

12 2025-09-22 | Week 5 | Lecture 12

This lecture is based on sections 2.1 and 2.2 in the textbook.

The nexus question of this lecture: What is a vector space?

12.1 Vector spaces

Definition 45 (Vector Space). A set V is called a **vector space** if there are operations called “vector addition” and “scalar multiplication” on V such that the following 2 “closure properties” properties hold

- C1. $u + v \in V$ whenever $u, v \in V$ (“ V is closed under vector addition”)
- C2. $cv \in V$ whenever $c \in \mathbb{R}$ and $v \in V$ (“ V is closed under scalar multiplication”)

and the following 8 algebraic properties hold:

- A1. $u + v = v + u$ for all $u, v \in V$ (“vector addition is commutative”)
- A2. $u + (v + w) = (u + v) + w$ for all $u, v, w \in V$ (“vector addition is associative”)
- A3. There is a “zero vector” $\vec{0} \in V$ such that $v + \vec{0} = v$ for all $v \in V$ (“there is a zero vector”)
- A4. For each $v \in V$ there is an element $-v$ such that $v + (-v) = \vec{0}$ (“every vector has an additive inverse”)
- A5. $c(u + v) = cu + cv$ for all $c \in \mathbb{R}$ and all $u, v \in V$ (“scalar multiplication distributes over vector addition”)
- A6. $(c + d)v = cv + dv$ for all $c, d \in \mathbb{R}$ and all $v \in V$ (“scalar multiplication distributes over scalar addition”)
- A7. $c(dv) = (cd)v$ for all $c, d \in \mathbb{R}$ and all $v \in V$ (“scalar multiplication is associative”)
- A8. $1 \cdot v = v$ for all $v \in V$. (“multiplying a vector by one doesn’t change it.”)

These 10 properties are called **the vector space axioms**. The elements of V are called **vectors**.

Note that C1 and C2 just say that (1) the sum of two vectors is itself a vector, and (2) the act of scaling a vector returns a vector. Axioms A1-A8 say that the usual rules of algebra apply.

At heart, this is just a list of the essential properties of vectors that we are familiar with.

Example 46 (The n -dimensional vector space \mathbb{R}^n). We regard \mathbb{R}^n as the set of $n \times 1$ column vectors with real entries:

$$\mathbb{R}^n = \left\{ \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} : x_1, \dots, x_n \in \mathbb{R} \right\}.$$

The “vector addition” is defined as *entrywise addition*

$$\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} := \begin{bmatrix} x_1 + y_1 \\ \vdots \\ x_n + y_n \end{bmatrix}$$

and the “scalar multiplication” is defined for every real number c as

$$c \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} := \begin{bmatrix} cx_1 \\ \vdots \\ cx_n \end{bmatrix}$$

To check that \mathbb{R}^3 is a vector space, we have to verify each of the 8 vector space axioms.

End of Example 46. \square

12.2 Properties of Vector Spaces

Theorem 47 (Basic properties of vector spaces). *Let V be a vector space. Then*

- The zero vector $\vec{0} \in V$ is unique.
- If $u + v = \vec{0}$ then $u = -v$ (i.e., the negative of v is unique.)
- For any $v \in V$, $0v = \vec{0}$.
- For any real number c , $c\vec{0} = \vec{0}$
- For any $v \in V$, $(-1)v = -v$.

Note about the proof of Theorem 47. The proofs of these properties are not so important, but it is worth thinking carefully about *why* basic properties like these need to be proved: while these properties might seem “obvious” for \mathbb{R}^n , general vector spaces may look very different from \mathbb{R}^n . See text for details. \square

12.3 Subspaces

Vector spaces can have smaller vector spaces sitting inside them.

Definition 48 (Subspace). A subset W of a vector space V is called a **subspace** of V if W is itself a vector space under the same operations of vector addition and scalar multiplication used by V .

We didn’t really need to check all 8 axioms to verify that W is a subspace V . The important criteria to check are summarized in the following theorem

Theorem 49. *Let W be a nonempty subset of a vector space V . Then W is a subspace iff the following conditions are satisfied:*

- (i.) $u + v \in W$ whenever $u, v \in W$.
- (ii.) $cu \in W$ whenever $c \in \mathbb{R}$ and $u \in W$.

Proof sketch. The proof consists of checking that W satisfies the vector space axioms. Since $W \subseteq V$, most of them are satisfied automatically because they are “inherited” from V . The ones that aren’t can be deduced from (i) and (ii). See text for details. \square

Example 50. Let W be the set

$$W := \left\{ \begin{bmatrix} x \\ y \\ 0 \end{bmatrix} : x, y \in \mathbb{R} \right\}$$

Then W is a subspace of \mathbb{R}^3 . By Theorem 49, to verify this, we first need to check that it is closed under vector addition and scalar multiplication.

End of Example 50. \square

Example 51. The set

$$A = \left\{ \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} : x, y \in \mathbb{R} \right\} \subseteq \mathbb{R}^3$$

does not form a subspace of \mathbb{R}^3 . This is because the sum of any two vectors in A has the form $\begin{bmatrix} * \\ * \\ 2 \end{bmatrix}$, which is not itself in A .

End of Example 51. \square

Example 52 (Important example). Let A be an $m \times n$ matrix. The solutions to the linear system

$$AX = 0$$

is a subspace in \mathbb{R}^n . We will check this example in the next class.

End of Example 52. \square