15 2025-02-14 | Week 5 | Lecture 14

Read 3.1-3.4 and 3.6

15.1 Poisson convergence

Fix $\lambda \in (0,1)$. Suppose have a sequence of magic coins $1, 2, \ldots$.

- Coin 1 has probability of heads λ
- Coin 2 has probability of heads $\frac{\lambda}{2}$
- Coin 3 has probability of heads $\frac{\lambda}{3}$
- And so forth.

Consider the following sequence of experiments:

- Experiment 1: Flip coin 1 once.
- Experiment 2: Flip coin 2 twice.
- Experiment 3: Flip coin 3 three times.
- Experiment n: Flip coin n, n times.

Now let us consider the nth experiment. Let $F_1, \ldots, F_n$ represent the n coin flips, with

$$F_1 = \begin{cases} 1 & : \text{ first coin flip is heads} \\ 0 & : \text{ first coin flip is tails} \end{cases}$$

$$F_2 = \begin{cases} 1 & : \text{ second coin flip is heads} \\ 0 & : \text{ second coin flip is tails} \end{cases}$$

and so forth for all n coin flips. So

$$X_n = F_1 + F_2 + \ldots + F_n$$

is the number of heads that you get.

Question: What is the expectation of X_n? We can use the linearity of expectation:

$$\begin{aligned} \mathbb{E}[X_n] &= \mathbb{E}[F_1 + F_2 + \ldots + F_n] \\ &= \mathbb{E}[F_1] + \mathbb{E}[F_2] + \ldots + \mathbb{E}[F_n] \\ &= \underbrace{\frac{\lambda}{n} + \frac{\lambda}{n} + \ldots + \frac{\lambda}{n}}_{n \text{ terms}} \\ &= \lambda. \end{aligned}$$

So the expected number of heads that we get doesn't change from experiment to experiment, even if we send $n \to \infty$. Think!—for each experiment, the probability of heads is $\frac{\lambda}{n}$ and that is getting smaller. However the total number of coin flips is getting larger and larger.

Question Is it possible that as $n \to \infty$, the random variable X_n converges to some new random variable, one that we haven't seen yet?

Yes.

Theorem 46 (The exponential function). *For any real number x, it holds that*

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}.$$

Definition 47 (Poisson ("Pwason")). Let $\lambda > 0$. A random variable X that takes on the values $0, 1, 2, \ldots$ is a *Poisson* random variable with parameter λ if its probability mass function is given by

$$p(k) := \mathbb{P}[X = k] = e^{-\lambda}\frac{\lambda^k}{k!} \quad \text{for } k = 0, 1, 2, \ldots.$$

Before explaining this, we should check that it is actually a probability mass function: remember, it needs to sum to 1:

$$\begin{aligned}
\sum_{k=0}^{\infty} p(k) &= \sum_{k=0}^{\infty} e^{-\lambda}\frac{\lambda^k}{k!} \\
&= e^{-\lambda}\sum_{k=0}^{\infty}\frac{\lambda^k}{k!} \\
&= e^{-\lambda}e^{\lambda} && \text{by Theorem 46} \\
&= 1.
\end{aligned}$$

The Poisson distribution is a common model since it models "accidents".

Example 48. Oahu has about a million cars and averages about 6 car accidents per day. The probability that any one car will have an accident is very small, and they are all (nearly) independent from each other, bus since there are so many cars, you expect 6 accidents, on average. But actual number of accidents each day is $X = 0$ or 1 or 2 or $\ldots$. The number of car accidents is thus poisson distributed with mean $\lambda = 6$ What is

$$F(2) = \mathbb{P}[X \leq 2]\,?$$

The probabilities of no accidents, one accident, and two accidents are

$$p(0) = e^{-6}\frac{6^0}{0!} = e^{-6} \approx 0.002$$

$$p(1) = e^{-6}\frac{6^1}{1!} = 6e^{-6} \approx 0.015$$

$$p(2) = e^{-6}\frac{6^2}{2!} = 18e^{-6} \approx 0.045$$

Therefore $F(2) = p(0) + p(1) + p(2) \approx 0.002 + 0.015 + 0.045 = 0.062$

End of Example 48. $\square$

Proposition 49. *Let $X \sim Pois(\lambda)$. Then*

$$\mathbb{E}[X] = \lambda \quad \textit{and} \quad \operatorname{Var}(X) = \lambda$$

That the variance and expected values are the same for the Poisson random variable is a neat coincidence.

Theorem 50 (Poisson approximation). *Recall that if $X \sim Binom(n, p)$, that means*

$$\mathbb{P}[X = k] = \binom{n}{k}p^k(1-p)^{n-k}, \quad \textit{for } k = 0, 1, 2, \ldots.$$

Let $\lambda = np$. When n is large and p is small (e.g., $n > 50$ and $\lambda < 5$), then we have the following approximation:

$$\mathbb{P}[X = k] \approx \frac{e^{-\lambda}\lambda^k}{k!} \quad \textit{for } k = 0, 1, 2, \ldots.$$

In other words, when n is large and p is small, X is well-approximated by a Poisson distribution with parameter $\lambda = np$.

This convergence/approximation idea is actually how I introduced the Poisson random variable in the first place, by considering the sequence of experiments involving magic coin flips.

Let's try an application of this:

Example 51 (Poisson approximation)**.** In *Dungeons and Dragons*, a 20-sided dice is used (aka the 'd20'). When you roll a 1 out of 20, that's called a *critical failure*, which usually results in something terrible happening.

Suppose that in the course of a game, a twenty side dice is rolled $n = 100$ times. Let X be the number of critical failures. The probability of a critical failure is $p = 1/20 = .05$. So

$$X \sim \text{Bin}\left(100, \frac{1}{20}\right).$$

So, for example, we have $\mathbb{E}[X] = 100 \times \frac{1}{20} = 5$.

The pmf of X is

$$\mathbb{P}[X = k] = \binom{100}{k}\left(\frac{1}{20}\right)^k\left(1 - \frac{1}{20}\right)^{100-k}, \text{ for } k = 0, 1, 2, \ldots, 100$$

Since $p = 0.05$ is small and $n = 100$ is large, we can approximate X with a Poisson random variable Y with parameter $\lambda = np = 5$. That is, $Y \sim \text{Pois}(5)$. The pmf of Y is

$$\mathbb{P}[Y = k] = e^{-5} \cdot \frac{5^k}{k!}, \text{ for } k = 0, 1, 2, \ldots$$

Let's compute some probabilities, rounding to three decimal places

x	$\mathbb{P}[X = k]$	$\mathbb{P}[Y = k]$
0	.006	.007
1	.031	.034
2	.081	.084
3	.140	.140
4	.178	.175

End of Example 51. □