

WORKSHEET 4: CHI-SQUARED TESTS

Useful Fact: The pdf of a chi-squared random variable with d degrees of freedom is

$$f(x) = \frac{1}{2^{d/2}\Gamma(d/2)} x^{\frac{d}{2}-1} e^{-x^2/2}, \quad x > 0.$$

Goodness-of-Fit Test

Suppose we have a multinomial random variable with k outcomes, labelled $1, \dots, k$, constructed from n trials such that in each trial outcome i occurs with probability p_i . The *chi-squared statistic* is the quantity

$$\chi^2 := \sum_{i=1}^k \frac{(N_i - E_i)^2}{E_i}$$

where E_i is the expected number of times outcome i occurs, and N_i is the number of times outcome i is observed. For a goodness of fit test using this statistic, χ^2 has a chi-squared distribution with $k - 1$ degrees of freedom.

Problem 1 (Goodness-of-fit test). According to the Mars Wrigley Confectionery Company, the color distribution of a bag of skittles is skittles, the null hypothesis

$$H_0 : p_{\text{green}} = p_{\text{yellow}} = p_{\text{red}} = p_{\text{orange}} = p_{\text{purple}} = 0.2$$

the alternative hypothesis is

H_1 : at least one of the probabilities is different from 0.2

(a) Assuming that H_0 is true, what is E_i for all $i \in \{\text{green, yellow, red, orange, purple}\}$?

$$E_i = 100 (0.2) = 20 \text{ for all } i. \quad (\text{Assuming } n=100 \text{ Skittles})$$

(b) Fill in the following table with expected and observed counts:

	green	yellow	red	orange	purple
expected	20	20	20	20	20
observed	16	18	24	21	21

(c) Compute the chi-squared statistic χ^2 for your bag of skittles

$$\chi^2 = \frac{(20-16)^2}{20} + \frac{(20-18)^2}{20} + \frac{(20-24)^2}{20} + \frac{(20-21)^2}{20} + \frac{(20-21)^2}{20}$$

(d) Circle the correct phrase to complete the sentence: *A larger chi-square value means the observed data is (very different from) / very similar to) the expected data.*

(e) From theorem stated in class, χ^2 has a chi-squared distribution with $k - 1$ degrees of freedom. Set up and evaluate an integral to determine the p-value $\mathbb{P}[\chi^2 \geq u]$, where u is the value you got in the previous part of the problem.

$$\text{Here } k=4, \text{ so } P[\chi^2 \geq u] = \int_u^{\infty} \frac{1}{4} x e^{-x^2/2} dx$$

(f) Write a sentence or two interpreting your result

Reject H_0 if P-value small

Fail to reject H_0 if P-value not small

Test of Independence

Problem 2 (Test of Independence). A 2007 study of $n = 186$ Japanese children examined a possible relationship between the onset of autism-related developmental regression and exposure to the measles mumps rubella (MMR) vaccine.¹ One of their analyses involved the following dataset:

Table of observed values

	Vaccinated	Not Vaccinated
Autism	15	39
No Autism	45	87

(a) Find the following values

- total number of children exhibiting autistic regression 54
- total number of children not exhibiting autistic regression 132
- total number of vaccinated children 60
- total number of unvaccinated children 126

(b) Let p be the probability that a child exhibits autistic regression, and let q be the probability that a child is vaccinated. Use your answers from the previous part to estimate p and q .

$$p = \frac{54}{186} = .29 \quad q = \frac{60}{186} = .323$$

(c) Under the assumption that autism and vaccination are independent, we can compute expected values for each of the four categories of outcomes, which we'll call e_{11} , e_{12} , e_{21} , and e_{22} , and arrange them in the following table:

Table of expected counts

	Vaccinated	Not Vaccinated
Autism	$e_{11} = 17.4$	$e_{12} = 42.6$
No Autism	$e_{21} = 36.6$	$e_{22} = 89.4$

Answers will vary
a bit due to rounding.

For example, under this assumption, the expected number of children who are both vaccinated and exhibit autistic regression is

$$e_{11} = npq$$

Fill in the values of the table.

