# Lecture Notes for Math 307:
# Linear Algebra and Differential Equations

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## Contents



## About this document

These lecture notes were prepared by Max Hill for a 16-week linear algebra course (MATH 307) at University of Hawaii at Manoa in Fall 2025.

The textbook used is *Linear Algebra and Differential Equations* (2002) by G. Peterson S. Sochacki, in which we cover primarily Chapters 1,2,5, and 6

# 0 Tentative Course Outline

- **Weeks 1-3: Matrices and determinants.** *(Systems of linear equations, matrices, matrix operations, inverse matrices, special matrices and their properties, and determinants.)*
  - Section 1.1: Systems of Linear Equations
  - Section 1.2: Matrices and Matrix Operations
  - Section 1.3: Inverses of Matrices
  - Section 1.4: Special Matrices and Additional Properties of Matrices
  - Section 1.5: Determinants
  - Section 1.6: Further Properties of Determinants
  - Section 1.7: Proofs of Theorems on Determinants
- **Weeks 4-6: Vector spaces.** *(Vector spaces, subspaces, spanning sets, linear independence, bases, dimension, null space, row and column spaces, Wronskian.)*
  - Section 2.1: Vector Spaces
  - Section 2.2: Subspaces and Spanning Sets
  - Section 2.3: Linear Independence and Bases
  - Section 2.4: Dimension; Nullspace, Rowspace, and Column Space
  - Section 2.5: Wronskians
- **Weeks 7-11: Linear transformations, spectral theory.** *(Linear transformation, eigenvalues and eigenvectors, algebra of linear transformations, matrices for linear transformations, eigenvalues and eigenvectors, similar matrices, diagonalization, Jordan normal form.)*
  - Section 5.1: Linear Transformations
  - Section 5.2: The Algebra of Linear Transformations
  - Section 5.3: Matrices for Linear Transformations
  - Section 5.4: Eigenvalues and Eigenvectors of Matrices
  - Section 5.5: Similar Matrices, Diagonalization, and Jordan Canonical Form
  - Section 5.6: Eigenvectors and Eigenvalues of Linear Transformations
- **Midterm Exam**
- **Weeks 12-14: Systems of differential equations.** *(Theory of systems of linear differential equations, homogeneous systems with constant coefficients, the diagonalizable case, nondiagonalizable case, nonhomogeneous linear systems, applications to $2 \times 2$ and $3 \times 3$ systems of nonlinear differential equations.)*
  - Section 6.1: The Theory of Systems of Linear Differential Equations
  - Section 6.2: Homogenous Systems with Constant Coefficients: The Diagonalizable Case
  - Section 6.3: Homogenous Systems with Constant Coefficients: The Nondiagonalizable Case
  - Section 6.4: Nonhomogeneous Linear Systems
  - Section 6.6: Applications Involving Systems of Linear Differential Equations
  - Section 6.7: $2 \times 2$ Systems of Nonlinear Differential Equations
- **Weeks 14-16: Other stuff if time allows.** *(Converting differential equations to first order systems (section 6.5), linearization of $2 \times 2$ nonlinear systems (???), stability and instability (section 6.7), predator-prey equations (section 6.7.1).)*
- **Final Exam**

# 1 2025-08-25 | Week 01 | Lecture 01

*This lecture is based on textbook section 1.1. Introduction to Systems of Linear Equations*

> ***The nexus question of this lecture:*** *What is a system of linear equations, and what does it mean to 'solve' a system of linear equations?*

## 1.1 A first example of a system of linear equations

We begin with something concrete.

