2 2025-01-15 | Week 1 | Class 2

Tell the students to read chapters 2.1 - 2.5

Example 1 (Powerball Lottery)**.** The a ticket for the powerball lottery costs \$2. There are two outcomes:

- You win \$300,000,000.
- You don't win any money.

The probability of winning is approximately $\frac{1}{300{,}000{,}000}$. Let X be the net payoff from the game, in dollars:

- If you lose, then $X = -2$, since you had to pay \$2 to play the game.
- If you win, then your net payoff is $X = 299,999,998$.

What is the expected value of X?

$$\begin{aligned}\mathbb{E}[X] &= \mathbb{P}[\text{win}] \cdot (-2) + \mathbb{P}[\text{lose}] \cdot (299,999,998) \\ &= \frac{299,999,999}{300,000,000}(-2) + \frac{1}{300,000,000}(299,999,999) \\ &= -1.\end{aligned}$$

Conclusion: you "expect" to lose \$1 every time you play. Similarly, if you play twice, you expet to lose \$2. If you play 10 times, you expect to lose \$10. Etc. This property is called *linearity* of expectation.

- Go over problem 4, parts (b) and (c). This is the problem where you look ath te expected value of 2 or 120 dice.
- Go over problem 6.
- I'll put problems 7 and 8 on the first homework, lol.
- "Elementary" doesn't mean easy.

2.1 Sample space

An *experiment* is an activity or process whose outcome is subject to uncertainty. Examples include flipping a coin, rolling a dice, measuring the size of a wave, or the amount of rainfall. WE USUALLY DENOTE RANDOM QUANTITIES WITH CAPITAL LETTERS, LIKE 'X'. (We also tend to denote *sets* with capital letters, so ask if you get confused).

The *sample space* of an experiment is the *set* of all possible outcomes.

Example 2 (Sample space)**.**

- If I roll dice, the sample space is
$$S = \{1, 2, 3, 4, 5, 6\}$$
- If I flip a coin, the sample space is
$$S = \{T, H\}$$
- The amount of daily rainfall is
$$S = \{x \in \mathbb{R} : x \geq 0\}$$
- Flip a coin repeatedly until you get heads. Then count the number of times you had to flip the coin. The sample space is
$$S = \{1, 2, 3, 4, \ldots\}.$$

The last two examples show that the sample space need not be finite.

The *elements* of S are called *outcomes*. A specified collection of outcomes is called an *event*. So we think of an event as being a *subset* of the sample space:

$$\text{an event} = \text{a set of outcomes} = \text{a subset of } S.$$

Example 3 (Rolling a dice)**.** To illustrate, suppose we roll a dice. Let X be the value of the dice roll. We know the sample space of the experiment is $S = \{1, 2, 3, 4, 5, 6\}$. Some possible events, along with the sets we identify them with, are shown below:

- "the dice roll is even" . . . . $\{2, 4, 6\}$
- "the dice roll is at least 3" . . . . $\{3, 4, 5, 6\}$
- "the dice roll equals 1" . . . . $\{1\}$

It's common to represent events using square brackets like this: [event description]. For example, the first event above could also be written as:

$$[X \text{ is even}] \quad \text{or} \quad [X \in \{2, 4, 6\}]\,.$$

And we might write the other two events as

$$[X \geq 3] \quad \text{and} \quad [X = 1]\,.$$

Example 4 (Rolling two dice)**.** When rolling a red and a blue dice, the sample space consists of 36 possible outcomes:

1 1	1 2	1 3	1 4	1 5	1 6
2 1	2 2	2 3	2 4	2 5	2 6
3 1	3 2	3 3	3 4	3 5	3 6
4 1	4 2	4 3	4 4	4 5	4 6
5 1	5 2	5 3	5 4	5 5	5 6
6 1	6 2	6 3	6 4	6 5	6 6

so we can write the sample space as:

$$\begin{aligned} S &= \{(x, y) : x, y \in \{1, 2, 3, 4, 5, 6\}\} \\ &= \{(1, 1), (1, 2), \ldots, (6, 6)\} \end{aligned}$$

The event that the dice are equal is

$$[\text{dice are equal}] = \{(1, 1), (2, 2), (3, 3), (4, 4), (5, 5), (6, 6)\}$$

We could have other events too, like that the dice sum to 3:

$$[\text{dice sum to } 4] = \{(1, 3), (2, 2), (3, 1)\}$$

All of the outcome pairs are equally likely, so we can compute the probabilities of by counting entires in the table. Let

$$Z = \text{the sum of the red dice and the blue dice}$$

By counting entries in our table, we see that

$$\mathbb{P}[Z = 4] = \frac{3}{36}.$$

Similarly,

$$\mathbb{P}[Z = 7] = \frac{6}{36} = \frac{1}{6}$$

and

$$\mathbb{P}[Z \leq 5] = \frac{10}{36}.$$

The previous example illustrates the critically imporant idea of **enumeration:**

> If your sample space is finite and consists of *equally likely outcomes*, then you can compute lots of probabilities easily by listing outcomes and counting them. To be prcecise, for any event A,
>
> $$\mathbb{P}[A] = \frac{|A|}{|S|} = \frac{\#\text{ of outcomes in } A}{\#\text{ of outcomes in } S}$$

I already had you do a bunch of these problems on Monday.
For the rainfall example,

$$[\text{between 1 and 2 inches of rain}] = \{x \in \mathbb{R} : 1 \leq x \leq 2\}.$$