

## 18 2025-10-06 | Week 07 | Lecture 18

*The nexus question of this lecture: What are the three fundamental linear subspaces associated with a matrix  $A$ ?*

### 18.1 Three fundamental subspaces

**Definition 81** (The fundamental subspaces of  $A$ ). Let  $A \in M_{m \times n}(\mathbb{R})$ . There are three important vector spaces associated with  $A$ :

- The **column space**, which is the subspace of  $M_{m \times 1}(\mathbb{R})$  spanned by the columns of  $A$ . Notation:  $CS(A)$
- The **row space**, which is the subspace of  $M_{1 \times n}(\mathbb{R})$  spanned by the rows of  $A$ . Notation  $RS(A)$ .
- The **null space** which is the subspace of  $\mathbb{R}^n$  of vectors  $x$  such that  $Ax = 0$ . Notation  $NS(A)$ . The nullspace and the kernel are the same thing.

**Remark 82** (Connection between column space and matrix multiplication). The idea of column space is natural. If

$$A = \left[ \begin{array}{c|c|c|c} | & | & \dots & | \\ A_1 & A_2 & & A_n \\ | & | & & | \end{array} \right] \in M_{m \times n}(\mathbb{R})$$

then for any vector  $x = (x_1, \dots, x_n)^\top \in \mathbb{R}^n$

$$Ax = A_1x_1 + A_2x_2 + \dots + A_nx_n$$

This is linear combination of the columns of  $A$ , so the output  $Ax$  is always an element of the column space. Another word for column space is the **image** or **range** of the [linear transformation of the] matrix  $A$ .

**Definition 83** (rank). The **rank** of a matrix is the dimension of its column space (= dim of row space).

**Theorem 84.** For  $A \in M_{m \times n}(\mathbb{R})$ , the dimensions satisfy

$$\dim RS(A) = \dim CS(A)$$

and

$$\underbrace{\dim CS(A)}_{\text{'rank'}} + \underbrace{\dim NS(A)}_{\text{'nullity'}} = \underbrace{n}_{\# \text{ cols of } A}$$

The second part is called the rank-nullity theorem. Noting that  $\text{rank}(A) = \dim CS(A)$  and that  $NS(A) = \ker(A)$ , we have

### 18.2 Some examples of computing bases for the three fundamental subspaces

**Example 85** (Row space, column space, null space). Let

$$A = \begin{bmatrix} 1 & 0 & -1 & 3 \\ 2 & 1 & 4 & -1 \\ 4 & 1 & 2 & 5 \end{bmatrix} \in M_{3 \times 4}(\mathbb{R}).$$

Find bases for

- (a)  $RS(A)$
- (b)  $NS(A)$

## Solution

(a) Wrong:

$$\begin{bmatrix} 1 & 0 & -1 & 3 \end{bmatrix}, \begin{bmatrix} 2 & 1 & 4 & -1 \end{bmatrix}, \begin{bmatrix} 4 & 1 & 2 & 5 \end{bmatrix}$$

Better approach:

*Idea: row reduction does not change row space, so row reduce until we get a linearly independent set.*

The reduced row echelon form is

$$A_{\text{RREF}} = \begin{bmatrix} 1 & 0 & -1 & 3 \\ 0 & 1 & 6 & -7 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad (15)$$

A basis of  $A_{\text{RREF}}$  is

$$\begin{bmatrix} 1 & 0 & -1 & 3 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 6 & 7 \end{bmatrix}$$

(since these are linearly independent). Moreover, we note that since row reduction doesn't change the row space,

$$RS(A) = RS(A_{\text{RREF}}).$$

and hence

$$\begin{bmatrix} 1 & 0 & -1 & 3 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 6 & 7 \end{bmatrix}$$

are a basis for  $RS(A)$  as well.

(b) We will use the fact that

$$NS(A) = NS(A_{\text{RREF}}).$$

So it suffices to find a basis for  $NS(A_{\text{RREF}})$ . Let's do a computation to see what  $NS(A_{\text{RREF}})$  looks like. Recall that  $NS(A_{\text{RREF}})$  consists of the vectors  $x$  satisfying

$$A_{\text{RREF}}x = 0 \quad (16)$$

If  $x = (x_1, \dots, x_5)^\top$  satisfies Eq. (16), then we have

$$\begin{bmatrix} 1 & 0 & -1 & 3 \\ 0 & 1 & 6 & -7 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

Writing this as equations, we have

$$\begin{aligned} x_1 - x_3 + 3x_4 &= 0 \\ x_2 + 6x_3 - 7x_4 &= 0 \\ 0 &= 0. \end{aligned}$$

Therefore we have:

$$\begin{aligned} x_1 &= x_3 - 3x_4 \\ x_2 &= -6x_3 + 7x_4 \end{aligned}$$

where  $x_3, x_4$  are free variables.

Therefore

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} x_3 - 3x_4 \\ -6x_3 + 7x_4 \\ x_3 \\ x_4 \end{bmatrix} = x_3 \begin{bmatrix} 1 \\ -6 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -3 \\ 7 \\ 0 \\ 1 \end{bmatrix}$$

This shows that the vectors

$$\begin{bmatrix} 1 \\ -6 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -3 \\ 7 \\ 0 \\ 1 \end{bmatrix}$$

span the nullspace of  $A$ . Moreover, they are also linearly independent (two vectors are linearly independent if and only if they are not scalar multiples of each other). Therefore

$$\begin{bmatrix} 1 \\ -6 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -3 \\ 7 \\ 0 \\ 1 \end{bmatrix}$$

form a basis for  $NS(A_{\text{RREF}})$ . Since  $NS(A_{\text{RREF}}) = NS(A)$ , they are a basis for  $NS(A)$  as well. Since there are two vectors in the basis,  $\dim NS(A) = 2$ .

End of Example 85.  $\square$