# 10 2025-02-03 | Week 4 | Lecture 9

*If you haven't yet, please finish reading sections 2.4 and 2.5.*

## 10.1 Law of Total Probability + Bayes' Theorem

**Theorem 31** (The Law of Total Probability)**.** *Let $A_1, \ldots, A_n$ be events which partition the sample space (i.e., they are mutually exclusive and $A_1 \cup \cdots \cup A_n = S$). Then*

$$\mathbb{P}[B] = \sum_{i=1}^{n} \mathbb{P}[B \mid A_i]\, \mathbb{P}[A_i]$$

*Proof.* Proof by picture. □

**Example 32** (Questing adventurers)**.** *[Similar to example 2.29 from text]* A large group of adventurers are given the choice of one of three quests: slay a dragon ($Q_1$), defeat the lich king ($Q_2$), or retrieve the long-lost scepter of fire in the underground kingdom of Avernus ($Q_3$). Out of this group, 50% of the adventurers undertake $Q_1$, 30% undertake $Q_2$, and the remaining 20% choose $Q_3$.

It is known that within the first year, 25% of adventurers who set out to slay a dragon become dragon food instead. Moreover, 20% of adventurers who attempt to stop the lich king end up joining the ranks of his growing undead army, and 10% of adventurers who descend into Avernus to retrieve the scepter of fire get slain by underground lizardmen.

(a) What is the probability that a randomly-selected adventurer undertakes quest (1) and gets eaten by a dragon?

*Solution:* In other words, we want to compute $\mathbb{P}\left[\text{☠} \cap Q_1\right]$. Using the multiplication rule, we have

$$\mathbb{P}\left[\text{☠} \cap Q_1\right] = \mathbb{P}\left[\text{☠} \mid Q_1\right] \cdot \mathbb{P}[Q_1] = \frac{1}{4} \cdot \frac{1}{2} = \frac{1}{8} = .125$$

(b) What is the probability that a randomly-selected adventurer meets an untimely demise?

*Solution:*

$$\begin{aligned}
\mathbb{P}\left[\text{☠}\right] &= \mathbb{P}\left[(Q_1 \cap \text{☠}) \cup (Q_2 \cap \text{☠}) \cup (Q_3 \cap \text{☠})\right] \\
&= \mathbb{P}\left[Q_1 \cap \text{☠}\right] + \mathbb{P}\left[Q_2 \cap \text{☠}\right] + \mathbb{P}\left[Q_3 \cap \text{☠}\right] \\
&= .125 + .06 + .02 \\
&= .205,
\end{aligned}$$

so we conclude that 20.5% of adventurers meet an untimely demise ☠ within the first year.

(c) If an adventurer meets an untimely demise, what is the probability that they undertook quest (1)? What about quests (2) and (3)?

*Solution:* In other words, we want to compute $\mathbb{P}\left[Q_i \mid \text{☠}\right]$ for $i = 1, 2, 3$. For this we use the definition of conditional probabiblity. For $i = 1$, we have:

$$\begin{aligned}
\mathbb{P}\left[Q_1 \mid \text{☠}\right] &= \frac{\mathbb{P}\left[Q_1 \cap \text{☠}\right]}{\mathbb{P}\left[\text{☠}\right]} \\
&= \frac{.125}{.205} && \text{by parts (a) and (b)} \\
&= \frac{25}{41} \\
&\approx 61\%.
\end{aligned}$$

We conclude that 61% of adventurers who met an untimely end were done in by a dragon.

End of Example 32. □

In part (b), we used the *law of total probability*:

$$\mathbb{P}[A] = \sum_i \mathbb{P}[A \cap B_i] = \sum_i \mathbb{P}[A \mid B_i]\,\mathbb{P}[B_i],$$

(which holds as long as the sequence $B_1, B_2, \ldots$ are mutually exclusive and exhaustive). In particular, we found $\mathbb{P}\left[☠\right]$ by taking $A = ☠$, along with $B_1 = Q_1$, $B_2 = Q_2$, and $B_3 = Q_3$.