**Example 1** (A first example of a *system of linear equations*)**.** Consider the following word problem:

> *A boat travels between two ports on a river 48 miles apart. When traveling downstream (i.e., with the current), the trip takes 4 hours, but when traveling upstream (i.e., fighting the current), the trip takes 6 hours.*
>
> *Assume that the boat and the current are both moving at a constant speed. What is the speed of the boat in still water, and what is the speed of the current?*

This problem is hard to reason through without writing something down, but becomes much simpler when we formalize it mathematically with equations. The unknowns are (1) **the speed of the boat in still water** and (2) **the speed of the current**. So let

$$\begin{aligned} x &:= (\text{the speed of the boat in still water}) \\ y &:= (\text{the speed of the current}). \end{aligned}$$

The speed of the boat going downstream is $x + y$. Therefore, since (speed) $\times$ (time) = (distance travelled), we have

$$4(x + y) = 48, \quad \text{or equivalently} \quad x + y = 12.$$

Similarly, the sped of the boat going upstream is $x - y$, so

$$6(x - y) = 48, \quad \text{or equivalently} \quad x - y = 8$$

Thus, we have the following *system of linear equations:*

$$\begin{cases} x + y = 12 \\ x - y = 8. \end{cases} \tag{1}$$

This system has **two equations** and **two variables** ($x$ and $y$). You have encountered systems of equations like this many times. With the help of the technology of algebra, solving this problem (namely, solving System (1)) is much easier than solving the original word problem.

- In this case, the problem can be easily solved **algebraically** using a substitution (e.g., plug $x = 8 + y$ into the first equation and solve for $y$, then solve for $x$ after finding $y$). This gives the solution $(x, y) = (10, 2)$. The speed of the boat in still water is 10mph. The speed of the river current is 2mph.
- We can conceive of another type of solution, which uses a **geometric**, rather than algebraic perspective: observe that each equation $x + y = 12$ and $x - y = 8$ represents a line on the $xy$-plane. Plot the lines. Their intersection is the point $(10, 2)$, which is the solution.
- However, solving systems of equations like in (1) becomes more cumbersome when there are lots of variables and equations. Doing substitutions and algebraic manipulations will still work, but will be tedius and difficult if you have many equations and variables.

  Later, we will introduce a general algorithm which can solve any such system. This algorithm is called ***Gauss-Jordan elimination***, and it will be one of the core techniques that we will use to solve many types of problems in this class.

End of Example 1. $\square$

## 1.2 Key definitions: linear systems and their solutions

In this section, we formalize the mathematical objects we are studying.

**Definition 2** (Linear equation)**.** A ***linear equation*** in the variables $x_1, \ldots, x_n$ is an equation that can be written in the form

$$a_1x_1 + a_2x_2 + \ldots + a_nx_n = b,$$

where $a_1, \ldots, a_n$ and $b$ are constants (e.g., fixed real numbers). The numbers $a_1, \ldots, a_n$ are called ***coefficients***.

Note that the variables $x_1, \ldots, x_n$ are not raised to any powers. That's what makes the equation *linear*. If we had squares or cubes of some of the $x_i$'s, or products like $x_1x_3$, then the equation would be quadratic or cubic, or something else, but not linear.

**Example 3** (Examples of linear equations)**.**

- The equation
$$2x - 3y = 1$$
is a linear equation in the variables $x$ and $y$. Its graph is a line on the $xy$-plane.

- The equation
$$3x - y + 2z = 9$$
is a linear equation in the variables $x, y$ and $z$. Its graph is a plane in 3-dimensional space (denoted $\mathbb{R}^3$).

- The equation
$$-x_1 + 5x_2 + \pi^2 x_3 + \sqrt{2}x_4 = e^2$$
is a linear equation in the variables $x_1, x_2, x_3$, and $x_4$. The coefficients are
$$a_1 = -1, \quad a_2 = 5, \quad a_3 = \pi, \quad \text{and} \quad a_4 = \sqrt{2}.$$
The graph of this linear system is a 3-dimensional hyperplane in 4d-space (i.e., $\mathbb{R}^4$).

