6 2026-01-26 | Week 03 | Lecture 06

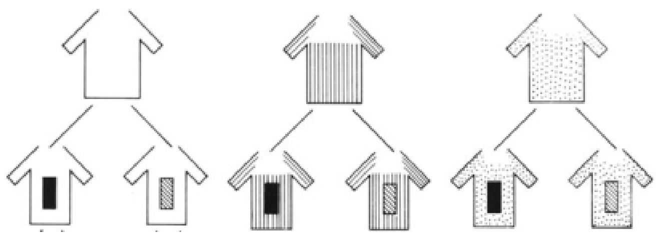
This lecture is based on section 2.3 in the textbook.

There are three main counting principles we'll look at:

- multiplication principle
- permutation principle
- combination principle

6.1 Multiplication principle

Example 12. A man has 3 shirts and two ties. How many ways can he dress himself?



We can schematize this as three shirts s_1, s_2, s_3 and two ties t_1, t_2, in which case the possible outfits are:

$$(s_1, t_1)\ (s_2, t_1)\ (s_3, t_1)$$
$$(s_1, t_2)\ (s_2, t_2)\ (s_3, t_2)$$

Of course, we don't need to write s and t; it is enough to write

$$(1, 1)\ (2, 1)\ (3, 1)$$
$$(1, 2)\ (2, 2)\ (3, 2)$$

where the first slot is "shirt" and the second is "tie". Thus the methematical way to name the collection of outfits is the set of all ordered couples (a, b) with $a = 1, 2$ and $b = 1, 2, 3$. So you can see the total number of outfits is $2 \times 3 = 6$.

In general we can talk about ordered k-tuples $(a_1, \ldots, a_k)$, where for each j from 1 to k, the symbol a_j is the assignment (choice) for the j^{th} slot and it may be denoted by a numeral between 1 and n_j. In our example, $k = 2$ and $n_1 = 3$ and $n_2 = 2$.

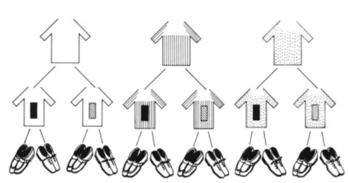
If the man also have two pairs of shoes, we can add a third slot for "shoes", so the question is asking to count the number of elements in the set

$$\{(a_1, a_2, a_3) : a_1 \in \{1, 2, 3\}, a_2, a_3 \in \{1, 2\}\}$$

In this case, $n_1 = 3$, $n_2 = 2$ and $n_3 = 2$, so the number of outfits is

$$n_1 n_2 n_3 = 3 \cdot 2 \cdot 2 = 12,$$

which is shown in the following picture:



End of Example 12. $\square$

In this example, we've used the "multiplication principle". Let's state this principle formally now. To do so, we need the following definition:

Definition 13 (k-tuple)**.** A ***k-tuple*** is an ordered list of k objects.

For example, $(1, 2, 5, 4)$ and $(3, 3, 5, 4)$ are both examples of 4-tuples. Note that repeats are allowed.

Theorem 14 (Multiplication Principle)**.** *Suppose S consist of k-tuples and there are n_1 choices for the first element, n_2 choices for the second, etc. Then there are*

$$n_1 \times n_2 \times \cdots \times n_k$$

possible k-tuples.

Example 15 (Application of the multiplication principle to probability)**.** Consider an experiment in which we roll a dice 5 times in a row. One possible outcome is

$$(1, 4, 6, 4, 4).$$

This is an ordered list with five entries, i.e., a 5-tuple.

Problem 1: What is the sample space? How many elements does it have?

- there are 6 choices for the first entry
- 6 choices for the second entry

  $\vdots$

- 6 choices for the fifth entry

so the number of possible outcomes is

$$6 \times 6 \times 6 \times 6 \times 6 = 6^5 = 7776.$$

Problem 2: Let $E = [\text{All 5 dice are less than or equal to 3}]$. What is $\mathbb{P}[E]$?

By the enumeration principle,

$$\mathbb{P}[E] = \frac{|E|}{|S|} = \frac{\#\text{ ways that all dice are} \leq 3}{7776}$$

now we use again the multiplication principle to count the k-tuples with all entries less than 3:

$$3 \times 3 \times 3 \times 3 \times 3 = 3^5 = 243$$

So

$$\mathbb{P}[E] = \frac{243}{7776} = \frac{1}{32}$$

End of Example 15. $\square$

6.2 Permutations and combinations

Example 16. How many ways are there to order the 4 letters A, B, C, D?

Use the multiplication principle

$$4 \times 3 \times 2 \times 1$$

End of Example 16. □

Proposition 17 (Counting permutations)**.** *The total number of ways to order n distinct objects is*

$$n! = n \times (n-1) \times (n-2) \times \cdots \times 2 \times 1.$$

(Note that we define $0! = 1$.)

Definition 18 (permutation and combination)**.** A ***set*** is an *unordered* collection of elements, all of which are *distinct.*

- $\{1, 2, 4, 5, 3\}$ is a set
- $\{1, 1, 2, 4, 5, 3\}$ is not a set

A ***permutation*** is an ordered subset. An unordered subset is called a ***combination***.