

13 2025-02-10 | Week 5 | Lecture 12

Today's lecture is on sections 3.3 and 3.4 in the textbook

13.1 Expectation

The *expected value* of a discrete random variable is

$$\mu = \mathbb{E}[X] = \sum_x x \cdot p(x)$$

where the x in the summation runs over all possible values of X .

In words, the expected value is the *long-run average*, meaning that if you repeated an experiment many times (independently), the long-run average would converge to $\mathbb{E}[X]$.

Textbook uses the notation μ or μ_X for $\mathbb{E}[X]$.

The following is a simple but important fact:

Proposition 39 (Linearity of Expectation). *Let a, b be numbers and X be a random variable. Then*

$$\mathbb{E}[aX + b] = a\mathbb{E}[X] + b$$

We saw this earlier, when we looked at the expected value of playing the lottery once ($-\$1$) and of playing 10 times ($-\$10$), etc.

Theorem 40 (Expectation of a function of X). *Let X be a discrete random variable with pmf $p(x)$. Let $h : \mathbb{R} \rightarrow \mathbb{R}$ be any function. Then*

$$\mathbb{E}[h(X)] = \sum_x h(x)p(x)$$

provided that the sum on the right hand side is absolutely convergent. As usual, the summation runs over all possible values of X .

Example 41. Suppose Y is a discrete random variable with pmf

$$\begin{aligned} p(-2) &= \mathbb{P}[Y = -2] = 0.1 \\ p(-1) &= \mathbb{P}[Y = -1] = 0.3 \\ p(1) &= \mathbb{P}[Y = 1] = 0.4 \\ p(2) &= \mathbb{P}[Y = 2] = 0.2 \end{aligned}$$

and such that $\mathbb{P}[Y = y] = 0$ if $y \notin \{-2, -1, 1, 2\}$. Find the following quantities:

- (a) $\mathbb{E}[Y]$
- (b) $\mathbb{E}[3Y + 7]$
- (c) $\mathbb{E}[Y^3 + 2Y]$
- (d) $\mathbb{E}[e^X]$

Solution to (a):

$$\begin{aligned} \mathbb{E}[Y] &= -2(0.1) + (-1)(0.3) + (1)(0.4) + (2)(0.2) \\ &= -.2 - .3 + .4 + .4 \\ &= .3 \end{aligned}$$

Solution to (b):

$$\begin{aligned} \mathbb{E}[3Y + 7] &= 3\mathbb{E}[Y] + 7 && \text{by Proposition 39} \\ &= 3(0.3) + 7 && \text{by our answer to part (a).} \\ &= 7.9. \end{aligned}$$

Solution to (c): Here we will apply Theorem 40 with $h(x) = x^3 + 2x$:

$$\begin{aligned}\mathbb{E}[Y^3 + 2Y] &= \sum_{y \in \{-2, -1, 1, 2\}} h(y)p(y) \\ &= h(-2)p(-2) + h(-1)p(-1) + h(1)p(1) + h(2)p(2) \\ &= (-12)(0.1) + (-3)(0.3) + (3)(0.4) + (12)(0.2) \\ &= 1.5.\end{aligned}$$

Solution to (d): Here we will apply Theorem 40 with $h(x) = e^x$:

$$\begin{aligned}\mathbb{E}[e^Y] &= \sum_{y \in \{-2, -1, 1, 2\}} h(y)p(y) \\ &= \sum_{y \in \{-2, -1, 1, 2\}} e^y p(y) \\ &= e^{-2}p(-2) + e^{-1}p(-1) + e^1p(1) + e^2p(2) \\ &= e^{-2} \cdot 0.1 + e^{-1} \cdot 0.3 + e^1 \cdot 0.4 + e^2 \cdot 0.2 \\ &\approx 2.689.\end{aligned}$$

End of Example 41. \square

13.2 Variance

The *variance* of a random variable X is

$$\text{Var}(X) := \sum_x (x - \mu)^2 p(x)$$

where $\mu = \mathbb{E}[X]$ and x ranges over all possible values that X can take. The variance of a random variable is often denoted σ^2 .

Important formula: There is a shortcut for computing variance

$$\text{Var}(X) = \mathbb{E}[X^2] - (\mathbb{E}[X])^2$$

Variance is a measure of how likely the value of a random variable is to be far from its expected value. In an ideal world, we would measure this by

$$\mathbb{E}[|X - \mu|] = (\text{expected distance of } X \text{ from its mean})$$

But unfortunately, the absolute value function is mathematically difficult to work with, so instead we use

$$\mathbb{E}[(X - \mu)^2] = \text{Var}(X)$$

which is mathematically easier to work with because it doesn't have an absolute value.

We'll do some examples next time