  **Observation:** In these examples, we observe a simple relationship between the number of variables and the dimension of the graph:
$$(\text{dimension of graph}) = (\#\text{ of variables}) - 1.$$
Here, the term ***dimension*** refers to the number of free variables. In the first equation (which is $2x - 3y = 1$), it's easy to see that if we know one of the variables, then the other one is automatically determined. So it makes sense that the graph of this equation is of dimension 1 (which it is, because it's a line). For the second equation, if we know any 2 of the variables, then the third variable is automatically determined, so it makes sense that the dimension of the graph is 2 (which it is, because planes are 2-dimensional). Etc.

End of Example 3. □

**Definition 4** (Linear system, solution of a linear system)**.** When considered together, a collection of $m$ linear equations

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{21}x_2 + \ldots + a_{2n}x_n = b_2 \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n = b_m \end{cases} \tag{2}$$

is called a ***system of linear equations***, or ***linear system*** for short. A ***solution*** to a system of linear equations is a set of values for $x_1, \ldots, x_n$ which satisfy all equations in system (2).

**Example 5** (A system of linear equations)**.** An example of a system of linear equations is

$$\begin{cases} \;\;\; x \;-\;\;\; y \;+\;\;\; z = \;\;\,0 \\ \;\;2x \;-\; 3y \;+\; 4z = -2 \\ -2x \;-\;\;\; y \;+\;\;\; z = \;\;\,7 \end{cases}$$

When a linear system like this walks in the door, we always first ask two basic questions: (1) 'how many equations does it have?' and (2) 'how many variables does it have?'. In this case, we have $m = 3$ equations and $n = 3$ variables.

End of Example 5. □

## 1.3 How to understand solutions of linear systems geometrically

Here is a very useful geometric perspective. In system (2), we have a system of $m$ equations expressed in $n$ variables $x_1, \ldots, x_n$. Each of the $m$ equations is the equation of some hyperplane[1] which lives in $n$-dimensional space ($\mathbb{R}^n$). *The solution to the linear system is the intersection of these hyperplanes.*

For example, in Example 5, the 'hyperplanes' were lines, and their intersection was the point $(x, y) = (10, 2)$.

We will spend a lot of time understanding what hyperplanes look like, and what intersections of hyperplanes look like.

[1]Note: Hyperplanes will be defined more formally later, but for now can be thought of as generalized lines or planes, since a 1-dimensional hyperplanes is a *line* and a 2-dimensional hyperplane is a *plane*.

# 2 2025-08-27 | Week 01 | Lecture 02

> ***The nexus question of this lecture:*** *What do solutions to linear systems look like?*

## 2.1 How to understand solutions of linear systems geometrically

Here is a very useful geometric perspective. In system (2), we have a system of $m$ equations expressed in $n$ variables $x_1, \ldots, x_n$. Each of the $m$ equations is the equation of some hyperplane[2] which lives in $n$-dimensional space ($\mathbb{R}^n$). *The solution to the linear system is the intersection of these hyperplanes.*

The clearest example of this can be seen in the linear system:

**Example 6** (The case with two variables)**.**

$$\begin{cases} a_{11}x + a_{12}y = b_1 \\ a_{21}x + a_{22}y = b_2 \end{cases} \tag{3}$$

where $a_{12}, a_{22} \neq 0$. (In this case, the "hyperplanes" are simply lines.) Here, the solutions to the first equation are the points on the line

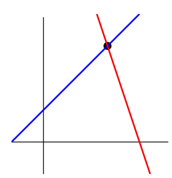
$$y = -\frac{a_{11}}{a_{12}}x + \frac{b_1}{a_{12}} \tag{4}$$

Similarly, the solutions to the second equation are the points on the line

$$y = -\frac{a_{21}}{a_{22}}x + \frac{b_2}{a_{22}}. \tag{5}$$

There are three possible things that can happen when we intersect the two lines in Eqs. (4) and (5):

- **Case 1.** The two line equations Eqs. (4) and (5) represent distinct lines and are not parallel. In this case, their intersection consists of a unique point, like this:

  In this case, the system (3) has **exactly one solution**—namely, the intersection of the two lines, just like we saw in the boat example.

