20 2025-10-10 | Week 07 | Lecture 20

> ***The nexus question of this lecture:*** *What is a linear transformation?*

This lecture is based on section 5.1 in the textbook.w

20.1 Function notation

When a function f goes from a set X to a set Y, we write

$$f : X \to Y$$

which is read as "f maps X to Y". The set X is the ***domain*** of f. The set Y is the ***codomain*** of f. The subset

$$\text{Range}(f) = \{f(x) \mid x \in X\}$$

is called the ***range*** of f.

Example 92.
- Let $f(x) = e^x$. Then $f : \mathbb{R} \to \mathbb{R}$. The range of this function is the set of positive real numbers.
- Let $g : \mathbb{R}^2 \to \mathbb{R}$ ge given by

$$g\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = x^2 - y^2$$

  The range of the function is $\mathbb{R}$, which is the same as its codomain.

End of Example 92. □

20.2 Linear Transformation

Definition 93. Let V, W be vector spaces and $T : V \to W$ a function. We say that T is a ***linear transformation*** if, for all vectors $u, v \in V$ and $c \in \mathbb{R}$, we have

(i) $T(u + v) = T(u) + T(v)$ (preserves addition)

(ii) $T(cV) = cT(v)$. (preserves scalar multiplication)

Sometimes people refer to linear transformations as ***linear operators***, which means the same thing.

You have seen linear transformations before, even if you didn't call them that at the time. For example, the derivative operator $\frac{d}{dx}$ is one such function.

Example 94. Let $P = \{\text{all polynomials in the variable } x\}$. So an arbitary element of P looks like

$$p = c_0 + c_1 x + c_2 x^2 + \ldots + c_r x^r$$

for some nonnegative integer r.

We know that P is a vector space. Define a function

$$T : P \to P$$

where $T(p) = p'$. In other words, $T = \frac{d}{dx}$.

To check that T is a linear transformation, let $p, q \in P$ and $c \in \mathbb{R}$. Then

- $T(p + q) = \frac{d}{dx}[p(x) + q(x)] = p'(x) + q'(x) = T(p) + T(q)$.
- $T(cp) = \frac{d}{dx}[cp(x)] = c\frac{d}{dx}[p(x)] = cT(p)$.

End of Example 94. □

Example 95. Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ be ggiven by

$$T \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x + y - z \\ x + 2y + x \end{bmatrix}$$

we can check that this is also a linear transformation.

First observe that we can write

$$T \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 & 1 & -1 \\ 1 & 2 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}.$$

Letting $A = \begin{bmatrix} 1 & 1 & -1 \\ 1 & 2 & 1 \end{bmatrix}$, we can write

$$T(v) = Av, \text{ for } v \in \mathbb{R}^3.$$

Let's check conditions (i) and (ii) in the definition of linear transformation:

- Proof of (i): Let $u, v \in \mathbb{R}^3$. Then

$$T(u + v) = A(u + v) = Au + Av = T(u) + T(v)$$

  This shows that condition (i) holds.

- Proof of (ii): Let $c \in \mathbb{R}$ and $u \in \mathbb{R}^3$. Then

$$T(cu) = A(cu) = c(Au) = cT(u)$$

  This shows that condition (ii) holds.

Therefore since both conditions are met, T is a linear transformation.

End of Example 95. □

The proofs from the previous example didn't depend on the specific form of A, only that A was a matrix. Thus we have the following theorem:

Theorem 96. *If A is an $m \times n$ matrix, then the function $T : \mathbb{R}^n \to \mathbb{R}^m$ given by*

$$T(X) = AX$$

is a linear transformation.

Linear transformations of this form are called ***matrix transformations***.

Theorem 97. *Suppose $T : V \to W$ is a linear transformation. Then*

(i) $T(0) = 0$.

(ii) For any vectors $v_1, \dots, v_n \in V$ and scalars $c_1, \dots, c_n \in \mathbb{R}$, we have

$$T(c_1 v_1 + \ldots + c_n v_n) = c_1 T(v_1) + \ldots + c_n T(v_n)$$

The second property says that linear transformations preserve linear combinations.

Proof. To show that $T(0) = 0$ involves a trick. Observe that

$$T(0) = T(0 + 0) = T(0) + T(0).$$

Subtracting $T(0)$ from both sides gives

$$T(0) = 0.$$

Proof of (ii) is omitted, but follows from the definition of linear transformation. (HW?) □

Definition 98. The ***kernel*** of a linear transformation $T : V \to W$ the set

$$\ker(T) = \{v \in V : T(v) = 0\}$$

The previous theorem shows that 0 is always in $\ker(T)$. Just like matrices. Huh.

Example 99. Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear transformation defined by

$$T \begin{bmatrix} x \\ y \\ z \end{bmatrix} := \underbrace{\begin{bmatrix} 1 & 1 & -1 \\ 1 & 2 & 1 \end{bmatrix}}_{A} \begin{bmatrix} x \\ y \\ z \end{bmatrix}.$$

Question: Find a basis for $\ker(T)$. [Equivalently: find a basis for $NS(A)$.]

Solution: *Idea: solve the homogeneous system $AX = 0$, then interpret the solution.*

The system $AX = 0$ is

$$\begin{bmatrix} 1 & 1 & -1 \\ 1 & 2 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

We can solve this with row reduction:

$$\left[\begin{array}{ccc|c} 1 & 1 & -1 & 0 \\ 1 & 2 & 1 & 0 \end{array}\right] \xrightarrow{\text{row reduction}} \left[\begin{array}{ccc|c} 1 & 0 & -3 & 0 \\ 0 & 1 & 2 & 0 \end{array}\right].$$

This corresponds to the equations

$$\begin{cases} x - 3z = 0 \\ y + 2z = 0 \end{cases}$$

or

$$\begin{cases} x = 3z \\ y = -2z \end{cases}$$

where z a free variable. Thus, every solution $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ to $AX = 0$ takes the form

$$\begin{bmatrix} 3z \\ -2z \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix} z$$

Therefore the vector $\begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix}$ form a basis for $\ker(T)$.

End of Example 99. $\square$