28 2025-10-29 | Week 10 | Lecture 28

> ***The nexus question of this lecture:*** *What are eigenvalues and eigenvectors and how do we find them?*

One way to understand what a linear linear transformation does is to undertand which parts of space are invariant — that is, which parts of space don't change.

28.1 Eigenvalues

Definition 124 (Eigenvalue,eigenvector)**.** If A is an $n \times n$ matrix, an ***eigenvector*** of A is a nonzero column vector $v \in \mathbb{R}^n$ such that

$$Av = \lambda v$$

for some scalar $\lambda \in \mathbb{C}$. The scalar λ is called an ***eigenvalue***.

Example 125. Let

$$A = \frac{1}{2}\begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$$

In Example 121, we say that this transformation corresponds to a dilation by a factor of 2 in the direction of 45°.

By geometric considerations, we can see that the line $y = x$ is invariant under this transformation (vectors along this line get scaled by 2 but don't jump off the line). Also we see that the line $y = -x$ is invariant (vectors along this line don't change at all).

The vector $v = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ lies on the line $y = x$. We see that

$$Av = 2v$$

so that $v = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is an eigenvector with eigenvalue 2.

Similarly, the following vector lies on the line $y = -x$:

$$v = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

It satisfies $Av = 1v$. Hence it is an eigenvector with eigenvalue 1.

End of Example 125. ☐

Example 126. Let

$$A = \begin{bmatrix} 1 & -3 \\ -2 & 2 \end{bmatrix}$$

has eigenvewctor

$$v = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

with eigenvalue 4.

End of Example 126. ☐

28.2 How do we find eigenvalues?

Idea: look to the system $AX = \lambda X$. This is equivalent to

$$(\lambda I - A)X = 0.$$

Since $X \neq 0$ (since by definition eigenvectors must be nonzero), we conclude that the matrix $\lambda I - A$ is singular, and hence by the key theorem, $\det(\lambda I - A) = 0$.

Theorem 127. *Let A be an $n \times n$ matrix. Then $\lambda \in \mathbb{R}$ is an eigenvalue of A if and only if* $\det(\lambda I - A) = 0$.

Definition 128. The ***characteristic equation of A*** is

$$\det(\lambda I - A) = 0.$$

When A is an $n \times n$ matrix, the left hand side of the characteristic equation is a polynomial in the variable λ of degree n, and is called the ***characteristic polynomial*** of A.

Example 129. Let

$$A = \begin{bmatrix} 1 & -3 \\ -2 & 2 \end{bmatrix}.$$

Then the characteristic polynomial of A is

$$\begin{aligned} \det(\lambda I - A) &= \det \begin{bmatrix} \lambda - 1 & 3 \\ 2 & \lambda - 2 \end{bmatrix} \\ &= (\lambda - 1)(\lambda - 2) - 6 \\ &= \lambda^2 - 3\lambda - 4 \\ &(\lambda - 4)(\lambda + 1) \end{aligned}$$

This is equal to zero if and only if $\lambda = 4$ or $\lambda = -1$. Therefore the eigenvalues of A are $\lambda = 4$ and $\lambda = -1$.

End of Example 129. □

28.3 How do we find eigenvectors?

Idea: First find the eigenvalues λ. Then for each eigenvalue λ, the eigenvectors are the nontrivial solutions of the homogeneous ssytem

$$(\lambda I - A)X = 0.$$

(This is a linear system which we can solve using row reduction.)

In other words, the eigenvectors are the nonzero vectors in the linear subspace

$$NS(\lambda I - A).$$

So we just need to compute a basis of this nullspace, which is called the ***eigenspace***. When we ask to find the eigenvalues, it is always enough to just compute the basis of the eigenspace.

Example 130. Find the eigenvectors of the matrix

$$A = \begin{bmatrix} 1 & -3 \\ -2 & 2 \end{bmatrix}$$

The equations we need to solve are

- **When $\lambda = 4$:** $4I - A = 0$ or

  $$\begin{bmatrix} 3 & 3 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

  Reducing find the nullspace is

  $$NS(4I - A) = \left\{ y \begin{bmatrix} -1 \\ 1 \end{bmatrix}, y \in \mathbb{R} \right\}$$

  Technically, all vectors in $NS(4I - A)$ are eigenvectors for $\lambda = 4$. To give a concrete example, we have eigenvector $v = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$.

End of Example 130. □