- **Case 2.** The two line equations Eqs. (4) and (5) represent two parallel but different lines. In this case, the two lines never intersect each other (i.e., there is no point that lies on both lines), so the system (3) has **no solutions**.

- **Case 3.** The two equations of lines are the same, so they represent the same line. Therefore the intersection of the two lines is the entire line. Therefore, there are **infinitely many solutions** to the linear system (3). Namely, any point $(x, y)$ on the line is a solution to the linear system.

End of Example 6. □

These three cases desribed in Example 6 constitute the following trichotomy:

**Theorem 7.** *A system of linear equations either has (1) exactly one solution, (2) no solution, or (3) infinitely many solutions.*

We haven't proven this fact, only illustrated it for systems of linear equations like (3) that have 2 equations and 2 variables. In fact, as we shall see, this fact always holds for all linear systems of the form given in (2), no matter how many equations and variables.

[2]Note: Hyperplanes will be defined more formally later, but for now can be thought of as generalized lines or planes, since a 1-dimensional hyperplanes is a *line* and a 2-dimensional hyperplane is a *plane*.

## 2.2 The planar case

Recall that, geometrically, a line is determined by two features:

1. A slope $m$ which determines the direction of the line

2. A point $(x_0, y_0)$ which the line passes through, as this determines where the line lives on the $xy$-plane

It is easy to see that these two things determine everything about a line because the equation of a line can be expressed as

$$y - y_0 = m(x - x_0)$$

and to write this down, all we need are $m$ and $(x_0, y_0)$.

Just like a line, a plane is determined by two things:

1. A normal vector $n = \langle A, B, C\rangle$ which determines the tilt of the plane. (Here, $A, B$, and $C$ are fixed constants)

2. A point $(x_0, y_0, z_0)$ which the plane passes through, as this determines where in 3-d space ($\mathbb{R}^3$) the plane lives.

To be prcise, a plane $\mathbb{P}$ consists of the set of points $(x, y, z)$ satisfying the following equation:

$$A(x - x_0) + B(y - y_0) + C(z - z_0) = 0 \tag{6}$$

This is the standard form equation of a plane, and we can write it down if we know both $n = \langle A, B, C\rangle$ and $(x_0, y_0, z_0)$. So if we know those two things, then we know the equation of the plane, meaning we know everything about it.

By a little bit of algebra, we can rewrite Eq. (6) as

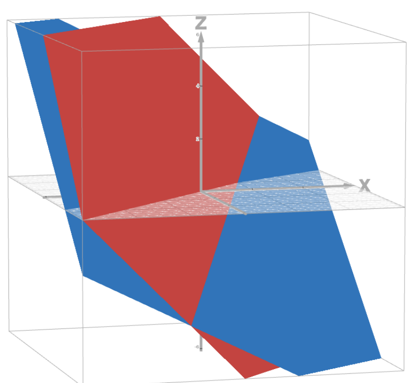
$$Ax + By + Cz = D$$

where $D = Ax_0 + By_0 + Cz_0$. This is a linear equation. Just like how the solutions to a linear equation with 2 variables form a line, the solutions to a linear equation with 3 variables form a plane.

**Example 8** (A system with three variables)**.** Suppose we wish to solve the linear system

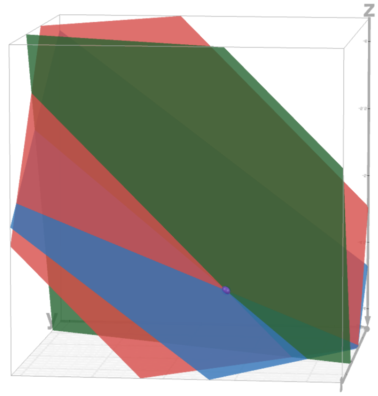
$$\begin{cases} \phantom{-2}x \; - \; y + \; z = \phantom{-}0 \\ \phantom{-}2x \; - \; 3y + \; 4z = -2 \\ -2x \; - \; y + \; z = \phantom{-}7 \end{cases}$$

In this case, each equation is the equation of a plane. The planes for the first two equations are the following:




The plane for the first equation is in red. The plane for the second equation is blue. Any point on the red plane is a solution to the first equation $x - y + z = 0$. Any point on the blue plane is a solution to the second equation $2x - 3y + 4z = -2$. The two planes interesect in a line. If I pick any point on this line, then it satisfies both equations.

