27 2025-10-27 | Week 10 | Lecture 27

> ***The nexus question of this lecture:*** *What do linear transformations of 2-dimensional space look like?*

Example 121. Find a linear transformation that consists of stretching space in the direction $\theta = 45°$ (i.e. $\theta = \pi/4$ radians) by a factor of 2.
Answer: The linear transformation we are looking for is

$$R_{-\frac{\pi}{4}} \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} R_{\frac{\pi}{4}}$$

since this rotates space 45° counterclockwise, then stretches space vertically by a factor of 2, and rotates space back 45° clockwise.

Now,

$$R_{\frac{\pi}{4}} = \begin{bmatrix} \cos(\frac{\pi}{4}) & -\sin(\frac{\pi}{4}) \\ \sin(\frac{\pi}{4}) & \cos(\frac{\pi}{4}) \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$$

and

$$R_{-\frac{\pi}{4}} = R_{\frac{\pi}{4}}^{\top} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$$

Doing the matrix multiplication we get:

$$\frac{1}{2} \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$$

This matrix represents the linear transformation which stretches space in the direction $\theta = 45°$.

End of Example 121. □

A matrix P is said to be a ***projection*** if $P^2 = P$. The simplest projection is the identity I, but usually projections collapse dimension. If, in addition, the matrix P is symmetric (that is, if $P = P^{\top}$), then P is called an ***orthogonal projection***. If $P^{\top} \neq P$ it is called an oblique projection.

Example 122 (Projections)**.** Consider the line passing through the origin and the point $(c, s) = (\cos(\theta), \sin(\theta))$. Let P be the projection onto the θ-line. This is given by the matrix

$$P = \begin{bmatrix} c^2 & cs \\ cs & s^2 \end{bmatrix}$$

This matrix has no inverse, because the linear transformation has no inverse (because it collapses dimension). Indeed, $\det P = c^2 s^2 - (cs)^2 = 0$

Points on the θ-line are pojected to themselves. So projecting twice is the same as projecting onces, so $P^2 = P$. This is easy to check using $s^2 + c^2 = 1$:

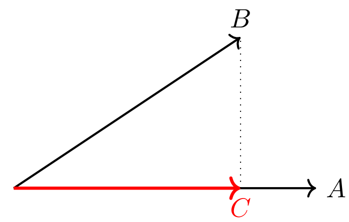
$$P^2 \begin{bmatrix} c^2 & cs \\ cs & s^2 \end{bmatrix} \begin{bmatrix} c^2 & cs \\ cs & s^2 \end{bmatrix} = \begin{bmatrix} c^2(c^2+s^2) & cs(c^2+s^2) \\ cs(c^2+s^2) & s^2(c^2+s^2) \end{bmatrix} = P$$

The nullspace of P is the line $y = -\frac{c}{s}x$. (picture)

Connection with vector projections: Recall from calculus class that the ***vector projection*** of B onto A is defined as the vector C given by

$$C = \underbrace{\left(\frac{A \cdot B}{|A|}\right)}_{\text{signed length of C}} \underbrace{\left(\frac{A}{|A|}\right)}_{\text{unit vector}} \tag{23}$$

This corresponds to the picuture



In other words, C is the shadow of B on A, if we shine a light directly above B. This is an orthogonal projection because the dotted line is orthogonal to A.

Let's work out an example. Suppose

$$\vec{A} = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} \quad \text{and} \quad \vec{B} = \begin{bmatrix} -2 \\ 3 \\ 1 \end{bmatrix}.$$

Then by Eq. (23), the vector projection is

$$C = \begin{bmatrix} 1/2 \\ 1/2 \\ 1 \end{bmatrix}.$$

Can we find a matrix P that effectuates this transformation? That is, such that $PB = C$?

In general, the matrix for an orthogonal projection can be written as

$$P = U(U^\top U)^{-1}U^\top, \tag{24}$$

where U is any matrix whose columns form a basis of the subspace onto which we are projecting. In the case of the vector projection of B onto A, we are projecting onto the line spanned by the vecctor A. Hence, we can take

$$U = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}.$$

Now let's compute P. First observe that

$$U^\top U = \begin{bmatrix} 1 & 1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} = 1 + 1 + 4 = 6,$$

so

$$(U^\top U)^{-1} = \frac{1}{6}.$$

Plugging this into Eq. (24), we get

$$P = \frac{1}{6} \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} \begin{bmatrix} 1 & 1 & 2 \end{bmatrix} = \frac{1}{6} \begin{bmatrix} 1 & 1 & 2 \\ 1 & 1 & 2 \\ 2 & 2 & 4 \end{bmatrix}.$$

Now we can check that $PB = C$:

$$PB = \frac{1}{6} \begin{bmatrix} 1 & 1 & 2 \\ 1 & 1 & 2 \\ 2 & 2 & 4 \end{bmatrix} \begin{bmatrix} -2 \\ 3 \\ 1 \end{bmatrix} = \frac{1}{6} \begin{bmatrix} -2+3+2 \\ -2+3+2 \\ -4+6+4 \end{bmatrix} = \frac{1}{6} \begin{bmatrix} 3 \\ 3 \\ 6 \end{bmatrix} = \begin{bmatrix} 1/2 \\ 1/2 \\ 1 \end{bmatrix} = C.$$

End of Example 122. □

Example 123 (Reflection)**.** Reflection across the θ-line (i.e. the line passing through the origin with angle θ) is given by the following matrix:

$$H = \begin{bmatrix} 2c^2 - 1 & 2cs \\ 2cs & 2s^2 - 1 \end{bmatrix}$$

where $c = \cos(\theta)$ and $s = \sin(\theta)$.

Observe

$$H^2 = H$$

$$H = 2P - I$$

For example, $\theta = 90°$ gives

$$H = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

End of Example 123. □