4 2025-09-03 | Week 02 | Lecture 04

> ***The nexus question of this lecture:*** *What is Gauss-Jordan elimination (aka: row reduction) and how do we use it to solve linear systems?*

Now I will present ***Gauss-Jordan elimination***. This is also called ***Gaussian elimination***, or more commonly, ***row reduction***. I will illustrate it by means of an example.

4.1 Using row reduction to solve a linear system with a unique solution

Suppose we wish to solve

$$\begin{cases} x - y + z = 0 \\ - y + 2z = -2 \\ -2x - y + z = 7 \end{cases} \tag{8}$$

Steps: We initialize the algorithm by setting up an ***augmented matrix*** corresponding to the system. For the system in (8), the augmented matrix is

$$\left[\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 2 & -3 & 4 & -2 \\ -2 & -1 & 1 & 7 \end{array}\right].$$

- The matrix to the left of the vertical row is the ***coefficient matrix.***
- A line of numbers going from left to right is called a ***row*** of the matrix. A line of numbers going down the matrix is a ***column.***

Gauss-Jordan elimination is like a game where the player has three possible moves, called ***row operations***.

1. Interchange two rows.
2. Multiply a row by a nonzero number.
3. Replace a row by itself plus a multiple of another row.

The player does row operations with the **goal** of making the diagonal entries of the coefficient matrix 1's and making as many of the other numbers zero, if possible. Here are the row operations for this example:

$$\left[\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 2 & -3 & 4 & -2 \\ -2 & -1 & 1 & 7 \end{array}\right] \xrightarrow{R_2-2R_1\to R_2} \left[\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ \mathbf{0} & \mathbf{-1} & \mathbf{2} & \mathbf{-2} \\ -2 & -1 & 1 & 7 \end{array}\right] \xrightarrow{R_3+2R_1\to R_3} \left[\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 0 & -1 & 2 & -2 \\ \mathbf{0} & \mathbf{-3} & \mathbf{3} & \mathbf{7} \end{array}\right]$$

$$\xrightarrow{R_1-R_2\to R_1} \left[\begin{array}{ccc|c} \mathbf{1} & \mathbf{0} & \mathbf{-1} & \mathbf{2} \\ 0 & -1 & 2 & -2 \\ 0 & -3 & 3 & 7 \end{array}\right] \xrightarrow{R_3-3R_2\to R_3} \left[\begin{array}{ccc|c} 1 & 0 & -1 & 2 \\ 0 & -1 & 2 & -2 \\ \mathbf{0} & \mathbf{0} & \mathbf{-3} & \mathbf{13} \end{array}\right] \xrightarrow{(-1)\cdot R_2\to R_2} \left[\begin{array}{ccc|c} 1 & 0 & -1 & 2 \\ \mathbf{0} & \mathbf{1} & \mathbf{-2} & \mathbf{2} \\ 0 & 0 & -3 & 13 \end{array}\right]$$

$$\xrightarrow{(-\frac{1}{3})\cdot R_3\to R_3} \left[\begin{array}{ccc|c} 1 & 0 & -1 & 2 \\ 0 & 1 & -2 & 2 \\ \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{-13/3} \end{array}\right] \xrightarrow{R_2+2R_3\to R_2} \left[\begin{array}{ccc|c} 1 & 0 & -1 & 2 \\ \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{-20/3} \\ 0 & 0 & 1 & -13/3 \end{array}\right] \xrightarrow{R_1+R_3\to R_1} \left[\begin{array}{ccc|c} \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{-7/3} \\ 0 & 1 & 0 & -20/3 \\ 0 & 0 & 1 & -13/3 \end{array}\right]$$

We now convert the augmented matrix back to a system of linear equations:

$$\begin{cases} 1x - 0y + 0z = -7/3 \\ 0x - 1y + 0z = -20/3 \\ 0x - 0y + 1z = -13/3 \end{cases}$$

or more simply,

$$\begin{aligned} x &= -7/3 \\ y &= -20/3 \\ z &= -13/3 \end{aligned}$$

We can check that this is a solution to the original system of equations (8).

4.2 The goal when doing row reduction

In the previoux example, we used row reduction Gauss-Jordan elimination to solve a linear system. The example we did had a unique solution. When that happens we can reduce the coefficient matrix to a matrix like

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

possibly with one or more rows of zeros at the bottom. (It may be larger or smaller depending on the number of equations and variables).

But in general, as we've seen, a linear system either has (1) one solution, (2) no solutions, or (3) infinitely many solutions. And if cases (2) or (3) happen, then we won't be able to do that. So we need to relax our "goal" when doing row reduction.

Our new goal is to reduce the coefficient matrix to ***reduced row-echelon form***, which in the case of a linear system with three equations and three variables, means it should look like one of these

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} 1 & 0 & \# \\ 0 & 1 & \# \\ 0 & 0 & 0 \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} 1 & \# & \# \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

where # denotes any arbitrary number.

Definition 11. More precisely, a coefficient matrix

$$\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}.$$

is said to be in ***reduced row echelon form*** if

1. Any rows of zeros appear at the bottom
2. The leftmost nonzero entry of all other rows equals 1 (the "leading 1's")
3. Each leading 1 of a nonzero row appears to the right of the leading row above it
4. All the other entries of a column containing a leading 1 are zero

This definition is *general*: it applies to any system with m equations and n variables. The pattern will become natural once you've worked a few (dozen?) examples.