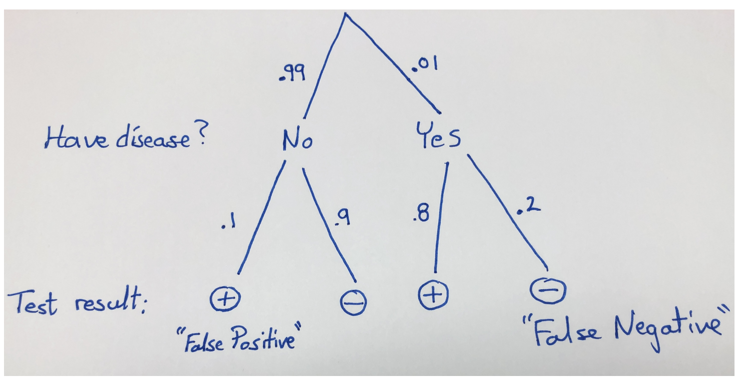
In part (c), we have used *Bayes' Theorem*, which is usually stated as

$$\mathbb{P}[B_i \mid A] = \frac{\mathbb{P}[A \cap B]}{\mathbb{P}[A]} = \frac{\mathbb{P}[A \mid B_i]\,\mathbb{P}[B_i]}{\sum_k \mathbb{P}[A \mid B_k]\,\mathbb{P}[B_k]}.$$

In part (c) of Example 32, we had $A = ☠$ and $B_i = Q_i$ for $i = 1, 2, 3$. We were able to interpet $\mathbb{P}\left[Q_1 \mid ☠\right]$ as "the proportion of dead aventurers who were done in by a dragon". More generally, the right-hand side of Bayes' theorem suggests that we can think of $\mathbb{P}[B_i \mid A]$ as "the contribution of $B_i$ to the total probability of $A$".

You can memorize the RHS of the above formula, but it should suffice to know how to use tree digrams like we did in this problem, which will work as long as you know the law of total probability and the definition of conditional probability.

**Example 33** (Bayes' Theorem)**.** Prevalance of a disease in a population is 1%. A new diagnostic test is advertised as having a false positive rate of .1 and a false negative rate of .2. (In other words, 10% of positive tests are wrong, and 20% of negative tests are wrong.) Suppose you are selected for a random screening, and you test posititve. What's the chance that you have the disease?




We want $\mathbb{P}[\text{no disease} \mid \text{test positive}]$. For brevity, let's let

$$D = [\text{have disease}] \quad \text{and} \quad T^+ = [\text{test positive}].$$

In other words, we want

$$\mathbb{P}\left[D \mid T^+\right]$$

By the definition of conditional probability,

$$\mathbb{P}\left[D \mid T^+\right] = \frac{\mathbb{P}[T^+ \cap D]}{\mathbb{P}[T^+]} \tag{3}$$

We will compute the numerator and denominator separately:

- (Numerator) By the multiplication rule for conditional probability ($\mathbb{P}[A \cap B] = \mathbb{P}[A|B]\,\mathbb{P}[B]$), we have

  $$\mathbb{P}\left[T^+ \cap D\right] = \mathbb{P}\left[T^+ \mid D\right]\mathbb{P}[D] = (.8)(.01) = 0.008. \tag{4}$$

  We've now computed the numerator of Eq. (3).

