12 2025-09-22 | Week 05 | Lecture 12

This lecture is based on sections 2.1 and 2.2 in the textbook.

> ***The nexus question of this lecture:*** *What is a vector space?*

12.1 Vector spaces

Definition 45 (Vector Space)**.** A set V is called a ***vector space*** if there are operations called "vector addition" and "scalar multiplication" on V such that the following 2 "closure properties" properties hold

C1. $u + v \in V$ whenever $u, v \in V$ ("V is closed under vector addition")

C2. $cv \in V$ whenever $c \in \mathbb{R}$ and $v \in V$ ("V is closed under scalar multiplication")

and the following 8 algebraic properties hold:

A1. $u + v = v + u$ for all $u, v \in V$ ("vector addition is commutative")

A2. $u + (v + w) = (u + v) + w$ for all $u, v, w \in V$ ("vector addition is associative")

A3. There is a "zero vector" $\vec{0} \in V$ such that $v + \vec{0} = v$ for all $v \in V$ ("there is a zero vector")

A4. For each $v \in V$ there is an element $-v$ such that $v + (-v) = 0$ ("every vector has an additive inverse")

A5. $c(u + v) = cu + cv$ for all $c \in \mathbb{R}$ and all $u, v \in V$ ("scalar multiplication distributes over vector addition")

A6. $(c + d)v = cv + dv$ for all $c, d \in \mathbb{R}$ and all $v \in V$ ("scalar multiplication distributes over scalar addition")

A7. $c(dv) = (cd)v$ for all $c, d \in \mathbb{R}$ and all $v \in V$ ("scalar multiplication is associative")

A8. $1 \cdot v = v$ for all $v \in V$. ("multiplying a vector by one doesn't change it.")

These 10 properties are called ***the vector space axioms.*** The elements of V are called ***vectors***.

Note that C1 and C2 just say that (1) the sum of two vectors is itself a vector, and (2) the act of scaling a vector returns a vector. Axioms A1-A8 say that the usual rules of algebra apply.

At heart, this is just a list of the essential properties of vectors that we are familiar with.

Example 46 (The n-dimensional vector space $\mathbb{R}^n$)**.** We regard $\mathbb{R}^n$ as the set of $n \times 1$ column vectors with real entries:

$$\mathbb{R}^n = \left\{ \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} : x_1, \dots, x_n \in \mathbb{R} \right\}.$$

The "vector addition" is defined as *entrywise addition*

$$\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} := \begin{bmatrix} x_1 + y_1 \\ \vdots \\ x_n + y_n \end{bmatrix}$$

and the "scalar multiplication" is defined for every real number c as

$$c \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} := \begin{bmatrix} cx_1 \\ \vdots \\ cx_n \end{bmatrix}$$

To check that $\mathbb{R}^3$ is a vector space, we have to verify each of the 8 vector space axioms.

End of Example 46. □

12.2 Properties of Vector Spaces

Theorem 47 (Basic properties of vector spaces)**.** *Let V be a vector space. Then*

- *The zero vector $\vec{0} \in V$ is unique.*
- *If $u + v = \vec{0}$ then $u = -v$ (i.e., the negative of v is unique.)*
- *For any $v \in V$, $0v = \vec{0}$.*
- *For any real number c, $c\vec{0} = \vec{0}$*
- *For any $v \in V$, $(-1)v = -v$.*

Note about the proof of Theorem 47. The proofs of these properties are not so important, but it is worth thinking carefully about *why* basic properties like these need to be proved: while these properties might seem "obvious" for $\mathbb{R}^n$, general vector spaces may look very different from $\mathbb{R}^n$. See text for details. ☐

12.3 Subspaces

Vector spaces can have smaller vector spaces sitting inside them.

Definition 48 (Subspace)**.** A subset W of a vector space V is called a ***subspace*** of V if W is itself a vector space under the same operations of vector addition and scalar multiplication used by V .

We didn't really need to check all 8 axioms to verify that W is a subspace V. The important criteria to check are summarized in the following theorem

Theorem 49. *Let W be a nonempty subset of a vector space V. Then W is a subspace iff the following conditions are satisfied:*

(i.) $u + v \in W$ *whenever* $u, v \in V$.

(ii.) $cu \in W$ *whenever* $c \in \mathbb{R}$ *and* $u \in W$.

Proof sketch. The proof consists of checking that W satisfies the vector space axioms. Since $W \subseteq V$, most of them are satisfied automatically becuase they are "inherited" from V. The ones that aren't can be deduced from (i) and (ii). See text for details. ☐

Example 50. Let W be the set

$$W := \left\{ \begin{bmatrix} x \\ y \\ 0 \end{bmatrix} : x, y \in \mathbb{R} \right\}$$

Then W is a subspace of $\mathbb{R}^3$. By Theorem 49, to verify this, we first need to check that it is closed under vector addition and scalar multiplication.

End of Example 50. ☐

Example 51. The set

$$A = \left\{ \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} : x, y \in \mathbb{R} \right\} \subseteq \mathbb{R}^2$$

does not form a subspace of $\mathbb{R}^3$. This is because the sum of any two vectors in A has the form $\begin{bmatrix} * \\ * \\ 2 \end{bmatrix}$, which is not itself in A.

End of Example 51. ☐

Example 52 (Important example)**.** Let A be an $m \times n$ matrix. The solutions to the linear system

$$AX = 0$$

is a subspace in $\mathbb{R}^n$. We will check this example in the next class.

End of Example 52. ☐