21 2025-10-13 | Week 08 | Lecture 21

The nexus question of this lecture: *Is a basis all you need?*

This lecture is based on sections 5.1 and 2.4

21.1 All you need is a basis

If we know how a linear transformation acts on a basis, then we know its values on all other vectors as well.

Example 100 (Cool example)**.** Suppose that $T : \mathbb{R}^3 \to \mathbb{R}^2$ is a linear transformation.

Note that the vectors

$$\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

form a basis of $\mathbb{R}^3$. (Check this).

Suppose that T is defined for each of these three basis vectors as

$$T \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} := \begin{bmatrix} 2 \\ 3 \end{bmatrix}, \quad T \begin{bmatrix} 0 \\ 1 \end{bmatrix} := \begin{bmatrix} 0 \\ 3 \end{bmatrix}, \quad \text{and} \quad T \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} := \begin{bmatrix} 0 \\ 2 \end{bmatrix}.$$

Question: What is $T \begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix}$?

Solution: First, we will write $\begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix}$ as a linear combination of the basis vectors. To do this, we solve the linear system

$$c_1 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} + c_3 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix}$$

which gives these unique solution $c_1 = 2$, $c_2 = 1$, $c_3 = -1$.

Then

$$\begin{aligned} T \begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix} &= T \left(2 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right) \\ &= 2T \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + T \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} - T \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \\ &= 2 \begin{bmatrix} 2 \\ 3 \end{bmatrix} + \begin{bmatrix} 0 \\ 3 \end{bmatrix} - \begin{bmatrix} 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 4 \\ 7 \end{bmatrix}. \end{aligned}$$

We could ask the same question for an arbitrary matrix, namely, what is $T \begin{bmatrix} x \\ y \\ z \end{bmatrix}$? See pg 237-238 in the textbook.

End of Example 100. □

This example shows that if we how a linear transform acts on a basis, then we know everything about it. This suggests that there is something very special about a basis, that it's a way to represent a vector space as a whole using just a finite set of vectors.

21.2 Unique basis representations

Theorem 101 (Unique basis representation)**.** *Let V be a vector space and let $v_1, \dots, v_n$ be a basis for V. Then every vector can be written as a unique linear combination of $v_1, \dots, v_n$.*

Proof. Let $u \in V$ be arbitrary. Since $v_1, \dots, v_n$ is a basis, it spans V. Therefore we can write u as a linear combination

$$u = c_1 v_1 + c_2 v_2 + \dots + c_n v_n \tag{17}$$

for some $c_1, \dots, c_n \in \mathbb{R}$. This shows that u can be written as a linear combination of $v_1, \dots, v_n$.

It remains to show that there is only one way to write u as a linear combination of $v_1, \dots, v_n$.

Suppose we have some *other* linear combination

$$u = c'_1 v_1 + c'_2 v_2 + \dots + c'_n v_n \tag{18}$$

where $c'_1, \dots, c_n \in \mathbb{R}$. Then

$$\begin{aligned} 0 &= u - u \\ &= (c_1 v_1 + c_2 v_2 + \dots + c_n v_n) - (c'_1 v_1 + c'_2 v_2 + \dots + c'_n v_n) \\ &= (c_1 - c'_1) v + (c_2 - c'_2) v + \dots + (c_n - c'_n) v \end{aligned}$$

Since $v_1, \dots, v_n$ are linearly independent, it follows that

$$c_1 = c'_1, \quad c_2 = c'_2, \quad \dots \quad c_n = c'_n$$

So the two linear combinations in Eqs. (17) and (18) are actually the same. This shows uniqueness. ☐

One consequence of Theorem 101 is that it allows us to translate any n-dimensional vector space, no matter how exotic, to the more familiar setting of $\mathbb{R}^n$. The idea is as follows:

1. Let V be an abstract vector space (functions, polynomials, whatever) of dimension n.
2. Pick a basis $v_1, \dots, v_n$ for V
3. Each $a \in V$ can be written as
$$a_1 v_1 + a_2 v_2 + \dots + a_n v_n$$
where $a_1, \dots, a_n \in \mathbb{R}$. In particular, Theorem 101 tells us that there is only one choice of $a_1, \dots, a_n$ that work. So we can write
$$a = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}$$

In other words, any n-dimensional vector space is just $\mathbb{R}^n$ in disguise. So from one perpsective, to study all (finite-dimensional) vector spaces, it's enough to just study $\mathbb{R}^n$.

But there's a catch. Doing this requires "picking" a basis—and if you and I pick different bases, then we will end up with different representations of the same fundamental objects. Consider the set of direction vectors that an electron could move. Clearly this "is" $\mathbb{R}^3$, but since there are no coordinate axes, we need to *choose* what the x, y, z directions are: I can just declare that a particular direction is the x direction, for example. But if you choose a different direction, then we'll end up with different ways of representing the same directions. My $\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$ might be your $\begin{bmatrix} 5 \\ -3 \\ 2 \end{bmatrix}$. This may be undesirable.

If you and I each choose different basis for $\mathbb{R}^3$, are we really studying the same space? Maybe we are (I think so!). This is because many properties in linear algebra are "basis invariant" in the sense that they don't depend on the basis you pick. An example is the dimension of a vector space or its linear subspaces. Another example is linear transformations: many geometric properties of linear transformations (e.g., whether they preserve orientation, how much they stretch space, whether they collapse the dimesnion) don't depend on how we choose to represent the space. But if that's the case, then maybe we shouldn't need to rely on picking bases to study these things.

So my answer to the nexus question, "is a basis all you need" is "mostly yes, but sometimes it's more than than you need."