  Similarly, recalling that $D^c = [\text{don't have disease}]$, we could compute

  $$\mathbb{P}\left[T^+ \cap D^c\right] = \mathbb{P}\left[T^+ \mid D^c\right]\mathbb{P}[D^c] = (.1)(.99) = 0.099. \tag{5}$$

- (Denominator) By the law of total probability,

$$\begin{aligned}\mathbb{P}\left[T^+\right] &= \mathbb{P}\left[T^+ \cap D\right] + \mathbb{P}\left[T^+ \cap D^c\right] \\ &= 0.008 + 0.099 && \text{by Eqs. (4) and (5)}\end{aligned}$$

- Finally, having done the necessary work, we can now plug our results into Eq. (3):

$$\mathbb{P}\left[H \mid T^+\right] = \frac{0.008}{0.099 + 0.008} \approx .075$$

We conclude that if test positive in a random screening, your probability of actually having the disease is about 7.5%. Is this result surprising?

End of Example 33. □

# 11 2025-02-05 | Week 4 | Lecture 10

## 11.1 Application of the Law of Total Probability: The probability of winning at craps

Recall the rules of craps:

1. You roll two dice and add them up.
   - If the sum is $2, 3$, or $12$, you lose ("craps")
   - If you roll 7 or 11, then you win ("natural")
   - Otherwise you establish "point", which is whatever number you got.
2. If you established point, then you must keep rolling until one of two events occurs:
   - You roll a 7: you lose
   - You roll your "point" number: you win.

Let's introduce some useful notation: Let

$$W = [\text{You win}],$$

and for each $k = 2, 3, 4, \ldots, 12$, define

$$F_k = [\text{Your first roll is } k].$$

**Question:** What is $\mathbb{P}[W]$?
**Solution:** By the *Law of Total Probability*, we have

$$\mathbb{P}[W] = \sum_{k=2}^{12} \mathbb{P}[W \mid F_k] \times \mathbb{P}[F_k] \tag{6}$$

In words,

$$\mathbb{P}[W \mid F_k] = \text{ probability of winning, given you rolled } n \text{ on the first roll}$$

| $k$ | $P[W \mid F_i]$ | $\mathbb{P}[F_k]$ |
|---|---|---|
| $2$ | 0 | 1/36 |
| $3$ | 0 | 2/36 |
| $4$ | 1/3 | 3/36 |
| 5 | 4/10 | 4/36 |
| 6 | 5/11 | 5/36 |
| 7 | 1 | 6/36 |
| 8 | 5/11 | 5/36 |
| 9 | 4/10 | 4/36 |
| 10 | 1/3 | 3/36 |
| 11 | 1 | 2/36 |
| 12 | 0 | 1/36 |

Therefore by Eq. (6)

$$\mathbb{P}[W] = 0 \cdot \frac{1}{36} + 0 \cdot \frac{2}{36} + \frac{1}{3} \cdot \frac{3}{36} + \frac{4}{10} \cdot \frac{4}{36} + \frac{5}{11} \cdot \frac{5}{36} + 1 \cdot \cdot \frac{6}{36} + \ldots + 0 \cdot \frac{1}{36} = 0.492999...$$

Your chance of winning is about 49.3%.

**Example 34** (Birthday problem)**.** Let

$$E = [\text{At least one birthday match in a room of } r \text{ people }]$$

Want to compute

$$\mathbb{P}[E].$$

By (iii.),

$$\mathbb{P}[E] = 1 - \mathbb{P}[E^c].$$

In words, $E^c = $ [no birthday match]. Let's assume there are 365 days in a year, and that all days are equally likely. Let

$$D_j = \mathbb{P}[j\text{-th person differs from all predecessors}]$$

Then

$$\begin{aligned}
\mathbb{P}[E] &= 1 - \mathbb{P}[E^c] \\
&= 1 - \mathbb{P}[D_1]\,\mathbb{P}[D_2 \mid D_1]\,\mathbb{P}[D_3 \mid D_1 D_2] \ldots \mathbb{P}[D_r \mid D_1 \cdots D_{r-1}] \\
&= 1 - \frac{365}{365} \cdot \frac{364}{365} \cdot \frac{363}{365} \cdots \frac{365-(r-1)}{365}
\end{aligned}$$

End of Example 34. $\square$