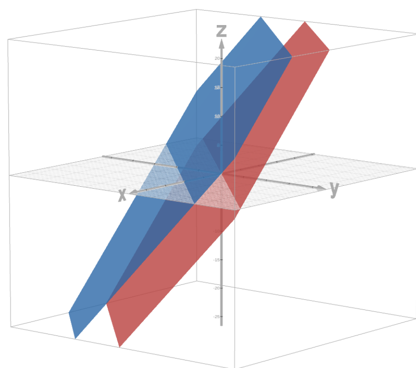
But our system has three equations, so we have a third plane, and the intersection of all three planes is a point, as shown:



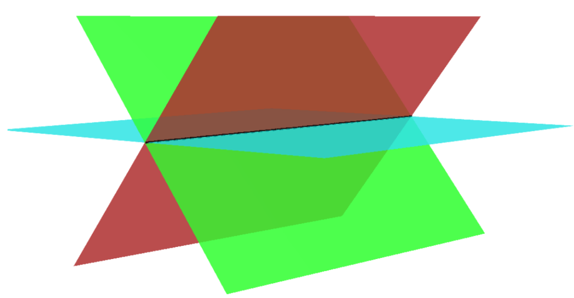
In this case, the system of equations has a unique solution, which is the unique point of intersection of the planes. Here's the desmos link to see the plots of these three planes, if you want to play around with them:

`https://www.desmos.com/3d/gpgtw2rjaf`

Of course there are other ways that three planes could have intersected. For example, two of the planes might be parallel, like the following picture, in which case the system will have no solutions:




Or the three planes could intersect in a line, like the following picture, in which case there are infinitely many solutions (credit Noah R. for the observation and picture):



There are other ways that three planes could intersect as well, but the trichotomy stated earlier always holds: their intersection either constists of (1) exactly one point, (2) infinitely many points, or (3) zero points.

End of Example 8. □

We've seen in this lecture that for systems of linear equations with two variables, the solutions are the intersection of lines. For systems of linear equations with three variables, the solutions are the intersections of planes. ... And for systems with $n > 3$ variables, the solutions are the intersections of hyperplanes.

## 2.3 Motivation for Gauss-Jordan Elimination

> ***The nexus question for the rest of this lecture:*** *How can we solve a linear system without resorting to substitution?*

In the next lecture, I will present ***Gauss-Jordan elimination***, a general algorithm which can be used to solve general systems of linear equations, which does not use substitution. For now, I will answer the above question by working out an example which motivates the main ideas, so that the Gauss-Jordan algorithm doesn't feel like its coming out of nowhere when we present it in the next lecture.

**Example 9** (Solving a linear system with elementary operations)**.** Suppose we wish to solve the following system:

$$\begin{cases} x - y + z = 0 & (E_1) \\ 2x - 3y + 4z = -2 & (E_2) \\ -2x - y + z = 7 & (E_3) \end{cases} \tag{7}$$

This system has 3 equations, labeled $E_1, E_2, E_3$, and 3 variables $x, y$ and $z$. Suppose that we know ahead of time that this system has a unique solution (it does). Then, in principle, we could solve this using substitution, but that would suck. Instead, I will illustrate an approach in which we iteratively transform this linear system into successively simpler systems until we get to a point where the solution is obvious.

To do this, we will play a game where there are three 'moves' available to us. The three moves are:

1. Interchange two equations in the system.
2. Multiply an equation by a nonzero number.
3. Replace an equation by itself plus a multiple of another equation.