(d) Compute the chi-squared statistic

$$\chi^2 = \sum_{\text{all } i,j} \frac{(o_{ij} - e_{ij})^2}{e_{ij}} = 0.7 \quad (\text{approximately})$$

where o_{11} , o_{12} , o_{21} , and o_{22} are the observed values.

(e) Set up and evaluate an integral to compute a p -value for your χ^2 statistic. Here $d=1$.

$$P[\chi^2 \geq 0.7] = \int_{\frac{1}{\sqrt{2\pi}}}^{\infty} e^{-x^2/2} dx \approx 0.2$$

(f) Write a sentence or two interpreting your result.

The p-value isn't very small so we fail to reject H_0 . The

test fails to detect a relationship between the MMR vaccine and autism.

¹Uchiyama et al, *J Autism Dev Disord*, 2007. <https://doi.org/10.1007/s10803-006-0157-3>

Problem 3 (Simpson's Paradox: The Perils of Aggregation). In this problem we compare the performance of two medical centers, Hospital A and Hospital B. We have the following survival data for surgery patients from these two hospitals from the last 6 weeks:

Outcome	Hospital A	Hospital B
Died	63	16
Survived	2037	784

- (a) Compare the survival rates of the two hospitals: what percentage of patients at Hospital A survive? What about Hospital B?

At hospital A, 97% survive, which is lower than 98% for hospital B.

- (b) Is there a relationship between patient survival and which hospital they go to? Do a chi-squared test of independence. What p -value do you get? In one or two sentences, state your null hypothesis and draw a conclusion about it. $H_0: \text{no relationship between hospital \& survival rate.}$

$\chi^2 \approx 1.8$ or $\chi^2 \approx 2.2$ are both okay. p isn't small \rightarrow don't reject, probably

- (c) We now consider a third factor: the condition of the patient when they are admitted for surgery. Patients are either classified as "poor condition" or "good condition". Here's the more detailed data:

Good Condition			Poor Condition		
Outcome	Hospital A	Hospital B	Outcome	Hospital A	Hospital B
Died	6	8	Died	57	8
Survived	594	592	Survived	1443	192

- (d) What fraction of Hospital A's patients were classified as good condition? What about Hospital B?

For A, $\frac{600}{2100} \approx 29\%$ Good Condition. For B, $\frac{600}{900} \approx 67\%$ Good Condition.

- (e) Out of the patients in **good condition** who went to hospital A, what fraction survived? What about hospital B? Compare these two survival rates.

A: $\frac{594}{600} = 99\%$ survive. B: $\frac{592}{600} \approx 98.67\%$ (A is better than B)

- (f) Out of the patients in **bad condition** who went to hospital A, what fraction survived? What about hospital B? Compare these two survival rates.

A: $\frac{1443}{1500} = 96.2\%$ survive. B: $\frac{192}{200} = 96\%$ survive (A better than B)

- (g) Compare your answers to parts (a) (e) (f). What you are observing is called *Simpson's Paradox*. How can this discrepancy be explained? If you are facing surgery, should you go to Hospital A or Hospital B?

A has lower survival rates in aggregate because it accepts more high risk patients. You should go to hospital A.

- (h) Fill in the following contingency table:

Outcome	Good Condition	Bad Condition
Died	14	65
Survived	1186	1635

- (i) Is there a relationship between patient survival and the condition that they are admitted for surgery. Perform a chi-squared test of independence and report a p -value.

$\chi^2 = 17.7$ or $\chi^2 = 18.7$ both okay.

p -Values are about $1 \times 10^{-5} \Rightarrow$ Reject Null!

Problem 4 (Another example of Simpson's Paradox). The influence of race on imposition of the death penalty for murder has been much studied and contested in the courts. The following three-way table classifies 326 cases in which the defendant was convicted of murder.² The three variables are the defendant's race, the victim's race, and whether or not the defendant was sentenced to death.

White Defendants			Black Defendants		
Victim Race	Yes	No	Victim Race	Yes	No
White	19	132	White	11	52
Black	0	9	Black	6	97

- (a) From these data make a 2-way table of defendant's race by death penalty. Does it look like there is a relationship between these two things?

