Lecture Notes for Math 307: Linear Algebra and Differential Equations

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Last updated: November 19, 2025

Contents

About this document

These lecture notes were prepared by Max Hill for a 16-week linear algebra course (MATH 307) at University of Hawaii at Manoa in Fall 2025.

The textbook used is *Linear Algebra and Differential Equations* (2002) by G. Peterson S. Sochacki, in which we cover primarily Chapters 1,2,5, and 6

0 Tentative Course Outline

- **Weeks 1-3: Matrices and determinants.** *(Systems of linear equations, matrices, matrix operations, inverse matrices, special matrices and their properties, and determinants.)*
  - Section 1.1: Systems of Linear Equations
  - Section 1.2: Matrices and Matrix Operations
  - Section 1.3: Inverses of Matrices
  - Section 1.4: Special Matrices and Additional Properties of Matrices
  - Section 1.5: Determinants
  - Section 1.6: Further Properties of Determinants
  - Section 1.7: Proofs of Theorems on Determinants
- **Weeks 4-6: Vector spaces.** *(Vector spaces, subspaces, spanning sets, linear independence, bases, dimension, null space, row and column spaces, Wronskian.)*
  - Section 2.1: Vector Spaces
  - Section 2.2: Subspaces and Spanning Sets
  - Section 2.3: Linear Independence and Bases
  - Section 2.4: Dimension; Nullspace, Rowspace, and Column Space
  - Section 2.5: Wronskians
- **Weeks 7-11: Linear transformations, spectral theory.** *(Linear transformation, eigenvalues and eigenvectors, algebra of linear transformations, matrices for linear transformations, eigenvalues and eigenvectors, similar matrices, diagonalization, Jordan normal form.)*
  - Section 5.1: Linear Transformations
  - Section 5.2: The Algebra of Linear Transformations
  - Section 5.3: Matrices for Linear Transformations
  - Section 5.4: Eigenvalues and Eigenvectors of Matrices
  - Section 5.5: Similar Matrices, Diagonalization, and Jordan Canonical Form
  - Section 5.6: Eigenvectors and Eigenvalues of Linear Transformations
- **Midterm Exam**
- **Weeks 12-14: Systems of differential equations.** *(Theory of systems of linear differential equations, homogeneous systems with constant coefficients, the diagonalizable case, nondiagonalizable case, nonhomogeneous linear systems, applications to 2×2 and 3×3 systems of nonlinear differential equations.)*
  - Section 6.1: The Theory of Systems of Linear Differential Equations
  - Section 6.2: Homogenous Systems with Constant Coefficients: The Diagonalizable Case
  - Section 6.3: Homogenous Systems with Constant Coefficients: The Nondiagonalizable Case
  - Section 6.4: Nonhomogeneous Linear Systems
  - Section 6.6: Applications Involving Systems of Linear Differential Equations
  - Section 6.7: 2×2 Systems of Nonlinear Differential Equations
- **Weeks 14-16: Other stuff if time allows.** *(Converting differential equations to first order systems (section 6.5), linearization of 2×2 nonlinear systems (???), stability and instability (section 6.7), predator-prey equations (section 6.7.1).)*
- **Final Exam**

1 2025-08-25 | Week 01 | Lecture 01

This lecture is based on textbook section 1.1. Introduction to Systems of Linear Equations

> ***The nexus question of this lecture:*** *What is a system of linear equations, and what does it mean to 'solve' a system of linear equations?*

1.1 A first example of a system of linear equations

We begin with something concrete.

Example 1 (A first example of a *system of linear equations*)**.** Consider the following word problem:

> *A boat travels between two ports on a river 48 miles apart. When traveling downstream (i.e., with the current), the trip takes 4 hours, but when traveling upstream (i.e., fighting the current), the trip takes 6 hours.*
>
> *Assume that the boat and the current are both moving at a constant speed. What is the speed of the boat in still water, and what is the speed of the current?*

This problem is hard to reason through without writing something down, but becomes much simpler when we formalize it mathematically with equations. The unknowns are (1) **the speed of the boat in still water** and (2) **the speed of the current**. So let

$$\begin{aligned} x &:= (\text{the speed of the boat in still water}) \\ y &:= (\text{the speed of the current}). \end{aligned}$$

The speed of the boat going downstream is $x + y$. Therefore, since (speed) $\times$ (time) = (distance travelled), we have

$$4(x + y) = 48, \quad \text{or equivalently} \quad x + y = 12.$$

Similarly, the sped of the boat going upstream is $x - y$, so

$$6(x - y) = 48, \quad \text{or equivalently} \quad x - y = 8$$

Thus, we have the following *system of linear equations:*

$$\begin{cases} x + y = 12 \\ x - y = 8. \end{cases} \tag{1}$$

This system has **two equations** and **two variables** (x and y). You have encountered systems of equations like this many times. With the help of the technology of algebra, solving this problem (namely, solving System (1)) is much easier than solving the original word problem.

- In this case, the problem can be easily solved **algebraically** using a substitution (e.g., plug $x = 8 + y$ into the first equation and solve for y, then solve for x after finding y). This gives the solution $(x, y) = (10, 2)$. The speed of the boat in still water is 10mph. The speed of the river current is 2mph.
- We can conceive of another type of solution, which uses a **geometric**, rather than algebraic perspective: observe that each equation $x + y = 12$ and $x - y = 8$ represents a line on the xy-plane. Plot the lines. Their intersection is the point $(10, 2)$, which is the solution.
- However, solving systems of equations like in (1) becomes more cumbersome when there are lots of variables and equations. Doing substitutions and algebraic manipulations will still work, but will be tedius and difficult if you have many equations and variables.

  Later, we will introduce a general algorithm which can solve any such system. This algorithm is called ***Gauss-Jordan elimination***, and it will be one of the core techniques that we will use to solve many types of problems in this class.

End of Example 1. □

1.2 Key definitions: linear systems and their solutions

In this section, we formalize the mathematical objects we are studying.

Definition 2 (Linear equation)**.** A ***linear equation*** in the variables $x_1, \ldots, x_n$ is an equation that can be written in the form

$$a_1x_1 + a_2x_2 + \ldots + a_nx_n = b,$$

where $a_1, \ldots, a_n$ and b are constants (e.g., fixed real numbers). The numbers $a_1, \ldots, a_n$ are called ***coefficients***.

Note that the variables $x_1, \ldots, x_n$ are not raised to any powers. That's what makes the equation *linear*. If we had squares or cubes of some of the x_i's, or products like x_1x_3, then the equation would be quadratic or cubic, or something else, but not linear.

Example 3 (Examples of linear equations)**.**

- The equation
$$2x - 3y = 1$$
is a linear equation in the variables x and y. Its graph is a line on the xy-plane.
- The equation
$$3x - y + 2z = 9$$
is a linear equation in the variables x, y and z. Its graph is a plane in 3-dimensional space (denoted $\mathbb{R}^3$).
- The equation
$$-x_1 + 5x_2 + \pi^2 x_3 + \sqrt{2}x_4 = e^2$$
is a linear equation in the variables x_1, x_2, x_3, and x_4. The coefficients are
$$a_1 = -1, \quad a_2 = 5, \quad a_3 = \pi, \quad \text{and} \quad a_4 = \sqrt{2}.$$
The graph of this linear system is a 3-dimensional hyperplane in 4d-space (i.e., $\mathbb{R}^4$).

  Observation: In these examples, we observe a simple relationship between the number of variables and the dimension of the graph:
$$(\text{dimension of graph}) = (\#\text{ of variables}) - 1.$$
Here, the term ***dimension*** refers to the number of free variables. In the first equation (which is $2x - 3y = 1$), it's easy to see that if we know one of the variables, then the other one is automatically determined. So it makes sense that the graph of this equation is of dimension 1 (which it is, because it's a line). For the second equation, if we know any 2 of the variables, then the third variable is automatically determined, so it makes sense that the dimension of the graph is 2 (which it is, because planes are 2-dimensional). Etc.

End of Example 3. □

Definition 4 (Linear system, solution of a linear system)**.** When considered together, a collection of m linear equations

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{21}x_2 + \ldots + a_{2n}x_n = b_2 \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n = b_m \end{cases} \tag{2}$$

is called a ***system of linear equations***, or ***linear system*** for short. A ***solution*** to a system of linear equations is a set of values for $x_1, \ldots, x_n$ which satisfy all equations in system (2).

Example 5 (A system of linear equations)**.** An example of a system of linear equations is

$$\begin{cases} \;\; x \; - \;\;\; y \; + \;\;\; z = \;\; 0 \\ \;\; 2x \; - \; 3y \; + \; 4z = -2 \\ -2x \; - \;\;\; y \; + \;\;\; z = \;\; 7 \end{cases}$$

When a linear system like this walks in the door, we always first ask two basic questions: (1) 'how many equations does it have?' and (2) 'how many variables does it have?'. In this case, we have $m = 3$ equations and $n = 3$ variables.

End of Example 5. $\square$

1.3 How to understand solutions of linear systems geometrically

Here is a very useful geometric perspective. In system (2), we have a system of m equations expressed in n variables $x_1, \ldots, x_n$. Each of the m equations is the equation of some hyperplane[1] which lives in n-dimensional space ($\mathbb{R}^n$). *The solution to the linear system is the intersection of these hyperplanes.*

For example, in Example 5, the 'hyperplanes' were lines, and their intersection was the point $(x, y) = (10, 2)$.

We will spend a lot of time understanding what hyperplanes look like, and what intersections of hyperplanes look like.

[1]Note: Hyperplanes will be defined more formally later, but for now can be thought of as generalized lines or planes, since a 1-dimensional hyperplanes is a *line* and a 2-dimensional hyperplane is a *plane*.

2 2025-08-27 | Week 01 | Lecture 02

> ***The nexus question of this lecture:*** *What do solutions to linear systems look like?*

2.1 How to understand solutions of linear systems geometrically

Here is a very useful geometric perspective. In system (2), we have a system of m equations expressed in n variables $x_1, \ldots, x_n$. Each of the m equations is the equation of some hyperplane[2] which lives in n-dimensional space ($\mathbb{R}^n$). *The solution to the linear system is the intersection of these hyperplanes.*

The clearest example of this can be seen in the linear system:

Example 6 (The case with two variables)**.**

$$\begin{cases} a_{11}x + a_{12}y = b_1 \\ a_{21}x + a_{22}y = b_2 \end{cases} \tag{3}$$

where $a_{12}, a_{22} \neq 0$. (In this case, the "hyperplanes" are simply lines.) Here, the solutions to the first equation are the points on the line

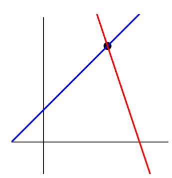
$$y = -\frac{a_{11}}{a_{12}}x + \frac{b_1}{a_{12}} \tag{4}$$

Similarly, the solutions to the second equation are the points on the line

$$y = -\frac{a_{21}}{a_{22}}x + \frac{b_2}{a_{22}}. \tag{5}$$

There are three possible things that can happen when we intersect the two lines in Eqs. (4) and (5):

- **Case 1.** The two line equations Eqs. (4) and (5) represent distinct lines and are not parallel. In this case, their intersection consists of a unique point, like this:

  In this case, the system (3) has **exactly one solution**—namely, the intersection of the two lines, just like we saw in the boat example.

- **Case 2.** The two line equations Eqs. (4) and (5) represent two parallel but different lines. In this case, the two lines never intersect each other (i.e., there is no point that lies on both lines), so the system (3) has **no solutions**.

- **Case 3.** The two equations of lines are the same, so they represent the same line. Therefore the intersection of the two lines is the entire line. Therefore, there are **infinitely many solutions** to the linear system (3). Namely, any point (x, y) on the line is a solution to the linear system.

End of Example 6. □

These three cases desribed in Example 6 constitute the following trichotomy:

Theorem 7. *A system of linear equations either has (1) exactly one solution, (2) no solution, or (3) infinitely many solutions.*

We haven't proven this fact, only illustrated it for systems of linear equations like (3) that have 2 equations and 2 variables. In fact, as we shall see, this fact always holds for all linear systems of the form given in (2), no matter how many equations and variables.

[2]Note: Hyperplanes will be defined more formally later, but for now can be thought of as generalized lines or planes, since a 1-dimensional hyperplanes is a *line* and a 2-dimensional hyperplane is a *plane*.

2.2 The planar case

Recall that, geometrically, a line is determined by two features:

1. A slope m which determines the direction of the line
2. A point (x_0, y_0) which the line passes through, as this determines where the line lives on the xy-plane

It is easy to see that these two things determine everything about a line because the equation of a line can be expressed as

$$y - y_0 = m(x - x_0)$$

and to write this down, all we need are m and (x_0, y_0).

Just like a line, a plane is determined by two things:

1. A normal vector $n = \langle A, B, C\rangle$ which determines the tilt of the plane. (Here, A, B, and C are fixed constants)
2. A point (x_0, y_0, z_0) which the plane passes through, as this determines where in 3-d space ($\mathbb{R}^3$) the plane lives.

To be precise, a plane $\mathbb{P}$ consists of the set of points (x, y, z) satisfying the following equation:

$$A(x - x_0) + B(y - y_0) + C(z - z_0) = 0 \tag{6}$$

This is the standard form equation of a plane, and we can write it down if we know both $n = \langle A, B, C\rangle$ and (x_0, y_0, z_0). So if we know those two things, then we know the equation of the plane, meaning we know everything about it.

By a little bit of algebra, we can rewrite Eq. (6) as

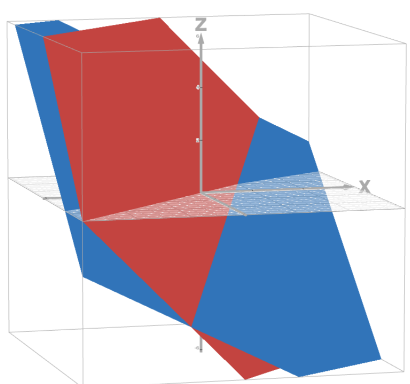
$$Ax + By + Cz = D$$

where $D = Ax_0 + By_0 + Cz_0$. This is a linear equation. Just like how the solutions to a linear equation with 2 variables form a line, the solutions to a linear equation with 3 variables form a plane.

Example 8 (A system with three variables)**.** Suppose we wish to solve the linear system

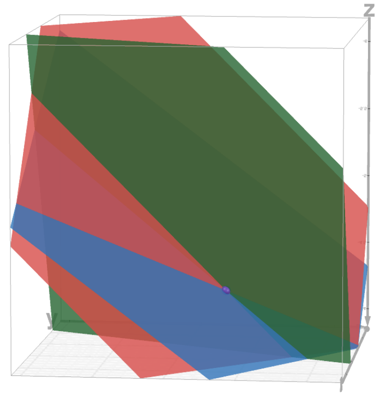
$$\begin{cases} x - y + z = 0 \\ 2x - 3y + 4z = -2 \\ -2x - y + z = 7 \end{cases}$$

In this case, each equation is the equation of a plane. The planes for the first two equations are the following:



The plane for the first equation is in red. The plane for the second equation is blue. Any point on the red plane is a solution to the first equation $x - y + z = 0$. Any point on the blue plane is a solution to the second equation $2x - 3y + 4z = -2$. The two planes interesect in a line. If I pick any point on this line, then it satisfies both equations.

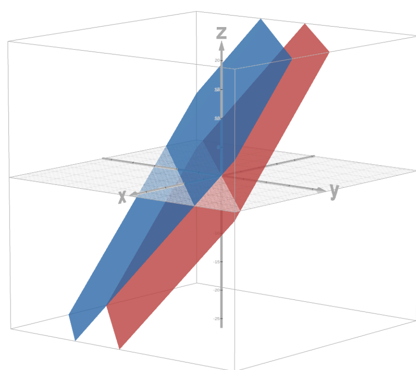
But our system has three equations, so we have a third plane, and the intersection of all three planes is a point, as shown:



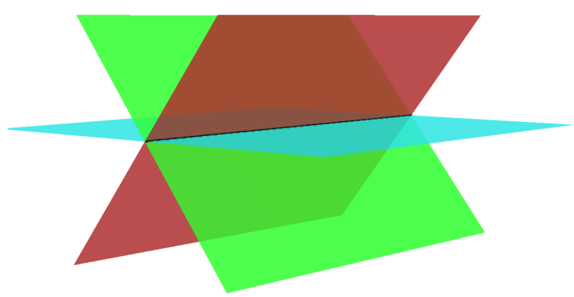
In this case, the system of equations has a unique solution, which is the unique point of intersection of the planes. Here's the desmos link to see the plots of these three planes, if you want to play around with them:

`https://www.desmos.com/3d/gpgtw2rjaf`

Of course there are other ways that three planes could have intersected. For example, two of the planes might be parallel, like the following picture, in which case the system will have no solutions:



Or the three planes could intersect in a line, like the following picture, in which case there are infinitely many solutions (credit Noah C. for the observation and picture):



There are other ways that three planes could intersect as well, but the trichotomy stated earlier always holds: their intersection either constists of (1) exactly one point, (2) infinitely many points, or (3) zero points.

End of Example 8. □

We've seen in this lecture that for systems of linear equations with two variables, the solutions are the intersection of lines. For systems of linear equations with three variables, the solutions are the intersections of planes. ... And for systems with $n > 3$ variables, the solutions are the intersections of hyperplanes.

3 2025-08-29 | Week 01 | Lecture 03

This lecture is based on section 1.1 in the textbook

> ***The nexus question of this lecture:*** *How can we solve a linear system without resorting to substitution?*

Recall in the boat example (Example 1), we had the system

$$\begin{cases} x + y = 12 \\ x - y = 8 \end{cases}$$

And we could solve this using subsitution. Another thing we could have done, would be to add the second equation to the first, giving us a new, simpler but equivalent system:

$$\begin{cases} 2x = 20 \\ x - y = 8 \end{cases}$$

Then divide the firs equation by two

$$\begin{cases} x = 10 \\ x - y = 8 \end{cases}$$

Then subtract the first equation from the second:

$$\begin{cases} x = 10 \\ -y = -2 \end{cases}$$

Then multiply the second equation by -1

$$\begin{cases} x = 10 \\ y = 2 \end{cases}$$

And tada! We have found our solution without doing substitution. But this example was very simple, so maybe it's special and we can't always do this sort of thing? Actually, we can. In the rest of the lecture, I'll try to formalize these sorts of steps we used here and apply them to a more complicated problem.

The reason I'm doing this is because, in the next lecture, I will begin to present ***Gauss-Jordan elimination*** (aka ***row reduction***), a general method which can be used to find the solutions of any system of linear equation which does not use substitution. For now, the we will work out an example which motivates the main ideas that will be used by Gauss-Jordan elimination.

3.1 Solving a linear system using via simplifying transformations

Example 9 (Solving a linear system with elementary operations)**.** Suppose we wish to solve the following system:

$$\begin{cases} x - y + z = 0 & (E_1) \\ 2x - 3y + 4z = -2 & (E_2) \\ -2x - y + z = 7 & (E_3) \end{cases} \tag{7}$$

This system has 3 equations, labeled E_1, E_2, E_3, and 3 variables x, y and z. Suppose that we know ahead of time that this system has a unique solution (we showed this graphically in Example 8). Then, in principle, we could solve this using substitution, but that would suck. Instead, I will illustrate an approach in which we iteratively transform this linear system into successively simpler systems until we get to a point where the solution is obvious.

To do this, we will play a game where there are three 'moves' available to us. The three moves are:

1. Interchange two equations in the system.

2. Multiply an equation by a nonzero number.

3. Replace an equation by itself plus a multiple of another equation.

These moves are called ***elementary operations***, and if we use them intelligently, they will allow us to transform the linear system into a simpler system.

Two systems of equations are said to be ***equivalent*** if they have the same solutions. Applying elementary operations always results in an equivalent system. Our goal will be to use some combination of elementary operations to produce a system of the form

$$\begin{cases} x = * \\ y = * \\ z = * \end{cases}$$

where each $*$ is a constant which we will have computed. This will be our solution to the linear system (7), because the two systems will be equivalent.

First, let's apply operation 3: specifically, by replacing E_2 with $E_2 - 2E_1$:

$$\begin{cases} x - y + z = 0 \\ \quad - y + 2z = -2 \\ -2x - y + z = 7 \end{cases}$$

We have eliminated the x from the second equation, yielding a simpler system. Let's keep doing this. To eliminate x from equation 3, let's apply operation 3 again: This time, replace E_3 with $E_3 + 2E_1$:

$$\begin{cases} x - y + z = 0 \\ \quad - y + 2z = -2 \\ \quad -3y + 3z = 7 \end{cases}$$

Apply operation 3, replace E_1 with $E_1 - E_2$. This will allow us to eliminate y from E_1:

$$\begin{cases} x \quad - z = 2 \\ \quad - y + 2z = -2 \\ \quad -3y + 3z = 7 \end{cases}$$

Apply operation 3, replace E_3 with $E_3 - 3E_2$. This will allow us to eliminate y from E_3:

$$\begin{cases} x \quad - z = 2 \\ \quad - y + 2z = -2 \\ \quad\quad -3z = 13 \end{cases}$$

Apply operation 2 twice: multiply both the first and second equations by 3:

$$\begin{cases} 3x \quad - 3z = 6 \\ \quad - 3y + 6z = -6 \\ \quad\quad -3z = 13 \end{cases}$$

Apply operation 3, twice. First, replace E_1 with $E_1 - E_3$. Then replace E_2 with $E_2 + 2E_3$. Doing both of these, we get:

$$\begin{cases} 3x = -7 \\ -3y = 20 \\ -3z = 13 \end{cases}$$

Apply operation 2 by multiplying the first equation by $1/3$. Then multiply the second and third equations both by $-1/3$:

$$\begin{cases} x = -7/3 \\ y = -20/3 \\ z = -13/3 \end{cases}$$

This is the solution to the original equation. We have used elementary operatons to reduce our original linear system Eq. (7) to the above system, which equivalent to the original system.

While solving this system was still a lot of (tedious) work, it was still probably simpler than doing substitution.

End of Example 9. □

3.2 Representing a linear system as an augmented matrix

In the procedure presented in Example 9, we didn't really need to track the variables, only the *coefficients* and the *quantities on the right hand sides* of the equations. Instead of working with the equations directly, it will be simpler to work with the following matrix, called the ***augmented matrix*** corresponding go Eq. (7):

$$\left[\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 2 & -3 & 4 & -2 \\ -2 & -1 & 1 & 7 \end{array}\right].$$

Comparing this with system (7), it becomes clear that the augmented matrix was obtained essentially by just erasing the variables x, y, and z in (7), and then placing what remains into an array. We also drew a vertical line to the separate the left- and right-hand sides of the equations. Inside the augmented matrix, the 3×3 submatrix of coefficients

$$\begin{bmatrix} 1 & -1 & 1 \\ 2 & -3 & 4 \\ -2 & -1 & 1 \end{bmatrix}$$

is called the ***coefficient matrix*** of the system.

More precise definitions are as follows:

Definition 10 (Augmented Matrix)**.** Given a linear system of the form (2), the ***augmented matrix*** is

$$\left[\begin{array}{cccc|c} a_{11} & a_{12} & \dots & a_{1n} & b_1 \\ a_{21} & a_{22} & \dots & a_{2n} & b_2 \\ \vdots & \vdots & & \vdots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} & b_m \end{array}\right]$$

and the ***coefficient matrix*** is the matrix

$$\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}.$$

This is our first application of

4 2025-09-03 | Week 02 | Lecture 04

> ***The nexus question of this lecture:*** *What is Gauss-Jordan elimination (aka: row reduction) and how do we use it to solve linear systems?*

Now I will present ***Gauss-Jordan elimination***. This is also called ***Gaussian elimination***, or more commonly, ***row reduction***. I will illustrate it by means of an example.

4.1 Using row reduction to solve a linear system with a unique solution

Suppose we wish to solve

$$\begin{cases} x - y + z = 0 \\ -y + 2z = -2 \\ -2x - y + z = 7 \end{cases} \tag{8}$$

Steps: We initialize the algorithm by setting up an ***augmented matrix*** corresponding to the system. For the system in (8), the augmented matrix is

$$\left[\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 2 & -3 & 4 & -2 \\ -2 & -1 & 1 & 7 \end{array}\right].$$

- The matrix to the left of the vertical row is the ***coefficient matrix.***
- A line of numbers going from left to right is called a ***row*** of the matrix. A line of numbers going down the matrix is a ***column.***

Gauss-Jordan elimination is like a game where the player has three possible moves, called ***row operations***.

1. Interchange two rows.
2. Multiply a row by a nonzero number.
3. Replace a row by itself plus a multiple of another row.

The player does row operations with the **goal** of making the diagonal entries of the coefficient matrix 1's and making as many of the other numbers zero, if possible. Here are the row operations for this example:

$$\left[\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 2 & -3 & 4 & -2 \\ -2 & -1 & 1 & 7 \end{array}\right] \xrightarrow{R_2-2R_1\to R_2} \left[\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ \mathbf{0} & \mathbf{-1} & \mathbf{2} & \mathbf{-2} \\ -2 & -1 & 1 & 7 \end{array}\right] \xrightarrow{R_3+2R_1\to R_3} \left[\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 0 & -1 & 2 & -2 \\ \mathbf{0} & \mathbf{-3} & \mathbf{3} & \mathbf{7} \end{array}\right]$$

$$\xrightarrow{R_1-R_2\to R_1} \left[\begin{array}{ccc|c} \mathbf{1} & \mathbf{0} & \mathbf{-1} & \mathbf{2} \\ 0 & -1 & 2 & -2 \\ 0 & -3 & 3 & 7 \end{array}\right] \xrightarrow{R_3-3R_2\to R_3} \left[\begin{array}{ccc|c} 1 & 0 & -1 & 2 \\ 0 & -1 & 2 & -2 \\ \mathbf{0} & \mathbf{0} & \mathbf{-3} & \mathbf{13} \end{array}\right] \xrightarrow{(-1)\cdot R_2\to R_2} \left[\begin{array}{ccc|c} 1 & 0 & -1 & 2 \\ \mathbf{0} & \mathbf{1} & \mathbf{-2} & \mathbf{2} \\ 0 & 0 & -3 & 13 \end{array}\right]$$

$$\xrightarrow{(-\frac{1}{3})\cdot R_3\to R_3} \left[\begin{array}{ccc|c} 1 & 0 & -1 & 2 \\ 0 & 1 & -2 & 2 \\ \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{-13/3} \end{array}\right] \xrightarrow{R_2+2R_3\to R_2} \left[\begin{array}{ccc|c} 1 & 0 & -1 & 2 \\ \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{-20/3} \\ 0 & 0 & 1 & -13/3 \end{array}\right] \xrightarrow{R_1+R_3\to R_1} \left[\begin{array}{ccc|c} \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{-7/3} \\ 0 & 1 & 0 & -20/3 \\ 0 & 0 & 1 & -13/3 \end{array}\right]$$

We now convert the augmented matrix back to a system of linear equations:

$$\begin{cases} 1x - 0y + 0z = -\ 7/3 \\ 0x - 1y + 0z = -20/3 \\ 0x - 0y + 1z = -13/3 \end{cases}$$

or more simply,

$$\begin{aligned} x &= -7/3 \\ y &= -20/3 \\ z &= -13/3 \end{aligned}$$

We can check that this is a solution to the original system of equations (8).

4.2 The goal when doing row reduction

In the previoux example, we used row reduction Gauss-Jordan elimination to solve a linear system. The example we did had a unique solution. When that happens we can reduce the coefficient matrix to a matrix like

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

possibly with one or more rows of zeros at the bottom. (It may be larger or smaller depending on the number of equations and variables).

But in general, as we've seen, a linear system either has (1) one solution, (2) no solutions, or (3) infinitely many solutions. And if cases (2) or (3) happen, then we won't be able to do that. So we need to relax our "goal" when doing row reduction.

Our new goal is to reduce the coefficient matrix to ***reduced row-echelon form***, which in the case of a linear system with three equations and three variables, means it should look like one of these

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} 1 & 0 & \# \\ 0 & 1 & \# \\ 0 & 0 & 0 \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} 1 & \# & \# \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

where # denotes any arbitrary number.

Definition 11. More precisely, a coefficient matrix

$$\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}.$$

is said to be in ***reduced row echelon form*** if

1. Any rows of zeros appear at the bottom
2. The leftmost nonzero entry of all other rows equals 1 (the "leading 1's")
3. Each leading 1 of a nonzero row appears to the right of the leading row above it
4. All the other entries of a column containing a leading 1 are zero

This definition is *general*: it applies to any system with m equations and n variables. The pattern will become natural once you've worked a few (dozen?) examples.

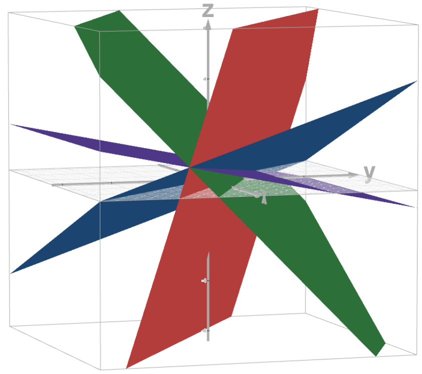
4.3 Using row reduction to solve a system with infitely many solutions

Here's an example which shows what happens when we try to solve a linear system with infinitely many solutions:

Example 12 (Using row reduction to solve a system with infitely many solutions)**.** We wish to solve the system

$$\begin{cases} 2x + 3y - z = 3 \\ -x - y + 3z = 0 \\ x + 2y + 2z = 3 \\ y + 5z = 3 \end{cases}$$

This system has 4 equations and 3 variables. Each equation represents a plane. The solutions, if there are any, will be the intersection of these four planes. I've plotted the planes in Desmos:



It looks like the four planes intersect in a line. So we should expect infinitely many solutions.