These moves are called ***elementary operations***, and if we use them intelligently, they will allow us to transform the linear system into a simpler system.

Two systems of equations are said to be ***equivalent*** if they have the same solutions. Applying elementary operations always results in an equivalent system. Our goal will be to use some combination of elementary operations to produce a system of the form

$$\begin{cases} x = * \\ y = * \\ z = * \end{cases}$$

where each $*$ is a constant which we will have computed. This will be our solution to the linear system (7), because the two systems will be equivalent.

First, let's apply operation 3: specifically, by replacing $E_2$ with $E_2 - 2E_1$:

$$\begin{cases} x - y + z = 0 \\ - y + 2z = -2 \\ -2x - y + z = 7 \end{cases}$$

We have eliminated the $x$ from the second equation, yielding a simpler system. Let's keep doing this. To eliminate $x$ from equation 3, let's apply operation 3 again: This time, replace $E_3$ with $E_3 + 2E_1$:

$$\begin{cases} x - y + z = 0 \\ - y + 2z = -2 \\ -3y + 3z = 7 \end{cases}$$

Apply operation 3, replace $E_1$ with $E_1 - E_2$. This will allow us to eliminate $y$ from $E_1$:

$$\begin{cases} x - z = 2 \\ - y + 2z = -2 \\ -3y + 3z = 7 \end{cases}$$

Apply operation 3, replace $E_3$ with $E_3 - 3E_2$. This will allow us to eliminate $y$ from $E_3$:

$$\begin{cases} x \quad\quad - \quad z = \quad 2 \\ \quad - \ y + \ 2\,z = - \ 2 \\ \quad\quad\quad - \ 3z = \quad 13 \end{cases}$$

Apply operation 2 twice: multiply both the first and second equations by 3:

$$\begin{cases} 3x \quad\quad\quad - 3z = 6 \\ \quad - \ 3y \ + 6z = -6 \\ \quad\quad\quad\quad - 3z = 13 \end{cases}$$

Apply operation 3, twice. First, replace $E_1$ with $E_1 - E_3$. Then replace $E_2$ with $E_2 + 2E_3$. Doing both of these, we get:

$$\begin{cases} 3x \quad\quad\quad = - 7 \\ \quad\quad 3y \quad\quad = \ 20 \\ \quad\quad\quad -3z = \ 13 \end{cases}$$

Apply operation 2 by dividing all three equations by 3:

$$\begin{cases} x \quad\quad\quad = - 7/3 \\ \quad\quad y \quad\quad = \ 20/3 \\ \quad\quad\quad -z = \ 13/3 \end{cases}$$

This is the solution to the original equation. We have used elementary operatons to reduce our original linear system Eq. (7) to the above system, which equivalent to the original system.

While solving this system was still a lot of (tedious) work, it was still probably simpler than doing substitution.

End of Example 9. □

**Remark 10** (Forget the variables)**.** In the procedure presented in , we didn't really need to track the variables, only the *coefficients* and the *quantities on the right hand sides* of the equations. Instead of working with the equations directly, it will be simpler to work with the following matrix, called the ***augmented matrix*** corresponding go Eq. (7):

$$\left[\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 2 & -3 & 4 & -2 \\ -2 & -1 & 1 & 7 \end{array}\right].$$

Comparing this with system (7), it becomes clear that the augmented matrix was obtained essentially by just erasing the variables $x, y$, and $z$ in (7), and then placing what remains into an array. We also drew a vertical line to the separate the left- and right-hand sides of the equations. Inside the augmented matrix, the $3 \times 3$ submatrix of coefficients

$$\begin{bmatrix} 1 & -1 & 1 \\ 2 & -3 & 4 \\ -2 & -1 & 1 \end{bmatrix}$$

is called the ***coefficient matrix*** of the system.

