3 2025-01-16 | Week 01 | Lecture 03

> ***The nexus question of this lecture:*** *What is the general framework for probability (continued)?*

- Go over problem 4, parts (b) and (c), and problem 6 on the worksheet.

3.1 Powerball

Example 3 (Powerball Lottery)**.** The a ticket for the powerball lottery costs \$2. There are two outcomes:

- You win \$300,000,000.
- You don't win any money.

The probability of winning is approximately $\frac{1}{300{,}000{,}000}$. Let X be the net payoff from the game, in dollars:

- If you lose, then $X = -2$, since you had to pay \$2 to play the game.
- If you win, then your net payoff is $X = 299,999,998$.

What is the expected value of X?

$$\begin{aligned}\mathbb{E}[X] &= \mathbb{P}[\text{win}] \cdot (-2) + \mathbb{P}[\text{lose}] \cdot (299,999,998) \\ &= \frac{299,999,999}{300,000,000}(-2) + \frac{1}{300,000,000}(299,999,999) \\ &= -1.\end{aligned}$$

Conclusion: you "expect" to lose \$1 every time you play. Similarly, if you play twice, you expet to lose \$2. If you play 10 times, you expect to lose \$10. Etc. This property is called *linearity* of expectation.

End of Example 3. □

3.2 Recap of terminology

Recall: The ***sample space*** of an experiment is the *set* of all possible ***outcomes***. An ***event*** is a collection of outcomes. That is,

$$\text{an event} = \text{a set of outcomes} = \text{a subset of the sample space } S.$$

Example 4 (Sum 2d4)**.** Roll a 4-sided dice twice (this is the **experiment**). There are 16 possible **outcomes**. The **sample space** is

$$S = \{(x, y) : x, y \in \{1, 2, 3, 4\}\},$$

which consists of 16 ordered pairs (x, y), the **sample points**. An **event** is any subset of these.

We'll give two examples of events.

Let X be the sum of the two rolls. We can represent X by the following table:

		Dice 2			
		1	2	3	4
	1	2	3	4	5
Dice 1	2	3	4	5	6
	3	4	5	6	7
	4	5	6	7	8

The event that $X = 6$ is:

$$[X = 6] = \{(1, 5), (2, 4), (3, 3), (4, 2), (5, 1)\}.$$

Here's another example of an event:

$$[X \text{ is even}] = \{(1,1),(1,3),(2,2),(2,4),(3,1),(3,3),(4,2),(4,4)\}.$$

The **enumeration principle** says that, when outcomes are equally likely, we can compute the probability of an event E by

$$\mathbb{P}[E] = \frac{|E|}{|S|}.$$

So

$$\mathbb{P}[X=6] = \frac{3}{16}$$

and

$$\mathbb{P}[X \text{ is even}] = \frac{8}{16} = \frac{1}{2}.$$

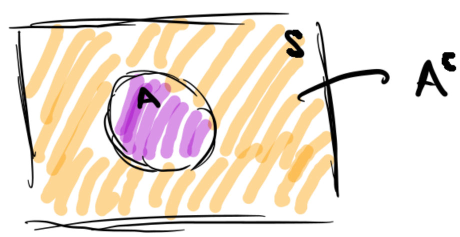
End of Example 4. □

Since probability theory is formalized in terms of sets, we need to have some intuition about set theory.

3.3 Understanding the "events" as sets

Let A and B be events. That is, A and B are subsets of S.

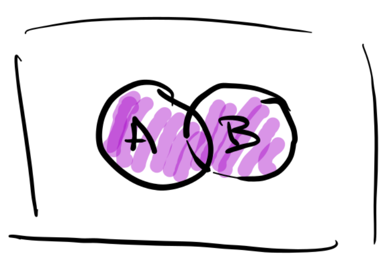
- Suppose A is an event. It's a subset of S, like this:



- The ***complement*** of A, denoted A^c is the set of points in S that are **not** in A.

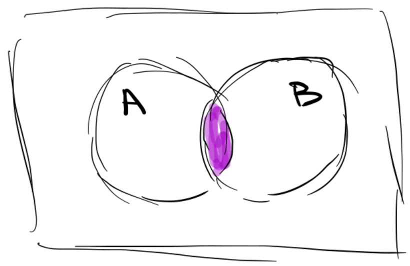
  We interpet A^c as the event that A didn't happen.

- The ***union*** of A and B, denoted $A \cup B$, to denote the set of all outcomes that are in A or B this includes outcomes that are in both):



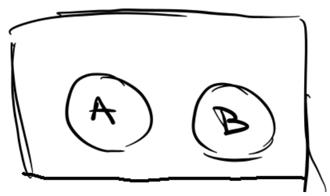
  We interpret $A \cup B$ as the event that either A or B (or both) occur.

- The ***intersection*** of A and B, denoted $A \cap B$ is the set of points that are contained in both A and B:



We interpret $A \cap B$ as the event that A and B both occur.

- What if A and B don't overlap at all? In that case, their intersection has nothing in it!



In this case, we say that $A \cap B$ is the "empty set", which is the set containing no elements. We use the symbol $\emptyset$ to represent the empty set; that is,

$$\emptyset := \{\}$$

This is sometimes called the ***null event***. If $A \cap B = \emptyset$, then we say that A and B are ***disjoint***. Disjoint events are ***mutually exclusive***, in the sense that they cannot happen simultaneously.

For example, when rolling a dice, the following two events are mutually exclusive:

$$[\text{dice is even}] \quad \text{and} \quad [\text{dice is odd}]$$

- An sequence of events $\{E_1, E_2, \ldots\}$ is said to be ***pairwise disjoint*** if and only if

$$E_i \cap E_j = \emptyset \quad \text{whenever } i \neq j.$$

In other words, none of the events in E "overlap" with any other events. For an example of this, the experiment from last lecture involving repeatedly drawing a ball from an urn with replacement until we get the gold ball. Let X be the number of balls we had to draw until we got the gold ball. Let

$$E_n = [X = n] \text{ for } n = 1, 2, 3, \ldots$$

Then $\{E_1, E_2, \ldots\}$ is a pairwise disjoint sequence of events.