Looks like white defendants are sentenced to death

slightly more often - (11.9% vs 10.2%)

	White defendant	Black defendant
Death penalty	19	17
No Death Penalty	141	149

- (b) Suppose you performed a chi-squared test of independence and for the table you just made. You get $p = .64$. How do you interpret this result?

H_0 : No relationship between defendant race and death penalty

$P = 0.64$ means we don't reject the null, or at least we don't have evidence to reject H_0 .

- (c) Show that Simpson's paradox holds: specifically, that

- a higher percent of white defendants are sentenced to death overall 11.9% vs 10.2%
- but for both black and white victims, a higher percent of black defendants are sentenced to death.

White victim : black defendants sentenced to death 17% of the time, vs 12.6% for white defendants

Black victim : whites are never sentenced to death for killing blacks in this data, vs Black defendants are sentenced to death 5.8% of the time.

- (d) Basing your reasoning on the data, explain why the paradox holds in language that a judge could understand.

Race of the victim matters when it comes to sentencing, and this is obscured in part (a).

²Data from M. Radelet, American Sociological Review, 1981. <https://doi.org/10.2307/2095088>

Problem 5 (One True Love). A poll is taken of $n = 2625$ individuals; people were surveyed on some demographic information and also asked whether or not they believed in "the one true love".

In this problem, we test whether or not there is a relationship between the following two factors

- Factor 1: education level
- Factor 2: sentiment about true love

We can construct a two-way contingency table for these two factors, as follows:

Response	HS	Some College	College	Total
Agree	363	176	196	735
Disagree	557	466	789	1812
Don't Know	20	26	32	78
Total	940	668	1017	2625

This contingency table has $r = 3$ rows and $c = 3$ columns (plus a row and column for totals). The data counts themselves take the form of a 3×3 matrix

$$\begin{matrix} Y_{11} & Y_{12} & Y_{13} & \cdots & Y_{1c} \\ Y_{21} & Y_{22} & Y_{23} & \cdots & Y_{2c} \\ \vdots & & & & \vdots \\ Y_{r1} & Y_{r2} & Y_{r3} & \cdots & Y_{rc} \end{matrix}$$

where Y_{ij} is the number of samples for which factor 1 is i and factor 2 is j . We've also added row and column totals in the margins. We can construct a corresponding table of "estimated expected values" of the form

$$\begin{matrix} E_{11} & E_{12} & E_{13} & \cdots & E_{1c} \\ E_{21} & E_{22} & E_{23} & \cdots & E_{2c} \\ \vdots & & & & \vdots \\ E_{r1} & E_{r2} & E_{r3} & \cdots & E_{rc} \end{matrix}$$

by the formula

$$(1) \quad E_{ij} = n \times \frac{(\text{sum of row } i)}{n} \times \frac{(\text{sum of column } j)}{n}$$

which gives us a new table of estimated expected values.

Fact: For contingency tables with r rows and c columns, the chi-squared statistic

$$\chi^2 := \sum_{i=1}^r \sum_{j=1}^c \frac{(Y_{ij} - E_{ij})^2}{E_{ij}}$$

has chi-squared distribution with $(r - 1) \times (c - 1)$ degrees of freedom.

Questions:

(a) Explain why the formula in Equation (1) holds. *By indep. we can multiply probabilities. Any explanation here is okay.*

(b) What is the degrees of freedom for the chi-squared statistic in this problem? Write down the formula for the pdf of the chi-squared statistic.

$$df = 4, \quad f(x) = \frac{1}{4} x e^{-x/2}, \quad x > 0$$

(c) Do the χ^2 test of independence (i.e., compute a table of expected values, find χ^2 , and use the known distribution of the statistic to compute a p -value).

$$\chi^2 \approx 94, \quad p\text{-value} \approx 10^{-19}$$

(d) Interpret your result.

Extremely small p-value \Rightarrow Reject H_0 .

There is probably a relationship between education level & sentiment about one true love.