passing through the origin, for $n > 3$.

We can also connect the notion of kernel back to our "key theorem", which now stands as the following:

Theorem 58 (The Key Theorem of Linear Algebra (partial version))**.** *Let A be an $n \times n$ matrix. Then the following are equivalent:*

(i.) *A is invertible (i.e., A^{-1} exists)*

(ii.) $\det A \neq 0$

(iii.) *The linear system $AX = B$ has a unique solution for each $B \in \mathbb{R}^n$.*

(iv.) *A is row equivalent to I*

(v.) *A is nonsingular (i.e., the only solution to the linear system $AX = \vec{0}$ is $X = \vec{0}$)*

(vi.) $\ker(A) = \left\{\vec{0}\right\}$