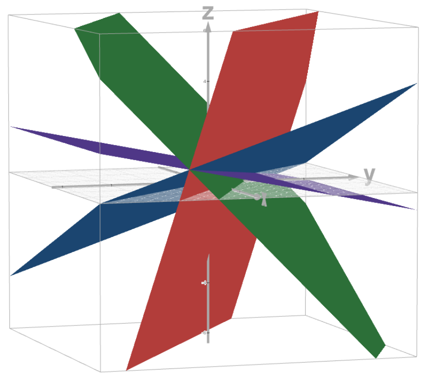
4.3 Using row reduction to solve a system with infitely many solutions

Here's an example which shows what happens when we try to solve a linear system with infinitely many solutions:

Example 12 (Using row reduction to solve a system with infitely many solutions)**.** We wish to solve the system

$$\begin{cases} 2x + 3y - z = 3 \\ -x - \ \ y + 3z = 0 \\ x + 2y + 2z = 3 \\ \quad\quad y + 5z \ = 3 \end{cases}$$

This system has 4 equations and 3 variables. Each equation represents a plane. The solutions, if there are any, will be the intersection of these four planes. I've plotted the planes in Desmos:



It looks like the four planes intersect in a line. So we should expect infinitely many solutions.

Step 1. Write down the augmented matrix of the system.

$$\left[\begin{array}{ccc|c} 2 & 3 & -1 & 3 \\ -1 & -1 & 3 & 0 \\ 1 & 2 & 2 & 3 \\ 0 & 1 & 5 & 3 \end{array}\right]$$

Step 2. Do some combination or row reduction steps until the coefficient matrix is in reduced row echelon form. (This is Example 3 in the textbook, refer there for the steps.)

Step 3. Our matrix is now in reduced row echelon form:

$$\left[\begin{array}{ccc|c} 1 & 0 & -8 & -3 \\ 0 & 1 & 5 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Let's convert it back into a linear system of equations:

$$\begin{aligned} x - 8z &= -3 \\ y + 5z &= 3 \\ 0 &= 0 \\ 0 &= 0 \end{aligned}$$

Therefore, we have

$$\begin{cases} x = -3 + 8z \\ y = 3 - 5z \end{cases} \tag{9}$$

where z is any real number. There are no restrictions on the value of z. Any choice of z gives us a valid solution to our original system of equations. If $z = 0$, we have the solution $(x, y, z) = (-3, 3, 0)$. If $z = 1$, then we have the solution $(x, y, z) = (5, -2, 1)$, and so forth.

Interpretation: In this case, z is called a ***free variable*** and x and y are called ***dependent variables***. The solutions to the original linear system consist of all points on a line which cuts through 3-dimensional space $\mathbb{R}^3$. Eq. (9) gives us a parametric equation of the line. The set of solutions is 1-dimensional, because it is a line.

End of Example 12. □

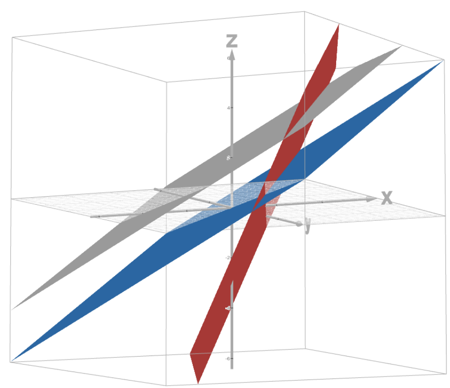
We've covered two of the three cases. For the last case, we consider the question *what happens if we attempt to solve a linear system that has no solutions?*

4.4 Using row reduction to attempt to solve a system with no solutions

Example 13 (Using row reduction to attempt to solve a system with no solutions)**.** Suppose we wish to solve the system

$$\begin{aligned} 2x + y - z &= 3 \\ -x - y + 2z &= 0 \\ -x - y + 2z &= 4 \end{aligned}$$

Here's a plot of the planes:



Their intersection if the three planes is *empty*: there is no point which lies on all three planes. So this system has no solution. What happens when we try to use row reduction?

Step 1. Write down the augmented matrix:

$$\left[\begin{array}{ccc|c} 2 & 1 & -1 & 3 \\ -1 & -1 & 2 & 0 \\ -1 & -1 & 2 & 4 \end{array}\right]$$

Step 2. Do row reduction to get to reduced row echelon form (I'm skipping steps here):

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 3 \\ 1 & -1 & 3 & 3 \\ 0 & 0 & 0 & 4 \end{array}\right]$$

Step 3. Convert back to a linear system:

$$\begin{cases} x \qquad + z = 3 \\ x - y + 3z = 3 \\ 0 \qquad\quad\;\; = 4 \end{cases}$$

The last equation is never true, no matter what values we choose for x, y and z. So the system has no solution

End of Example 13. □

4.5 Notes and resources for learning row reduction

NOTE: I'm not going to do a lot of examples of row reduction because it's so time consuming that it's not a great use of lecture time. So I expect you to teach yourself how to do row reduction. If you want to see more examples, some good online videos are

`https://www.youtube.com/watch?v=OP2aQUOevhI`
`https://www.youtube.com/watch?v=eDb6iugi6Uk`

There are also some nice online tools to help with row reduction, for example:

`https://textbooks.math.gatech.edu/ila/demos/rrinter.html`
`https://www.math.odu.edu/~bogacki/cgi-bin/lat.cgi?c=roc`

General advice:

1. Of course I'm going to ask you to row-reduce matrices on your exams, and for that you'll need to be able to do it by hand, without a calculator.
2. Write down all your steps, including a new matrix at every step in an organized way.
3. Use the notation $R_i \leftrightarrow R_j$ to indicate a swap of rows i and j; $cR_i \to R_i$ to indicate a multiplication of row i by a constant c; and, $R_i + cR_j \to R_i$ to indicate that you've added c times row j to row i.
4. Work left to right, top to bottom. Start by making the top left entry 1. Then use it to make all the numbers below it zero. Then go to the second column, second row, and make that 1. Then make everything beneath it zero. Etc.