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Recall, a *probability measure* $\mathbb{P}$ is a function which assigns to each possible event a probability, so that the probability of an event E is denoted $\mathbb{P}[E]$ or $\mathbb{P}(E)$. The following axioms hold:

A.1 (Nonnegativity) For every event E, we have

$$\mathbb{P}[E] \geq 0.$$

A.2 (Sum-to-one) $\mathbb{P}[S] = 1$

A.3 (Countable additivity) Let $E_1, E_2, \ldots$ be an infinite sequence of events. If the sequence is pairwise disjoint, then

$$\mathbb{P}[E_1 \cup E_2 \cup \ldots] = \mathbb{P}[E_1] + \mathbb{P}[E_2] + \ldots.$$

recall that paiwise dijoint means that there is no overlap.

Proposition 7 (Basic properties of probability measure)**.**

(i.) *(The null event has probability zero)* $\mathbb{P}[\emptyset] = 0$

(ii.) *(Finite additivity) Let $\{E_1, \ldots, E_n\}$ be a finite sequence of events. If the sequence is pairwise disjoint, then*

$$\mathbb{P}[E_1 \cup E_2 \cup \ldots \cup E_n] = \mathbb{P}[E_1] + \mathbb{P}[E_2] + \ldots + \mathbb{P}[E_n]$$

(iii.) *("With probability one, an event E either does occur or doesn't")* $\mathbb{P}[E^c] = 1 - \mathbb{P}[E]$

(iv.) *(Excision Property) If A, B are events and $A \subseteq B$, then*

$$\mathbb{P}[B \backslash A] = \mathbb{P}[B] - \mathbb{P}[A].$$

(v.) *("The particular is less likely than the general") If A, B are events and $A \subseteq B$, then* $\mathbb{P}[A] \leq \mathbb{P}[B]$

(vi.) *("Probabilities are between 0 and 1") For any event E,* $\mathbb{P}[E] \in [0, 1]$

Proof of Proposition 7. First we will prove (i.). Let $E_i = \emptyset$ for all $i = 1, 2, 3, ...$ Then

- $\{E_1, E_2, \ldots\}$ is a pairwise disjoint sequence.
- If $E_1 \cup E_2 \cup \ldots = \emptyset$.

Therefore,

$$\begin{aligned}
\mathbb{P}[\emptyset] &= \mathbb{P}[E_1 \cup E_2 \cup \ldots] \\
&= \mathbb{P}[E_1] + \mathbb{P}[E_2] + \ldots && \text{by } \textbf{A.3} \\
&= \mathbb{P}[\emptyset] + \mathbb{P}[\emptyset] + \ldots.
\end{aligned}$$

This equation implies $\mathbb{P}[\emptyset] = 0$. We have now proved (i.).

Next we prove (ii.). Let $E_1, \ldots, E_n$ be a finite sequence of events which are pairwise disjoint. Then expand the sequence by taking $E_i = \emptyset$ for all $i \in \{n+1, n+2, \ldots\}$. Then by **A.3**,

$$\begin{aligned}
\mathbb{P}\left[\bigcup_{i=1}^{n} E_i\right] \\
&= \mathbb{P}\left[\bigcup_{i=1}^{\infty} E_i\right] \\
&= \sum_{i=1}^{\infty} \mathbb{P}[E_i] \\
&= \mathbb{P}[E_1] + \ldots \mathbb{P}[E_n] + \underbrace{\mathbb{P}[\emptyset] + \mathbb{P}[\emptyset] + \ldots}_{=0 \text{byProposition 7 (i.)}} \\
&= \mathbb{P}[E_1] + \ldots \mathbb{P}[E_n].
\end{aligned}$$

Next we prove (iii.). Let E be any event. Then

$$\begin{aligned} 1 &= \mathbb{P}[S] && \text{by } \mathbf{A.2} \\ &= \mathbb{P}[E \cup E^c] && \text{since } S = E \cup E^c \\ &= \mathbb{P}[E] + \mathbb{P}[E^c] && \text{by Proposition 7 (ii.), since } E \cap E^c = \emptyset. \end{aligned}$$

This proves (iii.).