Step 1. Write down the augmented matrix of the system.

$$\left[\begin{array}{ccc|c} 2 & 3 & -1 & 3 \\ -1 & -1 & 3 & 0 \\ 1 & 2 & 2 & 3 \\ 0 & 1 & 5 & 3 \end{array}\right]$$

Step 2. Do some combination or row reduction steps until the coefficient matrix is in reduced row echelon form. (This is Example 3 in the textbook, refer there for the steps.)

Step 3. Our matrix is now in reduced row echelon form:

$$\left[\begin{array}{ccc|c} 1 & 0 & -8 & -3 \\ 0 & 1 & 5 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Let's convert it back into a linear system of equations:

$$\begin{aligned} x - 8z &= -3 \\ y + 5z &= 3 \\ 0 &= 0 \\ 0 &= 0 \end{aligned}$$

Therefore, we have

$$\begin{cases} x = -3 + 8z \\ y = 3 - 5z \end{cases} \tag{9}$$

where z is any real number. There are no restrictions on the value of z. Any choice of z gives us a valid solution to our original system of equations. If $z = 0$, we have the solution $(x, y, z) = (-3, 3, 0)$. If $z = 1$, then we have the solution $(x, y, z) = (5, -2, 1)$, and so forth.

Interpretation: In this case, z is called a ***free variable*** and x and y are called ***dependent variables***. The solutions to the original linear system consist of all points on a line which cuts through 3-dimensional space $\mathbb{R}^3$. Eq. (9) gives us a parametric equation of the line. The set of solutions is 1-dimensional, because it is a line.

End of Example 12. $\square$

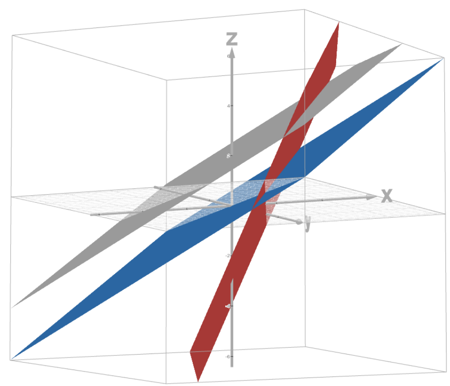
We've covered two of the three cases. For the last case, we consider the question *what happens if we attempt to solve a linear system that has no solutions?*

4.4 Using row reduction to attempt to solve a system with no solutions

Example 13 (Using row reduction to attempt to solve a system with no solutions)**.** Suppose we wish to solve the system

$$\begin{aligned} 2x + y - z &= 3 \\ -x - y + 2z &= 0 \\ -x - y + 2z &= 4 \end{aligned}$$

Here's a plot of the planes:



Their intersection if the three planes is *empty*: there is no point which lies on all three planes. So this system has no solution. What happens when we try to use row reduction?

Step 1. Write down the augmented matrix:

$$\left[\begin{array}{ccc|c} 2 & 1 & -1 & 3 \\ -1 & -1 & 2 & 0 \\ -1 & -1 & 2 & 4 \end{array}\right]$$

Step 2. Do row reduction to get to reduced row echelon form (I'm skipping steps here):

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 3 \\ 1 & -1 & 3 & 3 \\ 0 & 0 & 0 & 4 \end{array}\right]$$

Step 3. Convert back to a linear system:

$$\begin{cases} x \quad\quad + z = 3 \\ x - y + 3z = 3 \\ 0 \quad\quad\quad\;\; = 4 \end{cases}$$

The last equation is never true, no matter what values we choose for x, y and z. So the system has no solution

End of Example 13. $\square$

4.5 Notes and resources for learning row reduction

NOTE: I'm not going to do a lot of examples of row reduction because it's so time consuming that it's not a great use of lecture time. So I expect you to teach yourself how to do row reduction. If you want to see more examples, some good online videos are

`https://www.youtube.com/watch?v=OP2aQUOevhI`
`https://www.youtube.com/watch?v=eDb6iugi6Uk`

There are also some nice online tools to help with row reduction, for example:

`https://textbooks.math.gatech.edu/ila/demos/rrinter.html`
`https://www.math.odu.edu/~bogacki/cgi-bin/lat.cgi?c=roc`

General advice:

1. Of course I'm going to ask you to row-reduce matrices on your exams, and for that you'll need to be able to do it by hand, without a calculator.
2. Write down all your steps, including a new matrix at every step in an organized way.
3. Use the notation $R_i \leftrightarrow R_j$ to indicate a swap of rows i and j; $cR_i \to R_i$ to indicate a multiplication of row i by a constant c; and, $R_i + cR_j \to R_i$ to indicate that you've added c times row j to row i.
4. Work left to right, top to bottom. Start by making the top left entry 1. Then use it to make all the numbers below it zero. Then go to the second column, second row, and make that 1. Then make everything beneath it zero. Etc.

5 2025-09-05 | Week 02 | Lecture 05

> ***The nexus question of this lecture:*** *What is a matrix, and what are the fundamental algebraic operations we can do with it?*

This lecture is based on section 1.2 in the textbook.

5.1 Matrices and matrix notation

A ***matrix*** is a rectangular array of objects, usually numbers, which are called ***entries***. If the number of rows and the number of columns are equal, the matrix is said to be a ***square matrix***.

For example,

$$\begin{bmatrix} 1 & 0 & 3 \\ 2 & 5 & -3 \end{bmatrix} \quad \text{or} \quad \underbrace{\begin{bmatrix} 1 & 0 & -7 & 5 \end{bmatrix}}_{\text{a row vector}} \quad \text{or} \quad \underbrace{\begin{bmatrix} 1 \\ 4 \\ 0 \end{bmatrix}}_{\text{a column vector}} \quad \text{or} \quad \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

A matrix with m rows and n columns takes the form

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}.$$

Such a matrix is said to be an $m \times n$ ***matrix***. The pair (m, n) is called the ***dimensions*** of the matrix (i.e., the number of rows and number of columns). If $m = n$, then the matrix is said to be a ***square matrix***.

The set of all $m \times n$ matrices with real entries is denoted

$$M_{m \times n}(\mathbb{R})$$

For example, in set notation

$$M_{2\times 3}(\mathbb{R}) = \left\{ \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} : a_{11}, a_{12}, a_{13}, a_{21}, a_{22}, a_{23} \in \mathbb{R} \right\}$$

Notation 14. It is common to denote a matrix compactly using the notation

$$A = [a_{ij}] \quad \text{or} \quad A = (a_{ij})$$

To denote the entry at row i, column j, we write either

$$\text{ent}_{ij}(A) \quad \text{or more simply,} \quad a_{ij}$$

For example, if

$$A = (a_{ij}) = \begin{bmatrix} -1 & 2 & 1 \\ 5 & 4 & -9 \\ 3 & -4 & 7 \end{bmatrix}$$

then $a_{23} = \text{ent}_{23}(A) = -9$, and $a_{21} = \text{ent}_{21} = 5$.

Vectors are a special case of matrices. An n-dimensonal vector is an $n \times 1$ matrices (a column matrix, typically).

5.2 Matrix operations: scaling, addition and multiplication

We can ***multiply matrices by a scalar***, in the obvious way: $10 \times \begin{bmatrix} 3 & 4 \\ -2 & 0 \end{bmatrix} = \begin{bmatrix} 30 & 40 \\ -20 & 0 \end{bmatrix}$.

We can ***add matrices***, also in the obvious way:

$$\begin{bmatrix} 3 & 4 \\ -2 & 0 \end{bmatrix} + \begin{bmatrix} 1 & 5 \\ 2 & 10 \end{bmatrix} = \begin{bmatrix} 4 & 9 \\ 0 & 10 \end{bmatrix}.$$

But note that we can only add two matrices if they have the same dimensons:

$$\begin{bmatrix} 5 & 2 \\ -2 & -1 \end{bmatrix} + \begin{bmatrix} -1 & 10 \\ 4 & 7 \\ 2 & 5 \end{bmatrix} = \text{ undefined.}$$

We can also **multiply** matrices, but before defining matrix multiplication, it will be helpful to recall the notion of dot product. Suppose we have two vectors of the same length:

$$X = \begin{bmatrix} x_1 & x_2 & \cdots & x_n \end{bmatrix} \quad \text{and} \quad Y = \begin{bmatrix} y_1 & y_2 & \cdots & y_n \end{bmatrix},$$

then the ***dot product*** of X and Y, denoted $X \cdot Y$, is

$$X \cdot Y = x_1y_1 + x_2y_2 + \ldots + x_ny_n = \sum_{i=1}^{n} x_iy_i.$$

Matrix multiplication is defined in the following way:

Matrix Multiplication

If $A = (a_{ij})$ is a $p \times n$ matrix and $B = (b_{ij})$ is an $n \times q$ matrix, then we can can think of A and B as

$$A = \begin{bmatrix} - A_1 - \\ - A_2 - \\ \vdots \\ - A_p - \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} | & | & & | \\ B_1 & B_2 & \ldots & B_q \\ | & | & & | \end{bmatrix}$$

where $A_1, \ldots, A_p$ are the $1 \times n$ row vectors

$$A_i = \begin{bmatrix} a_{i1} & a_{i2} & \cdots & a_{in} \end{bmatrix} \quad (i = 1, \ldots, p)$$

and $B_1, \ldots, B_q$ are $n \times 1$ column vectors:

$$B_j = \begin{bmatrix} b_{1j} \\ b_{2j} \\ \vdots \\ b_{nj} \end{bmatrix} \quad (j = 1, 2, \ldots, q)$$

Then the product $P = AB$ is a $p \times q$ matrix $P = (p_{ij})$, whose entries are

$$\begin{aligned} p_{ij} &= A_i \cdot B_j \\ &= \sum_{k=1}^{n} a_{ik}b_{kj}. \end{aligned}$$

For example,

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix} = \begin{bmatrix} 1 \cdot 5 + 2 \cdot 7 & 1 \cdot 6 + 2 \cdot 8 \\ 3 \cdot 5 + 4 \cdot 7 & 3 \cdot 6 + 4 \cdot 8 \end{bmatrix} = \begin{bmatrix} 19 & 22 \\ 43 & 50 \end{bmatrix}$$

Dimensionality requirement for matrix multiplication: we can only multiply two matrices if their dimensions match in the right way. If A is a $p \times n$ matrix, and B is an $\tilde{n} \times q$ matrix, then the product AB is defined if and only if $n = \tilde{n}$. That is,

$$AB \text{ is } \begin{cases} \text{a } p \times q \text{ matrix} & \text{if} \quad n = \tilde{n} \\ \text{undefined} & \text{if} \quad n \neq \tilde{n} \end{cases}$$

Properties of matrix multiplication: Perhaps surprisingly, despite matrix multiplication's complicated definition, it nonetheless behaves sort of like regular multiplication in that the ***associative property*** and ***distributed property*** both hold. That is,

$$ABC = (AB)C = A(BC) \quad \text{(associative property)}$$

and

$$\begin{aligned} A(B+C) &= AB + BC \quad \text{(left distributive property)} \\ (B+C)A &= BA + CA \quad \text{(right distributive property)} \end{aligned}$$

It's not obvious why the properties always hold; they require proof. For now, we will take them as a given.

5.3 Special classes of matrices

5.3.1 Diagonal matrices

If $A = (a_{ij})$ is a square (i.e., $n \times n$) matrix, then the entries $a_{11}, a_{22}, a_{33}, \ldots, a_{nn}$ are called the ***diagonal entries***. A square matrix of the form

$$A = \begin{bmatrix} a_{11} & 0 & 0 & \ldots & 0 \\ 0 & a_{22} & 0 & \ldots & 0 \\ 0 & 0 & a_{33} & \ldots & 0 \\ \vdots & \vdots & & \ddots & \\ 0 & 0 & 0 & \ldots & a_{nn} \end{bmatrix}.$$

is called a ***diagonal*** matrix, and the notation used for such matrices is $A = \operatorname{diag}(a_{11}, \ldots, a_{nn})$.

5.3.2 Identity matrix

An ***identity matrix*** is a diagonal matrix with 1's on its diagonal (and 0's everywhere else). For example,

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \text{et cetera}$$

Identity matrices are always square. The notation for an $n \times n$ identity matrix is I_n, or more simply I.

An identity matrix has the property that when you multiply it by another matrix, it doesn't change the other matrix. For example,

$$\begin{bmatrix} 5 & 6 \\ 2 & 7 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 5 & 6 \\ 2 & 7 \end{bmatrix}$$

It's like multiplying a number by 1.

5.3.3 Zero matrices

A matrix whose entries are all zeros is called a ***zero matrix***, like

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{et cetera}$$

The book uses the notation $\mathbf{O}_{m \times n}$ to denote an $m \times n$ zero matrix, or sometimes even just $\mathbf{O}$.

6 2025-09-08 | Week 03 | Lecture 06

> ***The nexus question of this lecture:*** *How do we encode a linear system using matrices? And once thus encoded, what can we say about the solutions to the linear system just by looking at the matrix?*

This lecture is based on sections 1.2 and 1.3 of the textbook.

6.1 Inverting matrices

We begin this lecture by introducing the idea of a matrix inverse, which we will use to help answer the main question of the lecture.

Recall that the identity matrix is a matrix of the form

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \text{et cetera}$$

and we usually denote an identity matrix by the letter I_n (or just I if the dimensions are clear from context). Such a matrix behaves like the number 1, in the sense that $AI = A$ and $IA = A$ for any matrix A.

Every nonzero number a has an inverse $a^{-1} = \frac{1}{a}$ such that

$$a \cdot a^{-1} = a^{-1}a = 1.$$

It would natural to conjecture that every nonzero matrix A also has an inverse A^{-1} such that

$$AA^{-1} = A^{-1}A = I$$

In fact, this conjecture is false: we cannot do this with every matrix, but sometimes we can.

To be precise, if two matrices satisfy the property that $AB = BA = I$, then they are said to be ***inverses***. In this case, we write $B = A^{-1}$ (which doesn't mean $1/A$). This is analogous to when we multiply two numbers like $3 \cdot \frac{1}{3} = \frac{1}{3} \cdot 3 = 1$.

Example 15 (Matrix inverses)**.** An example are the matrices

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 5 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} -5 & 2 \\ 3 & -1 \end{bmatrix}$$

Because

$$\begin{aligned} AB &= \begin{bmatrix} 1 & 2 \\ 3 & 5 \end{bmatrix} \begin{bmatrix} -5 & 2 \\ 3 & -1 \end{bmatrix} \\ &= \begin{bmatrix} 1(-5) + 2 \cdot 3 & 1 \cdot 2 + 2(-1) \\ 3(-5) + 5 \cdot 3 & 2 \cdot 2 + 5(-1)) \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\ &= I_2 \end{aligned}$$

And similarly, $BA = I_2$. Therefor $A^{-1} = B$.

End of Example 15. ☐

As noted, not every matrix has an inverse. A matrix that doesn't have an inverse is called ***singular*** or ***noninvertible***. A matrix that has an inverse, is called ***nonsingular*** or ***invertible***. Generally in mathematics, "singular" means "bad".

Moreover, note that it is possible for $AB = I$ but $BA \neq I$, as the following example shows.

Example 16. Let

$$A = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} \quad \text{and} \quad B = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{bmatrix}$$

Then

$$AB = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{but} \quad BA = \frac{1}{2} \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

End of Example 16. □

Incidentally this example illustrates one imporant way that matrix multiplication is not like multiplication of numbers.

6.2 An aside about commutatitivity

Consider two real numbers $a, b \in \mathbb{R}$. Then

$$ab = ba.$$

This property is called ***commutatitivity***. One way that matrix multiplication differs from multiplication of real numbers is that if we consider two matrices A, B, then it is usually not the case that $AB = BA$. Whenever it is the case that $AB = BA$, we say that the matrices A and B ***commute***, but again, this usually doesn't happen. Here are some examples:

Example 17 (Commutativity and noncommutatitivity of matrix multiplication)**.** Let

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & 3 \\ 0 & 2 \end{bmatrix} \quad \text{and} \quad C = \begin{bmatrix} 2 & 0 \\ 3 & 1 \end{bmatrix}.$$

Direct computation shows that

$$AB = BA = \begin{bmatrix} 2 & 5 \\ 0 & 2 \end{bmatrix}.$$

In this case, we say that the matrices A and B commute.

On the other hand, the matrices A and C do not commute because

$$AC = \begin{bmatrix} 5 & 1 \\ 3 & 1 \end{bmatrix} \quad \text{but} \quad CA = \begin{bmatrix} 2 & 2 \\ 3 & 4 \end{bmatrix}$$

so $AC \neq CA$.

End of Example 17. □

6.3 Encoding a linear system via matrix equations

In this section, we answer the first part of the main question of lecture. Consider the 2×2 linear system

$$\begin{aligned} 3x_1 + 7x_2 &= 5 \\ 2x_1 - 6x_2 &= 1 \end{aligned}$$

Using matrix multiplication, we can write this system as

$$\begin{bmatrix} 3 & 7 \\ 2 & -6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 5 \\ 1 \end{bmatrix}$$

In fact, we can do something similar with any linear system, as we now show:

Suppose we have any linear system, like this:

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{21}x_2 + \ldots + a_{2n}x_n = b_2 \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n = b_m \end{cases}$$

Define the three matrices

$$A = \begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \ldots & a_{mn} \end{bmatrix}, \quad X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

Then we can represent the linear system compactly as the matrix equation

$$AX = B.$$

This answers the question *"how can we encode a linear system using matrices?"* (It alsow provides us with the beginnings of an answer to the question *"why is matrix multiplication defined the way it is?"*, which is that matrix multiplication allows us to compactly represent linear systems, though a more compelling answer to this question will come later in the course.)

6.4 What does A tell us about the linear system $AX = B$?

For the remainder of the lecture, we consider the question, *"What can we tell about the solutions to the linear system $AX = B$, just by looking at the matrix A?"* It is not obvious that we should be able to tell anything at all about the solutions just by looking at A—after all, that's, like trying to says something about the solutions to a system of equations *by only looking at the left-hand sides of the equations.*

Observe that if we know A^{-1}, then we can multiply both sides of this equation on the left by

$$A^{-1}AX = A^{-1}B$$

which simplifies to

$$X = A^{-1}B. \tag{10}$$

Thus, X is the unique solution to our original system. This motivates the following theorem:

Theorem 18. *A linear system $AX = B$ with n equations and n variables has exactly one solution if and only if A is invertible.*

This theorem provides a partial answer to the main question of the lecture. What is remarkable is that it says we don't need to know anything at all about the vector B to know whether the system has a unique solution. We only need to know whether A is singular or nonsingular. This hints a deeper structure in linear algebra that we will explore more fully throughout this course.

As an aside, note that when doing computations, it is usually easier to solve a linear system $AX = B$ directly using row reduction, rather than (1) finding the inverse of A and then (2) multiplying both sides by A^{-1}. So the latter approach isn't recommended for solving a system of linear equations. Just use row reduction.

We can actually push our answer to the lecture question a little bit further, by drawing a connection with one of your homework problems (homework 1, problem 3). In that problem, you showed that for the matrix $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$, the linear system

$$AX = B$$

has a unique solution if and only if the determinant of A (defined as $a_{11}a_{22} - a_{12}a_{21}$) is nonzero. That is,

$$\text{determinant of } A \text{ is nonzero} \iff AX = B \text{ has a unique solution}$$

Connecting this with Theorem 18, we have:

$$\text{determinant of } A \text{ is nonzero} \iff AX = B \text{ has a unique solution} \iff A \text{ is invertible}$$

Putting it together, we have the following theorem, which gives us an answer to the main question of the lecture:

Theorem 19. *Let A be an $n \times n$ matrix. Then the following are equivalent:*

(i.) *A is invertible.*

(ii.) *The determinant of A is nonzero.*

(iii.) *For any vector $B \in \mathbb{R}^n$ the linear system $AX = B$ has exactly one solution.*

We haven't proved this theorem for all matrices; we've only shown that it holds for 2×2 matrices. In fact, it holds for all matries, as we'll see later.

7 2025-09-10 | Week 03 | Lecture 07

This lecture isn't really based on any textbook section, but sections 1.5 and 1.6 cover determinants. Please read sections 1.3 and 1.4 for Friday.

> ***The nexus question of this lecture:*** *What is a determinant, geometrically?*

7.1 The determinant of 2×2 matrix

Recall that the determinant of a 2×2 matrix is defined as the quantity

$$\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc.$$

Later we'll see how to define determinants for $n \times n$ matrices, but for this lecture I'm going to focus on the 2×2 case in order to hopefully demonstrate why you should even care about determinants at all.

7.2 A matrix is a transformation of space

Consider the 2×2 matrix

$$A = \begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix}.$$

Preliminaries: Some elementary properties of A

Let us consider what this matrix does to the standard basis vectors $e_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $e_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$:

$$Ae_1 = \begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \end{bmatrix} \quad \text{and} \quad Ae_2 = \begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

In other words, multiplication by A sends the vector e_1 to the vector $\begin{bmatrix} 3 \\ 1 \end{bmatrix}$. It sends the vector e_2 to $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

Let's give these vectors names and compute some values that will be useful later:

$$u = \begin{bmatrix} 3 \\ 1 \end{bmatrix} \quad \text{and} \quad v = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

- The magnitudes of u and v are

$$|u| = \sqrt{3^2 + 1^2} = \sqrt{10} \quad \text{and} \quad |v| = \sqrt{2^2 + 1^2} = \sqrt{5}$$

- The cosine of angle θ between u and v can be computed using the dot product, by the formula

$$u \cdot v = |u||v| \cos(\theta).$$

  Doing the computation, we get

$$\cos(\theta) = \frac{1}{\sqrt{2}}. \tag{11}$$

The key idea

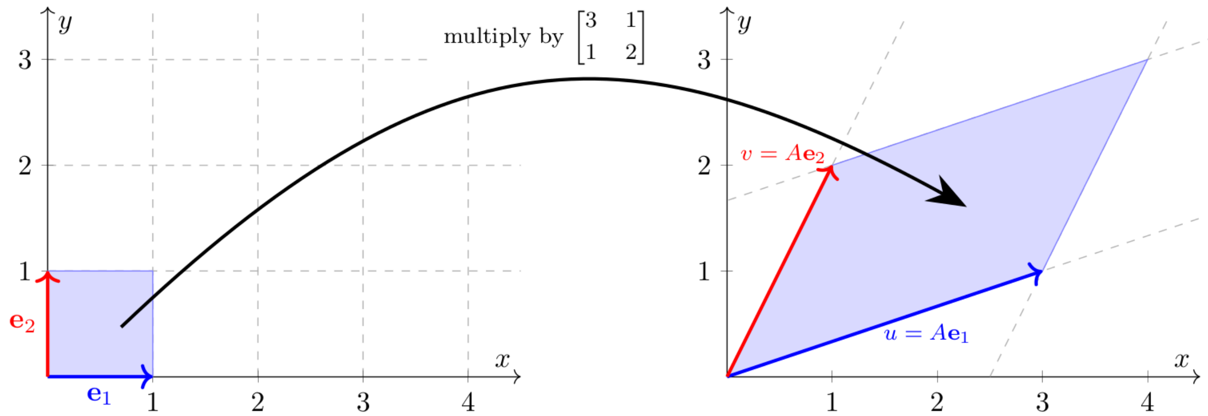
Suppose we decided to multiply *every* vector in the plane by the matrix $\begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix}$. In that case, we could conceive of the matrix transforming space (i.e., the plane) in some way. For a general vector in the plane $\begin{bmatrix} x \\ y \end{bmatrix}$, the action of A is the following:

$$\begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3x + y \\ x + 2y \end{bmatrix}$$

If we use "$\mapsto$" to mean "gets sent to", then we have

$$\begin{bmatrix} x \\ y \end{bmatrix} \mapsto \begin{bmatrix} 3x+y \\ x+2y \end{bmatrix}$$

The following plot shows the effect of multiplying *all* vectors in the unit square by A:

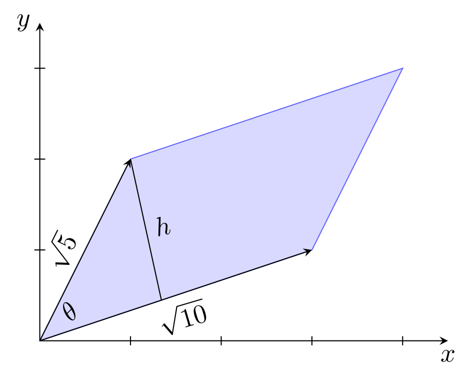


The effect of applying A is to transform space (i.e., the plane) by stretching it in some way. For this particular matrix, the unit square get mapped to the shown parallelogram. Squares adjacent to the unit square get sent to adjacent parallelograms.

7.3 The "volume scaling factor" of a transformation

Question: By what factor does the matrix $\begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix}$ scale the area of a region in space?

Answer: The unit square has area 1. What about the parallelogram?



The parallelogram has area

$$\text{(area of parallelogram)} = \text{(base)} \times \text{(height)} \tag{12}$$

In this case, the base of the parallelogram is $|u|= \sqrt{10}$. To find the height h, use the definition of $\sin(\theta)$:

$$\sin(\theta) = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{h}{\sqrt{5}}.$$

which implies

$$h = \sqrt{5}\sin(\theta)$$

We already know that $\cos(\theta) = \frac{1}{\sqrt{2}}$ by Eq. (11). Therefore

$$\begin{aligned}
\sin^2(\theta) &= 1 - \cos^2(\theta) && (\text{since } \sin^2(\theta) + \cos^2(\theta) = 1) \\
&= 1 - \frac{1}{2} \\
&= \frac{1}{2}.
\end{aligned}$$

Taking square roots, we get $\sin(\theta) = \frac{1}{\sqrt{2}}$. Hence $h = \frac{\sqrt{5}}{\sqrt{2}}$. Therefore, by Eq. (12),

$$(\text{area of parallelogram}) = \left(\sqrt{10}\right)\left(\frac{\sqrt{5}}{\sqrt{2}}\right) = 5.$$

To conclude, the linear transformation of space obtained by multiplying every vector by the matrix $A = \begin{bmatrix} 1 & 3 \\ 2 & 1 \end{bmatrix}$ took a region of area 1 (the unit square) and mapped it to a parallelogram of area 5. The scaling is uniform throughout the plane, so every region of area 1 gets mapped to a region of area 5. In other words, the transformation scales the volume by a factor 5.

Now let's look at the determinant of the matrix:

$$\det \begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix} = 3 \cdot 2 - 1 \cdot 1 = 5$$

The determinant is also 5. This is not a coincidence. **The determinant tells us how much space gets scaled by the linear transformation induced by the matrix.**

7.4 How do we interpret the determinant when it's zero or negative?

The determinant measures how much the linear transformation scales volume. But then what does it mean geometrically for a matrix to have determinant zero or negative?

- If the determinant is zero, that means regions with positive area get mapped to regions of zero area. **This always occurs as a result of dimension collapse.** For example, the following matrix has determinant zero:

  $$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

  Note that

  $$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ 0 \end{bmatrix}$$

  Geometrically, this matrix projects every point in the plane to its location on the x-axis. This means that every point in the plane (a 2-dimensional surface) gets mapped to the x-axis (a 1-dimensional line with zero area). The dimension collapse here is the reduction in dimension from 2D to 1D.

  Another example would be a transformation that projects every point in 3D space onto a specified plane; this is because mapping a 3D region onto a 2D plane destroys volume. An example of such a transformation is obtained from the matrix

  $$P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

  which projects points in 3D space onto the xy-plane. We haven't discussed how to define the determinant for 3×3 matrices, but based on our geometric intuition, we'd expect P to have determinant zero (it does).

- If the determinant is negative, that means the transformation reverses the orientation of space in the same way a mirror changes left and right hands. In 2D, this occurs when the transformation reflects the plane across a line. For example, the matrix

  $$R = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

  reflects the xy-plane across the line $y = x$, but doesn't stretch space at all. The determinant is -1. Under this transformation, regions don't stretch or shrink, but they do get flipped.

  Another example can be obtained if we decided to *combine* two transformations:

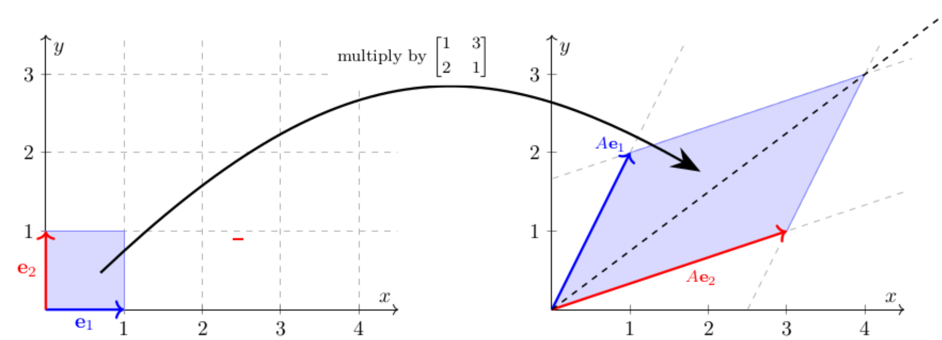
  - **First**, flip the xy-plane using R.
  - **Then**, transform space using the matrix A.

  To combine these two tranformations, we multiply the matrices like this:

  $$AR = \underbrace{\begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix}}_{2^{\text{nd}}} \underbrace{\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}}_{1^{\text{st}}} = \begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix}$$

  This new matrix has determinant -5. It is similar to the matrix A we considered earlier, but the columns are swapped. Geometrically, this matrix transforms space by first reflecting the plane across the line $y = x$ and then doing the same stretchy thing done by the previous matrix.

