29 2025-04-04 | Week 11 | Lecture 28

exam average was 30.75 (among those who took the exam)

29.1 The method of moments

Example 88 (Method of Moments)**.** Suppose $X_1, \ldots, X_n$ are the waiting times, in minutes, between customer orders at Raising Cane's at University and King St. Assume that the waiting times are IID exponentially distributed with some unknown rate parameter λ.

Suppose the first 10 waiting times are

$$0.2, 0.1, 1, 0.7, 0.8, 0.2, 0.5, 0.5, 0.4, 2$$

Question: On average, how many orders would you expect per minute?

So the average wait time is

$$\frac{0.2 + 0.1 + 1 + 0.7 + 0.8 + 0.2 + 0.5 + 0.5 + 0.4 + 2}{10} = .64 \text{ minutes/order}$$

So a good estimate for the number of orders per minute is

$$1/.64 \approx 1.56 \text{ orders/minute}$$

We have estimated the rate.

In other words, the pdf for every X_i is

$$f(x) = \lambda e^{-\lambda x}, \quad \lambda \geq 0$$

We have

$$\mathbb{E}\left[X_1\right] = \int_0^\infty x f(x) dx = \ldots = \frac{1}{\lambda}$$

' We also know by the Law of Large Numbers that

$$\overline{X} := \frac{X_1 + \ldots + X_n}{n} \to \mathbb{E}\left[X_1\right] \quad \text{as } n \to \infty$$

Thus, when n is large, one good way to estimate λ would be to set

$$\overline{X} = \frac{1}{\lambda}$$

and then solve for λ:

$$\lambda = \frac{1}{\overline{X}}.$$

This is precisely what we did. Thus we take as our estimator the function $\widehat{\lambda} = \frac{1}{\overline{X}}$.

This example was brought to you by Raising Cane's.

End of Example 88. □

This example illustrates one general method of estimation, called *the method of moments.* In this method, one has an IID random sample $X_1, \ldots, X_n$ all having distribution X. It is assumed that you know the type of the distribution of X (e.g., exponential, normal, binomial, whatever) but you don't know one or more numerical parameters $\theta_1, \ldots \theta_m$ (e.g., the mean, variance, rates, etc). You then consider one or more of equations of the following form:

$$\begin{aligned}
\frac{X_1 + \ldots + X_n}{n} &= \mathbb{E}\left[X\right] \\
\frac{X_1^2 + \ldots + X_n^2}{n} &= \mathbb{E}\left[X^2\right] \\
\frac{X_1^3 + \ldots + X_n^3}{n} &= \mathbb{E}\left[X^3\right] \\
&\vdots
\end{aligned}$$

In these equations, the LHS is what you observed in your data (e.g., $\overline{X} = .64$ orders per minute). The right-hand sides are the moments, which are known functions of the parameters $\theta_1, \ldots, \theta_m$. You then solve the system of equations for $\theta_1, \ldots, \theta_m$. This gives you numerical point estimates for the unknown parameters $\theta_1, \ldots, \theta_m$.

29.2 The Gamma Distribution

The *gamma function* is

$$\Gamma(\alpha) := \int_0^\infty x^{\alpha-1} e^{-x} dx, \quad \alpha > 0.$$

Fact: For any positive integer $n \geq 1$,

$$\Gamma(n) = (n-1)!$$

For any $\alpha > 1$,

$$\Gamma(\alpha) = (\alpha - 1)\Gamma(\alpha - 1)$$

We say that a random variable X has a *gamma distribution with shape parameter $\alpha > 0$ and scale parameter $\beta > 0$* if X has pdf of X is

$$f(x) = \begin{cases} \frac{1}{\beta^\alpha \Gamma(\alpha)} x^{\alpha-1} e^{-x/\beta} & : \quad x \geq 0 \\ 0 & : \quad x < 0 \end{cases}$$

