

22 2025-03-05 | Week 9 | Lecture 21

Please read 5.1-5.4

Example 71. Let X and Y be the lifetimes of two lightbulbs. Assume that $X \sim \exp(\alpha)$ and $Y \sim \exp(\beta)$, where α and β are positive numbers. It's reasonable to assume these as independent. Then the joint pdf is

$$\begin{aligned} f(x, y) &= f_X(x)f_Y(y) \\ &= (\alpha e^{-\alpha x} \mathbf{1}_{[x>0]}) (\beta e^{-\beta y} \mathbf{1}_{[y>0]}) \\ &= \alpha \beta e^{-\alpha x - \beta y} \mathbf{1}_{[x,y>0]} \\ &= \begin{cases} \alpha \beta e^{-\alpha x - \beta y} & : x, y > 0 \\ 0 & : \text{else} \end{cases} \end{aligned}$$

Suppose $\alpha = 1/1000$ and $\beta = 1/1200$. Then the expected lifetimes of the bulbs are 1000 hours and 1200 hours, respectively.

Question: What is the probability that both lightbulbs last at least 900 hours?

Want:

$$\begin{aligned} \mathbb{P}[X \geq 900, Y \geq 900] &= \int_{900}^{\infty} \int_{900}^{\infty} (\alpha e^{-\alpha x}) (\beta e^{-\beta y}) dx dy \\ &= \int_{900}^{\infty} \left[\int_{900}^{\infty} \alpha e^{-\alpha x} dx \right] (\beta e^{-\beta y}) dy \\ &= \left(\int_{900}^{\infty} \alpha e^{-\alpha x} dx \right) \left(\int_{900}^{\infty} \beta e^{-\beta y} dy \right) \\ &= \left([e^{-\alpha x}]_{x=900}^{x=\infty} \right) \left([e^{-\beta y}]_{y=900}^{y=\infty} \right) \\ &= (e^{-900\alpha} - 0) (e^{-900\beta} - 0) \\ &= e^{-900(\alpha+\beta)} \\ &= e^{-900(\frac{1}{1000} + \frac{1}{1200})} \\ &= e^{-1.65} \\ &\approx .2 \end{aligned}$$

So with probability about one fifth, both lightbulbs will last at least 900 hours.

A faster way to do this. Observe that the events $[X \geq 900]$ and $[Y \geq 900]$ are independent, because the random variables are. Then

$$\begin{aligned} \mathbb{P}[X \geq 900, Y \geq 900] &= \mathbb{P}[X \geq 900] \mathbb{P}[Y \geq 900] \\ &= e^{-900\alpha} e^{-900\beta} \\ &= e^{-1.65} \end{aligned}$$

End of Example 71. \square

Example 72 (Treasure chest). A randomly looted pirate chest contains treasure, consisting of a mix of gemstones, gold pieces, and valuable navigational charts. The weight of the treasure chest is 10lbs, but the weight contribution of each type of treasure is random. Let X be the proportion of the treasure consisting of gemstones (by weight). Let Y be the proportion which is gold pieces. And let Z the proportion which consists of charts.

Because the proportions sum to 1, it is enough to consider only two of them, X and Y .

Because X and Y are *proportions*, they take values in the following set:

$$D = \{(x, y) : 0 \leq x \leq 1, 0 \leq y \leq 1, x + y \leq 1\}$$

For this problem, we will assume the joint pdf of (X, Y) is

$$f(x, y) = \begin{cases} 24xy & : (x, y) \in D \\ 0 & : (x, y) \notin D \end{cases}$$

Observe that the density increases as x and y increase. So points near the diagonal boundary are *relatively more likely* than points in the bottom left corner. This is appropriate: since gold and gems are heavier than paper, we expect that most of the weight of the treasure will consist of these things, rather than navigational charts.

Question: What is the probability that more than half of the weight of the treasure comes from navigational charts, rather than gems and gold?

We want to compute

$$\mathbb{P}[Z \geq 1/2]$$

We know $X + Y + Z = 1$, so $Z = 1 - X - Y$. Therefore we want to compute

$$\mathbb{P}[1 - X - Y \geq 1/2]$$

or equivalently,

$$\mathbb{P}[X + Y \leq 1/2]$$

Let $A = \{(x, y) : 0 \leq x \leq 1, 0 \leq y \leq 1, x + y \leq .5\}$

$$\mathbb{P}[X + Y \leq 1/2] = \mathbb{P}[(X, Y) \in A] = \int_0^{.5} \int_0^{.5-x} f(x, y) dy dx$$

Some other things we can do:

- check of f is a pdf
- compute the marginal densities f_X and f_Y

End of Example 72. \square