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We continued the activity from Wednesday about shuffling cards and permutations. I numbered the cards $1, 2, \ldots$. Then I had each person write their name on the card. Then I collected and shuffled the cards and then dealt them back randomly to all the people.

Then we had everyone stand next to the person whose name is on their new card. (Usually, you don't get your own card back, because the cards were shuffled). This results in several distinct groups of people. These are called *cycles*. The *order* of a cycle is the number of people. A cycle of order 1 is called a *fixed point*. You are a fixed point if you got your original card back. In that case, you have to stand by yourself.

We can think of permutations as functions. Formally, a *permutation on the set* $\{1, 2, \ldots, n\}$ is a function

$$\pi : \{1, 2, \ldots, n\} \to \{1, 2, \ldots, n\}$$

such that π is both injective and surjective (i.e., is a bijection).

By shuffling the deck, we are generating a permutation uniformly at random. We did this experiment several times. The number of fixed points, cycles, and the size of the cycles are all random.

We also introduce *cycle notation*, which is described really good here, (up to about 7 minutes and 40 seconds):

`https://www.youtube.com/watch?v=MpKG6FmcIHk`

Question #1: Suppose we draw a permutation on the set $\{1, 2, \ldots, n\}$ uniformly at random. What's the expected number of fixed points? (That is, on average, how many people get their own card back?)

Let F_i be the event that i is a fixed point. Clearly,

$$\mathbb{P}[F_i] = \frac{1}{n}$$

since each person has a 1-in-n chance of getting their own card back.

Let N be the number of fixed points (this is a random variable). Then

$$N = \mathbf{1}_{F_1} + \mathbf{1}_{F_2} + \ldots + \mathbf{1}_{F_n}$$

Therefore

$$\begin{aligned}\mathbb{E}[N] &= \mathbb{E}[\mathbf{1}_{F_1} + \mathbf{1}_{F_2} + \ldots + \mathbf{1}_{F_n}] \\ &= \mathbb{E}[\mathbf{1}_{F_1}] + \mathbb{E}[\mathbf{1}_{F_2}] + \ldots + \mathbb{E}[\mathbf{1}_{F_n}] \\ &= \mathbb{P}[F_1] + \mathbb{P}[F_2] + \ldots + \mathbb{P}[F_n] \\ &= \frac{1}{n} + \frac{1}{n} + \ldots + \frac{1}{n} \\ &= 1.\end{aligned}$$

Therefore, on average, we expect 1 fixed point.

Question #2: What is the probability of having no fixed points?

By enumerating all the permutations, we showed that for $n = 1, 2, 3, 4$, the probability of no fixed points is

$$1, \quad 1/2, \quad 1/3, \quad 3/8$$

which suggests the following conjecture

$$\mathbb{P}[N = 0] = \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \frac{1}{5!} + \ldots + (-1)^n \frac{1}{n!}$$

This would mean that when n is large,

$$\mathbb{P}[N = 0] \approx \frac{1}{e} \approx 0.37$$