Fact: If X is gamma distributed with shape $\alpha > 0$ and scale $\beta > 0$, then

$$\mathbb{E}\left[X\right] = \alpha\beta \quad \text{and} \quad \mathbb{E}\left[X_1^2\right] = \alpha(\alpha+1)\beta^2 \quad \text{and} \quad \mathrm{Var}(X) = \alpha\beta^2$$

This is a very flexible class of distributions, often used in general practice for things like

- amount of degredation or wear
- random rates (e.g. cancer, dna mutation, etc)
- survival and waiting times (e.g. for a given number of events to occur, or how long you survive after being exposed to a high dose of radiation, etc)
- sum of IID exponential waiting times.
- has nice mathematical properties, so it's COMMONLY used in (advanced) Bayesian statistical techniques
- etc

Verification that f is a probability density function: Let $x \geq 0$. THen

$$f(x) = \frac{1}{\beta^\alpha \Gamma(\alpha)} x^{\alpha-1} e^{-x/\beta} \geq 0.$$

We need to show that

$$\int_0^\infty f(x) dx = 1$$

This can be shown using a u-subsitution. We will do the u-subsituttion $u = x/\beta$, so that $x = \beta u$ and

$dx = \beta du$

$$\begin{aligned}
\int_0^\infty f(x)dx &= \frac{1}{\beta^\alpha \Gamma(\alpha)} \int_0^\infty x^{\alpha-1} e^{-x/\beta} dx \\
&= \frac{1}{\beta^\alpha \Gamma(\alpha)} \int_0^\infty (\beta u)^{\alpha-1} e^{-u} \beta du \\
&= \frac{1}{\beta^\alpha \Gamma(\alpha)} \beta^{\alpha-1} \beta \int_0^\infty u^{\alpha-1} e^{-u} du \\
&= \frac{1}{\Gamma(\alpha)} \underbrace{\int_0^\infty u^{\alpha-1} e^{-u} du}_{=\Gamma(\alpha)} \\
&= 1.
\end{aligned}$$

We've shown that f is nonnegative and integrates to 1. Therefore it is a pdf.

Example 89. Suppose $X_1, \dots, X_n$ are IID gamma distributed random variables with parameters α and β which are unknown to us.
Know:

Can we estimate α and β from data? for example, set

$$\frac{X_1 + \ldots + X_n}{n} = \mathbb{E}\left[X_1\right]$$

and

$$\frac{X_1^2 + \ldots + X_n^2}{n} = \mathbb{E}\left[X_1^2\right]$$

We know that for a gamma-distributed random variable X with parameters $\hat{\alpha}$ and $\hat{\beta}$, we have $\mathbb{E}\left[X\right] = \hat{\alpha}\hat{\beta}$ and $\mathbb{E}\left[X_1^2\right] = \hat{\alpha}(\hat{\alpha}+1)\hat{\beta}^2$. So these equations become

$$\frac{X_1 + \ldots + X_n}{n} = \hat{\alpha}\hat{\beta}$$

and

$$\frac{X_1^2 + \ldots + X_n^2}{n} = \hat{\alpha}(\hat{\alpha}+1)\hat{\beta}^2$$

We now solve for $\hat{\alpha}$ and $\hat{\beta}$. Suppose that we sample data which is such that

$$\frac{X_1 + \ldots + X_n}{n} = c$$

and

$$\frac{X_1 + \ldots + X_n}{n} = d$$

where c, d are numbers. Then

$$\hat{\alpha}\hat{\beta} = c$$

and

$$d = (\hat{\alpha}\hat{\beta})^2 + (\hat{\alpha}\hat{\beta})\hat{\beta}$$

so that

$$d = c^2 + c\hat{\beta}$$

Solving for $\hat{\alpha}$ and $\hat{\beta}$ gives

$$\hat{\beta} = \frac{d - c^2}{c} \quad \text{and} \quad \hat{\alpha} = \frac{c^2}{d - c^2}.$$

End of Example 89. □