  The picture is almost the same as the previous one, but note how the red and blue vectors swapped compared to the first picture. The orientation has changed, which is why the determinant is negative.



8 2025-09-12 | Week 03 | Lecture 08

This lecture is based on sections 1.3 in your textbook

> ***The nexus question of this lecture:*** *What are the key properties of the matrix inverse?*

8.1 What is a matrix inverse?

Definition 20 (Matrix Inverse)**.** The matrix A is ***invertible*** if there is a matrix A^{-1} such that

$$A^{-1}A = I \quad \text{and} \quad AA^{-1} = I$$

If so we call A^{-1} the ***inverse*** of A.

Note:

- Invertible matrices are always square. Nonsquare matrices are not invertible.
- Whatever A does, A^{-1} undoes. If A stretches spaces, A^{-1} compresses it back. If A flips space, A^{-1} flips it back. If A rotates spaces, A^{-1} rotates it back, etc.

Theorem 21 (The socks and shoes property)**.** *If A and B are invertible then so is AB, and*

$$(AB)^{-1} = B^{-1}A^{-1}$$

If you put on socks and then shoes, what is the inverse of that? Take off the shoes, and then the socks.
Note that Theorem 21 has two conclusions: first, that products of matrices are invertible, and second the formula $(AB)^{-1} = B^{-1}A^{-1}$.

Proof of Theorem 21. It is sufficient to show that $B^{-1}A^{-1}$ is the inverse of AB, which we can do by direct computation:

$$(AB)(B^{-1}A^{-1}) = A(BB^{-1})A^{-1} = A(I)A^{-1} = AA^{-1} = I.$$

and

$$(B^{-1}A^{-1})(AB) = B^{-1}(A^{-1}A)B = B^{-1}B = I$$

□

Theorem 22. *Suppose that A and B are square matrices such that either $AB = I$ or $BA = I$. Then A is an invertible matrix and $A^{-1} = B$*

I'm going to omit the proof of this theorem, but I will introduce the general framework that that is used to prove it and results like it, because it is based on an important idea.

8.2 Row reduction is multiplication by elementary matrices

Section 1.3 in the textbook provides a framework, based on row reduction, for deducing properties of inverses. Recall that in row reduction, you have three basic operations:

- swapping two rows
- multiplying a row by a number
- adding a multiple of a row to another row.

These operations can all be done using matrix multiplication, using matrices called ***elementary matrices***. Here are some examples:

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} c & d \\ a & b \end{bmatrix} \qquad R_1 \leftrightarrow R_2$$

$$\begin{bmatrix} 5 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 5a & 5b \\ c & d \end{bmatrix} \qquad 5R_1 \to R_1$$

$$\begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} c & d \\ 3a+c & 3b+d \end{bmatrix} \qquad 3R_1 + R_2 \to R_2$$

Fact: Elementary matrices are always invertible, because one can always undo a row operation. Moreover, by Theorem 21 (which tells us that products of invertible matrices are invertible), we also know that products of elementary matrices are also invertible.

What does it all mean? If you can row reduce a matrix A to I, that means there exists some sequence of elementary matrices $E_1, \ldots, E_m$ such that

$$E_1 E_2 E_3 \cdots E_m A = I.$$

Letting $M = E_1 E_2 E_3 \cdots E_m$, we have

$$MA = I.$$

By Theorem 22, this is enough to conclude that A is invertible and

$$A^{-1} = M.$$

We've just proven part of the following theorem:

Theorem 23. *Let A be a square matrix. The following are equivalent:*

(i.) A can be row-reduced to the identity matrix I.

(ii.) A is invertible.

In particular, we've shown that (*i.*) implies (*ii.*). The reverse direction (that (*ii.*) implies (*i.*)) can also be proved using this framework of elementary matrices, but I'd rather spend the time showing an example of how to compute the inverse of a matrix using row reduction.

8.3 Computing matrix inverses using row-reduction

The next example will illustrate a general approach for computing a matrix inverse.

Example 24 (Computing the inverse of a matrix)**.** In the previous lecture, we considered the matrix

$$A = \begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix}$$

which stretches space in some way, scaling area by a factor of 5. Suppose we wish to find A^{-1}.

Let

$$X = \begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{bmatrix}$$

We need X to satisfy

$$AX = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Writing out AX, we have

$$AX = \begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{bmatrix} = \begin{bmatrix} 3x_{11} + x_{21} & 3x_{12} + x_{22} \\ x_{11} + 2x_{21} & x_{12} + 2x_{22} \end{bmatrix} \overset{\text{set}}{=} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

By setting AX equal to I, we have a system of four linear equations in four variables $x_{11}, x_{12}, x_{21}, x_{22}$.

We have a nice way to solve linear systes: row reduction. Moreover, for finding inverses, there is a clever way to do it, as I now show.

Step 1. Set up an augmented matrix of the following form:

$$\left[\begin{array}{cc|cc} 3 & 1 & 1 & 0 \\ 1 & 2 & 0 & 1 \end{array}\right]$$

Step 2. Completely row reduce the matrix until the left part is the identity matrix. (I'm skipping these steps). After doing this for this example, we get:

$$\left[\begin{array}{cc|cc} 1 & 0 & 2/5 & -1/5 \\ 0 & 1 & -1/5 & 3/5 \end{array}\right]$$

Step 3. Draw the conclusion that A^{-1} is the the matrix to the right of the bar:

$$A^{-1} = \begin{bmatrix} 2/5 & -1/5 \\ -1/5 & 3/5 \end{bmatrix}$$

End of Example 24. □

Remark 25. The approach in Example 24 works for larger matrices as well, see Example 1 in Section 1.3 of the textbook (p.31).

Remark 26. Recall that from the previous lecture, we saw that the matrix $A = \begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix}$ corresponds to a transformation of space that somehow stretched space in a way that scaled area by a factor of 5, because $\det(A) = 5$ (see Section 7.3). Since A^{-1} undoes whatever A did, so it must shrink space by a factor of 5. Indeed,

$$\det(A^{-1}) = \left(\frac{2}{5}\right)\left(\frac{3}{5}\right) - \left(-\frac{1}{5}\right)\left(-\frac{1}{5}\right) = \frac{6}{25} - \frac{1}{25} = \frac{1}{5}.$$

This illustrates the following theorem:

Theorem 27. *If A^{-1} exists then*

$$\det(A^{-1}) = \frac{1}{\det(A)}.$$

Now that we know the determinant of a matrix measures how space is scaled by the transformation, the geometric reason why this theorem is true is obvious: the matrix A transforms space in some way, and then A^{-1} undoes that transformation. So we are left with the ***identity transformation***, which doesn't change space at all. So if A scales space by a factor of a then A^{-1} must scale it by a factor of $\frac{1}{a}$. We'll probably give an actual proof later, but that's the idea.

9 2025-09-15 | Week 04 | Lecture 09

This lecture is based on sections 1.5 and 1.6 in the textbook. We are going to skip section 1.7

> ***The nexus question of this lecture:*** *How do we understand (and compute) the determinant, algebraically?*

9.1 Review of the "key theorem" of linear algebra

Theorem 28 (The Key Theorem of Linear Algebra (partial version))**.** *Let A be an $n \times n$ matrix. Then the following are equivalent:*

(i.) A^{-1} *exists (i.e., A is invertible)*

(ii.) $\det A \neq 0$

(iii.) The linear system $AX = B$ has a unique solution for each $B \in \mathbb{R}^n$.

(iv.) A is row equivalent to I

(v.) ...

Property (iv.) says we can row reduce A into I. The term for this is "row equivalence". Precisely, If A and B are matrices, we say that A is ***row equivalent*** to B if there is a sequence of elementary row operations which if applied to A will result in B.

9.2 Definition of the determinant

Consider a square matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix}.$$

Given an entry a_{ij}, the ***minor*** of a_{ij}, denoted M_{ij}, is the matrix obtained from A by deleting row i and column j of A. For example, if

$$A = \begin{bmatrix} a_{11}, a_{12}, a_{13} \\ a_{21}, a_{22}, a_{23} \\ a_{31}, a_{32}, a_{33} \end{bmatrix}$$

then some minors are

$$M_{11} = \begin{bmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{bmatrix}, \quad M_{12} = \begin{bmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{bmatrix}, \quad \text{and} \quad M_{32} = \begin{bmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{bmatrix}.$$

Definition 29 (Determinant)**.** For a 1×1 matrix, $A = [a_{11}]$, we define $\det(A) = 1$. If A is an $n \times n$ matrix with $n \geq 2$, we define the determinant recursively as

$$\det(A) = \sum_{j=1}^{n} (-1)^{j+1} a_{1j} \det(M_{1j}). \tag{13}$$

Note: Note that the determinant is defined only for square matrices.

Geometrically, the determinant is the (signed) volume scaling factor of the transformation of space, which is very useful to keep in mind. Definition 29 is also useful because it allows us to see how to actually compute determinants.

For a 2×2 matrix, Definition 29 simplifies to

$$\det\left(\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}\right) = a_{11}a_{22} - a_{12}a_{21}.$$

To simplify notation, we use vertical bars to denote determinant $|A| := \det(A)$, or something like this:

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} := \det\left(\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}\right).$$

Example 30 (Computing the determinant of a 3×3 matrix)**.**

$$\begin{aligned} \begin{vmatrix} 2 & 3 & -2 \\ -1 & 6 & 3 \\ 4 & -2 & 1 \end{vmatrix} &= 2\begin{vmatrix} 6 & 3 \\ -2 & 1 \end{vmatrix} - 3\begin{vmatrix} -1 & 3 \\ 4 & 1 \end{vmatrix} + (-2)\begin{vmatrix} -1 & 6 \\ 4 & -2 \end{vmatrix} \\ &= 2\left[6 \cdot 1 - 3(-2)\right] - 3\left[(-1)1 - 3 \cdot 4\right] - 2\left[(-1)(-2) - 6 \cdot 4\right] \\ &= 107. \end{aligned}$$

End of Example 30. □

9.3 Computing determinants using cofactor expansions

The formula for the determinant in Definition 29 is called a ***cofactor expansion*** (there are other formulas). A ***cofactor*** of an entry a_{ij} is the quantity

$$C_{ij} = (-1)^{i+j}\det(M_{ij})$$

so our formula in Eq. (13) can be written as

$$\det(A) = \sum_{j=1}^{n} a_{1j}C_{1j}.$$

This is called the ***cofactor expansion about the first row.*** The next theorem tell us that, in fact, we could have picked *any* row or column and done a similar calculation to get the determinant:

Theorem 31 (Cofactor Expansion)**.** *If A is an $n \times n$ matrix with $n \geq 2$, then*

(i.) For any fixed $i = 1, 2, \ldots, n$, we have

$$\det(A) = \sum_{j=1}^{n} a_{ij}C_{ij} \quad \textit{(cofactor expansion about the } i^{\text{th}} \textit{ row)}$$

(ii.) For any fixed $j = 1, 2, \ldots, n$, we have

$$\det(A) = \sum_{i=1}^{n} a_{ij}C_{ij} \quad \textit{(cofactor expansion about the } j^{\text{th}} \textit{ column)}$$

This theorem is proved by induction on n in section 1.7, but the proof is technical, so we'll skip it. Two examples will illustrate this theorem.

Example 32 (Alternative cofactor expansions)**.** Let's compute the determinant of the matrix from Example 30 in two different ways, using Theorem 31:

The cofactor expansion about the third row:

$$\begin{aligned}\begin{vmatrix}2 & 3 & -2\\ -1 & 6 & 3\\ 4 & -2 & 1\end{vmatrix} &= a_{31}C_{31} + a_{32}C_{32} + a_{33}C_{33}\\ &= 4\begin{vmatrix}3 & -2\\ 6 & 3\end{vmatrix} - (-2)\begin{vmatrix}2 & -2\\ -1 & 3\end{vmatrix} + 1\begin{vmatrix}2 & 3\\ -1 & 6\end{vmatrix}\\ &= 107.\end{aligned}$$

The cofactor expansion about the second column:

$$\begin{aligned}\begin{vmatrix}2 & 3 & -2\\ -1 & 6 & 3\\ 4 & -2 & 1\end{vmatrix} &= a_{21}C_{21} + a_{22}C_{22} + a_{32}C_{32}\\ &= -3\begin{vmatrix}-1 & 3\\ 4 & 1\end{vmatrix} + 6\begin{vmatrix}2 & -2\\ 4 & 1\end{vmatrix} - (-2)\begin{vmatrix}2 & -2\\ -1 & 3\end{vmatrix}\\ &= 107.\end{aligned}$$

The -3 at the beginning of this is not a typo. It's -3 rather than 3 because of the -1 introduced by the cofactor C_{21}, which is $C_{21} = (-1)^{2+1}\begin{vmatrix}-1 & 3\\ 4 & 1\end{vmatrix} = -\begin{vmatrix}-1 & 3\\ 4 & 1\end{vmatrix}$.

End of Example 32. □

9.4 Key properties of the determinant

Theorem 33. *If I is an $n \times n$ identity matrix, then* $\det(I) = 1$. *If* $\mathbf{O}$ *is an $n \times n$ zero matrix, then* $\det(\mathbf{O}) = 0$.

I claim that when this theorem is considered geometrically, its truth becomes obvious. Why is it obvious?

- Because the identity matrix I corresponds to the transformation of space that *doesn't change anything*. This is called the ***identity transformation***. This doesn't stretch (or reflect) space at all, and hence areas/volumes are not changed at all. So the determinant, being the volume scaling factor of the transformation, is 1.
- And because the zero matrix $\mathbf{O}$ maps every vector to the vector $\vec{0} = (0, 0, \ldots, 0)$. This means $\mathbf{O}$ collapses the entirety of n-dimensional space into a single point (i.e., the origin), which has dimension zero. The dimensional collapse means that area/volume gets destroyed, and hence $\det(\mathbf{O}) = 0$.

The next theorem says that the determinant "preserves multiplication".

Theorem 34 (The determinant preserves multiplication)**.** *If A and B are $n \times n$ matrices, then*

$$\det(AB) = \det(A)\det(B).$$

The textbook provides a nice algebraic proof of this in terms of the row reduciton framework presented in Section 8.2 [namely, the proof of Theorem 1.24 in section 1.6 (p.52-53), which I encourage you to read]. But I claim that this theorem is *obvious* when its geometric meaning is considered (i.e., in terms of matrices as transformations of space). We'll start with this idea next lecture.

10 2025-09-17 | Week 04 | Lecture 10

This lecture is based on sections 1.5 and 1.6 in the textbook.

> ***The nexus question of this lecture:*** *What are some useful connections between the geometric and algebraic interpretations of the the determinant?*

Recall that for $A \in M_{n\times n}$, we have

$$\det(A) = \begin{pmatrix}\text{the (signed) volume scaling} \\ \text{factor of the transformation}\end{pmatrix} = \sum_{j=1}^{n} a_{1j}(-1)^{1+j}\det(M_{1j})$$

These are the geometric interpretation (left) and algebraic interpretation (right) of the determinant.

10.1 Determinants preserve multiplication

Theorem 35 (The determinant preserves multiplication)**.** *If A and B are $n \times n$ matrices, then*

$$\det(AB) = \det(A)\det(B).$$

Key background - Matrices as tranformations: The geometric idea of understanding matrices as transformations of space makes this theorem obvious. Let $P = AB$. The transformation of space given by P is

- first, do the transformation of B
- then, do the transformation of A.

Why is it in this order? To see why, let $P = AB$, and consider how P acts on a vector X:

$$PX = (AB)X = A(BX)$$

The placement of the parentheses means we first transform X with B. Then, whatever we get from that, we transform with A. In other words, the product P acts on X by first applying B and then applying A.

In other words, matrix multiplication can be understood as *function composition* of the transformations of A and B.

Explanation of why Theorem 35 is true: When two transformations are composed (by multiplying the matrices), the total scaling is the product of the scalings of each transformation. If B scales volume by a factor of 2, and A scales it again by a factor of 5, then the final scaling induced by $P = AB$ will be 10. In symbols

$$\det(AB) = 10 = 5 \cdot 2 = \det(A)\det(B).$$

The idea is similar when the negative determinants (i.e., corresponding to transformations which include some sort of reflection) are used.

One consequence of Theorem 35 is the following relationship:

Theorem 36 (Determinant of an inverse)**.** *If A is invertible, then*

$$\det(A^{-1}) = \frac{1}{\det(A)}$$

Proof.

$$\begin{aligned}
\det(A)\det(A^{-1}) &= \det(AA^{-1}) && \text{(by Theorem 35)} \\
&= \det(I) && \text{(since } AA^{-1} = I\text{)} \\
&= 1 && \text{(since the determinant of the identity matrix is always 1).}
\end{aligned}$$

Dividing both sides by $\det(A^{-1})$ gives the result. ☐

10.2 Determinants and dimension collapse

The next theorem describes a class of matrices which have dimension zero because they collapse space:

Theorem 37 (Zero row/column)**.** *Let A be an $n \times n$ matrix. If A has a row of zeros (or a column of zeroes), then the determinant is zero.*

Proof. Suppose that the k^{th} row of A has only zero entries. That means

$$a_{k1} = 0, \quad a_{k2} = 0, \quad a_{k3} = 0, \quad \ldots \quad \text{and} \quad a_{kn} = 0$$

By Theorem 31(i.),

$$\det(A) = \sum_{j=1}^{n} a_{ij} C_{ij}$$

for any choice of $j \in \{1, 2, \ldots, n\}$. Picking $i = k$, we get

$$\begin{aligned} \det(A) &= \sum_{j=1}^{n} a_{kj} C_{kj} \\ &= \sum_{j=1}^{n} (0) C_{kj} \\ &= 0. \end{aligned}$$

The proof for the case where A has a column of zero entries is similar. □

To see why Theorem 37 is true geometrically, it suffices to consider an example

Example 38 (A row of zeroes implies zero determinant)**.**

$$A = \begin{bmatrix} 4 & 5 & -1 \\ 6 & 2 & 3 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 4 & 5 & -1 \\ 6 & 2 & 3 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4x + 5y - z \\ 6x + 2y + 3z \\ 0 \end{bmatrix}$$

As a transformation of space, this matrix sends every point $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ to another point of the form $\begin{bmatrix} * \\ * \\ 0 \end{bmatrix}$. Therefore every point gets mapped to a point in the set

$$\left\{(x, y, z) \in \mathbb{R}^3 : z = 0\right\}$$

This is the plane $z = 0$. So the matrix A effectuates a dimension collapse, from 3 dimensonal space into to a 2-dimensional plane. This destroys volume, so $\det(A) = 0$.

End of Example 38. □

Example 39 (Remark on Theorem 37)**.** Of course, not every matrix that has determinant zero has a row or column of zeroes. For example, one can check that

$$\begin{vmatrix} 0 & 1 & 2 \\ 3 & 1 & 2 \\ 5 & 2 & 4 \end{vmatrix} = 0.$$

Yet this matrix doesn't have a row or column of zeroes. Notice that the second and third columns are colinear: one is a scalar multiple of the other. This has something to do with it.

End of Example 39. □

10.3 Determinants and axis-aligned stretching

A special case of matrices are those which ***triangular***. A matrix is said to be ***upper triangular***, if all entries below the main diagonal are equal to zero. Like this:

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 5 & 6 & 7 \\ 0 & 0 & 8 & 9 \\ 0 & 0 & 0 & 10 \end{bmatrix}$$

Similarly, a matrix is said to be ***lower triangular*** if all the entries above the main diagonal are zero.

Triangular matrices always correspond to some combination of the following two transformations:

- stretch space in the direction of one or more the coordinate axes (e.g., in the direction of the x-axis or y-axis or z-axis, etc.)
- possibly also one or more "shear transforms"

Shear transformations don't stretch space at all, so the determinant of a triangular matrix is determined only by how much it stretches space in the directions of the coordinate axes x, y, and z directions. And it turns out, these stretchings are easy to see just by looking at the matrix.

Here's an example. The matrix

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{bmatrix}$$

stretches space in the x direction by 1, stretches spaces in the y direction by 4, and stretches space in the z direction by 6. The overall stretching of volume is therefore $24 = 1 \cdot 4 \cdot 6$. So

$$\det(A) = 1 \cdot 4 \cdot 6 = 24.$$

This is the idea behind the following theorem:

Theorem 40 (Determinants of triangular matrices)**.** *Let $T = (t_{ij}) \in M_{n\times n}(\mathbb{R})$ be a triangular matrix . Then*

$$\det(T) = t_{11}t_{22}\cdots t_{nn}.$$

In words, the determinant equals the product of the diagonal entries.

This theorem can be proved by induction on n.

11 2025-09-19 | Week 04 | Lecture 11

This lecture is based on sections 2.1 and 2.2 in the textbook.

> ***The nexus question of this lecture:*** *What are the essential properties of vectors?*

So far we have defined a "vector" as any $n \times 1$ column matrix

$$x = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$$

whose entries are real numbers. We can think of the vector x in several ways

- as a data structure consisting of an ordered list of entries
- as an arrow in space, possessing both magnitude and direction (e.g., a velocity)
- as a "point" in the n-dimensional space $\mathbb{R}^n$.

In some sense, these different notions are just superficially different ways of thinking about the same fundamental underlying mathematical object.

So what are the *essential* properties of vectors?

11.1 The criteria for vectorhood

Thinking abstractly, I propose that any definition of vectors should capture three key properties:

1. **Vector addition:** we have to be able to add vectors together, which always gives us another vector (as opposed to some other sort of mathematical object)

$$\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} = \begin{bmatrix} x_1 + y_1 \\ \vdots \\ x_n + y_n \end{bmatrix}$$

2. **Scalar multiplication:** We can *scale* vectors, by multiplying by a scalar $c \in \mathbb{R}$

$$c \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} cx_1 \\ \vdots \\ cx_n \end{bmatrix}$$

   and the act of scaling a vector always returns a vector.

3. **The algebra:** When doing algebra with vectors, the algebra should behave mostly how we'd expect (e.g., if u, v are vectors then $u + v = v + u$, and if c is a scalar then $c(u + v) = cu + cv$, etc.). The exception is that you don't need to be able to multiply or divide vectors.

We'll offer a more precise definition later, but in some sense, these three things are really all it should take to define a "vector". But understanding vectors in this way opens up a lot of room for things that don't necessarily "look like" the vectors in $\mathbb{R}^n$, but nonetheless behave precisely in the ways that vectors should behave.

11.2 Some new examples of vectors

Example 41 (The infinite-dimensional vector space of sequences)**.**

$$\mathbb{R}^{\mathbb{N}} := \{(a_1, a_2, a_3, \ldots) : a_1, a_2, \ldots \in \mathbb{R}\}$$

Here, the "vectors" are actually infinite sequences, like

$$(1, 1/2, 1/3, 1/4, \ldots) \quad \text{or} \quad (0, 1, 0, 1, 0, 1, \ldots) \quad \text{or} \quad (0, 0, 0, 0, 0, \ldots) \quad \text{or} \quad (1, 2, 4, 8, 16, 32, \ldots)$$

Of course we can add two sequences

$$(a_1, a_2, a_3, \ldots) + (b_1, b_2, b_3, \ldots) = (a_1 + b_1, a_2 + b_2, a_3 + b_3, \ldots)$$

and we can scale them

$$c(a_1, a_2, \ldots) = (ca_1, ca_2, \ldots)$$

and there's no reason to think that arithmetic would behave differently with infinite sequences than with finite column vectors.

While $\mathbb{R}^n$ is n-dimensional space with vectors of length n, $\mathbb{R}^{\mathbb{N}}$ is infinite-dimensional with vectors which are infinitely long.

End of Example 41. ☐

Polynomials seem to fit the same criteria for vectorhood as well:

Example 42 (Polynomials of degree at most 2)**.**

$$\mathbb{R}[x]_{\leq 2} = \left\{a_0 + a_1 x + a_2 x^2 : a_0, \ldots, a_2 \in \mathbb{R} \text{ and } x \text{ is an symbolic variable}\right\}$$

Here, the "vectors" are polynomials like

$$1 + x + 3x^2 \quad \text{or} \quad x - x^2 \quad \text{or} \quad -1 + x.$$

Here, we can add the "vectors" of this set in the usual way. If

$$u = a_0 + a_1 x + a_2 x^2 \quad \text{and} \quad v = b_0 + b_1 x + b_2 x^2,$$

Then the vector $u + v$ is

$$(a_0 + a_0) + (a_1 + a_1)x + (a_2 + a_2)x^2$$

which is again a polynomial.

We also have a notion of scalar multiplication. If $c \in \mathbb{R}$, then

$$\begin{aligned} cu &= c\left(a_0 + a_1 x + a_2 x^2\right) \\ &= ca_0 + ca_1 x + ca_2 x^2 \end{aligned}$$

which is still a polynomial of degree at most 2.

Question: What is the dimension of the "space" $\mathbb{R}[x]_{\leq 2}$?

To specify an element of $\mathbb{R}[x]_{\leq 2}$, which takes the form

$$a_0 + a_1 x + a_2 x^2,$$

we need to know three numbers: a_0, a_1, a_2. So there are three independent variables, meaning the dimension is 3. We can see this by "rewriting" the elements of $\mathbb{R}[x]_{\leq 2}$ as

$$a_0 + a_1 x + a_2 x^2 \quad \leftrightarrow \quad \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix}$$

in which it becomes clear that the dimension really is 3.

We can define $\mathbb{R}[x]_{\leq n}$ similarly to $\mathbb{R}[x]_{\leq 2}$, as

$$\mathbb{R}[x]_{\leq n} = \left\{a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n : a_0, a_1, \ldots, a_n \in \mathbb{R} \text{ and } x \text{ is a symbolic variable}\right\}$$

in which case the dimension is $n + 1$ (because you need to specify $n + 1$ coefficients).

End of Example 42. ☐

But why stop at polynomials? We can regard functions as "vectors" too!

Example 43 (Real-valued continuous functions on the closed unit interval)**.** Let

$$C[0,1] = \{f : [0,1] \to \mathbb{R} \mid f \text{ is continuous}\}$$

For example, this includes functions like

$$\sin(x), \quad e^x, \quad 4x^2 - 1, \quad 4$$

Here "vector addition" is defined in the usual way of adding functions (i.e., $f+g$ is the function $f(x)+g(x)$), and "scalar multiplication" as well (i.e., af is the function $af(x)$).
Question: What is the dimension of $C[0,1]$?

Consider that $\mathbb{R}[x]_{\leq 2} \subseteq C[0,1]$ (since polynomials are continuous). Therefore, $C[0,1]$ contains a set of dimension 2, so its dimension is at least 2.

Similarly, for every positive integer n (no matter how large), we have $\mathbb{R}[x]_{\leq n} \subseteq C[0,1]$. Hence, $C[0,1]$ contains a set of dimension at least $n+1$, for every $n \geq 1$. So $C[0,1]$ must be infinite-dimensional.

End of Example 43. □

Example 44 (Possible velocities of a particle)**.** Consider the set of possible velocities of an electron in space. This is a vector space. Clearly this is 3-dimensional (assuming space itself is 3-dimensional).

But it's not immediately clear how to represent these velocity vectors in the form

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

since in real life, space doesn't have coordinate axis.

End of Example 44. □

Although these examples don't necessarily "look like" sets of vectors (at least superficially), they all share the common essential structure embodied in properties 1,2, and 3.

We can finally define our subject matter:

Linear algebra is the study of finite dimensional "spaces" of vectors.

12 2025-09-22 | Week 05 | Lecture 12

This lecture is based on sections 2.1 and 2.2 in the textbook.

> ***The nexus question of this lecture:*** *What is a vector space?*

12.1 Vector spaces

Definition 45 (Vector Space)**.** A set V is called a ***vector space*** if there are operations called "vector addition" and "scalar multiplication" on V such that the following 2 "closure properties" properties hold

C1. $u + v \in V$ whenever $u, v \in V$ ("V is closed under vector addition")

C2. $cv \in V$ whenever $c \in \mathbb{R}$ and $v \in V$ ("V is closed under scalar multiplication")

and the following 8 algebraic properties hold:

A1. $u + v = v + u$ for all $u, v \in V$ ("vector addition is commutative")

A2. $u + (v + w) = (u + v) + w$ for all $u, v, w \in V$ ("vector addition is associative")

A3. There is a "zero vector" $\vec{0} \in V$ such that $v + \vec{0} = v$ for all $v \in V$ ("there is a zero vector")

A4. For each $v \in V$ there is an element $-v$ such that $v + (-v) = 0$ ("every vector has an additive inverse")

A5. $c(u + v) = cu + cv$ for all $c \in \mathbb{R}$ and all $u, v \in V$ ("scalar multiplication distributes over vector addition")

A6. $(c + d)v = cv + dv$ for all $c, d \in \mathbb{R}$ and all $v \in V$ ("scalar multiplication distributes over scalar addition")

A7. $c(dv) = (cd)v$ for all $c, d \in \mathbb{R}$ and all $v \in V$ ("scalar multiplication is associative")

A8. $1 \cdot v = v$ for all $v \in V$. ("multiplying a vector by one doesn't change it.")

These 10 properties are called ***the vector space axioms.*** The elements of V are called ***vectors***.

Note that C1 and C2 just say that (1) the sum of two vectors is itself a vector, and (2) the act of scaling a vector returns a vector. Axioms A1-A8 say that the usual rules of algebra apply.

At heart, this is just a list of the essential properties of vectors that we are familiar with.

Example 46 (The n-dimensional vector space $\mathbb{R}^n$)**.** We regard $\mathbb{R}^n$ as the set of $n \times 1$ column vectors with real entries:

$$\mathbb{R}^n = \left\{ \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} : x_1, \dots, x_n \in \mathbb{R} \right\}.$$

The "vector addition" is defined as *entrywise addition*

$$\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} := \begin{bmatrix} x_1 + y_1 \\ \vdots \\ x_n + y_n \end{bmatrix}$$

and the "scalar multiplication" is defined for every real number c as

$$c \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} := \begin{bmatrix} cx_1 \\ \vdots \\ cx_n \end{bmatrix}$$

To check that $\mathbb{R}^3$ is a vector space, we have to verify each of the 8 vector space axioms.