More precise definitions are as follows:

**Definition 11** (Augmented Matrix)**.** Given a linear system of the form (2), the ***augmented matrix*** is

$$\left[\begin{array}{cccc|c} a_{11} & a_{12} & \dots & a_{1n} & b_1 \\ a_{21} & a_{22} & \dots & a_{2n} & b_2 \\ \vdots & \vdots & & \vdots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} & b_m \end{array}\right]$$

and the ***coefficient matrix*** is the matrix

$$\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}.$$

# 3 2025-08-29 | Week 01 | Lecture 03

***The nexus question of this lecture:*** *What is Gauss-Jordan Elimination?*

**Steps:** We initialize the algorithm by setting up an ***augmented matrix*** corresponding to the system. For the system in (**??**), the augmented matrix is

$$\left[\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 2 & -3 & 4 & -2 \\ -2 & -1 & 1 & 7 \end{array}\right].$$

- The matrix to the left of the vertical row is called the ***coefficient matrix.***
- A line of numbers going from left to right is called a ***row*** of the matrix. A line of numbers going down the matrix is a ***column.***

Gauss-Jordan elimination is like a game where the player has three possible moves, called ***row operations***:

1. Interchange two rows.
2. Multiply a row by a nonzero number.
3. Replace a row by itself plus a multiple of another row.

The player does row operations with the **goal** of making as many numbers in the coefficient matrix zero as possible.

$$\left[\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 2 & -3 & 4 & -2 \\ -2 & -1 & 1 & 7 \end{array}\right] \xrightarrow{R_2-2R_1} \left[\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ \mathbf{0} & \mathbf{-1} & \mathbf{2} & \mathbf{-2} \\ -2 & -1 & 1 & 7 \end{array}\right] \xrightarrow{R_3+2R_1} \left[\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 0 & -1 & 2 & -2 \\ \mathbf{0} & \mathbf{-3} & \mathbf{3} & \mathbf{7} \end{array}\right]$$

$$\xrightarrow{R_1-R_2} \left[\begin{array}{ccc|c} \mathbf{1} & \mathbf{0} & \mathbf{-1} & \mathbf{2} \\ 0 & -1 & 2 & -2 \\ 0 & -3 & 3 & 7 \end{array}\right] \xrightarrow{R_3-3R_2} \left[\begin{array}{ccc|c} 1 & 0 & -1 & 2 \\ 0 & -1 & 2 & -2 \\ \mathbf{0} & \mathbf{0} & \mathbf{-3} & \mathbf{13} \end{array}\right] \xrightarrow{(-1)\cdot R_2} \left[\begin{array}{ccc|c} 1 & 0 & -1 & 2 \\ \mathbf{0} & \mathbf{1} & \mathbf{-2} & \mathbf{2} \\ 0 & 0 & -3 & 13 \end{array}\right]$$

$$\xrightarrow{(-\frac{1}{3})\cdot R_3} \left[\begin{array}{ccc|c} 1 & 0 & -1 & 2 \\ 0 & 1 & -2 & 2 \\ \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{-13/3} \end{array}\right] \xrightarrow{R_2+2R_3} \left[\begin{array}{ccc|c} 1 & 0 & -1 & 2 \\ \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{-20/3} \\ 0 & 0 & 1 & -13/3 \end{array}\right] \xrightarrow{R_1+R_3} \left[\begin{array}{ccc|c} \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{-7/3} \\ 0 & 1 & 0 & -20/3 \\ 0 & 0 & 1 & -13/3 \end{array}\right]$$

We now convert the augmented matrix back to a system of linear equations:

$$\begin{cases} 1x \; - \; 0y + 0z = - \;\; 7/3 \\ 0x \; - \; 1y + 0z = -20/3 \\ 0x \; - \; 0y + 1z = -13/3 \end{cases}$$

or more simply,

$$\begin{aligned} x &= -7/3 \\ y &= -20/3 \\ z &= -13/3 \end{aligned}$$

We can check that this is a solution to the original system of equations (**??**).