Next we prove (iv.). Observe that

$$\begin{aligned} B &= (B \cap A) \cup (B \cap A^c) \\ &= A \cup B \cap A^c && \text{since } A \subseteq B. \end{aligned}$$

This is a disjoint union. Therefore by (ii.),

$$\begin{aligned} \mathbb{P}[B] &= \mathbb{P}[A] + \mathbb{P}[B \cap A^c] \\ &= \mathbb{P}[A] + \mathbb{P}[B \backslash A]. \end{aligned}$$

Rearranging terms proves (iii.). Next we prove (v.). Assume $A \subseteq B$. Then by (iv.),

$$\mathbb{P}[B \backslash A] = \mathbb{P}[B] - \mathbb{P}[A]$$

Moreover, by **A.1**, $\mathbb{P}[B \backslash A] \geq 0$. Hence

$$0 \leq \mathbb{P}[B] - \mathbb{P}[A]$$

and this implies (v.).

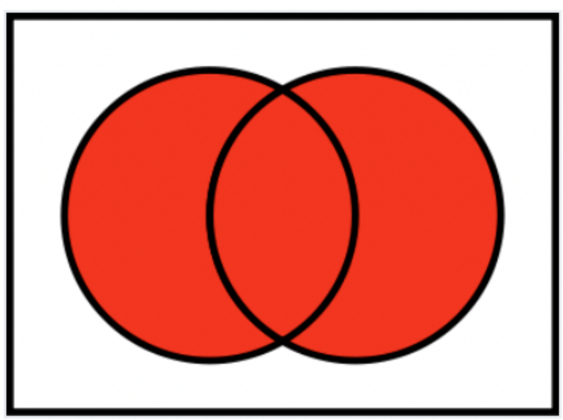
Finally, we will show (vi.). Let E be an event. Then

$$\begin{aligned} 0 \leq \mathbb{P}[E] &\leq \mathbb{P}[S] \leq && \text{by } \mathbf{A.1} \\ &\leq \mathbb{P}[S] && \text{by (v.)} \\ &= 1 && \text{by } \mathbf{A.2}. \end{aligned}$$

$\square$

Proposition 8 (Inclusion-Exclusion Prinicple)**.** $\mathbb{P}[A \cup B] = \mathbb{P}[A] + \mathbb{P}[B] - \mathbb{P}[A \cap B]$

Idea:



Proposition 9 (De Morgan's Laws)**.** *The following equalities hold:*

$$(A \cup B)^c = A^c \cap B^c \quad \textit{and} \quad (A \cap B)^c = A^c \cap B^c.$$

Proposition 10 (Distributive Laws)**.** *The following equalities hold:*

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$

and

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$

4.1 Counting techniques

4.1.1 Tuples / permutations + multiplication rule

A *k-tuple* is an ordered list of k numbers.

Example 11 (tuples)**.** Consider the experiment in which you roll a dice 5 times in a row:

$$(1, 4, 6, 4, 4)$$

is a 5-tuple.
Problem: What is the sample space? How many elements does it have?
Idea: we will use the *the product rule for k-tuples*:

- there are 6 choices for the first entry
- 6 choices for the second entry
- $\vdots$
- 6 choices for the fifth entry

so the number of possible outcomes is

$$6 \times 6 \times 6 \times 6 \times 6 = 6^5 = 7776.$$

Problem: Let $E =$ [All 5 dice are less than or equal to 3]. What is $\mathbb{P}[E]$?
By the counting principle,

$$\mathbb{P}[E] = \frac{|E|}{|S|} = \frac{\#\text{ ways that all dice are} \leq 3}{7776}$$

now we use again the product rule to count the k-tuples with all entries less than 3:

$$3 \times 3 \times 3 \times 3 \times 3 = 3^5 = 243$$

So

$$\mathbb{P}[E] = \frac{243}{7776} = \frac{1}{32}$$

End of Example 11. □