End of Example 46. □

12.2 Properties of Vector Spaces

Theorem 47 (Basic properties of vector spaces)**.** *Let V be a vector space. Then*

- *The zero vector $\vec{0} \in V$ is unique.*
- *If $u + v = \vec{0}$ then $u = -v$ (i.e., the negative of v is unique.)*
- *For any $v \in V$, $0v = \vec{0}$.*
- *For any real number c, $c\vec{0} = \vec{0}$*
- *For any $v \in V$, $(-1)v = -v$.*

Note about the proof of Theorem 47. The proofs of these properties are not so important, but it is worth thinking carefully about *why* basic properties like these need to be proved: while these properties might seem "obvious" for $\mathbb{R}^n$, general vector spaces may look very different from $\mathbb{R}^n$. See text for details. □

12.3 Subspaces

Vector spaces can have smaller vector spaces sitting inside them.

Definition 48 (Subspace)**.** A subset W of a vector space V is called a ***subspace*** of V if W is itself a vector space under the same operations of vector addition and scalar multiplication used by V .

We didn't really need to check all 8 axioms to verify that W is a subspace V. The important criteria to check are summarized in the following theorem

Theorem 49. *Let W be a nonempty subset of a vector space V. Then W is a subspace iff the following conditions are satisfied:*

(i.) $u + v \in W$ *whenever* $u, v \in V$.

(ii.) $cu \in W$ *whenever* $c \in \mathbb{R}$ *and* $u \in W$.

Proof sketch. The proof consists of checking that W satisfies the vector space axioms. Since $W \subseteq V$, most of them are satisfied automatically becuase they are "inherited" from V. The ones that aren't can be deduced from (i) and (ii). See text for details. □

Example 50. Let W be the set

$$W := \left\{ \begin{bmatrix} x \\ y \\ 0 \end{bmatrix} : x, y \in \mathbb{R} \right\}$$

Then W is a subspace of $\mathbb{R}^3$. By Theorem 49, to verify this, we first need to check that it is closed under vector addition and scalar multiplication.

End of Example 50. □

Example 51. The set

$$A = \left\{ \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} : x, y \in \mathbb{R} \right\} \subseteq \mathbb{R}^2$$

does not form a subspace of $\mathbb{R}^3$. This is because the sum of any two vectors in A has the form $\begin{bmatrix} * \\ * \\ 2 \end{bmatrix}$, which is not itself in A.

End of Example 51. □

Example 52 (Important example)**.** Let A be an $m \times n$ matrix. The solutions to the linear system

$$AX = 0$$

is a subspace in $\mathbb{R}^n$. We will check this example in the next class.

End of Example 52. □

13 2025-09-24 | Week 05 | Lecture 13

This lecture is based on section 2.2 in the text.

> ***The nexus question of this lecture:*** *What do linear subspaces look like?*

13.1 Subspaces

Recall that given a vector space V, a nonempty subset W is a ***linear subspace*** if W is a vector space (under the same vector addition and scalar multiplication as V.)

Recall also the following theorem from last time:

Theorem 53. *Let W be a nonempty subset of a vector space V. Then W is a linear subspace iff the following conditions are satisfied:*

(i.) $u + v \in W$ *whenever* $u, v \in V$.

(ii.) $cu \in W$ *whenever* $c \in \mathbb{R}$ *and* $u \in W$.

Example 54. Do the vectors of the form

$$W = \left\{ \begin{bmatrix} x \\ y \\ x - 2y \end{bmatrix} : x, y, z \in \mathbb{R} \right\} \subseteq \mathbb{R}^3$$

form a subspace of $\mathbb{R}^3$?

Yes. To check this, we will apply Theorem 53.

- **Condition (i):** Let $u, v \in W$. We need to show that $u + v \in W$. First, write

$$u = \begin{bmatrix} x_1 \\ y_1 \\ x_1 - 2y_1 \end{bmatrix} \quad \text{and} \quad v = \begin{bmatrix} x_2 \\ y_2 \\ x_2 - 2y_2. \end{bmatrix}$$

  Then

$$u + v = \begin{bmatrix} x_1 + x_2 \\ y_1 + y_2 \\ (x_1 + x_2) - 2(y_1 + y_2) \end{bmatrix}$$

  And we observe that this vector is in W. So $u + v \in W$, as desired.

- **Condition (ii):** Let $u \in W$ and $c \in \mathbb{R}$. We need to show that $cu \in W$.

$$cu = c \begin{bmatrix} x_1 \\ y_1 \\ x_1 - 2y_1 \end{bmatrix} = \begin{bmatrix} cx_1 \\ cy_1 \\ cx_1 - 2cy_1 \end{bmatrix}$$

  Observe that this vector is in W (with $x = cx_1$ and $y = cy_1$). So $cu \in W$, as desired.

End of Example 54. □

Example 55. Let $\mathbb{R}[x]$ be the set of all polynomials in the variable x. And let $\mathbb{R}[x]_{\leq n}$ be the set of all polynomials of degree $\leq n$. That is,

$$\mathbb{R}[x]_{\leq n} = \left\{ a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n : a_0, \ldots, a_n \in \mathbb{R} \right\}$$

Then $\mathbb{R}[x]_{\leq n}$ is a subspace of $\mathbb{R}[x]$. To see why, we just need to check the conditions of Theorem 53.

- closure under addition
- closure under scalar multiplication

See Example 42.

End of Example 55. □

13.2 The kernel of a matrix

Given an $m \times n$ matrix A, the ***kernel*** of A, denoted $\ker(A)$, is the set

$$\ker(A) = \left\{ X = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \in \mathbb{R}^n : AX = \vec{0} \right\}$$

Geometrically, the kernel is the set of of points in space that get A sent to the origin under the transormation of A. Moreover, since the kernel is the solution set to the linear system $AX = \vec{0}$, we know that it must either

- consist of a unique point (i.e., the linear system has exactly one solution)
- consist infinitely many points. (i.e., the system as infinitely many solutions)
- consist of zero points (i.e., the system has no solutions)

But note that no matter what the matrix A is, the linear system $AX = \vec{0}$ always has at least one solution, namely $X = \vec{0}$. Thus, $\ker(A)$ either consists of the single point $X = \vec{0}$, or it contains infinitely many points.

In fact, we can say more than that: the kernel of a matrix is always a linear subspace.

Theorem 56 (Kernels are subspaces)**.** *Let A be an $m \times n$ matrix. The solutions to the linear system*

$$AX = \vec{0}$$

is a subspace in $\mathbb{R}^n$. In other words, $\ker(A)$ is a linear subspace of $\mathbb{R}^n$.

Proof. To prove this theorem, we will apply Theorem 53. It will suffice to show that (i) if $u, v \in \ker(A)$ then $u + v \in \ker(A)$, and (ii) if $c \in \mathbb{R}$ then $cu \in \ker(A)$.

- **Proof of (i):** Let $u, v \in \ker(A)$. Then

  $$Au = \vec{0} \quad \text{and} \quad Av = \vec{0}.$$

  Then

  $$A(u + v) = Au + Av = \vec{0} + \vec{0} = \vec{0}$$

  Therefore $u + v \in \ker(A)$.

- **Proof of (ii):** Let $c \in \mathbb{R}$ and $u \in \ker(A)$. Then

  $$A(cu) = cAu = c(\vec{0}) = \vec{0}.$$

  This shows that $cu \in \ker(A)$.

□

Theorem 56 is only half the story; the converse is also true

Theorem 57 (Every subspace is a kernel)**.** *Every subspace of $\mathbb{R}^n$ is the kernel of some matrix.*

We don't yet have the technical machinery to express why this is true, but we'll encounter it in the coming weeks (linear spans, linear independence, and bases).

If we accept Theorem 57 on face for now, it tells us some useful geometric information about linear subspaces of $\mathbb{R}^n$. Namely, a linear subspaces of $\mathbb{R}^n$ is always one of the following:

- the point $\left\{\vec{0}\right\}$
- a line passing through the origin
- a plane passing through the origin
- an n-dimensional "plane" passing through the origin, for $n > 3$.

We can also connect the notion of kernel back to our "key theorem", which now stands as the following:

14 2025-09-26 | Week 05 | Lecture 14

This lecture is based on section 2.2 in the text.

> ***The nexus question of this lecture:*** *How can we build linear subspaces from vectors?*

14.1 Linear Span

We begin by showing how to construct a linear subspace from a collection of vectors.

Definition 58 (Linear combination)**.** Let V be a vector space and let $v_1, \ldots, v_n \in V$. An expression of the form

$$c_1 v_1 + c_2 v_2 + \ldots + c_n v_n \quad (c_1, \ldots, c_n \in \mathbb{R})$$

is called a ***linear combination*** of $v_1, \ldots, v_n$. The linear combination with $c_1 = c_2 = \ldots = c_n = 0$ is called the ***trivial linear combination***. If at leat one of the c_i's is nonzero we say that the linear combination is ***nontrivial***.

Definition 59 (Span)**.** Given a set of vectors $S = \{v_1, \ldots, v_n\}$, the set of all their linear combinations is called the ***span of S***, and is denoted $\mathrm{Span}(S)$. In set notation,

$$\begin{aligned}\mathrm{Span}(S) &= \mathrm{Span}\{v_1, \ldots, v_n\} \\ &= \{c_1 v_1 + \ldots + c_n v_n : c_1, \ldots, c_n \in \mathbb{R}\} \subseteq V\end{aligned}$$

Note that $\mathrm{Span}(S)$ always contains the point $\vec{0}$, which is achieved by the trivial linear combination.

If $\mathrm{Span}(S) = V$ then we say that S ***spans*** V. This means that every vector in V can be written as a linear combination of vectors in S.

Example 60. Is the vector

$$v = \begin{bmatrix} 2 \\ -5 \\ 1 \\ 10 \end{bmatrix} \quad \text{in} \quad \mathrm{Span}\left\{ \begin{bmatrix} 1 \\ -1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ -2 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \\ 3 \end{bmatrix} \right\}?$$

To check this, we need to determine if there exists constants $c_1, c_2, c_3 \in \mathbb{R}$ such that

$$c_1 \begin{bmatrix} 1 \\ -1 \\ 2 \\ 3 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ -2 \\ -1 \\ 2 \end{bmatrix} + c_3 \begin{bmatrix} -1 \\ 0 \\ 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 2 \\ -5 \\ 1 \\ 10 \end{bmatrix}.$$

If there exists at least one choice of c_1, c_2, c_3 such that the above holds, then v is in the span.

Converting this to a system of equations, we have

$$\begin{aligned} c_1 + c_2 - c_3 &= 2 \\ -c_1 - 2c_2 &= -5 \\ 2c_1 - c_2 + c_3 &= 1 \\ 3c_1 + 2c_2 + 3c_3 &= 10. \end{aligned}$$

We can solve this by setting up an augmented matrix and row-reducing. Doing this we get

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

which tells us that there is a solution: $(c_1, c_2, c_3) = (1, 2, 1)$. Hence v is in the span.

End of Example 60. □

Theorem 61. *If V is a vector space and $v_1, \ldots, v_n \in V$, then* $\operatorname{Span}(v_1, \ldots, v_n)$ *is a subspace of V.*

Proof. Follows from an application of Theorem 53. (You should check this – hw problem, probably). $\square$

Example 62. Let $S = \left\{ \begin{bmatrix} 1 \\ -2 \end{bmatrix}, \begin{bmatrix} 2 \\ -4 \end{bmatrix} \right\}$. Does S span $\mathbb{R}^2$? In other words, we are asking whether we write every vector $\begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^2$ as a linear combination of the form

$$c_1 \begin{bmatrix} 1 \\ -2 \end{bmatrix} + c_2 \begin{bmatrix} 2 \\ -4 \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$$

for some values of c_1 and c_2?

Converting this linear system into an augmented matrix, we have

$$\left[\begin{array}{cc|c} 1 & 2 & x \\ -2 & -2 & y \end{array}\right] \xrightarrow{R_2+2R_1 \to R_2} \left[\begin{array}{cc|c} 1 & 2 & x \\ 0 & 0 & y+2x \end{array}\right]$$

From this form, we see that there is a solution if and only if $y = -2x$. So for example, if we pick the vector $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ then there is no solution, meaning we cannot find c_1, c_2 such that

$$c_1 \begin{bmatrix} 1 \\ -2 \end{bmatrix} + c_2 \begin{bmatrix} 2 \\ -4 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

Therefore $\operatorname{Span}(S) \neq \mathbb{R}^2$. (This answers the question, but we can push a little bit further.)

Looking at the reduced form of the augmented matrix, we see that there is a solution for any vector $\begin{bmatrix} x \\ y \end{bmatrix}$ with $y = -2x$ (that is, for any vector of the form $\begin{bmatrix} x \\ -2x \end{bmatrix}$). In particular, we can take $c_1 = x$ and $c_2 = 0$. Therefore such vectors are in $\operatorname{Span}(S)$. This tells us that

$$\operatorname{Span}(S) = \left\{ \begin{bmatrix} x \\ -2x \end{bmatrix} : x \in \mathbb{R} \right\}$$

This is the line $y = -2x$. Note that this line passes through the origin, which we know from the last lecture is one of the possible forms a subspace can take.

End of Example 62. $\square$

We've shown how to build a subspace using a set of vectors: namely, take the span of the vectors. This is nice, but insufficient. It is of interest to know what is the *minimal* number of vectors needed to build a given subspace? That is, how can we build a subspace with as few vectors as possible? And how can we know that we can't use fewer vectors? To answer these question, we need the notion of *linear independence*, which will be next lecture.

15 2025-09-29 | Week 06 | Lecture 15

> ***The nexus question of this lecture:*** *How do we know if a linear system is minimal —i.e., that it doesn't have any redundant equations?*

(Investigating this question will help set us up to answer the question of how to build a linear subspace with a **minimal** set of vectors.)

15.1 Some review

Theorem 63 (The Key Theorem of Linear Algebra (partial version))**.** *Let A be an $n \times n$ matrix. Then the following are equivalent:*

(i.) *A is invertible (i.e., A^{-1} exists)*

(ii.) $\det A \neq 0$

(iii.) *The linear system $AX = B$ has a unique solution for each $B \in \mathbb{R}^n$.*

(iv.) *A is row equivalent to I*

(v.) *The only solution to $AX = \vec{0}$ is $X = \vec{0}$ (i.e., A is nonsingular)*

(vi.) $\ker(A) = \left\{\vec{0}\right\}$

(vii.) *???*

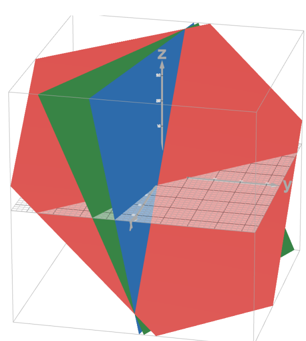
Review definition of kernel, too.

15.2 Dependent linear systems

Consider the linear system

$$\begin{cases} E_1: & x + y + z = 5 \\ E_2: & x + 5y + z = 6 \\ E_3: & 3x + 7y + 3z = 16 \end{cases}$$

The set of solutions to this system is the the intersection of three planes in 3d space, one for each equation. In principle, each equation imposes some restriction on what the solution set can be. For examplep, E_1 and E_2 intersect to form a line, so solutions must lie on that line. E_3 is the equation of another plane, and while it is not parallel to either E_1 or E_2, when we plot all three planes, we see that the intersection is the same line we get by just intersecting E_1, E_2:



Hence, the solutions to the system of equations form a line. The third plane in our system failed to to cut the line of intersection down to a single point. The equation E_3 didn't impose any additional restrictions on the solution set that weren't already imposed by E_1 and E_2. This is because $E_3 = 2E_1 + E_2$. In some sense, E_3 is just E_1 and E_2 in disguise.

Indeed, if we row reduce, we get a row of zeros:

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 5 \\ 1 & 5 & 1 & 6 \\ 3 & 7 & 3 & 16 \end{array}\right] \xrightarrow{R_3 - R_2 \to R_3} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 5 \\ 1 & 5 & 1 & 6 \\ 2 & 2 & 1 & 10 \end{array}\right] \xrightarrow{R_3 - 2R_1 \to R_3} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 5 \\ 1 & 5 & 1 & 6 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

This implies that the solutions of the system can be expressed as the intersection of only the first two planes.

How can we predict exactly when this will happen? To understand exactly when a linear system yields exactly one solution vs. infinitely many solutions, we need the concept of "linear independence".

15.3 Linear independence

Definition 64. Let V be a vector space and let $v_1, \ldots, v_n \in V$. We say that the set $\{v_1, \ldots, v_n\}$ is ***linearly dependent*** if there are scalars $c_1, c_2, \ldots, c_n$ *not all zero* such that

$$c_1 v_1 + c_2 v_2 + \ldots + c_n v_n = \vec{0}.$$

If the vectors $v_1, \ldots, v_n$ are not linearly dependent, we say that they are ***linearly independent.***

Geometrically, we can visualize linear dependence in the following way. For vectors in 3-dimensional space, two vectors are linearly dependent if they lie on the same line. *Three vectors are linearly dependent if they lie in the same plane.*

Example 65. Are the vectors

$$\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \\ 5 \end{bmatrix}$$

linearly dependent or linearly independent?

To determine the answer we need to solve the linear system, to see if there are any solutions other than $c_1 = c_2 = c_3 = 0$.

$$\begin{bmatrix} 1 & 3 & -1 \\ 2 & 2 & 2 \\ 3 & 1 & 5 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Row reducing, we get

$$\begin{bmatrix} 1 & 3 & -1 & 0 \\ 2 & 2 & 2 & 0 \\ 3 & 1 & 5 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Convert back to equations

$$\begin{aligned} c_1 + 2c_3 &= 0 \\ c_2 - c_3 &= 0 \\ 0 &= 0 \end{aligned}$$

We can write this as

$$\begin{aligned} c_1 &= -2c_3 \\ c_2 &= c_3 \end{aligned}$$

with free variable $c_3 \in \mathbb{R}$. Therefore, there are infinitely many solutions. For example, if $c_3 = 1$, we get the solution

$$\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$$

We've shown that there is a nontrivial linear combination of the vectors which equals zero. Therefore they are linearly dependent.

End of Example 65. □

This example illustrates a general method: To check any set of vectors $v_1, \dots, v_n$ for independence, put them in the columns of A Then solve the system $Ac = \vec{0}$. The vectors are dependent if there is a solution other than $c = \vec{0}$

To relate today's discussion of linear independence back to our nexus question, we utilize our key theorem. Let A be an $n \times n$ matrix, and let $B \in \mathbb{R}^n$ be an arbitrary vector. We wish to know whether the linear system $AX = B$ has a unique solution or not. (After all, if it has infinitely many solutions, that means we have at least one redundant equation.)

Then the following statements are equivalent:

$$\begin{gathered}
\text{The columns of } A \text{ are linearly independent.} \\
\Updownarrow \\
\text{The only solution to } AX = \vec{0} \text{ is } X = \vec{0}. \\
\Updownarrow \\
A \text{ is nonsingular} \\
\Updownarrow \\
\text{The equation } AX = B \text{ has exactly one solution for any } B \in \mathbb{R}^n.
\end{gathered}$$

16 2025-10-01: Week 06 | Lecture 16

> ***The nexus question of this lecture:*** *How can we build linear subspaces from a* ***minimal*** *set of vectors?*

16.1 Linear Indepdenence

Definition 66. Let V be a vector space and let $v_1, \ldots, v_n \in V$. We say that the set $\{v_1, \ldots, v_n\}$ is ***linearly dependent*** if there are scalars $c_1, c_2, \ldots, c_n$ *not all zero* such that

$$c_1 v_1 + c_2 v_2 + \ldots + c_n v_n = \vec{0}.$$

If the vectors $v_1, \ldots, v_n$ are not linearly dependent, we say that they are ***linearly independent.***

General method: To check any set of vectors $v_1, \ldots, v_n$ for independence, put them in the columns of A Then solve the system $Ac = \vec{0}$. The vectors are dependent if there is a solution other than $c = \vec{0}$

Example 67. Are the vectors

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} \right\}$$

linearly independent?

Solution: We need to check if the linear system

$$\begin{bmatrix} 1 & 1 & -1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

has any nontrivial solutions.

Using row reduction, we have

$$\left[\begin{array}{ccc|c} 1 & 1 & -1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \end{array}\right]$$

can be row reduced to

$$\left[\begin{array}{ccc|c} 1 & 1 & -1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 2 & 0 \end{array}\right]$$

We could go further, but this is sufficient. Converting back to a system of equations,

$$\begin{aligned} c_1 + c_2 - c_3 &= 0 \\ c_2 + c_3 &= 0 \\ 2c_3 &= 0 \end{aligned}$$

Back substitution gives $c_1 = c_2 = c_3 = 0$. Thus, the only solution to the linear system is the trival solution. Therefore the set of vectors is linearly independent.

End of Example 67. □

Example 68. Show that the columns of the following matrix are linearly independent:

$$\begin{bmatrix} 3 & 4 & 2 \\ 0 & 1 & 5 \\ 0 & 0 & 2 \end{bmatrix}$$

Look for a linear combination that makes zero:

$$c_1 \begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 4 \\ 1 \\ 0 \end{bmatrix} + c_3 \begin{bmatrix} 2 \\ 5 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

We need to show that c_1, c_2, c_3 are all forced to be zero.

Indeed, the third row forces c_3 to be zero. Then the second row forces c_2 to be zero. Then the first row forces c_1 to be zero. Hence, the columns are linearly independent

End of Example 68. □

16.2 Basis

Linear independence allows us to make precise the notion of what it means for a spaning set of vectors to be minimal. Such a set is called a basis:

Definition 69 (Basis)**.** Let V be a vector space and let $v_1, \ldots, v_n \in V$. We say that $v_1, \ldots, v_n$ are a ***basis*** for V if both the following conditions are satisfied:

(i.) $v_1, \ldots, v_n$ span V.

(ii.) $v_1, \ldots, v_n$ are linearly independent.

Condition (i.) says that you have enough vectors to generate the linear space V, and condition (ii.) says that *you don't have too many vectors* (i.e., minimality). To answer the motivating question of the last few lectures, a basis is a minimal generating set for a linear subspace (which may include be whole space V).

This latter point about minimality is related to the following theorem

Theorem 70. *Let V be a vector space, and let $v_1, \ldots, v_n \in V$. Then the $v_1, \ldots, v_n$ are linearly dependent if and only if one of the $v_1, \ldots, v_n$ is a linear combination of the others.*

Example 71 (The standard basis vectors)**.** The vectors

$$e_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad e_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad e_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

are a basis for $\mathbb{R}^3$.

To check this, observe that

(i.) (linear independence) It is clear that equality holds in the equation

$$c_1 e_1 + c_2 e_2 + c_3 e_3 = 0$$

only if $c_1 = c_2 = c_3 = 0$. That is, there is no nontrivial linear combination of e_1, e_2, e_3 which equals the zero vector. So e_1, e_2, e_3 are linearly independent.

(ii.) (spanning) If $\begin{bmatrix} a \\ b \\ c \end{bmatrix}$ is an arbitrary vector in $\mathbb{R}^3$, then

$$ae_1 + be_2 + ce_3 = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

Therefore $\mathrm{Span}(e_1, e_2, e_3)$ includes every vector in $\mathbb{R}^3$.

End of Example 71. □

Example 72. Show that

$$\begin{bmatrix}1\\0\\1\end{bmatrix},\quad \begin{bmatrix}1\\1\\1\end{bmatrix},\quad \begin{bmatrix}-1\\1\\1\end{bmatrix}$$

form a basis for $\mathbb{R}^3$.

Solution: We already checked that these vectors are linearly independent in Example 67. To it suffices to show that we can write any vector $\begin{bmatrix}z\\y\\z\end{bmatrix}$ in $\mathbb{R}^3$ as a linear combination of the three vectors.

To see this, set up the system

$$\begin{bmatrix}1 & 1 & -1\\0 & 1 & 1\\1 & 1 & 1\end{bmatrix}\begin{bmatrix}c_1\\c_2\\c_3\end{bmatrix} = \begin{bmatrix}x\\y\\z\end{bmatrix}.$$

If this system always has a solution, then the vectors span $\mathbb{R}^3$. If there is some vector $\begin{bmatrix}x\\y\\z\end{bmatrix}$ such that the above equation has no solutions, then the vectors do not span $\mathbb{R}^3$. We can check it by (you guessed it) row reducing.

$$\left[\begin{array}{ccc|c}1 & 1 & -1 & x\\0 & 1 & 1 & y\\1 & 1 & 1 & z\end{array}\right] \xrightarrow{R_3-R_1\to R_3} \left[\begin{array}{ccc|c}1 & 1 & -1 & x\\0 & 1 & 1 & y\\0 & 0 & 2 & z-x\end{array}\right] \xrightarrow{R_1-R_2\to R_1} \left[\begin{array}{ccc|c}1 & 0 & -2 & x-y\\0 & 1 & 1 & y\\0 & 0 & 2 & z-x\end{array}\right] \tag{14}$$

$$\xrightarrow{R_1+R_3\to R_1} \left[\begin{array}{ccc|c}1 & 0 & 0 & z-y\\0 & 1 & 1 & y\\0 & 0 & 2 & z-x\end{array}\right] \xrightarrow{R_2-\frac{1}{2}R_3\to R_2} \left[\begin{array}{ccc|c}1 & 0 & 0 & z-y\\0 & 1 & 0 & y-\frac{1}{2}(z-x)\\0 & 0 & 2 & z-x\end{array}\right] \xrightarrow{\frac{1}{2}R_3\to R_3} \left[\begin{array}{ccc|c}1 & 0 & 0 & z-y\\0 & 1 & 0 & y-\frac{1}{2}(z-x)\\0 & 0 & 1 & \frac{1}{2}(z-x)\end{array}\right]$$

Therefore, the solution is $c_1 = z-y$, $c_2 = y+\frac{x}{2}-\frac{z}{2}$, $c_3 = \frac{z}{2}-\frac{x}{2}$. Since this solution exists for any x, y, z, we have shown that the vectors form a basis.

End of Example 72. $\square$

17 2025-10-03 | Week 06 | Lecture 17

Sub lecture by R. Willett.

Lecture on section 2.4

If V is a vector space, a ***basis*** for V is a collection $v_1, \ldots, v_n \in V$ such that

- $v_1, \ldots, v_n$ span V
- $v_1, \ldots, v_n$ are linearly independent

Example 73 (Bases)**.** $\mathbb{R}^3$ has 'standard basis'

$$\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix},$$

but there may be others, e.g.,

$$\begin{bmatrix}3\\0\\0\end{bmatrix}, \begin{bmatrix}4\\1\\0\end{bmatrix}, \begin{bmatrix}2\\5\\2\end{bmatrix},$$

or

$$\begin{bmatrix}1\\0\\1\end{bmatrix}, \begin{bmatrix}1\\1\\1\end{bmatrix}, \begin{bmatrix}-1\\1\\1\end{bmatrix},$$

End of Example 73. □

Example 74. $M_{m\times n}(\mathbb{R})$ has basis $E_{11}, E_{12}, \ldots, E_{1n}, E_{21}, \ldots, E_{2n}, \ldots, E_{m1}, \ldots E_{mn}$ where E_{ij} is an $m \times n$ matrix with 1 at position (i, j) and zero else.

Again, there are others, e.g.

$$\begin{bmatrix}1 & 0\\0 & -1\end{bmatrix}, \begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}, \begin{bmatrix}0 & 1\\1 & 0\end{bmatrix}, \begin{bmatrix}0 & -1\\1 & 0\end{bmatrix}$$

End of Example 74. □

Theorem 75 (Theorem 2.9 in textbook – not obvious!)**.** *If $v_1, \ldots, v_n$ and $w_1, \ldots, w_m$ are both bases for the same vector space, then $m = n$.*

Definition 76 (Dimension)**.** The ***dimension*** of a vector space is the number of elements in a basis. Notation for the dimension of V:

$$\dim(V).$$

Example 77 (Dimension)**.**

- $\dim(\mathbb{R}^3) = 3$ and more generally $\dim(\mathbb{R}^n) = n$.
- $\dim(\mathbb{R}[x]_{\leq 2}) = 3$ and more generally $\dim(\mathbb{R}[x]_{\leq n}) = n + 1$

Comments:

- Some vector spaces have bases with infinitely many vectors (e.g. a basis for $\mathbb{R}[x]$ is $1, x, x^2, x^3, \ldots$). In this case, the dimension of V is infinite. Notation: $\dim(V) = \infty$.

  (In this course, you will mainly focus on finite dimensional vector spaces.)

- If V is the 0 vector space, we write $\dim(V) = 0$.

End of Example 77. □

Theorem 78 (Some important properties of dimension (see 2.11 and 2.12))**.** *Let V be a vector space with $\dim(V) = n$. Then*

(a) *If $v_1, \dots, v_k \in V$ are linearly independent, then $k \leq n$ and there are $v_{k+1}, v_{k+2}, \dots, v_n$ with $v_1, \dots, v_k, v_{k+1}, \dots, v_n$ a basis ("linear independent collections cannot be too big")*

(b) *If $v_1, \dots, v_k \in V$ span V, then $k \geq n$, and some collection of n vectors from $v_1, \dots, v_k$ is a basis '("spanning collections collections cannot be too small")*

Theorem 79 (Dimension-basis)**.** *Suppose $v_1, \dots, v_n \in V$, where $\dim(V) = n$. Then*

- *If $v_1, \dots v_n$ span V, then they are a basis.*
- *If $v_1, \dots, v_n$ are linearly independent, they are a basis.*

Example 80. Which (if any) of the following collections is a basis for $\mathbb{R}^3$?

