3 2025-01-17 | Week 1 | Class 3

3.1 Cool coin flipping experiment

Example 5 (Cool coin flipping experiment)**.** Consider the following experiment:

1. Flip a coin.
2. If the coin is tails, go back to step 1. Otherwise, stop.

In other words, we flip a coin repeatedly until we get a heads.
Output: the number of times we flipped the coin.

This experiment is very similar to the example from last lecture in which we had an urn with 300,000,000 balls, exactly one of which was gold.

The sample space is

$$S = \{1, 2, 3, 4, \ldots\}$$

since there is no limit to the number of times we might have to flip the coin before getting a heads!

Let X be the number of times we flipped the coin. Then

$$\mathbb{P}[X = 1] = \mathbb{P}[\text{first coin flip was heads}] = \frac{1}{2}$$

and

$$\mathbb{P}[X = 2] = \mathbb{P}[\text{first flip was tails and second was heads}] = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}.$$

Similarly, for any $n = 1, 2, 3, 4, \ldots$, we have

$$\mathbb{P}[X = n] = \left(\frac{1}{2}\right)^n \tag{1}$$

How many times do we expect to flip the coin? I mean, what is $\mathbb{E}[X]$?

We use the formula

$$\mathbb{E}[X] = \sum_{n=1}^{\infty} n\mathbb{P}[X = n]$$

which gives

$$\begin{aligned}
\mathbb{E}[X] &= \sum_{n=1}^{\infty} n \left(\frac{1}{2}\right)^n \\
&= \frac{1}{2} + 2\left(\frac{1}{2}\right)^2 + 3\left(\frac{1}{2}\right)^3 + 4\left(\frac{1}{2}\right)^4 + \ldots \\
&= \frac{1}{2} + 2 \cdot \frac{1}{4} + 3 \cdot \frac{1}{8} + 4 \cdot \frac{1}{16} + \ldots \\
&= \frac{1}{2} + \left(\frac{1}{4} + \frac{1}{4}\right) + \left(\frac{1}{8} + \frac{1}{8} + \frac{1}{8}\right) + \left(\frac{1}{16} + \frac{1}{16} + \frac{1}{16} + \frac{1}{16}\right) + \ldots \\
&= \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \ldots \\
&\qquad + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \ldots \\
&\qquad\qquad + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \ldots \\
&\qquad\qquad\qquad + \frac{1}{16} + \frac{1}{32} + \ldots \\
&\qquad\qquad\qquad\qquad + \frac{1}{32} + \ldots
\end{aligned}$$

The first row adds up to 1 by the geometric series formula (which says that $\sum_{n=1}^{\infty} r^n = \frac{r}{1-r}$). The second row adds up to 1/2 (because it's 1/2 less than the first row). The third row adds up to 1/4. The fourth adds up to 1/8. etc. So

$$\mathbb{E}[X] = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \dots$$

and applying the geometric series formula again, we see that

$$\mathbb{E}[X] = 2.$$

In other words, we expect to flip the coin twice.

This problem was easy to state (and some people guessed the correct answer ahead of time!), but it took a lot of technical work to compute the answer. (This sort of interplay between our intuition about games of chance, and the sometimes difficult technical work needed to answer questions conclusively, is a large part of why I think this topic is so interesting.)

End of Example 5. □

3.2 Set theory

Recall: The *sample space* of an experiment is the *set* of all possible *outcomes*. An *event* is a collection of outcomes.

an event = a set of outcomes = a subset of the sample space S

Roll a red and a blue dice. The **sample space** is the set

$$S = \{(1,1), (1,2), \dots (6,6)\}$$

with 36 ordered pairs. Each ordered pair is an **outcome**. The **event** that the dice are equal is the set

$$E = \{(1,1), (2,2), (3,3), (4,4), (5,5), (6,6)\}$$

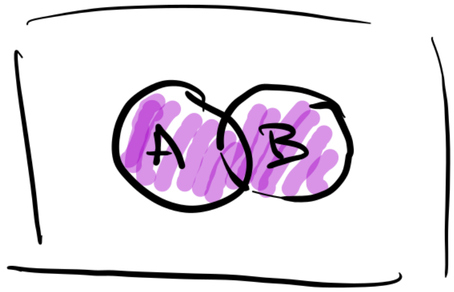
And the enumeration principle says that

$$\mathbb{P}[E] = \frac{|E|}{|S|} = \frac{6}{36} = \frac{1}{6}.$$

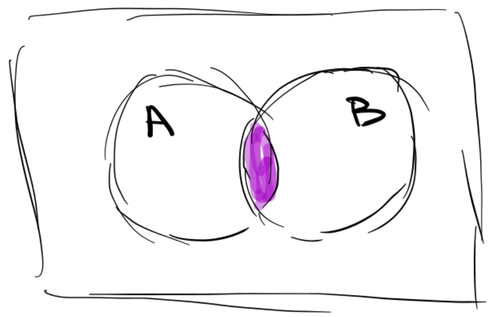
Since probability theory is formalized in terms of sets, we need to have some intuition about set theory.

