

Math 307: Homework 08

Due Friday, October 31. This homework draws on sections 5.1, 5.2 and 5.3 in the textbook.

Problem 1. Let $S, T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be the linear transformations

$$T \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x + 3y \\ x - y \end{bmatrix} \quad \text{and} \quad S \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2x - y \\ x + 2y \end{bmatrix}$$

- (a) Find matrices A, B such that T and S are expressed as the matrix transformations $T(X) = AX$ and $S(X) = BX$.
- (b) Find the matrix C such that the composition $S \circ T$ is expressed in the form $(S \circ T)(X) = CX$. Then verify that $C = BA$.
- (c) Find the matrix D such that the composition $T \circ S$ is expressed in the form $(T \circ S)(X) = DX$. Then verify that $D = AB$.

Problem 2. Let

$$A = \begin{bmatrix} 4 & -1 & 2 & 1 \\ 2 & 3 & -1 & -2 \\ 0 & 7 & -4 & -5 \\ 2 & -11 & 7 & 8 \end{bmatrix}$$

For the following parts, you must fully justify your work to receive credit—this includes providing at least one or two complete sentences explaining why your calculations justify your answer.

- (a) Find a basis for $NS(A)$
- (b) Find a basis for $RS(A)$
- (c) Find a basis for $CS(A)$
- (d) Determine the rank of A .
- (e) Determine whether the matrix is invertible

Problem 3. If S and T are linear transformations with $S(v) = T(v) = v$, then $S(T(v)) = v$ or v^2 ?

Problem 4. True or false, with counterexample if false:

- (a) If the vectors x_1, \dots, x_m span a subspace S , then $\dim S = m$.
- (b) The intersection of two subspaces of a vector space cannot be empty.
- (c) If $Ax = Ay$, then $x = y$.
- (d) The row space of A has a unique basis that can be computed by reducing A to reduced row-echelon form.
- (e) If a square matrix A has independent columns, then so does A^2 .
- (f) Any two bases of a linear subspace have the same number vectors.

Problem 5. Find all solutions of

$$\begin{cases} x_1 + x_2 + x_3 + x_4 = 0 \\ x_1 + x_2 + x_3 - x_4 = 4 \\ x_1 + x_2 - x_3 + x_4 = -4 \\ x_1 - x_2 + x_3 + x_4 = 2 \end{cases}$$

Problem 6. A linear transformation must leave the zero vector fixed: $T(0) = 0$. Prove this from $T(u+v) = T(u) + T(v)$ by choosing $v = \underline{\hspace{1cm}}$. Prove it also from the requirement $T(cv) = cT(v)$ by choosing $c = \underline{\hspace{1cm}}$.

Problem 7. Every straight line remains straight after a linear transformation. If z is halfway between x and y , show that Az is halfway between Ax and Ay

Problem 8. Let $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be the linear transformation

$$T \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 + x_2 \\ x_1 - x_2 \end{bmatrix}.$$

- (a) Find $[T]_{\alpha}^{\alpha}$ where α is the standard basis for \mathbb{R}^2 .
- (b) Let β be the basis for \mathbb{R}^2 consisting of $\begin{bmatrix} 1 \\ -1 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \end{bmatrix}$. Find the change of basis matrix from α to β .
- (c) Find the change of basis matrix from β to α .
- (d) Find $[T]_{\beta}^{\beta}$.
- (e) Find $[v]_{\beta}$ for $v = \begin{bmatrix} -2 \\ 3 \end{bmatrix}$.
- (f) Find $[T(v)]_{\beta}$.
- (g) Use the result of part (f) to find $T(v)$.

Problem 9. Let β be the basis of \mathbb{R}^2 consisting of the vectors $\begin{bmatrix} 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

- (a) Write the vectors $\begin{bmatrix} 5 \\ -4 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \end{bmatrix}$, and $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ in terms of the basis β . Please write your final answers as column vectors of the form $\begin{bmatrix} x \\ y \end{bmatrix}_{\beta}$.
- (b) Let T be the linear transformation $T = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 5x_1 + 3x_2 \\ -6x_1 - 4x_2 \end{bmatrix}$.
- (c) Find $[T]_{\alpha}^{\alpha}$, where α is the standard basis of \mathbb{R}^2 consisting of the vectors $\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.
- (d) Find $[T]_{\beta}^{\beta}$.

Problem 10. Let c be a scalar, and let $T : V \rightarrow W$ be a linear transformation. Verify that cT is a linear transformation.

Problem 11. The matrix $A = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$ produces a *stretching* in the x -direction. Draw the circle $x^2 + y^2 = 1$. What happens to this circle when space is transformed by A ? Illustrate your answer with a sketch.

Problem 12.

- (a) What 2×2 matrix has the effect of rotating every vector counterclockwise 90° and then projecting the result onto the x -axis?
- (b) What 2×2 matrix represents projection onto the x -axis followed by projection onto the y -axis?

Problem 13. The matrix $A = \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}$ yields a *shearing* transformation, which leaves the y -axis unchanged. Sketch its effect on the x -axis, by indicating what happens to the points $(x, y) = (1, 0), (2, 0)$, and $(-1, 0)$ —and how the whole axis is transformed.

Problem 14. The 4-Hadamard matrix is

$$H = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}$$

Find H^{-1} and write $v = \begin{bmatrix} 7 \\ 5 \\ 3 \\ 1 \end{bmatrix}$ as a linear combination of the columns of H .