(a) $\begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix}, \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$

(b) $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}, \begin{bmatrix} 7 \\ 8 \\ 9 \end{bmatrix}$

(c) $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}, \begin{bmatrix} 7 \\ 8 \\ 9 \end{bmatrix}, \begin{bmatrix} 10 \\ 11 \\ 12 \end{bmatrix}$

Solution:

(a) No: has too few vectors (so cannot span)

(c) No: has too many vectors (so cannot be linearly independeent)

(b) To check linear independence, we need to check whether

$$c_1 \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + c_2 \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} + c_3 \begin{bmatrix} 7 \\ 8 \\ 9 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

has nontrivial solutions.

Check linear independence by row reducing:

$$\begin{aligned} \begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{bmatrix} &\longrightarrow \begin{bmatrix} 1 & 4 & 7 \\ 0 & -3 & -6 \\ 0 & -6 & -12 \end{bmatrix} && R_2 - 2R_1 \text{ and } R_3 - 3R_1 \\ &\longrightarrow \begin{bmatrix} 1 & 4 & 7 \\ 0 & -3 & -6 \\ 0 & 0 & 0 \end{bmatrix} && R_3 - 2R_2 \end{aligned}$$

As we have a row of zeroes, there are no conditions on c_3, and we see that there are infinitely many solutions.

We conclude that the collection is not a basis (since it's not linearly independent.)

End of Example 80. □

17.1 Null space, row space, column space

Let $A \in M_{m\times n}(\mathbb{R})$. There are three important vector spaces associated with A :

- The ***column space***, which is the subspace of $M_{m\times 1}(\mathbb{R})$ spanned by the columns of A. Notation: $CS(A)$
- The ***row space***, which is the subspace of $M_{1\times n}(\mathbb{R})$ spanned by the rows of A. Notation $RS(A)$.
- The ***null space*** (aka: kernel) which is the subspace of R^n of vectors x such that $Ax = 0$. Notation $NS(A)$ or $\ker(A)$.

18 2025-10-06 | Week 07 | Lecture 18

> ***The nexus question of this lecture:*** *What are the three fundamental linear subspaces associated with a matrix A?*

18.1 Three fundamental subspaces

Definition 81 (The fundamental subspaces of A)**.** Let $A \in M_{m\times n}(\mathbb{R})$. There are three important vector spaces associated with A :

- The ***column space***, which is the subspace of $M_{m\times 1}(\mathbb{R})$ spanned by the columns of A. Notation: $CS(A)$
- The ***row space***, which is the subspace of $M_{1\times n}(\mathbb{R})$ spanned by the rows of A. Notation $RS(A)$.
- The ***null space*** which is the subspace of R^n of vectors x such that $Ax = 0$. Notation $NS(A)$. The nullspace and the kernel are the same thing.

Remark 82 (Connection between column space and matrix multiplication)**.** The idea of column space is natural. If

$$A = \begin{bmatrix} | & | & & | \\ A_1 & A_2 & \dots & A_n \\ | & | & & | \end{bmatrix} \in M_{m\times n}(\mathbb{R})$$

then for any vector $x = (x_1, \dots, x_n)^\top \in \mathbb{R}^n$

$$Ax = A_1x_1 + A_2x_2 + \ldots + A_nx_n$$

This is linear combination of the columns of A, so the output Ax is always an element of the column space. Another word for column space is the ***image*** or ***range*** of the [linear transformation of the] matrix A.

Definition 83 (rank)**.** The ***rank*** of a matrix is the dimension of its column space (= dim of row space).

Theorem 84. *For $A \in M_{m\times n}(\mathbb{R})$, the dimensions satisfy*

$$\dim RS(A) = \dim CS(A)$$

and

$$\underbrace{\dim CS(A)}_{\text{'rank'}} + \underbrace{\dim NS(A)}_{\text{'nullity'}} = \underbrace{n}_{\#\text{ cols of } A}$$

The second part is called the rank-nullity theorem. Noting that $\operatorname{rank}(A) = \dim CS(A)$ and that $NS(A) = \ker(A)$, we have

18.2 Some examples of computing bases for the three fundamental subspaces

Example 85 (Row space, column space, null space)**.** Let

$$A = \begin{bmatrix} 1 & 0 & -1 & 3 \\ 2 & 1 & 4 & -1 \\ 4 & 1 & 2 & 5 \end{bmatrix} \in M_{3\times 4}(\mathbb{R}).$$

Find bases for

(a) $RS(A)$

(b) $NS(A)$

Solution

(a) Wrong:

$$\begin{bmatrix}1 & 0 & -1 & 3\end{bmatrix}, \begin{bmatrix}2 & 1 & 4 & -1\end{bmatrix}, \begin{bmatrix}4 & 1 & 2 & 5\end{bmatrix}$$

Better approach:

Idea: row reduction does not change row space, so row reduce until we get a linearly independent set.

The reduced row echelon form is

$$A_{\text{RREF}} = \begin{bmatrix}1 & 0 & -1 & 3 \\ 0 & 1 & 6 & -7 \\ 0 & 0 & 0 & 0\end{bmatrix} \tag{15}$$

A basis of A_{RREF} is

$$\begin{bmatrix}1 & 0 & -1 & 3\end{bmatrix},\ \begin{bmatrix}0 & 1 & 6 & 7\end{bmatrix}$$

(since these are linearly independent). Moreover, we note that since row reduction doesn't change the row space,

$$RS(A) = RS(A_{\text{RREF}}).$$

and hence

$$\begin{bmatrix}1 & 0 & -1 & 3\end{bmatrix},\ \begin{bmatrix}0 & 1 & 6 & 7\end{bmatrix}$$

are a basis for $RS(A)$ as well.

(b) We will use the fact that

$$NS(A) = NS(A_{\text{RREF}}).$$

So it suffices to find a basis for $NS(A_{\text{RREF}})$. Let's do a computation to sese what $NS(A_{\text{RREF}})$ looks like. Recall that $NS(A_{\text{RREF}})$ consists of the vectors x satisfying

$$A_{\text{RREF}}x = 0 \tag{16}$$

If $x = (x_1, \ldots, x_5)^\top$ satisfies Eq. (16), then we have

$$\begin{bmatrix}1 & 0 & -1 & 3 \\ 0 & 1 & 6 & -7 \\ 0 & 0 & 0 & 0\end{bmatrix}\begin{bmatrix}x_1 \\ x_2 \\ x_3 \\ x_4\end{bmatrix} = \begin{bmatrix}0 \\ 0 \\ 0 \\ 0\end{bmatrix}.$$

Writing this as equations, we have

$$\begin{aligned} x_1 - x_3 + 3x_4 &= 0 \\ x_2 + 6x_3 - 7x_4 &= 0 \\ 0 &= 0. \end{aligned}$$

Therfore we have:

$$\begin{aligned} x_1 &= x_3 - 3x_4 \\ x_2 &= -6x_3 + 7x_4 \end{aligned}$$

where x_3, x_4 are free variables.

Therefore if $x \in RS(A_{\text{RREF}})$ (equivlently, if x satisfies Eq. (16)), then it has the following form

$$\begin{bmatrix}x_1 \\ x_2 \\ x_3 \\ x_4\end{bmatrix} = \begin{bmatrix}x_3 - 3x_4 \\ -6x_3 + 7x_4 \\ x_3 \\ x_4\end{bmatrix} = x_3\begin{bmatrix}1 \\ -6 \\ 1 \\ 0\end{bmatrix} + x_4\begin{bmatrix}-3 \\ 7 \\ 0 \\ 1\end{bmatrix}.$$

This shows that the vectors

$$\begin{bmatrix} 1 \\ -6 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 7 \\ 0 \\ 1 \end{bmatrix}$$

span $NS(A)$. Moreover, they are also linearly independent (since two vectors are linearly dependent if and only if they are multiples of each other, which these are clearly not). Therefore

$$\begin{bmatrix} 1 \\ -6 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 7 \\ 0 \\ 1 \end{bmatrix}$$

form a basis for $NS(A_{\text{RREF}})$. Since $NS(A_{\text{RREF}}) = NS(A)$, they are a basis for $NS(A)$ as well. Since there are two vectors in the basis, $\dim NS(A) = 2$.

End of Example 85. $\square$

19 2025-10-08 | Week 07 | Lecture 19

Topics: section 2.4 - fundamental subspaces, rank nullity, rank, and dimension

19.1 An alternative characterization of linear dependence

The following theorem gives us another quite useful characterization of linear dependence:

Theorem 86. *Let V be a vector space, and let $v_1, \dots, v_n \in V$. Then the $v_1, \dots, v_n$ are linearly dependent if and only if one of the $v_1, \dots, v_n$ is a linear combination of the others.*

Taking $n = 2$ in the above theorem gives a useful consequence: two u, v, are linearly dependent if and only if they are scalar multiples of each other.

For example, the vectors $\begin{bmatrix}1\\2\\3\end{bmatrix}, \begin{bmatrix}3\\6\\9\end{bmatrix}$. are scalar multiples of each other, and hence are dependent. On the other hand, the vectors $\begin{bmatrix}1\\2\\3\end{bmatrix}, \begin{bmatrix}3\\6\\10\end{bmatrix}$. are not scalar multiples of each other, so they are linearly independent.

19.2 Rank-Nullity Theorem

Definition 87 (Rank)**.** The ***rank*** of a matrix A, denoted $\operatorname{rank}(A)$, is the dimension of the column space of A. That is,

$$\operatorname{rank}(A) = \dim CS(A) \quad (= \dim RS(A))$$

Theorem 88 (Rank-nullity)**.** *Let A be an $m \times n$ matrix. Then*

$$\underbrace{\dim CS(A)}_{\operatorname{rank}(A)} + \underbrace{\dim NS(A)}_{\text{nullity(A)}} = n$$

19.3 More examples of computing bases for the three fundamental subspaces

Example 89 (Example 85 continued)**.** Consider the matrix

$$A = \begin{bmatrix} 1 & 0 & -1 & 3 \\ 2 & 1 & 4 & -1 \\ 4 & 1 & 2 & 5 \end{bmatrix} \in M_{3\times 4}(\mathbb{R}).$$

Previously, we showed the following:

- The vectors $\begin{bmatrix}1 & 0 & -1 & 3\end{bmatrix}$, $\begin{bmatrix}0 & 1 & 6 & 7\end{bmatrix}$ form a basis for $RS(A)$. Hence, $\dim RS(A) = 2$.
- The vectors $\begin{bmatrix}1\\-6\\0\\1\end{bmatrix}, \begin{bmatrix}-3\\7\\0\\1\end{bmatrix}$ form a basis for $NS(A)$. Hence $\dim NS(A) = 2$.

Question: What is $\operatorname{rank}(A)$? In other words, what is $\dim CS(A)$?

Solution 1 (general method): *Idea: to find the column space, take the transpose, row-reduce to find a basis for $RS(A^\top)$, then transpose back.*

$$A^\top = \begin{bmatrix} 1 & 2 & 4 \\ 0 & 1 & 1 \\ -1 & 4 & 2 \\ 3 & -1 & 5 \end{bmatrix} \xrightarrow{\text{row reduction}} \begin{bmatrix} 1 & 2 & 4 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Thus, the vectors

$$\begin{bmatrix}1 & 2 & 4\end{bmatrix}, \begin{bmatrix}0 & 1 & 1\end{bmatrix}$$

form a basis for $RS(A^\top)$. Transposing back, it follows that the vectors $\begin{bmatrix}1\\2\\4\end{bmatrix}, \begin{bmatrix}0\\1\\1\end{bmatrix}$ form a basis for $CS(A)$. Since there are two vectors in the basis, it follows that $\dim CS(A) = 2$.

Solution 2: By the Rank-Nullity Theorem (Theorem 88), we have

$$\dim CS(A) + \dim NS(A) = 4$$

Since $\dim NS(A) = 2$ (because 2 vectors in the basis), it follows that

$$\dim CS(A) = 2.$$

Solution 3: In the last lecture, we showed that $\dim RS(A) = 2$. Recall that $\text{rank}(A) := \dim CS(A) = \dim RS(A)$. Therefore $\dim CS(A) = 2$.

End of Example 89. □

Example 90. Find the column space of

$$A = \begin{bmatrix}1 & 1 & 2\\-1 & 1 & 0\\0 & 1 & 1\end{bmatrix}.$$

Solution: We will again use the general method for finding the column space from the previous example. The three steps are: (1) take the transpose, (2) row-reduce to find a basis for $RS(A^\top)$, then (3) transpose back.

We row reduce $A^\top$ which gives

$$A^\top = \begin{bmatrix}1 & -1 & 0\\1 & 1 & 1\\2 & 0 & 1\end{bmatrix} \xrightarrow{\text{row reduction}} \begin{bmatrix}1 & -1 & 0\\0 & 2 & 1\\0 & 0 & 0\end{bmatrix}.$$

The nonzero rows are a basis for $R(A^\top)$. Transposing back, we have

$$\begin{bmatrix}1\\-1\\0\end{bmatrix} \quad \text{and} \quad \begin{bmatrix}0\\2\\1\end{bmatrix}$$

This is a basis for the colum space of A.

End of Example 90. □

19.4 Connection between rank and invertibility

Theorem 91. *Let A be an $n \times n$ matrix. Then A is invertible if and only if rank$(A) = n$.*

One way to see why this is true involves thinking of A as a transformation of n-dimensional space $\mathbb{R}^n$. Observe that

- A is non-invertible exactly when the transformation collapses the dimension.
- The rank of A is the dimension of its range (since range = column space).
- So if $\text{rank}(A) < n$ then a dimension collapse occurs, in which case A is not invertible. But if $\text{rank}(A) = n$, then no dimension collapse occurs, so A is invertible in that case.

Thus, Theorem 91 gives another condition we can add to our key theorem.

20 2025-10-10 | Week 07 | Lecture 20

The nexus question of this lecture: *What is a linear transformation?*

This lecture is based on section 5.1 in the textbook.w

20.1 Function notation

When a function f goes from a set X to a set Y, we write

$$f : X \to Y$$

which is read as "f maps X to Y". The set X is the ***domain*** of f. The set Y is the ***codomain*** of f. The subset

$$\text{Range}(f) = \{f(x) \mid x \in X\}$$

is called the ***range*** of f.

Example 92.

- Let $f(x) = e^x$. Then $f : \mathbb{R} \to \mathbb{R}$. The range of this function is the set of positive real numbers.
- Let $g : \mathbb{R}^2 \to \mathbb{R}$ ge given by

$$g\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = x^2 - y^2$$

The range of the function is $\mathbb{R}$, which is the same as its codomain.

End of Example 92. □

20.2 Linear Transformation

Definition 93. Let V, W be vector spaces and $T : V \to W$ a function. We say that T is a ***linear transformation*** if, for all vectors $u, v \in V$ and $c \in \mathbb{R}$, we have

(i) $T(u + v) = T(u) + T(v)$ (preserves addition)

(ii) $T(cV) = cT(v)$. (preserves scalar multiplication)

Sometimes people refer to linear transformations as ***linear operators***, which means the same thing.

You have seen linear transformations before, even if you didn't call them that at the time. For example, the derivative operator $\frac{d}{dx}$ is one such function.

Example 94. Let $P = \{\text{all polynomials in the variable } x\}$. So an arbitary element of P looks like

$$p = c_0 + c_1 x + c_2 x^2 + \ldots + c_r x^r$$

for some nonnegative integer r.

We know that P is a vector space. Define a function

$$T : P \to P$$

where $T(p) = p'$. In other words, $T = \frac{d}{dx}$.

To check that T is a linear transformation, let $p, q \in P$ and $c \in \mathbb{R}$. Then

- $T(p + q) = \frac{d}{dx}[p(x) + q(x)] = p'(x) + q'(x) = T(p) + T(q)$.
- $T(cp) = \frac{d}{dx}[cp(x)] = c\frac{d}{dx}[p(x)] = cT(p)$.

End of Example 94. □

Example 95. Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ be ggiven by

$$T \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x + y - z \\ x + 2y + x \end{bmatrix}$$

we can check that this is also a linear transformation.

First observe that we can write

$$T \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 & 1 & -1 \\ 1 & 2 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}.$$

Letting $A = \begin{bmatrix} 1 & 1 & -1 \\ 1 & 2 & 1 \end{bmatrix}$, we can write

$$T(v) = Av, \text{ for } v \in \mathbb{R}^3.$$

Let's check conditions (i) and (ii) in the definition of linear transformation:

- Proof of (i): Let $u, v \in \mathbb{R}^3$. Then

  $$T(u + v) = A(u + v) = Au + Av = T(u) + T(v)$$

  This shows that condition (i) holds.

- Proof of (ii): Let $c \in \mathbb{R}$ and $u \in \mathbb{R}^3$. Then

  $$T(cu) = A(cu) = c(Au) = cT(u)$$

  This shows that condition (ii) holds.

Therefore since both conditions are met, T is a linear transformation.

End of Example 95. □

The proofs from the previous example didn't depend on the specific form of A, only that A was a matrix. Thus we have the following theorem:

Theorem 96. *If A is an $m \times n$ matrix, then the function $T : \mathbb{R}^n \to \mathbb{R}^m$ given by*

$$T(X) = AX$$

is a linear transformation.

Linear transformations of this form are called ***matrix transformations***.

Theorem 97. *Suppose $T : V \to W$ is a linear transformation. Then*

(i) $T(0) = 0$.

(ii) For any vectors $v_1, \ldots, v_n \in V$ and scalars $c_1, \ldots, c_n \in \mathbb{R}$, we have

$$T(c_1 v_1 + \ldots + c_n v_n) = c_1 T(v_1) + \ldots + c_n T(v_n)$$

The second property says that linear transformations preserve linear combinations.

Proof. To show that $T(0) = 0$ involves a trick. Observe that

$$T(0) = T(0 + 0) = T(0) + T(0).$$

Subtracting $T(0)$ from both sides gives

$$T(0) = 0.$$

Proof of (ii) is omitted, but follows from the definition of linear transformation. (HW?) □

Definition 98. The ***kernel*** of a linear transformation $T : V \to W$ the set

$$\ker(T) = \{v \in V : T(v) = 0\}$$

The previous theorem shows that 0 is always in $\ker(T)$. Just like matrices. Huh.

Example 99. Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear transformation defined by

$$T \begin{bmatrix} x \\ y \\ z \end{bmatrix} := \underbrace{\begin{bmatrix} 1 & 1 & -1 \\ 1 & 2 & 1 \end{bmatrix}}_{A} \begin{bmatrix} x \\ y \\ z \end{bmatrix}.$$

Question: Find a basis for $\ker(T)$. [Equivalently: find a basis for $NS(A)$.]

Solution: *Idea: solve the homogeneous system $AX = 0$, then interpret the solution.*
The system $AX = 0$ is

$$\begin{bmatrix} 1 & 1 & -1 \\ 1 & 2 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

We can solve this with row reduction:

$$\left[\begin{array}{ccc|c} 1 & 1 & -1 & 0 \\ 1 & 2 & 1 & 0 \end{array}\right] \xrightarrow{\text{row reduction}} \left[\begin{array}{ccc|c} 1 & 0 & -3 & 0 \\ 0 & 1 & 2 & 0 \end{array}\right].$$

This corresponds to the equations

$$\begin{cases} x - 3z = 0 \\ y + 2z = 0 \end{cases}$$

or

$$\begin{cases} x = 3z \\ y = -2z \end{cases}$$

where z a free variable. Thus, every solution $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ to $AX = 0$ takes the form

$$\begin{bmatrix} 3z \\ -2z \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix} z$$

Therefore the vector $\begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix}$ form a basis for $\ker(T)$.

End of Example 99. □

21 2025-10-13 | Week 08 | Lecture 21

The nexus question of this lecture: *Is a basis all you need?*

This lecture is based on sections 5.1 and 2.4

21.1 All you need is a basis

If we know how a linear transformation acts on a basis, then we know its values on all other vectors as well.

Example 100 (Cool example)**.** Suppose that $T : \mathbb{R}^3 \to \mathbb{R}^2$ is a linear transformation.

Note that the vectors

$$\begin{bmatrix}1\\1\\0\end{bmatrix}, \begin{bmatrix}0\\1\\1\end{bmatrix}, \begin{bmatrix}1\\0\\1\end{bmatrix}$$

form a basis of $\mathbb{R}^3$. (Check this).

Suppose that T is defined for each of these three basis vectors as

$$T\begin{bmatrix}1\\1\\0\end{bmatrix} := \begin{bmatrix}2\\3\end{bmatrix}, \quad T\begin{bmatrix}0\\1\\1\end{bmatrix} := \begin{bmatrix}0\\3\end{bmatrix}, \quad \text{and} \quad T\begin{bmatrix}1\\0\\1\end{bmatrix} := \begin{bmatrix}0\\2\end{bmatrix}.$$

Question: What is $T\begin{bmatrix}1\\3\\0\end{bmatrix}$?

Solution: First, we will write $\begin{bmatrix}1\\3\\0\end{bmatrix}$ as a linear combination of the basis vectors. To do this, we solve the linear system

$$c_1\begin{bmatrix}1\\1\\0\end{bmatrix} + c_2\begin{bmatrix}0\\1\\1\end{bmatrix} + c_3\begin{bmatrix}1\\0\\1\end{bmatrix} = \begin{bmatrix}1\\3\\0\end{bmatrix}$$

which gives these unique solution $c_1 = 2$, $c_2 = 1$, $c_3 = -1$.

Then

$$\begin{aligned}
T\begin{bmatrix}1\\3\\0\end{bmatrix} &= T\left(2\begin{bmatrix}1\\1\\0\end{bmatrix} + \begin{bmatrix}0\\1\\1\end{bmatrix} - \begin{bmatrix}1\\0\\1\end{bmatrix}\right)\\
&= 2T\begin{bmatrix}1\\1\\0\end{bmatrix} + T\begin{bmatrix}0\\1\\1\end{bmatrix} - T\begin{bmatrix}1\\0\\1\end{bmatrix}\\
&= 2\begin{bmatrix}2\\3\end{bmatrix} + \begin{bmatrix}0\\3\end{bmatrix} - \begin{bmatrix}0\\2\end{bmatrix} = \begin{bmatrix}4\\7\end{bmatrix}.
\end{aligned}$$

We could ask the same question for an arbitrary matrix, namely, what is $T\begin{bmatrix}x\\y\\z\end{bmatrix}$? See pg 237-238 in the textbook.

End of Example 100. □

This example shows that if we how a linear transform acts on a basis, then we know everything about it. This suggests that there is something very special about a basis, that it's a way to represent a vector space as a whole using just a finite set of vectors.

21.2 Unique basis representations

Theorem 101 (Unique basis representation)**.** *Let V be a vector space and let $v_1, \ldots, v_n$ be a basis for V. Then every vector can be written as a unique linear combination of $v_1, \ldots, v_n$.*

Proof. Let $u \in V$ be arbitrary. Since $v_1, \ldots, v_n$ is a basis, it spans V. Therefore we can write u as a linear combination

$$u = c_1 v_1 + c_2 v_2 + \ldots + c_n v_n \tag{17}$$

for some $c_1, \ldots, c_n \in \mathbb{R}$. This shows that u can be written as a linear combination of $v_1, \ldots, v_n$.

It remains to show that there is only one way to write u as a linear combination of $v_1, \ldots, v_n$.

Suppose we have some *other* linear combination

$$u = c_1' v_1 + c_2' v_2 + \ldots + c_n' v_n \tag{18}$$

where $c_1', \ldots, c_n \in \mathbb{R}$. Then

$$\begin{aligned} 0 &= u - u \\ &= (c_1 v_1 + c_2 v_2 + \ldots + c_n v_n) - (c_1' v_1 + c_2' v_2 + \ldots + c_n' v_n) \\ &= (c_1 - c_1')v + (c_2 - c_2')v + \ldots + (c_n - c_n')v \end{aligned}$$

Since $v_1, \ldots, v_n$ are linearly independent, it follows that

$$c_1 = c_1', \quad c_2 = c_2', \quad \ldots \quad c_n = c_n'$$

So the two linear combinations in Eqs. (17) and (18) are actually the same. This shows uniqueness. □

One consequence of Theorem 101 is that it allows us to translate any n-dimensional vector space, no matter how exotic, to the more familiar setting of $\mathbb{R}^n$. The idea is as follows:

1. Let V be an abstract vector space (functions, polynomials, whatever) of dimension n.
2. Pick a basis $v_1, \ldots, v_n$ for V
3. Each $a \in V$ can be written as
   $$a_1 v_1 + a_2 v_2 + \ldots + a_n v_n$$
   where $a_1, \ldots, a_n \in \mathbb{R}$. In particular, Theorem 101 tells us that there is only one choice of $a_1, \ldots, a_n$ that work. So we can write
   $$a = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}$$

In other words, any n-dimensional vector space is just $\mathbb{R}^n$ in disguise. So from one perpsective, to study all (finite-dimensional) vector spaces, it's enough to just study $\mathbb{R}^n$.

But there's a catch. Doing this requires "picking" a basis—and if you and I pick different bases, then we will end up with different representations of the same fundamental objects. Consider the set of direction vectors that an electron could move. Clearly this "is" $\mathbb{R}^3$, but since there are no coordinate axes, we need to *choose* what the x, y, z directions are: I can just declare that a particular direction is the x direction, for example. But if you choose a different direction, then we'll end up with different ways of representing the same directions. My $\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$ might be your $\begin{bmatrix} 5 \\ -3 \\ 2 \end{bmatrix}$. This may be undesirable.

If you and I each choose different basis for $\mathbb{R}^3$, are we really studying the same space? Maybe we are (I think so!). This is because many properties in linear algebra are "basis invariant" in the sense that they don't depend on the basis you pick. An example is the dimension of a vector space or its linear subspaces. Another example is linear transformations: many geometric properties of linear transformations (e.g., whether they preserve orientation, how much they stretch space, whether they collapse the dimesnion) don't depend on how we choose to represent the space. But if that's the case, then maybe we shouldn't need to rely on picking bases to study these things.

So my answer to the nexus question, "is a basis all you need" is "mostly yes, but sometimes it's more than than you need."

22 2025-10-15 | Week 08 | Lecture 22

> ***The nexus question of this lecture:*** *Why do the row space and column space always have the same dimension?*

This lecture is based on sections 5.1

22.1 The dimension of the column space equals the dimension of the row space

Consider the 3×2 matrix

$$\begin{bmatrix} 1 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

The row space is the vector space $\left\{ \begin{bmatrix} x & x \end{bmatrix} : x \in \mathbb{R} \right\}$. The column space is the vector space $\left\{ \begin{bmatrix} x \\ 0 \\ 0 \end{bmatrix} : x \in \mathbb{R} \right\}$. These are not the same. But they have the same dimesnion: 1.

The fact that the dimensions are equal is not a coincidence.

Theorem 102. *Given any $m \times n$ matrix A, the row space and column space always have the same dimesion. That is,* $\dim RS(A) = \dim CS(A)$.

22.2 Preliminaries

We begin with the following theorem.

Theorem 103. *Let X, Y be vector spaces. If $T : X \to Y$ is a linear transformation, then* $\ker(T)$ *is a subspace of X and* $\operatorname{range}(T)$ *is a subspace of Y.*

Proof. The fact that $\ker(T)$ is a subspace follows similarly to the proof of Theorem 56 (the "kernels are subspaces" theorem from lecture 13).

To show that $\operatorname{range}(T)$ is a subspace of Y, we need only verify two facts:

(i) $y + y' \in \operatorname{range}(T)$ whenever $y, y' \in \operatorname{range}(T)$

(ii) $cy \in \operatorname{range}(T)$ whenever $y \in Y$ and $c \in \mathbb{R}$

- **Proof of (i).** Let $y, y' \in Y$. Then there exist x, x' such that $T(x) = y$ and $T(x') = y'$. Since X is closed under vector addition, $x + x' \in X$. Moreover,

  $$T(x + x') = T(x) + T(x') = y + y'$$

  Therefore $y + y' \in \operatorname{range}(T)$.

- **Proof of (ii).** Let $y \in \operatorname{range}(T)$. Then there exists $x \in X$ such that $T(x) = y$. Therefore

  $$\begin{aligned} cy &= cT(x) \\ &= T(cx) \end{aligned}$$

  Since X is closed under scalar multiplication $cx \in X$. Therefore $cy \in \operatorname{range}(T)$.

$\square$

Because $\ker(T), \operatorname{range}(T)$ are subspaces by Theorem 103, they each have bases, a fact which we will need to use to prove the following theorem.

22.3 Rank-nullity, again

Theorem 104 (Rank Nullity Theorem – Theorem 5.4 in textbook)**.** *If $T : V \to W$ is a linear transformation where V is a finite dimensional vector space, then*

$$\dim \ker(T) + \dim \operatorname{range}(T) = \dim V$$

If T is an $m \times n$ matrix A, then $A : \mathbb{R}^n \to R^m$, and we can restate this as

$$\dim NS(A) + \dim CS(A) = n \tag{19}$$

(recall that the column space is the range, and that nullspace and kernel are synonyms).

Proof sketch. Suppose $\dim \ker(T) = k$ and $\dim V = n$ (note: $k \leq n$). Then with some work (omitted, see p242), one can find vectors $v_1, \dots, v_n \in V$ which form a basis for V with the property that

- $v_1, \dots, v_k$ is a basis for $\ker(T)$; and,
- $\underbrace{T(v_{k+1}), T(v_{k+1}), \dots, T(v_n)}_{n-k \text{ vectors}}$ is a basis for $\operatorname{range}(T)$.