- Suppose A is an event. It's a subset of S, like this:

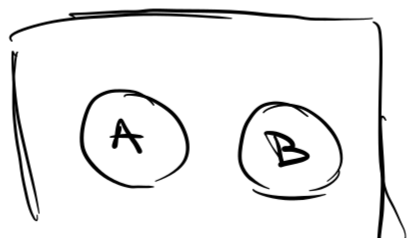
- Define the complmenent of A: A' or A^c. [see diagram above]
- Suppose B is also an event. We write $A \cup B$ to denote the set of all outcomes that are in A or B this includes outcomes that are in both):



- Do the same with $A \cap B$



- What if A and B don't overlap at all? In that case, their intersection has nothing in it!



  In this case, we say that $A \cap B$ is the "empty set", which is the set containing no elements. We use the symbol $\emptyset$ to represent the empty set; that is,

$$\emptyset := \{\}$$

  This is sometimes called the *null event*. If $A \cap B = \emptyset$, then we say that A and B are *mutually exclusive*, or *disjoint*. For example, if I roll a dice, the events

$$[\text{dice is even}] \quad \text{and} \quad [\text{dice is odd}]$$

  are mutually exclusive: they cannot happen simultaneously.

- A sequence of events $\{E_1, E_2, \ldots\}$ is said to be *pairwise disjoint* if and only if

$$E_i \cap E_j = \emptyset \quad \text{whenever } i \neq j.$$

  In other words, none of the events in E "overlap" with any other events. For an example of this, consider the experiment of Example 5, where X is the number of times we flip the coin in the experiment. Let

$$E_n = [X = n] \text{ for } n = 1, 2, 3, \ldots$$

  Then $\{E_1, E_2, \ldots\}$ is a pairwise disjoint sequence of events.

3.3 Probability Axioms

A *probability measure* $\mathbb{P}$ is a function which assigns to each event a probability. We denote the probability of an event E by

$$\mathbb{P}[E] \quad \text{or} \quad \mathbb{P}(E).$$

To be a *probability measure*, $\mathbb{P}$ must satisfy the following three axioms:

A.1 (Nonnegativity) For every event E, we have

$$\mathbb{P}[E] \geq 0.$$

A.2 (Sum-to-one) $\mathbb{P}[S] = 1$

A.3 (Countable additivity) Let $E_1, E_2, \ldots$ be an infinite sequence of events. If the sequence is pairwise disjoint, then

$$\mathbb{P}[E_1 \cup E_2 \cup \ldots] = \mathbb{P}[E_1] + \mathbb{P}[E_2] + \ldots.$$

Example 6 (Example 5 continued)**.** As an example of **A.3**, consider the experiment from Example 5, where we had

$$X = (\text{number of coin flips}).$$

For each $n = 1, 2, \ldots$, define the event E_n as

$$E_n = [X = n].$$

We already know that $\mathbb{P}[E_n] = \left(\frac{1}{2}\right)^n$ from Eq. (1). Also, you should verify for yourself that the sequence

$$\{E_1, E_2, E_3, \ldots\}$$

is pairwise disjoint.

Now, let's say we want to know the probability that X is even. Observe that

$$[X \text{ is even}] = E_2 \cup E_4 \cup E_6 \cup E_8 \cup \ldots$$

Next we will use **A.3** to compute the probability of this event:

$$\begin{aligned}
\mathbb{P}[X \text{ is even}] &= \mathbb{P}[E_2 \cup E_4 \cup E_6 \cup E_8 \cup \ldots] \\
&= \mathbb{P}[E_2] + \mathbb{P}[E_4] + \mathbb{P}[E_6] + \mathbb{P}[E_8] + \ldots \qquad (\text{by } \textbf{A.3}) \\
&= \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^4 + \left(\frac{1}{2}\right)^6 + \left(\frac{1}{2}\right)^8 + \ldots \qquad (\text{by Eq. (1)}) \\
&= \frac{1}{4} + \left(\frac{1}{4}\right)^2 + \left(\frac{1}{4}\right)^3 + \left(\frac{1}{4}\right)^4 + \ldots \\
&= \frac{\frac{1}{4}}{1 - \frac{1}{4}} \qquad \text{by the geometric series formula } \sum_{n=1}^{\infty} r^n = \frac{r}{1-r} \\
&= \frac{1}{3}.
\end{aligned}$$

So the probability that you flip the coin an *even* number of times is only 1/3.

End of Example 6. □