

### 3 2025-01-16 | Week 01 | Lecture 03

*The nexus question of this lecture: What is the general framework for probability (continued)?*

- Go over problem 4, parts (b) and (c), and problem 6 on the worksheet.

#### 3.1 Powerball

**Example 3** (Powerball Lottery). The a ticket for the powerball lottery costs \$2. There are two outcomes:

- You win \$300,000,000.
- You don't win any money.

The probability of winning is approximately  $\frac{1}{300,000,000}$ . Let  $X$  be the net payoff from the game, in dollars:

- If you lose, then  $X = -2$ , since you had to pay \$2 to play the game.
- If you win, then your net payoff is  $X = 299,999,998$ .

What is the expected value of  $X$ ?

$$\begin{aligned}\mathbb{E}[X] &= \mathbb{P}[\text{win}] \cdot (-2) + \mathbb{P}[\text{lose}] \cdot (299,999,998) \\ &= \frac{299,999,999}{300,000,000}(-2) + \frac{1}{300,000,000}(299,999,999) \\ &= -1.\end{aligned}$$

*Conclusion:* you “expect” to lose \$1 every time you play. Similarly, if you play twice, you expect to lose \$2. If you play 10 times, you expect to lose \$10. Etc. This property is called *linearity* of expectation.

End of Example 3.  $\square$

#### 3.2 Recap of terminology

*Recall:* The **sample space** of an experiment is the *set* of all possible **outcomes**. An **event** is a collection of outcomes. That is,

an event = a set of outcomes = a subset of the sample space  $S$ .

**Example 4** (Sum 2d4). Roll a 4-sided dice twice (this is the **experiment**). There are 16 possible **outcomes**. The **sample space** is

$$S = \{(x, y) : x, y \in \{1, 2, 3, 4\}\},$$

which consists of 16 ordered pairs  $(x, y)$ , the **sample points**. An **event** is any subset of these.

We'll give two examples of events.

Let  $X$  be the sum of the two rolls. We can represent  $X$  by the following table:

		Dice 2			
		1	2	3	4
Dice 1	1	2	3	4	5
	2	3	4	5	6
	3	4	5	6	7
	4	5	6	7	8

The event that  $X = 6$  is:

$$[X = 6] = \{(1, 5), (2, 4), (3, 3), (4, 2), (5, 1)\}.$$

Here's another example of an event:

$$[X \text{ is even}] = \{(1, 1), (1, 3), (2, 2), (2, 4), (3, 1), (3, 3), (4, 2), (4, 4)\}.$$

The **enumeration principle** says that, when outcomes are equally likely, we can compute the probability of an event  $E$  by

$$\mathbb{P}[E] = \frac{|E|}{|S|}.$$

So

$$\mathbb{P}[X = 6] = \frac{3}{16}$$

and

$$\mathbb{P}[X \text{ is even}] = \frac{8}{16} = \frac{1}{2}.$$

End of Example 4.  $\square$

Since probability theory is formalized in terms of sets, we need to have some intuition about set theory.

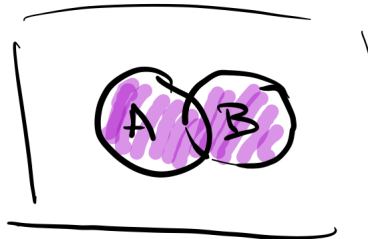
### 3.3 Understanding the “events” as sets

Let  $A$  and  $B$  be events. That is,  $A$  and  $B$  are subsets of  $S$ .

- Suppose  $A$  is an event. It's a subset of  $S$ , like this:

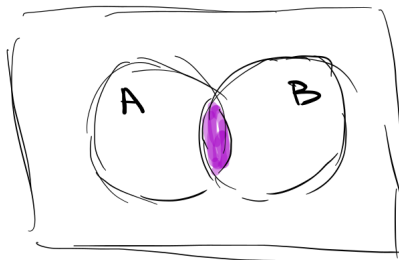


- The **complement** of  $A$ , denoted  $A^c$  is the set of points in  $S$  that are **not** in  $A$ . We interpret  $A^c$  as the event that  $A$  didn't happen.
- The **union** of  $A$  and  $B$ , denoted  $A \cup B$ , to denote the set of all outcomes that are in  $A$  or  $B$  this includes outcomes that are in both):



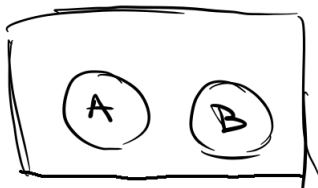
We interpret  $A \cup B$  as the event that either  $A$  or  $B$  (or both) occur.

- The **intersection** of  $A$  and  $B$ , denoted  $A \cap B$  is the set of points that are contained in both  $A$  and  $B$ :



We interpret  $A \cap B$  as the event that  $A$  and  $B$  both occur.

- What if  $A$  and  $B$  don't overlap at all? In that case, their intersection has nothing in it!



In this case, we say that  $A \cap B$  is the “empty set”, which is the set containing no elements. We use the symbol  $\emptyset$  to represent the empty set; that is,

$$\emptyset := \{\}$$

This is sometimes called the **null event**. If  $A \cap B = \emptyset$ , then we say that  $A$  and  $B$  are **disjoint**. Disjoint events are **mutually exclusive**, in the sense that they cannot happen simultaneously.

For example, when rolling a dice, the following two events are mutually exclusive:

$$[\text{dice is even}] \quad \text{and} \quad [\text{dice is odd}]$$

- A sequence of events  $\{E_1, E_2, \dots\}$  is said to be **pairwise disjoint** if and only if

$$E_i \cap E_j = \emptyset \quad \text{whenever } i \neq j.$$

In other words, none of the events in  $E$  “overlap” with any other events. For an example of this, the experiment from last lecture involving repeatedly drawing a ball from an urn with replacement until we get the gold ball. Let  $X$  be the number of balls we had to draw until we got the gold ball. Let

$$E_n = [X = n] \quad \text{for } n = 1, 2, 3, \dots$$

Then  $\{E_1, E_2, \dots\}$  is a pairwise disjoint sequence of events.