This shows that $\dim \operatorname{range}(T) = n - k$. Therefore we have:

- $\dim \ker(T) = k$
- $\dim \operatorname{range}(T) = n - k$
- $\dim V = n$

Putting these together implies

$$\dim \ker(T) + \dim \operatorname{range}(T) = \dim(V).$$

□

22.4 Dimension of the nullspace

Theorem 105. *Let A be an $n \times m$ matrix. Then*

$$\dim NS(A) = n - r,$$

where $r = \dim RS(A)$.

An example will illustrate why Theorem 105 is true. Suppose we have a 4×5 matrix A and we want to solve the linear system $AX = 0$. Suppose that row-reducing the augmented matrix gives

$$\left[\begin{array}{ccccc|c} 1 & 0 & 2 & 3 & 4 & 0 \\ 0 & 1 & 5 & 6 & 7 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

This has two nonzero zero rows after row-reduction, so $r = 2$. Theorem 105 says that

$$\dim NS(A) = n - r = 5 - 2 = 3.$$

We can check this. The augmented matrix above corresponds to the linear system

$$\begin{aligned} x_1 + 2x_3 + 3x_4 + 4x_5 &= 0 \\ x_2 + 5x_3 + 6x_4 + 7x_5 &= 0 \end{aligned}$$

or

$$x_1 = -2x_3 - 3x_4 - 4x_5$$
$$x_2 = -5x_3 - 6x_4 - 7x_5$$

and this has three free variables x_3, x_4, x_5. Therefore the solutions to $AX = 0$ are of the form

$$\begin{bmatrix} -2x_3 - 3x_4 - 4x_5 \\ -5x_3 - 6x_4 - 7x_5 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = x_3 \begin{bmatrix} -2 \\ -5 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -3 \\ -6 \\ 0 \\ 1 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} -4 \\ -7 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

Therefore the nullspace has dimension 3 because

$$NS(A) = \text{Span}\left\{ \begin{bmatrix} -2 \\ -5 \\ 1 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} -3 \\ -6 \\ 0 \\ 1 \\ 0 \end{bmatrix} \begin{bmatrix} -4 \\ -7 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

22.5 Proof that $\dim CS(A) = \dim RS(A)$

We can now prove the following theorem, which says that the row space and column space of a matrix are the same thing.

Theorem 106. *If A is any $m \times n$ matrix, then*

$$\dim RS(A) = \dim NS(A)$$

Proof. By Theorem 104,

$$\dim NS(A) + \dim CS(A) = n. \tag{20}$$

By Theorem 105,

$$\dim NS(A) = n - \dim RS(A) \tag{21}$$

Plugging Eq. (21) into Eq. (20) gives

$$n - \dim RS(A) + \dim CS(A) = n$$

wich simplifies to

$$\dim RS(A) = \dim CS(A).$$

□

Theorem 106 says that we are justified in writing

$$\text{rank}(A) = \dim NS(A) = \dim CS(A)$$

and hence that

$$\text{rank}(A) + \text{nullity}(A) = n$$

for any $m \times n$ matrix A.

23 2025-10-17 | Week 08 | Lecture 23

> ***The nexus question of this lecture:*** *Why is matrix multiplication defined the way it is?*

This lecture is based on section 5.2

23.1 Combining linear transformations: the geometric perspective

Let V, W be vector spaces. A ***linear transformation*** is a function $T : V \to W$ given by with the property that

$$T(av + bv') = aT(v) + bT(v')$$

for any $v, v' \in V$ and $a, b \in \mathbb{R}$.

If $V = \mathbb{R}^2$ then linear transformations include things like

- rotations about the origin
- reflections across any line of the form $y = mx$
- projections of the plane onto a line of the form $y = mx$
- uniform stretching of space
- any shearing of the plane (i.e., which slants the plane)
- any combinations of these

(The story is similar for higher dimensions, but we get some weirder transformations: for example in $\mathbb{R}^4$, one can rotate space about two axes at once.)

The last bullet—that we can combine linear transformations—is significant. In math-speak, let

$$S, R : \mathbb{R}^2 \to \mathbb{R}^2$$

be linear transformations. If we first apply the transformation S and then apply R, the resulting transformation of space is the function composition $R \circ S$.

Example 107. For example, the following matrix rotates space clockwise about the origin by 30° (i.e., $\pi/6$ radians):

$$R = \begin{bmatrix} \cos(\pi/6) & \sin(\pi/6) \\ -\sin(\pi/6) & \cos(\pi/6) \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix}$$

and the following matrix stretches space in the x-direction by a factor of 2:

$$S = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$$

Then rotating and then stretching is achieved by doing S and then R. That is, by $R \circ S$. I claim this is given by the matrix product

$$RS = \begin{bmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \sqrt{3} & \frac{1}{2} \\ -1 & \frac{\sqrt{3}}{2} \end{bmatrix}$$

The core idea of this lecture is that matrix multiplication is defined in such a way that this always happens: i.e., that the product RS corresponds to doing transformation S and then doing transformation T.

In other words, the idea of composing linear transformations comes first. The reason why people defined matrix multiplication the way it is, is because they wanted it to represent composition of linear functions. We make this idea more precise in the next section.

End of Example 107. □

23.2 Matrix multiplication is composition of linear transformations

Recall that if $f : X \to Y$ and $g : Y \to Z$ are functions, we define the composition $g \circ f$ as

$$g \circ f(x) := g(f(x)),\ x \in X.$$

Function composition can be thought of a form of "multiplication" (sort of).

Theorem 108 (Composition of linear transformations)**.** *If $T : V \to W$ and $S : W \to U$ are linear transformations, then the composite function $ST = S \circ T$ is a linear transformation $ST : V \to U$ is a linear transformation.*

Claim 1: Every linear transformation can be represented by a matrix.

Proof of Claim 1. This is the subject of the next lecture. $\square_{\text{Claim}}$

Theorem 109. *When we compose linear transformations T and S, this corresponds to multiplying their matrices.*

We will prove the case of 2×2 matrices. For a proof of the general case (which holds for any matrices), see Theorem 5.10 in the textbook (p 258).

Proof. Suppose $T, S : \mathbb{R}^2 \to \mathbb{R}^2$ are linear transformations. By Claim 1 both T and S can be represented by 2×2 matrices. That is, there exist $a, b, c, d, A, B, C, D \in \mathbb{R}$ such that

$$T \begin{bmatrix} x \\ y \end{bmatrix} := \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} ax + by \\ cx + dy \end{bmatrix} \quad \text{and} \quad S \begin{bmatrix} x \\ y \end{bmatrix} := \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} Ax + By \\ Cx + Dy \end{bmatrix}.$$

The matrix product is

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \underbrace{\begin{bmatrix} Aa + Bc & Ab + Bd \\ Ca + Dc & Cb + Dd \end{bmatrix}}_{\text{call this } M}$$

To show that $S \circ T$ corresponds exactly to matrix multiplication, we need to show that

$$S \circ T \left(\begin{bmatrix} x \\ y \end{bmatrix} \right) = M \begin{bmatrix} x \\ y \end{bmatrix} \tag{22}$$

for all $\begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^2$. This is because left-hand side of the above equation is the composition $S \circ T$. The right hand side is the product of the matrices for S and T. By showing that the equality holds, we will show that they are the same thing.

Indeed,

$$\begin{aligned} S \circ T \left(\begin{bmatrix} x \\ y \end{bmatrix} \right) &= S \left(T \begin{bmatrix} x \\ y \end{bmatrix} \right) \\ &= S \left(\begin{bmatrix} ax + by \\ cx + dy \end{bmatrix} \right) \\ &= \begin{bmatrix} A(ax + by) + B(cx + dy) \\ C(ax + by) + D(cx + dy) \end{bmatrix} \\ &= \begin{bmatrix} (Aa + Bc)x + (Ab + Bd)y \\ (Ca + Dc)x + (Cb + Dd)y \end{bmatrix} \\ &= M \begin{bmatrix} x \\ y \end{bmatrix}. \end{aligned}$$

We have demonstrated that Eq. (22) holds. We are done. :) $\square$

24 2025-10-20 | Week 09 | Lecture 24

> ***The nexus question of this lecture:*** *How do we find a matrix to represent a linear transformation?*

This lecture is based on section 5.2 and 5.3

24.1 The algebra of linear transformations

24.1.1 Algebraic operations

Let V and W be vector spaces. Consider two linear transformations $T : V \to W$ and $S : V \to W$. The function $T + S$ is defined as

$$(T + S)(v) := T(v) + S(v),\ v \in V$$

And for $c \in \mathbb{R}$, cT is defined as

$$(cT)(v) := cT(v),\ v \in V$$

Theorem 110. *If $T, S : V \to W$ are linear transformations, then so are $T + S$ and cT.*

In other words, the set of linear transformations $T : V \to W$ is closed under addition and scalar multiplication. This suggests that set of linear transformations from V to W forms a vector space. (It does. The rabbit hole goes deep...)

Example 111. Suppose

$$S \begin{bmatrix} x \\ y \end{bmatrix} := \begin{bmatrix} 2x - y \\ x + 2y \end{bmatrix} \quad \text{and} \quad T \begin{bmatrix} x \\ y \end{bmatrix} := \begin{bmatrix} x + y \\ x - y \end{bmatrix}$$

End of Example 111. □

Question 1: What is $S + 3T$?

$$\begin{aligned} (S + 3T)\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) &= S\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) + 3T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) \\ &= \begin{bmatrix} 2x - y \\ x + 2y \end{bmatrix} + 3\begin{bmatrix} x + y \\ x - y \end{bmatrix} \\ &= \begin{bmatrix} 2x - y \\ x + 2y \end{bmatrix} + \begin{bmatrix} 3x + 3y \\ 3x - 3y \end{bmatrix} \\ &= \begin{bmatrix} 5x + 2y \\ 4x - y \end{bmatrix} \end{aligned}$$

Question 2: What is $S \circ T$?

$$\begin{aligned} S \circ T \begin{bmatrix} x \\ y \end{bmatrix} &= S\left(T \begin{bmatrix} x \\ y \end{bmatrix}\right) \\ &= S \begin{bmatrix} x + y \\ x - y \end{bmatrix} \\ &= \begin{bmatrix} 2(x + y) - (x - y) \\ (x + y) + 2(x - y) \end{bmatrix} \\ &= \begin{bmatrix} x + 3y \\ 3x - y \end{bmatrix} \end{aligned}$$

At this point we note that we can represent

$$S = \begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix}, \quad T = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}, \quad S \circ T = \begin{bmatrix} 1 & 3 \\ 3 & -1 \end{bmatrix}, \quad S + 3T = \begin{bmatrix} 5 & 2 \\ 4 & -1 \end{bmatrix}$$

Notice that

$$\underbrace{\begin{bmatrix}2 & -1\\1 & 2\end{bmatrix}}_{S}\underbrace{\begin{bmatrix}1 & 1\\1 & -1\end{bmatrix}}_{T}=\underbrace{\begin{bmatrix}1 & 3\\3 & -1\end{bmatrix}}_{S\circ T}\quad\text{and that}\quad\underbrace{\begin{bmatrix}2 & -1\\1 & 2\end{bmatrix}}_{S}+3\underbrace{\begin{bmatrix}1 & 1\\1 & -1\end{bmatrix}}_{T}=\underbrace{\begin{bmatrix}5 & 2\\4 & -1\end{bmatrix}}_{S+3T}$$

24.1.2 Algebraic rules

The standard algebraic rules work with linear transformations:

Theorem 112. *Let R, S, T be linear transformations and $c, d \in R$. Then*

- $S + T = T + S$
- $R + (S + T) = (R + S) + T$
- $c(dT) = (cd)T$
- $c(S + T) = cS + cT$
- $R(ST) = (RS)T$
- $R(S + T) = RS + RT$
- $(R + S)T = RT + ST$
- $c(ST) = (cS)T = S(cT)$

Observe the glaring lack of the rule that $ST = TS$, which doesn't hold in general. In fact, all of the above rules are are identical to the rules for matrix algebra (i.e., if S, T, and R are matrices).

This makes sense because, as we show in the next section, linear transformations can be encoded with matrices.

24.2 Representing a linear transformation with a matrix

Let $T : V \to W$ be a linear transformation.
Goal: Find a matrix to represent T.

We'll start by choosing a basis for V and W. Let $\alpha = \{v_1, \ldots, v_n\}$ be a basis for V and $\beta = \{w_1, \ldots, w_m\}$ be a basis for W.

Following the ideas of Example 100, it is enough to understand how T acts on the basis $v_1, \ldots, v_n$.

Observe that

- Every vector in W can be written as a unique linear combination of the basis vectors $w_1, \ldots, w_m$ (because these form a basis).
- The vectors $T(v_1), \ldots, T(v_n)$ are all vectors in W.

From these two observations, we can find scalars $a_{ij} \in \mathbb{R}$ such that

$$\begin{aligned}T(v_1) &= a_{11}w_1 + a_{21}w_2 + \ldots + a_{m1}w_m\\T(v_2) &= a_{12}w_1 + a_{22}w_2 + \ldots + a_{m2}w_m\\&\vdots\\T(v_n) &= a_{1n}w_1 + a_{2n}w_2 + \ldots + a_{mn}w_m.\end{aligned}$$

We can change notation by writing thes as ***coordinate vectors***:

$$T(v_1) = \begin{bmatrix}a_{11}\\a_{21}\\\vdots\\a_{m1}\end{bmatrix}_\beta \quad T(v_2) = \begin{bmatrix}a_{12}\\a_{22}\\\vdots\\a_{m2}\end{bmatrix}_\beta \quad \cdots \quad T(v_n) = \begin{bmatrix}a_{1n}\\a_{2n}\\\vdots\\a_{mn}\end{bmatrix}_\beta$$

Here, the subscript β indicates that these vectors represent linear combinations of the basis $\beta = \{w_1, \ldots, w_m\}$. That is, we are using the notation

$$\begin{bmatrix} a_{1i} \\ a_{2i} \\ \vdots \\ a_{mi} \end{bmatrix}_\beta = a_{1i}w_1 + a_{2i}w_2 + \ldots + w_{mi}w_m, \quad i = 1, \ldots, n.$$

We can then represent T simply as the $m \times n$ matrix

$$[T]_\alpha^\beta = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

whose columns are the above vectors. We call this matrix ***the matrix of T with respect to the bases α and β.*** It is denoted $[T]_\alpha^\beta$.

Example 113. Let $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x+y \\ x-y \end{bmatrix}$. Then we can write $v_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $v_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

Here, we take as our basis $\alpha = \beta = \{v_1, v_2\}$. Observe that

- $T(v_1) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$
- $T(v_2) = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

Then

$$[T]_\alpha^\beta = \begin{bmatrix} | & | \\ T(v_1) & T(v_2) \\ | & | \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

Which makes sense, because

$$T\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x+y \\ x-y \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}.$$

End of Example 113. $\square$

25 2025-10-22 | Week 09 | Lecture 25

> ***The nexus question of this lecture:*** *The matrix we use to represent a linear transformation depends on a choice of bases. How does it change when whe choose different bases?*

This lecture is based on section 5.3 in the textbook

25.1 Introduction to change of basis

The following theorem summarizes the main ideas from the previous lecture.

Theorem 114 (Change of Basis I)**.** *Let $\mathbb{A}$ and $\mathbb{B}$ be vector spaces. If $T : \mathbb{A} \to \mathbb{B}$ is a linear map, $\alpha = \{\alpha_1, \ldots, \alpha_n\}$ is a basis of $\mathbb{A}$, and $\beta = \{\beta_1, \ldots, \beta_m\}$ is a basis of $\mathbb{B}$, then we can represent T by the matrix*

$$[T]_\alpha^\beta = \begin{bmatrix} | & | & & | \\ T(\alpha_1) & T(\alpha_2) & \ldots & T(\alpha_n) \\ | & | & & | \end{bmatrix}$$

where $T(\alpha_i) = A\alpha_i = a_{1i}\beta_1 + a_{2i}\beta_2 + \ldots + a_{mi}\beta_m$ for each $i = 1, \ldots, n$.
(Here, $\alpha_1, \ldots, \alpha_n$ are vectors in $\mathbb{A}$ and $\beta_1, \ldots, \beta_n$ are vectors in $\mathbb{B}$.)

Example 115 (Example 1 in Section 5.3)**.** Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear transformation

$$T \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 5x + z \\ 3x + 2y - 3z \\ 5x \end{bmatrix}$$

Let $\alpha = \{e_1, e_2, e_3\}$, where

$$e_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad e_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad e_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

Clearly α is a basis of $\mathbb{R}^3$.

(a) **Question:** Find the matrix of T with respect to the standard basis α, that is, find $[T]_\alpha^\alpha$.
Solution: Observe that

$$T(e_1) = T \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 5 \\ 3 \\ 5 \end{bmatrix} = 5e_1 + 3e_2 + 5e_3$$

Similarly,

$$T(e_2) = \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix} = 0e_1 + 2e_2 + 0e_3 \quad \text{and} \quad T(e_3) = \begin{bmatrix} 1 \\ -3 \\ 0 \end{bmatrix} = 1e_1 - 3e_2 + 0e_3$$

These becomes the columns of the matrix

$$[T]_\alpha^\alpha = \begin{bmatrix} 5 & 0 & 1 \\ 3 & 2 & -3 \\ 5 & 0 & 0 \end{bmatrix}$$

(b) **Question:** Let $\beta = \{\beta_1, \beta_2, \beta_3\}$, where

$$\beta_1 = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}, \quad \beta_2 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, \quad \beta_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

This is a basis of $\mathbb{R}^3$. Find $[T]_\beta^\beta$, the matrix of T with respect to the basis β.

(The point of this example is to show you how to find $[T]_\beta^\beta$ mechanistically; I'm not trying to illustrate why this particular choice of basis β is good or meaningful—it's not.)

Solution: The columns of $[T]_\beta^\beta$ are the vectors $T(\beta_1), T(\beta_2), T(\beta_3)$ expressed in terms of the vectors in β. By plugging $\beta_1, \beta_2, \beta_3$ into T, we see that

$$T(\beta_1) = \begin{bmatrix} 7 \\ -1 \\ 5 \end{bmatrix}, \quad T(\beta_2) = \begin{bmatrix} 6 \\ -2 \\ 5 \end{bmatrix}, \quad T(\beta_3) = \begin{bmatrix} 6 \\ 2 \\ 5 \end{bmatrix}$$

But in the above equations, the right hand sides are all expressed in terms of e_1, e_2, e_3. For example,

$$T(\beta_1) = \begin{bmatrix} 7 \\ -1 \\ 5 \end{bmatrix} = 7e_1 - e_2 + 5e_3.$$

This is no good: we need to express them using in basis β. Observe that

- $T(\beta_1) = \begin{bmatrix} 7 \\ -1 \\ 5 \end{bmatrix} \overset{*}{=} -2 \underbrace{\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}}_{\beta_1} +4 \underbrace{\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}}_{\beta_2} +5 \underbrace{\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}}_{\beta_3} = -2\beta_1 + 4\beta_2 + 5\beta_3 = \begin{bmatrix} -2 \\ 4 \\ 5 \end{bmatrix}_\beta$

  Similarly,
- $T(\beta_2) = \begin{bmatrix} 6 \\ -2 \\ 5 \end{bmatrix} \overset{*}{=} -1 \underbrace{\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}}_{\beta_1} +4 \underbrace{\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}}_{\beta_2} +3 \underbrace{\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}}_{\beta_3} = \begin{bmatrix} -1 \\ 4 \\ 3 \end{bmatrix}_\beta$
- $T(\beta_3) = \begin{bmatrix} 6 \\ 2 \\ 5 \end{bmatrix} \overset{*}{=} -1 \underbrace{\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}}_{\beta_1} +2 \underbrace{\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}}_{\beta_2} +5 \underbrace{\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}}_{\beta_3} = \begin{bmatrix} -1 \\ 2 \\ 5 \end{bmatrix}_\beta$

Therefore by Theorem 114

$$[T]_\beta^\beta = \begin{bmatrix} | & | & | \\ T(\beta_1) & T(\beta_2) & T(\beta_3) \\ | & | & | \end{bmatrix} = \begin{bmatrix} -2 & -1 & -1 \\ 4 & 4 & 2 \\ 5 & 3 & 5 \end{bmatrix}$$

This matrix represents the same linear transformation, T, but now we are using β as the basis for $\mathbb{R}^3$ rather than α as in part (a). For some linear transformations, the is a "best" basis to use (and often it is not the standard basis!)

There is also a

The starred equalities above

Note that the equalities marked with a $(*)$ required solving a linear system. In the $T(\beta_1)$ case, for example, we needed to solve the linear system

$$\begin{bmatrix} 7 \\ -1 \\ 5 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} + c_3 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix},$$

which has solution $c_1 = -2$, $c_2 = 4$, $c_3 = 5$.

End of Example 115. □

25.2 The change of basis matrix

Suppose $\alpha = \{\alpha_1, \ldots, \alpha_n\}$ and $\beta = \{\beta_1, \ldots, \beta_n\}$ are bases for the same vector space V.

We can write the vectors of basis β in terms of the vectors of basis α:

$$\begin{aligned}
\beta_1 &= p_{11}\alpha_1 + p_{21}\alpha_2 + \ldots + p_{n1}\alpha_n \\
\beta_2 &= p_{12}\alpha_1 + p_{22}\alpha_2 + \ldots + p_{n2}\alpha_n \\
&\vdots \\
\beta_n &= p_{1n}\alpha_1 + p_{2n}\alpha_2 + \ldots + p_{nn}\alpha_n.
\end{aligned}$$

where the p's are all scalars. In other notation, we can write the above equations as

$$\beta_1 = \begin{bmatrix} p_{11} \\ p_{21} \\ \vdots \\ p_{n1} \end{bmatrix}_\alpha, \quad \beta_2 = \begin{bmatrix} p_{12} \\ p_{22} \\ \vdots \\ p_{n2} \end{bmatrix}_\alpha, \quad \cdots \beta_n = \begin{bmatrix} p_{1n} \\ p_{2n} \\ \vdots \\ p_{nn} \end{bmatrix}_\alpha.$$

Concatenating these column vectors, we obtain the matrix

$$P = \begin{bmatrix} p_{11} & p_{12} & \cdots & p_{1n} \\ p_{21} & p_{22} & \cdots & p_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ p_{n1} & p_{n2} & \cdots & p_{nn} \end{bmatrix}$$

This matrix is called ***the change of basis matrix from α to β***.

The following theorem is Corollary 5.13 in the textbook.

Theorem 116 (Change of basis II)**.** *If $T : V \to V$ is a linear transformation, α and β are bases of V, and P is the change of basis matrix from α to β, then $[T]_\beta^\beta = P^{-1}[T]_\alpha^\alpha P$.*

Example 117 (Using a change of basis matrix)**.** In Example 115, we had

$$[T]_\alpha^\alpha = \begin{bmatrix} 5 & 0 & 1 \\ 3 & 2 & -3 \\ 5 & 0 & 0 \end{bmatrix} \quad \text{and} \quad [T]_\beta^\beta = \begin{bmatrix} -2 & -1 & -1 \\ 4 & 4 & 2 \\ 5 & 3 & 5 \end{bmatrix}$$

One can compute the change of basis matrix P and its inverse P^{-1} for α and β. (For details, see Example 2 pages 260-261.) These are

$$P = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 2 & 1 & 1 \end{bmatrix} \quad \text{and} \quad P^{-1} = \begin{bmatrix} -1 & 0 & 1 \\ 1/2 & -1/2 & 0 \\ 3/2 & 1/2 & -1 \end{bmatrix}.$$

Thus we could solve part (b) using the formula in Theorem 116:

$$\begin{aligned}
[T]_\beta^\beta &= P^{-1}[T]_\alpha^\alpha P \\
&= \begin{bmatrix} -1 & 0 & 1 \\ 1/2 & -1/2 & 0 \\ 3/2 & 1/2 & -1 \end{bmatrix} \begin{bmatrix} 5 & 0 & 1 \\ 3 & 2 & -3 \\ 5 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 2 & 1 & 1 \end{bmatrix} \\
&= \begin{bmatrix} -2 & -1 & -1 \\ 4 & 4 & 2 \\ 5 & 3 & 5 \end{bmatrix}.
\end{aligned}$$

This computation agrees with the answer for $[T]_\beta^\beta$ that we obtained in Example 115.

End of Example 117. □

26 2025-10-24 | Week 09 | Lecture 26

The nexus question of this lecture: *What do linear transformations of 2-dimensional space look like?*

Example 118 (Horizontal and Vertical Dilations)**.** Scale space by $a > 0$ in the x direction and $b > 0$ in the y-direction

$$\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$$

This can be undone by

$$\begin{bmatrix} \frac{1}{a} & 0 \\ 0 & \frac{1}{b} \end{bmatrix}$$

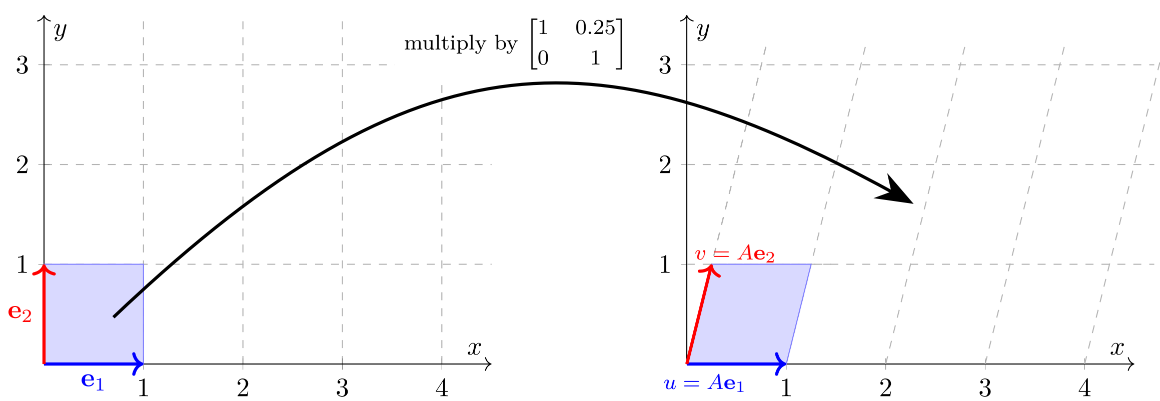
End of Example 118. □

Example 119 (Shear)**.**

$$A = \begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & 0 \\ b & 1 \end{bmatrix}$$

These have inverses

$$A^{-1} = \begin{bmatrix} 1 & -a \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad B^{-1} = \begin{bmatrix} 1 & 0 \\ -b & 1 \end{bmatrix}$$



This transformation is a ***horizontal shear***. It does not change the height of any point. Points above the x-axis get shifted right (because $a = 0.25$ is positive) and points below the x-axis get shifted left. The further away from the x-axis, the greater the horizontal shift. The x-axis is not changed at all by this transformation—it is ***invariant*** under the linear transformation. This transformation can be undone by $\begin{bmatrix} 1 & -0.25 \\ 0 & 1 \end{bmatrix}$, which is the inverse.

The transformation

$$\begin{bmatrix} 1 & 0 \\ b & 1 \end{bmatrix}$$

is similar, but effectuates a ***vertical shear***.

End of Example 119. □

A square matrix A is said to be an ***orthogonal matrix*** is A and $A^\top$ are inverses.

The determinant of an orthogonal matrix is alwasy ± 1 because

$$1 = \det(I) = \det(A^\top A) = \det(A^\top)\det(A) = (\det(A))^2\,.$$

Orthogonal matrices has the property that they *preserve distances*, i.e., that

$$\operatorname{dist}(x, y) = \operatorname{dist}(Ax, Ay).$$

In words, if you choose any two points, their distance doesn't change under the linear transformation—the points may get sent to new coordinates, but the distance between them doesn't change.

If A is an orthogonal matrix and $\det(A) = 1$, then we say that A is a ***rotation***.

Example 120 (Rotation)**.** Let $R_\theta : \mathbb{R}^2 \to \mathbb{R}^2$ be a rotation of the plane about origin by θ radians counter-clockwise. This linear transformation is represented by the matrix

$$R_\theta = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$$

Using the trig identities

$$\cos(a+b) = \cos(a)\cos(b) - \sin(a)\sin(b)$$
$$\sin(a+b) = \sin(a)\cos(b) + \cos(a)\sin(b)$$

we can show that

$$R_a R_b = R_{a+b}$$

That says that, eg if you rotate by say $10°$ and then by $24°$, the result is a rotation by $34°$.

Observe that $R_\theta^\top = R_{-\theta}$. In other words,

$$R_\theta^\top R_\theta = R_{-\theta} R_\theta = R_0 = I$$

Therefore R and $R^\top$ are inverses.

End of Example 120. $\square$

27 2025-10-27 | Week 10 | Lecture 27

The nexus question of this lecture: *What do linear transformations of 2-dimensional space look like?*

Example 121 (Composing transformations)**.** Find a linear transformation that consists of stretching space in the direction $\theta = 45°$ (i.e. $\theta = \pi/4$ radians) by a factor of 2.
Answer: The linear transformation we are looking for is

$$R_{-\frac{\pi}{4}} \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} R_{\frac{\pi}{4}}$$

since this rotates space 45° counterclockwise, then stretches space vertically by a factor of 2, and rotates space back 45° clockwise.

Now,

$$R_{\frac{\pi}{4}} = \begin{bmatrix} \cos(\frac{\pi}{4}) & -\sin(\frac{\pi}{4}) \\ \sin(\frac{\pi}{4}) & \cos(\frac{\pi}{4}) \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$$

and

$$R_{-\frac{\pi}{4}} = R_{\frac{\pi}{4}}^{\top} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$$

Doing the matrix multiplication we get:

$$\frac{1}{2} \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$$

This matrix represents the linear transformation which stretches space in the direction $\theta = 45°$.

