23 2025-03-07 | Week 8 | Lecture 22

Please read sections 5.1-5.4

Theorem 73 (Expectation of functions of random vectors)**.** *Let (X, Y) be a random vector with pmf $p(x, y)$ if X, Y are discrete or pdf $f(x, y)$ if X, Y are continuous. Let $h : \mathbb{R}^2 \to \mathbb{R}$ be a function. [Note: then $h(X, Y)$ is a random variable.] Then*

$$\mathbb{E}\left[h(X,Y)\right] \begin{cases} \sum_{all\ x,y} h(x,y)p(x,y) & : \textit{discrete case} \\ \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(x,y)f(x,y)dxdy & : \textit{continuous case} \end{cases}$$

Example 74. Gemstones are worth \$22,500 per pound, gold is worth \$15,000 per pound, and navigational charts are worth \$7500 per pound. The value of a random pound of treasure is

$$15,000X + 22,500Y + 7,500(1 - X - Y)$$

Recalling that the chest is 10lbs, the total value of the contents of the treasure chest is

$$\begin{aligned} h(X,Y) &= 150000X + 225000Y + (75000)(1 - X - Y) \\ &= 75000 + 75000X + 150000Y \end{aligned}$$

The expected total value is therefore

$$\mathbb{E}\left[h(X,Y)\right] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(x,y)f(x,y)dxdy = ...\$1,650,000$$

End of Example 74. □

One important function for Theorem 73 is when $h(X, Y) = (X - \mu_X)(Y - \mu_Y)$, where $\mu_X = \mathbb{E}[X]$ and $\mu_Y = \mathbb{E}[Y]$. That gives us a formula for what is called the covariance:

$$\mathrm{Cov}(X,Y) = \mathbb{E}\left[(X - \mu_X)(Y - \mu_Y)\right]$$

We also have

$$\mathrm{Cov}(X,Y) = \mathbb{E}\left[XY\right] - \mu_X\mu_Y$$

For example, we can compute the covariance of X and Y from the treasure example.

$$\mathrm{Cov}(X,Y) = \mathbb{E}\left[XY\right] - \mu_X\mu_Y$$

To compute $\mathbb{E}[XY]$, let's take $h(x, y) = xy$. Recalling that $f(x, y) = 24xy$ for $(x, y) \in D$, we have

$$\begin{aligned} \mathbb{E}\left[XY\right] &= \int_D h(x,y)f(x,y)dxdy \\ &= \int_D (xy)24xydxdy \\ &= 24\int_D x^2y^2dxdy \\ &= 24\int_0^1\int_0^{1-y} x^2y^2dxdy \\ &= 24\int_0^1 y^2\left[\int_0^{1-y} x^2dx\right]dy \end{aligned}$$

and so forth.

Example 75. Five friends purchased tickets to a concert. The tickets are all in the same row, next to each other, and numberd 1 through 5. What is the expected number of seats separating any two friends?

Let X, Y be the seat number of the first and second individual (chosen randomly). Possible (X, Y) pairs are

$$\{(1,2),(1,3),\ldots,(5,4)\}$$

and the joint pmf is

$$p(x,y) = \begin{cases} \frac{1}{20} & : \quad x, y \in \{1,2,3,4,5\} \text{ and } x \neq y \\ 0 & : \quad else \end{cases}$$

The number of seats separating individuals X and Y is $h(X,Y) = |X - Y| - 1$.

(Make a table)

Then

$$\mathbb{E}\left[h(X,Y)\right] = \sum_{(x,y):x\neq y} h(x,y)p(x,y) = \sum_{x=1}^{5} \sum_{\substack{y=1 \\ y\neq x}}^{5} \left(|x-y|-1\right) \frac{1}{20} = 1.$$

End of Example 75. □