7 2026-01-28 | Week 03 | Lecture 07

This lecture is based on section 2.3 in the textbook.

7.1 Review of multiplication principle

Definition 19 (k-tuple)**.** A ***k-tuple*** is an ordered list of k objects.

Theorem 20 (Multiplication Principle)**.** *Suppose S consist of k-tuples and there are n_1 choices for the first element, n_2 choices for the second, etc. Then there are*

$$n_1 \times n_2 \times \cdots \times n_k$$

possible k-tuples.

7.2 The permutation principle and the combination principle

Proposition 21 (Counting permutations)**.** *The total number of ways to order n distinct objects is*

$$n! = n \times (n-1) \times (n-2) \times \cdots \times 2 \times 1.$$

(Note that we define $0! = 1$.)

Definition 22. A ***set*** is an *unordered* collection of elements, all of which are *distinct.* A ***permutation*** is an ordered subset. An unordered subset is called a ***combination***.

Theorem 23 (The permutation principle)**.** *The number of permutations of size k that can be formed from n objects is*

$$(n)_k := n \cdot (n-1) \cdot (n-2) \cdots (n-k+1) = \frac{n!}{(n-k)!}.$$

There are k terms in this product. (The book uses the notation $P_{k,n}$ instead of $(n)_k$.)

Proof. Suppose our n objects are n balls in an urn, labeled $1, \ldots, n$. We draw n balls sequentially, each ball drawn is left out of the urn. The number permutations is the number of ways that k balls can be drawn out of the urn. We are dealing with ordered k-tuples

$$(a_1, \ldots, a_k)$$

where $a_1, \ldots, a_k \in \{1, \ldots, n\}$ are integers between 1 and n that must all be different. (Obviously, $k \leq n$ since there are n balls in the urn and we seek to draw out k of them).

For the first ball, we have n choices. For the second ball we have $n-1$ choice (since there remain only $n-1$ balls in the urn after our first draw). For the third ball we have $n-2$ choices, and so forth. By the k^{th} ball (our final draw), there are $n-k+1$ balls to choose from in the urn.

Therefore, by the multiplication principle,

$$\text{the number of permutations} = n \cdot (n-1) \cdot (n-2) \cdots (n-k+1) = \frac{n!}{(n-k)!}.$$

$\square$

Theorem 24 (The combination principle)**.** *Let S be a set with n elements. The number of (unordered) subsets of size k is*

$$\binom{n}{k} = \frac{n!}{k!\,(n-k)!}$$

Proof. To understand why the formula of Theorem 24 is correct, we begin by making two observations:

- **Observation 1.** For every unordered subset of size k, there are $k!$ ways to order it by Proposition 17. Therefore the following equation holds:

$$k! \times (\text{number of unordered subsets of size } k) = (\text{number of ordered subsets of size } k)$$

- **Observation 2.** The permutation principle (Theorem 23) tells us that

$$(\text{number of ordered subsets of size } k) = \frac{n!}{(n-k)!}.$$

We now put these two observations together. Let X and Y be the following quantities:

$$\begin{aligned} X &= (\text{number of ordered subsets of size } k) \\ Y &= (\text{number of unordered subsets of size } k). \end{aligned}$$

Then

$$\begin{aligned} k! \cdot Y &= X && \text{by Observation 1} \\ &= \frac{n!}{(n-k)!} && \text{by Observation 2} \end{aligned}$$

Dividing both sides by $k!$ gives

$$Y = \frac{n!}{k!\,(n-k)!},$$

and this proves Theorem 24. □

7.3 Examples of the three combinatorial principles

Example 25. A dessert company has 32 distinct desserts on its menu: 6 types of pie, 9 types of cake, and 17 flavors of ice cream.

(a) **Question:** After purchasing 4 different desserts, a customer decides he will eat them all in a row, and he cares about the order in which he eats them. How many different ways can he eat his 4 desserts?

Solution: Here we are counting the number of ways to order set of four objects. This number is $4!$ by Proposition 17.

(b) **Question:** How many ways are there for the customer to choose four desserts and then eat them in a particular order of his choice?

Solution 1: By the combination principle, there are $\binom{32}{4}$ combinations of desserts that the customer could choose. Moreover, by part (a), for each combination of 4 desserts, he can order them $4!$ ways. Therefore by the multiplication principle, there are

$$\binom{32}{4} \times 4! = 32 \cdot 31 \cdot 30 \cdot 29$$

ways that the customer could choose and then eat four desserts.

Solution 2: Another solution would be to note that we want to count the number of permutations of size $k = 4$ from a set of $n = 32$ distinct objects. (Think: the customer is selecting 4 desserts from an urn with 32 distinct desserts in it). By the permutation principle, this number is

$$(32)_4 = 32 \cdot 31 \cdot 30 \cdot 29$$

which agrees with our answer from solution 1.

(c) **Question:** If 9 desserts are randomly selected, how many ways are there to obtain 3 deserts of each type (pie, cake, and ice cream)?

Solution: For pie, there are $\binom{6}{3}$ combinations possible. For cake, there are $\binom{9}{3}$ combinations possible, and for ice cream, there are $\binom{17}{3}$.

By the multiplication principle, there are

$$\binom{6}{3} \times \binom{9}{3} \times \binom{17}{3}$$

ways to simultaneously select 3 pies, 3 cakes, and 3 ice cream flavors.

End of Example 25. □