End of Example 121. □

A matrix P is said to be a ***projection*** if $P^2 = P$. The simplest projection is the identity I, but usually projections collapse dimension. If, in addition, the matrix P is symmetric (that is, if $P = P^{\top}$), then P is called an ***orthogonal projection***. If $P^{\top} \neq P$ it is called an oblique projection.

Example 122 (Projections)**.** Consider the line passing through the origin and the point $(c, s) = (\cos(\theta), \sin(\theta))$. Let P be the projection onto the θ-line. This is given by the matrix

$$P = \begin{bmatrix} c^2 & cs \\ cs & s^2 \end{bmatrix}$$

This matrix has no inverse, because the linear transformation has no inverse (because it collapses dimension). Indeed, $\det P = c^2 s^2 - (cs)^2 = 0$

Points on the θ-line are pojected to themselves. So projecting twice is the same as projecting onces, so $P^2 = P$. This is easy to check using $s^2 + c^2 = 1$:

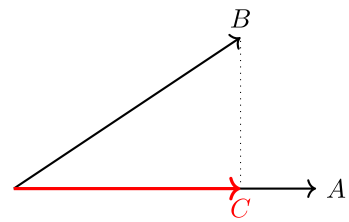
$$P^2 \begin{bmatrix} c^2 & cs \\ cs & s^2 \end{bmatrix} \begin{bmatrix} c^2 & cs \\ cs & s^2 \end{bmatrix} = \begin{bmatrix} c^2(c^2+s^2) & cs(c^2+s^2) \\ cs(c^2+s^2) & s^2(c^2+s^2) \end{bmatrix} = P$$

The nullspace of P is the line $y = -\frac{c}{s}x$. (picture)

Connection with vector projections: Recall from calculus class that the ***vector projection*** of B onto A is defined as the vector C given by

$$C = \underbrace{\left(\frac{A \cdot B}{|A|}\right)}_{\text{signed length of C}} \underbrace{\left(\frac{A}{|A|}\right)}_{\text{unit vector}} \tag{23}$$

This corresponds to the picicture



In other words, C is the shadow of B on A, if we shine a light directly above B. This is an orthogonal projection because the dotted line is orthogonal to A.

Let's work out an example. Suppose

$$\vec{A} = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} \quad \text{and} \quad \vec{B} = \begin{bmatrix} -2 \\ 3 \\ 1 \end{bmatrix}.$$

Then by Eq. (23), the vector projection is

$$C = \begin{bmatrix} 1/2 \\ 1/2 \\ 1 \end{bmatrix}.$$

Can we find a matrix P that effectuates this transformation? That is, such that $PB = C$?

In general, the matrix for an orthogonal projection can be written as

$$P = U(U^\top U)^{-1}U^\top, \tag{24}$$

where U is any matrix whose columns form a basis of the subspace onto which we are projecting. In the case of the vector projection of B onto A, we are projecting onto the line spanned by the vecctor A. Hence, we can take

$$U = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}.$$

Now let's compute P. First observe that

$$U^\top U = \begin{bmatrix} 1 & 1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} = 1 + 1 + 4 = 6,$$

so

$$(U^\top U)^{-1} = \frac{1}{6}.$$

Plugging this into Eq. (24), we get

$$P = \frac{1}{6} \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} \begin{bmatrix} 1 & 1 & 2 \end{bmatrix} = \frac{1}{6} \begin{bmatrix} 1 & 1 & 2 \\ 1 & 1 & 2 \\ 2 & 2 & 4 \end{bmatrix}.$$

Now we can check that $PB = C$:

$$PB = \frac{1}{6} \begin{bmatrix} 1 & 1 & 2 \\ 1 & 1 & 2 \\ 2 & 2 & 4 \end{bmatrix} \begin{bmatrix} -2 \\ 3 \\ 1 \end{bmatrix} = \frac{1}{6} \begin{bmatrix} -2+3+2 \\ -2+3+2 \\ -4+6+4 \end{bmatrix} = \frac{1}{6} \begin{bmatrix} 3 \\ 3 \\ 6 \end{bmatrix} = \begin{bmatrix} 1/2 \\ 1/2 \\ 1 \end{bmatrix} = C.$$

End of Example 122. $\square$

Example 123 (Reflection)**.** Reflection across the θ-line (i.e. the line passing through the origin with angle θ) is given by the following matrix:

$$H = \begin{bmatrix} 2c^2 - 1 & 2cs \\ 2cs & 2s^2 - 1 \end{bmatrix}$$

where $c = \cos(\theta)$ and $s = \sin(\theta)$.

Observe

$$H^2 = H$$

$$H = 2P - I$$

For example, $\theta = 90°$ gives

$$H = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

End of Example 123. □

28 2025-10-29 | Week 10 | Lecture 28

The nexus question of this lecture: *What are eigenvalues and eigenvectors and how do we find them?*

One way to understand what a linear linear transformation does is to undertand which parts of space are invariant — that is, which parts of space don't change.

28.1 Eigenvalues

Definition 124 (Eigenvalue, eigenvector)**.** If A is an $n \times n$ matrix, an ***eigenvector*** of A is a nonzero column vector v such that

$$Av = \lambda v$$

for some scalar $\lambda \in \mathbb{C}$. The scalar λ is called an ***eigenvalue***.

Example 125. Let

$$A = \frac{1}{2}\begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$$

In Example 121, we say that this transformation corresponds to a dilation by a factor of 2 in the direction of 45°.

By geometric considerations, we can see that the line $y = x$ is invariant under this transformation (vectors along this line get scaled by 2 but don't jump off the line). Also we see that the line $y = -x$ is invariant (vectors along this line don't change at all).

The vector $v = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ lies on the line $y = x$. We see that

$$Av = 2v$$

so that $v = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is an eigenvector with eigenvalue 2.

Similarly, the following vector lies on the line $y = -x$:

$$v = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

It satisfies $Av = 1v$. Hence it is an eigenvector with eigenvalue 1.

End of Example 125. □

Example 126. Let

$$A = \begin{bmatrix} 1 & -3 \\ -2 & 2 \end{bmatrix}$$

has eigenvewctor

$$v = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

with eigenvalue 4.

End of Example 126. □

28.2 How do we find eigenvalues?

Idea: look to the system $AX = \lambda X$. This is equivalent to

$$(\lambda I - A)X = 0.$$

Since $X \neq 0$ (since by definition eigenvectors must be nonzero), we conclude that the matrix $\lambda I - A$ is singular, and hence by the key theorem, $\det(\lambda I - A) = 0$.

Theorem 127. *Let A be an $n \times n$ matrix. Then $\lambda \in \mathbb{C}$ is an eigenvalue of A if and only if* $\det(\lambda I - A) = 0$.

Definition 128. The ***characteristic equation of A*** is

$$\det(\lambda I - A) = 0.$$

When A is an $n \times n$ matrix, the left hand side of the characteristic equation is a polynomial in the variable λ of degree n, and is called the ***characteristic polynomial*** of A.

Example 129. Let

$$A = \begin{bmatrix} 1 & -3 \\ -2 & 2 \end{bmatrix}.$$

Then the characteristic polynomial of A is

$$\begin{aligned} \det(\lambda I - A) &= \det \begin{bmatrix} \lambda - 1 & 3 \\ 2 & \lambda - 2 \end{bmatrix} \\ &= (\lambda - 1)(\lambda - 2) - 6 \\ &= \lambda^2 - 3\lambda - 4 \\ &(\lambda - 4)(\lambda + 1) \end{aligned}$$

This is equal to zero if and only if $\lambda = 4$ or $\lambda = -1$. Therefore the eigenvalues of A are $\lambda = 4$ and $\lambda = -1$.

End of Example 129. $\square$

28.3 How do we find eigenvectors?

Idea: First find the eigenvalues λ. Then for each eigenvalue λ, the eigenvectors are the nontrivial solutions of the homogeneous ssytem

$$(\lambda I - A)X = 0.$$

(This is a linear system which we can solve using row reduction.)

In other words, the eigenvectors are the nonzero vectors in the linear subspace

$$NS(\lambda I - A).$$

So we just need to compute a basis of this nullspace, which is called the ***eigenspace***. When we ask to find the eigenvalues, it is always enough to just compute the basis of the eigenspace.

Example 130. Find the eigenvectors of the matrix

$$A = \begin{bmatrix} 1 & -3 \\ -2 & 2 \end{bmatrix}$$

The equations we need to solve are

- **When $\lambda = 4$:** $4I - A = 0$ or

  $$\begin{bmatrix} 3 & 3 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

  Reducing find the nullspace is

  $$NS(4I - A) = \left\{ y \begin{bmatrix} -1 \\ 1 \end{bmatrix}, y \in \mathbb{R} \right\}$$

  Technically, all vectors in $NS(4I - A)$ are eigenvectors for $\lambda = 4$. To give a concrete example, we have eigenvector $v = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$.

End of Example 130. $\square$

29 2025-10-31 | Week 10 | Lecture 29

Examples of computing eigenvectors and eigenvalues

29.1 Recall definitions

Definition 131 (Eigenvalue,eigenvector)**.** If A is an $n \times n$ matrix, an ***eigenvector*** of A is a nonzero column vector v such that

$$Av = \lambda v$$

for some scalar $\lambda \in \mathbb{C}$. The scalar λ is called an ***eigenvalue***.

Definition 132. The ***characteristic polynomial of A*** is

$$\det(\lambda I - A).$$

Recall Theorem 127:

Theorem 133. *Let A be an $n \times n$ matrix. Then $\lambda \in \mathbb{C}$ is an eigenvalue of A if and only if* $\det(\lambda I - A) = 0$.

29.2 Examples of eigenvector/eigenvalue computations

Idea: Find the eigenvalues before finding the eigenvectors. Then for each eigenvalue λ, find the nullspace $NS(\lambda I - A)$. The vectors in the nullspace are the eigenvectors corresponding to λ. (Usually we just pick out a basis.)

Example 134. Find the eigenvectors of the matrix

$$A = \begin{bmatrix} 1 & -3 \\ -2 & 2 \end{bmatrix}$$

Last class, we showed that A has eigenvalues $\lambda = 4$ and $\lambda - 1$ The equations we need to solve are

- **When $\lambda = 4$:** we computed the nullsapce of the matrix $4I - A$, which gave

  $$NS(4I - A) = \left\{ y \begin{bmatrix} -1 \\ 1 \end{bmatrix}, y \in \mathbb{R} \right\}$$

  So $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$ is an eigenvector of A corresponding to $\lambda = 4$

- **When $\lambda = -1$:** $-I - A = 0$ Here we get a NS generated by a single basis element $\begin{bmatrix} 3 \\ 2 \end{bmatrix}$ so this is an eigenvector as well.

End of Example 134. □

Example 135. Find the eigenvaluess and eigenvectors of

$$A = \begin{bmatrix} 2 & -1 & 3 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

First we compute the characteristic polynomial:

$$\begin{aligned} \det(\lambda I - A) &= \det \begin{bmatrix} \lambda - 2 & 1 & -3 \\ 0 & \lambda + 1 & 0 \\ 0 & 0 & \lambda + 1 \end{bmatrix} \\ &= (\lambda - 2)(\lambda + 1)^2 \end{aligned}$$

This equals zero iff $\lambda = 2$ or $\lambda = -1$. These are the eigvenvalues.

To find eigenvectors, we need to find a basis for the nullspaces $NS(2I - A)$ and $NS(-I - A)$.

- $\lambda = 2$. Need to find $NS(2I - A)$.

  Row reducing the augmented matrix

$$\left[\begin{array}{ccc|c} 0 & 1 & -3 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 3 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

  This corresponds to the system with $x_2 = 0$, $x_3 = 0$, and x_1 a free variable. That is,

$$NS(2I - A) = \left\{ \begin{bmatrix} x_1 \\ 0 \\ 0 \end{bmatrix} : x_1 \in \mathbb{R} \right\} = \left\{ x_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} : x_1 \in \mathbb{R} \right\}$$

  So $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ is a basis for the nullspace. It is an eigenvector. Indeed, $A \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = 2 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$.

- $\lambda = -1$. Need to find $NS(-I - A)$.

End of Example 135. $\square$

30 2025-11-03 | Week 11 | Lecture 30

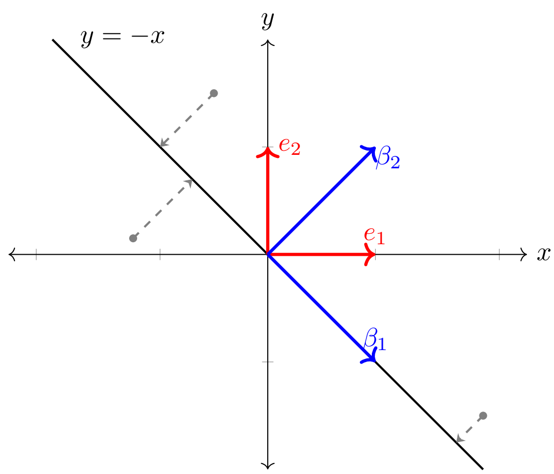
> ***The nexus question of this lecture:*** *What can we say about two matrices that represent the same linear transformation?*

This lecture draws from G. Strang's textbook "Linear Algebra and its Applications"

30.1 Similarity

Example 136 (Two matrices that represent the same projection)**.** Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the projection onto the line $y = -x$. We don't need a basis to describe this transformation. Even without a matrix, we can observe geometric properties of T:

- the dimension of its range is 1, so it has rank 1
- the dimension of its kernel (the line $y = x$) is 1, so the nullity is 1
- it collapses space, so has determinant 0
- it has two eigenvalues: 1 (with eigenvector β_1) and 0 (with eigenvector β_2)



Recall we can always represent a linear transformation $T : V \to V$ in the form of a matrix, but this representation depends on a choice of basis for V (see Section 24.2). We'll compare two choices of basis, $\alpha = \{e_1, e_2\}$ and $\beta = \{\beta_1, \beta_1\}$ (shown in the picture above).

- **The matrix $[T]_\beta^\beta$.** Note that β_1 and β_2 are eigenvectors of T. They are linearly independent and therefore form a basis (this doesn't always happen, but when it does, eigenvectors are the best basis). Using the notation $\begin{bmatrix} a \\ b \end{bmatrix}_\beta = a\beta_1 + b\beta_2$, observe from the figure that

$$T(\beta_1) = \beta_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}_\beta \quad \text{and} \quad T(\beta_2) = 0 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}_\beta .$$

Hence

$$[T]_\beta^\beta = \begin{bmatrix} | & | \\ T(\beta_1) & T(\beta_2) \\ | & | \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}.$$

This is the matrix of T with respect to the basis β. The first column come from the first basis vector (projected onto itself). The second colunn comes from the basis vector that is projected to zero.

- **The matrix $[T]_\alpha^\alpha$.** We'll compute this matrix using a change-of-basis matrix.

  Let P be the change-of-basis matrix from α to β. To compute P, write β_1, β_2 in terms of α_1, α_2:
$$\beta_1 = e_1 - e_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}_\alpha \quad \text{and} \quad \beta_2 = e_1 + e_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}_\alpha$$
  Therefore the change of basis matrix from α to β is
$$P = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$$
  This matrix has inverse
$$P^{-1} = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}.$$
  Now we compute $[T]_\alpha^\alpha$. By Theorem 116,
$$[T]_\beta^\beta = P^{-1}[T]_\alpha^\alpha P, \tag{25}$$
  which implies
$$[T]_\alpha^\alpha = P[T]_\alpha^\alpha P^{-1}.$$
  Therefore
$$\begin{aligned} [T]_\alpha^\alpha &= \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} \\ &= \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix}. \end{aligned}$$

Conclusion: the matrices
$$[T]_\alpha^\alpha = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix} \quad \text{and} \quad [T]_\beta^\beta = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$
both represent the same linear transformation T (a projection), but with different choice of bases. The linear transformation T has many geometric properties which don't depend on the choice of basis:

- the dimension of its range (the 'rank')
- the dimension of its kernel (the 'nullity')
- the determinant
- its eigenvalues

These properties don't hinge on how we choose to represent T as a matrix. Therefore although the matrices $[T]_\alpha^\alpha$ and $[T]_\beta^\beta$ are different, they share the same rank, determinant, nullity, etc.

End of Example 136. □

The two matrices $[T]_\alpha^\alpha, [T]_\beta^\beta$ from Example 136 are an example of *similar matrices*, which we define next.

Definition 137 (Similar)**.** Let $A, B \in \mathcal{M}_{n\times n}$. We say that B is ***similar*** to A if there exists an invertible matrix P such that
$$B = P^{-1}AP. \tag{26}$$

Question What does it *mean* for two matrices to be similar?

Answer: It means that they represent the same linear transformation with respect to different bases. This is because the formula Eq. (26) is the change-of-basis formula from Theorem 116 (like we used in Eq. (25)).

To be precise, if $A, B \in \mathcal{M}_{n\times n}(\mathbb{R})$ are similar, then there is a linear transformation $T : \mathbb{R}^n \to \mathbb{R}^n$ such that
$$A = [T]_\alpha^\alpha \quad \text{and} \quad B = [T]_\beta^\beta$$
for some choice of bases α and β of $\mathbb{R}^n$. Since quantities like rank, nullity, determinant, trace, and eigenvalues all represent geometric properties inherent to the linear transformation T, and since A and B are merely different representations of this transformation, the following theorem holds:

Theorem 138. *If A and B are similar, then they have the same rank, nullity, determinant, trace, eigenvalues, and characteristic polynomial.*

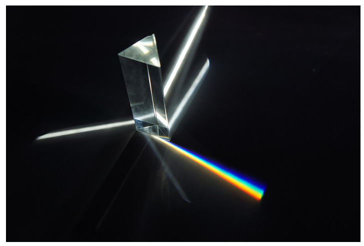
31 2025-11-05 | Week 11 | Lecture 31

The nexus question of this lecture: *When is a matrix similar to a diagonal matrix?*

Recall that two square matrices A, B are ***similar*** if $B = P^{-1}AP$ for some invertible matrix P.

31.1 Prism Analogy

The following analogy goes back to David Hlbert (early 1900s) and Wilhelm Wirtinger (1897). White light consists of a mixture of wavelengths. These can be seen clearly by passing the light through a prism that separates those wavelengths into a *spectrum of colors*:



Source: *https://en.wikipedia.org/wiki/Prism_(optics)*

The same can be done with matrices. For a matrix, the ***spectrum*** is the list of eigenvalues.

Example 139 (First diagonalization)**.** Consider the diagonal matrix

$$D = \begin{bmatrix} 4 & 0 \\ 0 & -1 \end{bmatrix}$$

This matrix has two pure colors: 4 and -1, which are the eigenvalues. The linear transformation does exactly two things: it stretches space by a factor of 4 in the x-direction (since it sends e_1 to $4e_1$) and it reflects space across the y-axis (since it sends e_2 to $-e_2$).

Now consider the matrix

$$A = \begin{bmatrix} 1 & -3 \\ -2 & 2 \end{bmatrix}$$

It's harder to see what the linear transformation of A does; it's like complicated white light consisting of several wavelengths that are hard to separate. We need a prism, and for that we'll choose $P = \begin{bmatrix} -1 & \frac{3}{2} \\ 1 & 1 \end{bmatrix}$. Now, one can check that

$$\underbrace{\begin{bmatrix} -\frac{2}{5} & \frac{3}{5} \\ \frac{2}{5} & \frac{2}{5} \end{bmatrix}}_{P^{-1}} \underbrace{\begin{bmatrix} 1 & -3 \\ -2 & 2 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} -1 & \frac{3}{2} \\ 1 & 1 \end{bmatrix}}_{P} = \underbrace{\begin{bmatrix} 4 & 0 \\ 0 & -1 \end{bmatrix}}_{D}.$$

We have passed the matrix A through a prism. Immediately, we can see from this that the eigenvalues of A are 4 and -1 (since A and D are similar). We now see the spectrum of A.

Where did P come from? The columns of P are eigenvectors of A:

$$V_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \quad \text{and} \quad V_2 = \begin{bmatrix} \frac{3}{2} \\ 1 \end{bmatrix}.$$

Interpreting A through the prism. In addition to knowing the eigenvalues of A, we can now more easily see how it transforms space. Let T be the linear transformation of A, and let $\beta = \{V_1, V_2\}$. P is the change-of-basis matrix from the standard basis $\{e_1, e_2\}$ to β. Since $D = P^{-1}AP$, we have shown that

$$[T]_\beta^\beta = \begin{bmatrix} 4 & 0 \\ 0 & -1 \end{bmatrix}.$$

We can interpret this as follows: T stretches space in the direction of V_1 by a factor of 4 and flips space so that V_2 points in the opposite direction.

End of Example 139. □

31.2 Diagonalizability

Definition 140 (Diagonalizable)**.** A square matrix is ***diagonalizable*** if it is similar to a diagonal matrix.

In Example 162,, we were able to form an invertible matrix P whose columns consisted of eigenvectors of A. The eigenvectors formed an ***eigenbasis***, or a basis consisting of eigenvectors. When this occurs, something special happens: $P^{-1}AP$ becomes a diagonal matrix.

Theorem 141. *A matrix $A \in \mathcal{M}_{n\times n}$ is diagonalizable if and only if there is a basis for $\mathbb{R}^n$ consisting of eigenvectors of A.*

This theorem answers the question of the lecture. (In fact, we could say a lot more. For example, every symmetric matrix $A \in \mathcal{M}_{n\times n}(\mathbb{R})$ is diagonalizable.)

32 2025-11-07 | Week 11 | Lecture 32

> ***The nexus question of this lecture:*** *How can we test whether a matrix is diagonalizable?*

Definition 142. Given a matrix A with eigenvalue λ, the ***eigenspace*** corresponding to λ is the subspace $NS(\lambda I - A)$. We denote this by E_λ.

Remark 143. E_λ is the subspace spanned by the eigenvectors corresponding to λ. It consists of all eigenvectors for λ plus the zero vector.

The following theorem gives a criterion for determining whether a matrix is diagonalizable.

Theorem 144 (Eigenspace dimension criterion)**.** *Suppose A is an $n \times n$ matrix with distinct eigenvalues $r_1, \ldots, r_k$. Let E_r be the eigenspace of r. Then A is diagonalizable if and only if*

$$\dim(E_{r_1}) + \dim(E_{r_2}) + \ldots + \dim(E_{r_k}) = n.$$

Remark 145. The core idea of Theorem 144 is this: in order for a matrix to be diagonalizable, it has to have enough eigenvectors.

The next two examples will illustrate the use of Theorem 144. In both examples, I'm going to skip most of the computational details.

Example 146 (Example 3 in section 5.4)**.** Let $A = \begin{bmatrix} 2 & -1 & 3 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ Then

$$A \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = 2 \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

and

$$A \begin{bmatrix} 1/3 \\ 1 \\ 0 \end{bmatrix} = (-1) \begin{bmatrix} 1/3 \\ 1 \\ 0 \end{bmatrix} \quad \text{and} \quad A \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} = (-1) \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

This shows that E_2 is spanned by $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ and E_{-1} is spanned by $\left\{ \begin{bmatrix} 1/3 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\}$.

Therefore by Theorem 144,

$$\dim(E_2) + \dim(E_{-1}) = 1 + 2 = 3$$

and hence A is diagonalizable.

We can say a bit more: we actually now have enough information to diabonalize A. In particular, we can take our 3 linearly independent eigenvectors and form a "prism" matrix P (technically, a change-of-basis matrix):

$$P = \begin{bmatrix} 1 & 1/3 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Since the three columns are linearly independent, the key theorem implies that P is invertible. Computing the inverse P^{-1} (not shown), it then follows that

$$P^{-1}AP = \begin{bmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}.$$

The right-hand side is a diagonal matrix, and so we have diagonalized A.

End of Example 146. □

Example 147 (Example 4 in section 5.4)**.** The matrix $A = \begin{bmatrix} 1 & -2 & -6 \\ -2 & 2 & -5 \\ 2 & 1 & 8 \end{bmatrix}$ has characteristic polynomial

$$\det(\lambda I - A) = (\lambda - 3)^2(\lambda - 5).$$

Therefore A has two eigenvalues: $\lambda = 3$ and $\lambda = 5$. Computing E_3 and E_5, we obtain:

$$E_3 = \left\{ c \begin{bmatrix} -2 \\ -1 \\ 1 \end{bmatrix} : c \in \mathbb{R} \right\} \quad \text{and} \quad E_5 = \left\{ c \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} : c \in \mathbb{R} \right\}$$

Both E_3 and E_5 are one-dimensional subspaces of $\mathbb{R}^3$ (since they are lines passing through the origin). Therefore, since $\dim(E_3) + \dim(E_5) = 2 \neq 3$, Theorem 144 implies that A is not diagonalizable.

This makes sense: to diagonalize A, we need to form an invertible "prism" matrix P by taking 3 linearly independent eigenvectors of A. But there aren't three linearly independent eigenvectors, there are only two. So A isn't diagonalizable.

End of Example 147. □

Theorem 148 (The determinant is the product of eigenvalues)**.** *Let A be an $n \times n$ matrix, and let $\lambda_1, \ldots, \lambda_n$ be the eigenvalues of A (possibly with repetitions). Then*

$$\det(A) = \lambda_1 \lambda_2 \ldots \lambda_n.$$

Proof. Let A be an $n \times n$ matrix. Then $\det(A - \lambda I)$ is a polynomial of degree n in the variable λ. The eigenvalues $\lambda_1, \ldots, \lambda_n$ of A are the roots of this polynomial. Therefore by the fundamental theorem of algebra, we can write

$$\det(A - \lambda I) = (\lambda_1 - \lambda)(\lambda_2 - \lambda) \cdots (\lambda_n - \lambda)$$

Taking $\lambda = 0$ implies $\det(A) = \lambda_1 \lambda_2 \cdots \lambda_n$. □

The next lecture will address the topic of what can be done with matrices which are not diagonalizable.

33 2025-11-10 | Week 12 | Lecture 33

substitute lecture on Jordan canonical form

33.1 Section 2

For a change of basis, the formula is

$$[T]_\beta^\beta = P^{-1}[T]_\alpha^\alpha P.$$

Let $A = [T]_\alpha^\alpha$ and $B = [T]_\beta^\beta$. Then, A is similar (i.e., equivalent) to B if $B = P^{-1}AP$. The change of basis matrix is an equivalence relation, so regardless of which basis we took, we want

- reflexivity: $X = X$ (X is equivalent to itself)
- symmetry: if $X = Y$ then $Y = X$
- transitivity: if $X = Y$ and $Y = Z$, then $X = Z$.

Suppose $\alpha = \{v_1, \dots, v_n\}$ is a basis of V and that B is a diagonal matrix. Then

$$B = \begin{bmatrix} d_1 & 0 & \cdots & 0 \\ 0 & d_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & d_n \end{bmatrix} = \begin{bmatrix} [T(v_1)]_\beta & 0 & \cdots & 0 \\ 0 & [T(v_2)_\beta & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & [T(v_n)]_\beta \end{bmatrix}$$

A is similar to the diagonal matrix B if and only if there is a basis of V consisting of eigenvectors

$$A = PBP^{-1} \quad \text{or} \quad B = P^{-1}AP$$

where B is the diagonal matrix.

If λ is an eigenvalue of T, denote the corresponding eigenspace by E_λ. Then T is diagonalizable if and only if

$$\sum_\lambda \dim(E_\lambda) = \dim(V) \tag{27}$$

Example 149. Let $A = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 2 & 0 \\ 3 & 0 & 3 \end{bmatrix}$ This has characteristic polynomial

$$p(\lambda) = (\lambda - 1)(\lambda - 2)(\lambda - 3)$$

so the eigenvalues are

$$\lambda_1 = 1, \quad \lambda_2 = 2, \quad \lambda_3 = 3.$$

We know that $\dim(E_1) \geq 1$ (it has to be at least one), $\dim(E_2) \geq 1$, and $\dim(E_3) \geq 1$ (by similar reasoning). Since $\dim(A) \leq 3$, and knowing $\dim(E_1), \dim(E_2)$, and $\dim(E_3)$, Eq. (27) gives $1+1+1 = 3$, so A is similar to the diagonal matrix

$$B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

End of Example 149. □

Example 150. Let $A = \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix}$ The characteristic polynomial is $p(\lambda) = (\lambda - 1)^3$. Thus, the only eigenvalue is $\lambda = 1$. Moreover,

$$\dim(E_1) = 1$$

Therefore by Eq. (27), A is not diagonalizable.

End of Example 150. □

If a matrix isn't diagonalizable, we have something a little weaker. A ***jordan block*** is a matrix of the form

$$\begin{bmatrix} \lambda & 1 & 0 & \cdots & 0 & 0 \\ 0 & \lambda & 1 & \cdots & 0 & 0 \\ 0 & 0 & \lambda & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda & 1 \\ 0 & 0 & 0 & \cdots & 0 & \lambda \end{bmatrix}$$

which may have any dimension. For example

$$\begin{bmatrix} 3 & 1 & 0 \\ 0 & 3 & 1 \\ 0 & 0 & 3 \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} 3 & 1 \\ 0 & 3 \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} 3 \end{bmatrix}$$

A matrix is in ***Jordan normal form*** if it has the form

$$\begin{bmatrix} J_1 & & & \\ & J_2 & & \\ & & \ddots & \\ & & & J_k \end{bmatrix}$$

where $J_1, J_2, \ldots, J_r$ are all Jordan blocks, and all other entries are zero.

Theorem 151. *Over $\mathbb{C}$, every matrix is equivalent to one in Jordan form or Jordan noarmal form.*

33.2 Section 1

Topic: Jordan Canonical Form

Setup $V = \mathbb{C}^n$, $M : V \to V$, M an $n \times n$ matrix. What basis of V makes M the "simplest"?

