31 2025-04-09 | Week 12 | Lecture 30

32 Simple random walk

Let's track the value of a stock. Suppose that every minute, the stock value goes up \$1 or down \$1. Then the change in stock value from the start of the day is

$$S_n = X_1 + \ldots + X_n$$

where $X_1, \ldots, X_n$ are IID random variables with

$$X_i = \begin{cases} +1 & : \text{ with probability } \theta \\ -1 & : \text{ with probability } 1 - e\theta \end{cases}$$

Suppose we track the value of the stock over the course of 400 minutes, (i.e., about 7 hours). And we observe that the value of the stock is down \$68.

The standard assumption is that the stock price fluctuates randomly, with no tendency to go up or down. That is, the null hypothesis is that $\theta = \frac{1}{2}$. Is the observed drop in stock price consistent with this hypothesis? Or, alternatively, is there a reason that the stock is down (the alternative hypothesis)?

Let's compute the p-value. We want to compute

$$\mathbb{P}\left[S_{400} \leq -68 \mid \theta = \frac{1}{2}\right]$$

First observe that

$$\mu = \mathbb{E}\left[X\right] = 0 \quad \text{and} \quad \sigma^2 = \mathbb{E}\left[X^2\right] = 1.$$

By the central limit theorem,

$$\frac{S_n}{n} \approx \mu + \frac{\sigma}{\sqrt{n}} Z$$

where Z is a standard normal. Taking $n = 400$, $\mu = 0$, and $\sigma = 1$, we get

$$\frac{S_{400}}{400} \approx \frac{1}{20} Z$$

Therefore

$$\begin{aligned}
\mathbb{P}\left[S_{400} \leq -68 \mid \theta = \frac{1}{2}\right] &= \mathbb{P}\left[\frac{S_{400}}{400} \leq -\frac{68}{400} \mid \theta = \frac{1}{2}\right] \\
&\approx \mathbb{P}\left[\frac{Z}{20} \leq -\frac{68}{400}\right] \\
&= \mathbb{P}\left[Z \leq -\frac{68}{20}\right] \\
&= \mathbb{P}\left[Z \leq -3.4\right] \\
&= \int_{-\infty}^{-3.4} \frac{1}{\sqrt{2\pi}} e^{-y^2/2} dy \\
&\approx 0.0003.
\end{aligned}$$

In this case, observing such a precipitous drop is very unlikely under the assumption of the null hypothesis. So we might reasonably reject the null hypothesis. It is much more likely that something caused the price drop.

32.1 The Matching Problem

Suppose there are n people in class. Everyone writes their name on one playing card. The cards are then shuffled, and dealt out again, so that each person gets a new (random) card. What is the probability that at least one person gets their original card?

By enumerating possibilities, we calculated that this has probability $2/3$ when there are $n = 3$ people.

32.2 Permutations:

- When we shuffle the cards, we permute their order. This induces a reordering of the cards, called a *permutation.*
- Introduce cycle notation
- Define *cycle*
- Define *order* a cycle
- Define *fixed point* of a permutatiion

Question 1: What is the probability that at least one person gets their original card?
Question 2: What is the expected number of fixed points?
Question 3: What is the expected number of cycles of order 2, 3,... ?
Question 4: What is the expected number of cycles?