Example 152. Imagine M is a 3×3 matrix with characteristic polynomial

$$\chi_M = (\lambda - \sqrt{2})^3$$

Then $\sqrt{2}$ is the only eigenvalue and it has algebraic multiplicity 3. There are three possibilities, depending on the dimension of the eigenspace $E_{\sqrt{2}}$:

- If the eigenspace is 3-dimensional, then A is similar to

$$\begin{bmatrix} \sqrt{2} & 0 & 0 \\ 0 & \sqrt{2} & 0 \\ 0 & 0 & \sqrt{2} \end{bmatrix}$$

- If the eigenspace is 2-dimensional, then A is similar to

$$\begin{bmatrix} \sqrt{2} & 0 & 0 \\ 0 & \sqrt{2} & 1 \\ 0 & 0 & \sqrt{2} \end{bmatrix}$$

- If the eigenspace is 1-dimensional, then A is similar to

$$\begin{bmatrix} \sqrt{2} & 1 & 0 \\ 0 & \sqrt{2} & 1 \\ 0 & 0 & \sqrt{2} \end{bmatrix}$$

End of Example 152. □

Theorem 153. *If M is an $n \times n$ matrix with distinct eigenvalues $\lambda_1, \ldots, \lambda_n$, then M is diagonalizable.*

Proof. There exists at least one eigenvector jkl □

34 2025-11-12 | Week 12 | Lecture 34

34.1 Multiplicity

Given a polynomial $p(x)$ with root r, the ***multiplicity*** of r in $p(x)$ is the highest power of $(x-r)$ which divides $p(x)$. For example, if

$$p(x) = (x-3)(x-7)^4(x+1)^2(x-1/2)^{10}.$$

then $p(x)$ four roots, $x=3$, $x=7$, $x=-1$, and $x=1/2$. The multiplicities of these roots are 1, 4, 2, and 10 respectively.

Definition 154 (Algebraic and geometric multiplicity)**.** Let A be an $n\times n$ matrix, and let λ be an eigenvalue of A. The ***algebraic multiplicity (AM)*** of λ is its multiplicity in the characteristic polynomial $\det(\lambda I - A)$. The ***geometric multiplicity (GM)*** of λ is the dimension of the eigenspace $E_\lambda = NS(\lambda I - A)$.

Theorem 155. *Given an eigenvalue λ of a matrix, we have*

$$1 \leq GM \leq AM$$

Theorem 156. *Let A be an $n \times n$ matrix. The following are equivalent:*

- *For every eigenvalue of A, the algebraic multiplicity equals the geometric multiplicity.*
- *A is diagonalizable.*

The next example shows that the second inequality can be strict.

Example 157 (A nondiagonalizable linear transformation)**.** Consider the linear transformation given by the matrix

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

This is a shear transform. We will use Theorem 156 to show that it is not diagonalizable.

Obser that A has only one eigenvalue, namely $\lambda = 1$. We can see this by computing the characteristic polynomial:

$$\det(A - \lambda I) = \det \begin{bmatrix} 1-\lambda & 1 \\ 0 & 1-\lambda \end{bmatrix} = (1-\lambda)^2$$

The algebraic multiplicity of $\lambda = 1$ is 2. The geometric multiplicity is the dimension of the nullspace of

$$B = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

This matrix has rank 1. By the rank nullity theorem, we have

$$\operatorname{rank}(B) + \dim NS(B) = \text{number of columns} = 2$$

Therefore

$$\dim NS(B) = 1.$$

So the geometric multiplicy of the eigenvalue $\lambda = 1$ is 1.

In this case, the geometric multiplicy (2) is less than the algebraic multiplicity (1). So A is not diagonalizable.

End of Example 157. □

34.2 An application of diagonalizability

Example 158. Let $A = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix}$. Suppose that we want to compute A^4 or A^{100}, but we don't want to do that much matrix multiplication.

First observe that

$$A = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

(You should check this). This equation is of the form

$$A = PDP^{-1}$$

where $D = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ is a diagonal matrix.

Fact: If $D = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$ then $D^k = \begin{bmatrix} a^k & 0 \\ 0 & b^k \end{bmatrix}$. And a similar equation holds for $n \times n$ diagonal matrices.

By this fact, we have $D^4 = D$. (Check this.)

Since $A = PDP^{-1}$, we have

$$\begin{aligned}
A^4 &= (PDP^{-1})^4 \\
&= (PDP^{-1})(PDP^{-1})(PDP^{-1})(PDP^{-1}) \\
&= PD(P^{-1}P)D(P^{-1}P)D(P^{-1}P)DP^{-1} && \text{by associativity} \\
&= PD^4P^{-1} && \text{this is the key step I wanted to show you} \\
&= PDP^{-1} && \text{since } D^4 = D \\
&= A.
\end{aligned}$$

That's neat: $A^4 = A$.

End of Example 158. $\square$

35 2025-11-14 | Week 12 | Lecture 35

The nexus question of this lecture: *What are the eigenvectors of a rotation?*

Suppose $T(x) = Ax$ is a linear transformation, and v is an eigenvector of T. This means that there exists a scalar $\lambda \in \mathbb{C}$ such that.

$$T(v) = \lambda v.$$

Geometrically, this equation says that an eigenvector is a direction that the transformation leaves intact: when T acts on v, it may stretch, shrink, or flip that vector, but doesn't rotate it into a new direction.

This naturally raises a striking question: what are the eigenvectors of a rotation? A rotation of the plane visibly changes *every* real direction in the plane. Every arrow in the plane is dragged away from where it once pointed. So if eigenvectors are "directions left intact", then how can a rotation possibly have any?

The short answer to this question is that we get complex eigenvectors. The rest of the lecture will consist of an example of a linear transformation of $\mathbb{R}^2$ involving a rotation which illustrates how this happens.

35.1 Eigenvectors and eigenvalues of a rotation

Example 159 (Eigenvalues and eigenvectors of a rotation)**.** Let

$$A = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}.$$

This is a counterclockwise rotation by 45° composed with a scaling by $\sqrt{2}$. That is, $A = \sqrt{2}R_{45^\circ}$.

Question: what are the eigenvalues and eigenvectors of A?

Solution: The characteristic polynomial is

$$\det \begin{bmatrix} \lambda-1 & 1 \\ -1 & \lambda-1 \end{bmatrix} = (\lambda-1)^2+1 = \lambda^2-2\lambda+2 = (\lambda-(1+i))(\lambda-(1-i)).$$

This has solutions $\lambda = 1+i$ and $\lambda = 1-i$. These two complex numbers are the eigenvalues of A.

- For $\lambda = 1+i$, we want to find the eigenspace $E_{1+i} = NS((1+i)I - A)$.

  First observe that $(1+i)I - A = \begin{bmatrix} 1+i & 0 \\ 0 & 1+i \end{bmatrix} - \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} i & 1 \\ -1 & i \end{bmatrix}$. We wish to solve

  $$\begin{bmatrix} i & 1 \\ -1 & i \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

  Next, set up and row reduce the augmented matrix:

  $$\left[\begin{array}{cc|c} i & 1 & 0 \\ -1 & i & 0 \end{array}\right] \xrightarrow{R_2 - iR_1 \to R_2} \left[\begin{array}{cc|c} i & 1 & 0 \\ 0 & 0 & 0 \end{array}\right] \xrightarrow{(-i)R_1 \to R_1} \left[\begin{array}{cc|c} 1 & -i & 0 \\ 0 & 0 & 0 \end{array}\right]$$

  This corresponds to the system

  $$\begin{cases} x - iy = 0 \\ 0 = 0 \end{cases}$$

  Hence $x = iy$ and y is a free variable. Therefore

  $$E_{1+i} = \left\{ \begin{bmatrix} iy \\ y \end{bmatrix} : y \in \mathbb{C} \right\} = \left\{ y \begin{bmatrix} i \\ 1 \end{bmatrix} : y \in \mathbb{C} \right\}$$

  In words, the eigenspace E_{1+i} is spanned by the vector $\begin{bmatrix} i \\ 1 \end{bmatrix}$.

- For $\lambda = 1 - i$, similar calculation shows that the eigenspace $E_{1-i} = NS((1-i)I - A)$ is spanned by the eigenvector
$$\begin{bmatrix} -i \\ 1 \end{bmatrix}.$$

End of Example 159. $\square$

Example 159 illustrates the following theorem:

Theorem 160. *Let A be an $n \times n$ matrix with real entries. If $\lambda = a + bi$ is an eigenvalue of A (here $a, b \in \mathbb{R}$), then $\bar{\lambda} = a - bi$ is also an eigenvalue.*

Remark 161. Every complex number $z \in \mathbb{C}$ can be written as $z = a + bi$ for some choice of real numbers a and b. The terminology used is that $a + bi$ and $a - bi$ are called ***complex conjugates***.

35.2 Diagonalization of a rotation

We can diagonalize the rotation.

Example 162. Let us consider again the matrix
$$A = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}.$$
This is the rotation from Example 159; there, we shows that this matrix has two eigenvectors:
$$\begin{bmatrix} i \\ 1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} -i \\ 1 \end{bmatrix}.$$
Theorem 144 implies that A is diagonalizable. The matrix $P = \begin{bmatrix} i & -i \\ 1 & 1 \end{bmatrix}$ (whose columns are the two eigenvectors) is invertible (because the eigenvectors are linearly independent). This is a change-of-basis matrix (our "prism"), and we can use it to diagonalize A as follows:
$$P^{-1}AP = \begin{bmatrix} 1+i & 0 \\ 0 & 1-i \end{bmatrix} \tag{28}$$
(see below for details to justify this computation). The main diagonal entries on the right hand side are the eigenvalues of A. Somehow, in rotating the plane 45° (and scaling it by a factor of $\sqrt{2}$), one of the complex eigenvectors gets multiplied by $1 + i$ and the other by $1 - i$.

Here are the details which justify Eq. (28): first observe that we have
$$P = \begin{bmatrix} i & -i \\ 1 & 1 \end{bmatrix} \quad \text{and} \quad P^{-1} = \frac{1}{2i} \begin{bmatrix} 1 & i \\ -1 & i \end{bmatrix}.$$
Therefore
$$\begin{aligned} P^{-1}AP &= \left(\frac{1}{2i} \begin{bmatrix} 1 & i \\ -1 & i \end{bmatrix} \right) \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} i & -i \\ 1 & 1 \end{bmatrix} \\ &= \left(\frac{1}{2i} \begin{bmatrix} 1 & i \\ -1 & i \end{bmatrix} \right) \begin{bmatrix} i-1 & -i-1 \\ i+1 & -i+1 \end{bmatrix} \\ &= \frac{1}{2i} \begin{bmatrix} 2i-2 & 0 \\ 0 & 2i+2 \end{bmatrix} \\ &= \begin{bmatrix} 1 - \frac{1}{i} & 0 \\ 0 & 1 + \frac{1}{i} \end{bmatrix} \\ &= \begin{bmatrix} 1+i & 0 \\ 0 & 1-i \end{bmatrix} && \text{since } \frac{1}{i} = -i, \end{aligned}$$
and the right-hand side is B. In this calculation, we used the fact that $\frac{1}{i} = -i$ (which holds because $(-i)i = 1$).

End of Example 162. $\square$

36 2025-11-17 | Week 13 | Lecture 36

We begin Section 6.1. This lecture is based on Section 5.4 in Gilbert Strang's Linear Algebra and its Applications)

36.1 Examples of systems of first-order linear differential equations

Example 163. An ant is moving in the xy-plane with velocity vector

$$v = \begin{bmatrix} 2x - 5y \\ x - 2y \end{bmatrix}$$

Suppose the ant starts at $(1, 1)$.

Question: Find the position of the ant at time $t > 0$
Solution: First recognize that

$$v = \begin{bmatrix} \frac{dx}{dt} \\ \frac{dy}{dt} \end{bmatrix}$$

and therefore we have the following system of equations:

$$\begin{cases} \frac{dx}{dt} = 2x - 5y \\ \frac{dy}{dt} = x - 2y \end{cases} \tag{29}$$

with initial conditions $x(0) = 1$ and $y(0) = 1$.

A solution to this system would consist of functions of the form

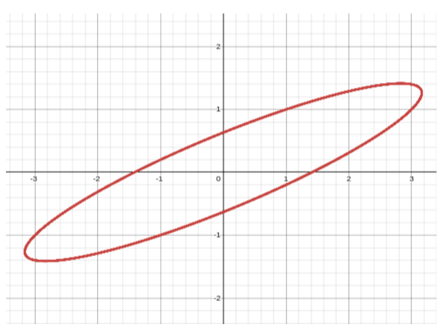
$$x = x(t) \quad \text{and} \quad y = y(t)$$

which satisfy the system of equations and initial conditions.

Without showing how we got it, here is one possible solution to the differential equation given in Eq. (29):

$$\begin{cases} x(t) = \cos(t) - 3\sin(t) \\ y(t) = \cos(t) - \sin(t) \end{cases}$$

These two equations parameterize the following curve:



To verify that this is indeed a solution, we can check first that $x(0) = 1$ and $y(0) = 1$, and that the right hand sides of Eq. (29) actually equal $x'(t)$ and $y'(t)$. To simplify, let $c = \cos(t)$ and $s = \sin(t)$:

$$2x - 5y = 2(c - 3s) - 5(c - s) = -s - 3c = x'(t)$$

and

$$x - 2y = (c - 3s) - 2(c - s) = -c - s = y'(t).$$

End of Example 163. □

Example 164 (Strang's example)**.** Consider the differential equation:

$$\begin{bmatrix} y_1' \\ y_2' \end{bmatrix} \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}. \tag{30}$$

Here, y_1, y_2 are both functions of t ("time"). This differential equation has the form

$$Y' = AY$$

where A is an $n \times n$ matrix and $Y = Y(t) = \begin{bmatrix} y_1(t) \\ \vdots \\ y_n(t) \end{bmatrix}$.

The only thing we need is the following:

Theorem 165 (Pure exponential solutions)**.** *Let A be an $n \times n$ matrix, let v be an eigenvector of A with eigenvalue λ, and let c be any scalar. Then*

$$Y(t) = ce^{\lambda t}v$$

is a solution of the system $Y' = AY$.

Proof.

$$\begin{aligned} Y'(t) &= \frac{d}{dt}\left[ce^{\lambda t}v\right] \\ &= ce^{\lambda t}\lambda v \\ &= ce^{\lambda t}Av \\ &= A\left(ce^{\lambda t}v\right) \\ &= AY(t) \end{aligned}$$

So $Y' = AY$, meaning $Y(t) = ce^{\lambda t}v$ is a solution of the differential equation. □

In our problem Eq. (30), the first step is to find the eigenvalues and eigenvectors:

$$A\begin{bmatrix} 1 \\ 1 \end{bmatrix} = (-1)\begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad \text{and} \quad A\begin{bmatrix} 1 \\ -1 \end{bmatrix} = (-3)\begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

By Theorem 165, we have two solution:

$$Y_1(t) = c_1e^{-t}\begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad \text{and} \quad Y_2(t) = c_2e^{-3t}\begin{bmatrix} 1 \\ -1 \end{bmatrix},$$

where c_1, c_2 are arbitrary constants. Combining these, we get the general solution

$$Y(t) = c_1e^{-t}\begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2e^{-3t}\begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

If we further know that $Y(0) = B = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$, we can solve for c_1, c_2 by observing that plugging $t = 0$ into the above formula gives

$$\begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = c_1e^{-t}\begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2e^{-3t}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

We can write this as

$$\begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

which implies that

$$\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}^{-1}\begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

(Note that the matrix $\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$ is invertible since its columns are linearly independent eigenvectors of A.)

End of Example 165. □

37 2025-11-19 | Week 13 | Lecture 37

topic: continuation of introduction to systems of linear differential equations

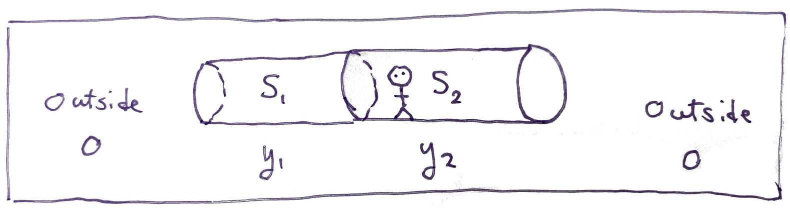
Theorem 166 (Pure exponential solutions)**.** *Let A be an $n \times n$ matrix, let v be an eigenvector of A with eigenvalue λ, and let c be any scalar. Then*

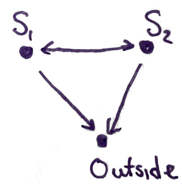
$$Y(t) = ce^{\lambda t}v$$

is a solution of the system $Y' = AY$.

37.1 Diffusion example

Example 167 (Diffusion)**.** A person is standing in a large pipe drops a flask of chemicals, releasing a chemical gas. The tube has 2 chambers, S_1 and S_2, as shown:





Let

$$\begin{aligned} y_1(t) &= \text{ concentration of chemical in chamber } S_1 \text{ at time } t \\ y_2(t) &= \text{ concentration of chemical in chamber } S_2 \text{ at time } t \end{aligned}$$

and assume that the concentration outside is always zero (the gas gets immediately blown away by the wind).

We make the following assumption:

At each time t, the diffusion rate between adjacent areas is the difference in concentrations.

Based on this assumption, y_1 (the concentration in S_1) is changing due to two factors:

- the diffusion to the outside, which is $(0 - y_1)$
- the diffusion into or out of S_2, which is $(y_2 - y_1)$.

Hence

$$\frac{dy_1}{dx} = (y_2 - y_1) + (0 - y_1) = -2y_1 + y_2.$$

Similar reasoning implies that

$$\frac{dy_2}{dx} = (y_1 - y_2) + (0 - y_2) = y_1 - 2y_2.$$

This gives us the following system of differential equations:

$$\begin{aligned} \frac{dy_1}{dt} &= -2y_1 + y_2 \\ \frac{dy_2}{dt} &= y_1 - 2y_2 \end{aligned}$$

Writing this in matrix form gives the matrix equation

$$\begin{bmatrix} y_1'(t) \\ y_2'(t) \end{bmatrix} \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} \tag{31}$$

This is a differential equation of the form

$$Y' = AY.$$

Observe that A has eigenvalue -1 with eigenvector $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and eigenvalue -3 with eigenvector $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$. Therefore by Theorem 166, our system of differential equations has solutions two pure exponential solutions:

$$Y_1(t) = c_1 e^{-t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad \text{and} \quad Y_2(t) = c_2 e^{-3t} \begin{bmatrix} 1 \\ -1 \end{bmatrix},$$

where c_1, c_2 are arbitrary constants. By superposition, the general solution is

$$Y(t) = c_1 e^{-t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 e^{-3t} \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

We can write this as

$$Y(t) = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} e^{-t} & 0 \\ 0 & e^{-3t} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}. \tag{32}$$

Further, suppose we are given the initial condition

$$Y(0) = B = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}.$$

(For our problem, a reasonable initial condition would be $Y(0) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, since the chemical gas gets released in section 2 of the pipe at time $t = 0$, but it hasn't reached S_1 yet.)

Plugging $t = 0$ into Eq. (32) gives

$$B = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

or equivalently

$$\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}^{-1} B$$

Plugging this back into Eq. (31) yeilds

$$Y(t) = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} e^{-t} & 0 \\ 0 & e^{-3t} \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}^{-1} B$$

Writing

$$P = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}, \quad \Lambda = \begin{bmatrix} -1 & 0 \\ 0 & -3 \end{bmatrix}, \quad \text{and} \quad e^{\Lambda t} = \begin{bmatrix} e^{-t} & 0 \\ 0 & e^{-3t} \end{bmatrix},$$

we see that our solution takes the form

$$Y(t) = P e^{\Lambda t} P^{-1} B. \tag{33}$$

If we take our initial condition to be $Y(0) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, as noted earlier, then we have $b_1 = 0$ and $b_2 = 1$. In that case, the matrix multiplication gives

$$\begin{aligned} Y(t) &= P e^{\Lambda t} P^{-1} B \\ &= \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & -3 \end{bmatrix} \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\ &= \frac{1}{2} \begin{bmatrix} e^{-t} + e^{-3t} & e^{-t} - e^{-3t} \\ e^{-t} - e^{-3t} & e^{-t} + e^{-3t} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} e^{-t} - e^{-3t} \\ e^{-t} + e^{-3t} \end{bmatrix}. \end{aligned}$$

In other words, at time t, the concentration of gas in S_1 is $y_1(t) = e^{-t} - e^{-3t}$ and the concentration of gas in S_2 is $y_2(t) = e^{-t} + e^{-3t}$. One can check that this satisfies

$$\begin{cases} Y'(t) = \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix} Y(t) \\ Y(0) = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \end{cases}$$

(check this!), so it is a solution to the differential equation. Phsycially, it also makes since, since it indicates that both concentrations are positive and tend to 0 as $t \to \infty$ as the gas diffuses. From the equations we can also see that diffusion causes the gas concentration to decay exponentially in time.

End of Example 167. □

37.2 General solution to $Y' = AY$

In Example 167 we obtained the fundamental solution shown in Eq. (33). The form that it took illustrate the following theorem:

Theorem 168. *If A can be diagonalized $A = P\Lambda P^{-1}$, then*

$$Y = Pe^{\Lambda t}P^{-1}B$$

is a solution to the differential equation

$$\begin{cases} Y' = Ay \\ Y(0) = B \end{cases}$$

Remark 169. In the above theorem, the columns of P are the eigenvectors $v_1, \ldots, v_n$ of A. So the formula is for the solution can be written as

$$Y(t) = \begin{bmatrix} | & & | \\ v_1 & \cdots & v_n \\ | & & | \end{bmatrix} \begin{bmatrix} e^{\lambda_1 t} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & e^{\lambda_n t} \end{bmatrix} P^{-1}Y(0),$$

Another way to write the solution: Let $c = (c_1, \ldots, c_n)^\top = P^{-1}Y(0)$, in which case we get

$$Y(t) = c_1 e^{\lambda_1 t} v_1 + \ldots + c_n e^{\lambda_n t} v_n.$$

Remark 170. When A is diagonalizable (as is assumed in Theorem 174), we have

$$Pe^{\Lambda t}P^{-1} = e^{At}$$

so the formula for the solution can be written in the wonderfully simple form

$$Y(t) = e^{At}Y(0).$$

In fact, this formula holds even if A isn't diagonalizable, but to understand that case, we'll need to first understand that e^{At} even means.

38 2025-11-21 | Week 13 | Lecture 38

38.1 The matrix exponential

In Example 164, we wrote doen $e^{\Lambda t}$ but Λt is a matrix. So we need to define the ***exponential of a matrix***:

- If Λ is diagonal $\begin{bmatrix} \lambda_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \lambda_n \end{bmatrix}$ then it's easy:

$$e^{\Lambda t} = \begin{bmatrix} e^{\lambda_1 t} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & e^{\lambda_n t} \end{bmatrix}$$

- If A is not a diagonal matrix, the idea is to imitate the power series $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots$. Take $x = At$:

$$e^{At} = 1 + At + \frac{(At)^2}{2!} + \frac{(At)^3}{3!} + \ldots$$

  This series always converges, and satisfies:

  * $e^{At} = e^{As} = e^{A(t+s)}$
  * $e^{At}e^{-tA} = I$ (so e^{At} is always invertible)
  * $\frac{d}{dt}\left[e^{At}\right] = Ae^{At}$

  If $A = P\Lambda P^{-1}$ for diagonal Λ then

$$A^k = P\Lambda^k P^{-1}.$$

  Then

$$\begin{aligned} e^{At} &= \sum_{k=0}^{\infty} \frac{(At)^k}{k!} \\ &= \sum_{k=0}^{\infty} \frac{\left(P\Lambda^k P^{-1}\right) t^k}{k!} \\ &= P\left(\sum_{k=0}^{\infty} \frac{\Lambda^k t^k}{k!}\right) P^{-1} \\ &= P\left(\sum_{k=0}^{\infty} \frac{(\Lambda t)^k}{k!}\right) P^{-1} \\ &= Pe^{\Lambda t}P^{-1}. \end{aligned}$$

  We can write the RHS of Eq. (33) as $e^{At}B$.

38.2 Definition of FOLDE

A ***system of first order linear differential equations*** takes the form

$$\begin{cases} y_1' = a_{11}y_1 + a_{12}y_2 + \ldots + a_{1n}y_n + g_1 \\ y_2' = a_{21}y_1 + a_{22}y_2 + \ldots + a_{2n}y_n + g_2 \\ \quad\vdots \\ y_n' = a_{n1}y_1 + a_{n2}y_2 + \ldots + a_{nn}y_n + g_n \end{cases}$$

Note: $y_1, \ldots, y_n$ are all functions of $x \in \mathbb{R}$. So are $y_1', \ldots, y_n'$. In matrix notation, we write

$$Y = Y(x) = \begin{bmatrix} y_1(x) \\ \vdots \\ y_n(x) \end{bmatrix}, \quad G = \begin{bmatrix} g_1 \\ \vdots \\ g_n \end{bmatrix}, \quad A = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix}$$

To be fully general, one can consider A and G as functions of x. I don't think we'll do that, but Y and Y' are functions of x, even if we don't write $Y(x)$ and $Y'(x)$.

We also specify that at some ***initial point*** $x_0 \in \mathbb{R}$, we assume that

$$Y(x_0) = \begin{bmatrix} y_1(x_0) \\ \vdots \\ y_n(x_0) \end{bmatrix} = \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix}$$

for some fixed $b_1, \ldots, b_n \in \mathbb{R}$. This is called the ***initial condition.***

With this notation, we can write a system of first order linear differential equations compactly as

$$Y' = AY + G \tag{34}$$

A special cas is when $G(x) = 0$ for all x. In that case, the system takes the form

$$Y' = AY. \tag{35}$$

Such a system is said to be ***homogeneous***. Otherwise it is ***nonhomogenous***.

38.3 Important theorems

Theorem 171 (Existence and uniqueness)**.** *If x_0 lies in an interval (a, b), and if $a_{ij}(x)$ and $g_i(x)$ are continuous on that interval for all i and j, then the initial value problem*

$$\begin{cases} Y' = AY + G \\ Y(x_0) = (b_1, \ldots, b_n)^\top \end{cases}$$

has a unique solution on the interval (a, b).

For a homogeneous system with no intial conditions, we may have many solutions. In fact

Theorem 172 (Solutions form a subspace)**.** *The solutions to a system of the form $Y' = AX$ (with n equations) form a vector space of dimension n.*

Here, the vectors are functions. The big vector space is

$$V = \left\{ \begin{bmatrix} f_1(x) \\ \vdots \\ f_n(x) \end{bmatrix} : f_i \in C^1(a, b) \right\},$$

where

$$C^1(a, b) = \{f : (a, b) \to \mathbb{R} \mid f' \text{ exists and is continuous}\}$$

Theorem 172 says that the solutions to $Y' = AX$ form an n-dimensional subspace of V.

This means that if Y_1 and Y_2 are solutions to Eq. (35), then $\alpha Y_1 + \beta Y_2$ is also a solution as well, for any scalars α, β.

If $Y_1, \ldots, Y_n$ are a set of n linearly independent solutions to Eq. (35), we call them a ***fundamental set of solutions***, and the ***general solution*** is

$$Y_H = c_1 Y_1 + \ldots + c_n Y_n$$

where $c_1, \ldots, c_n$ are scalars. The ***matrix of fundamental solutions*** is

$$Y_H = \begin{bmatrix} | & & | \\ c_1 Y_1 & \cdots & c_n Y_n \\ | & & | \end{bmatrix}$$

Theorem 173. *Suppose $Y_1, \ldots, Y_n$ form a fundamental set of solutions to $Y' = AY$, and that Y_P is a solution to*

$$Y' = AY + G \tag{36}$$

. Then every solution to Eq. (36) *has the form*

$$Y = Y_H + Y_P$$

where $Y_H = c_1Y_1 + \ldots + c_nY_n$.

The function Y_P is called a ***particular solution*** to the inhomogeneous system.

Theorem 174. *Let $k \in \mathbb{C}$. Then the differential equation $y'(x) = ky(x)$ has solution $y = ce^{kx}$, where c is any constant.*

Proof. If $y = ce^{kx}$ then $y' = cke^{kx} = ky$. □

Example 175. Consider the homogeneous differential equation

$$\begin{bmatrix} y_1' \\ y_2' \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \tag{37}$$

Here A does not depend on x (in other problems we'll see, it will). This has form $Y' = AY$.

We have two equations

$$\begin{cases} y_1' = -y_1 \\ y_2' = 3y_2 \end{cases}$$

By Theorem 174, this implies that y_1, y_2 take the form

$$y_1 = c_1e^{-x} \quad \text{and} \quad y_2 = c_2e^{3x}$$

Then

$$Y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} c_1e^{-x} \\ c_2e^{3x} \end{bmatrix}$$

is a solution to the differential equation in Eq. (37).

Write

$$Y = c_1 \begin{bmatrix} e^{-x} \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ e^{3x} \end{bmatrix} = \begin{bmatrix} e^{-x} & 0 \\ 0 & e^{3x} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

We can futher observe that

$$Y = c_1Y_1 + c_2Y_2$$

where $Y_1 = \begin{bmatrix} e^{-x} \\ 0 \end{bmatrix}$ and $Y_2 = \begin{bmatrix} 0 \\ e^{3x} \end{bmatrix}$. Together, Y_1, Y_2 form a fundamental set of solutions. The matrix of fundamental solutions is

$$M = \begin{bmatrix} e^{-x} & 0 \\ 0 & e^{3x} \end{bmatrix}$$

In this example $A = \begin{bmatrix} -1 & 0 \\ 0 & 3 \end{bmatrix}$ was a diagonal matrix; compare this with M.

End of Example 175. □