8 2025-01-29 | Week 3 | Lecture 7

8.1 Review of Set Operations

Students can contact the grader (name: Hunter) directly `hvt7hawaii.edu` with any questions.

Example 22 (Set Operations)**.** Let's review set operations. Suppose

$$A = [\text{I brought a black marker to class}] \quad \text{and} \quad B = [\text{I brought a blue marker to class}]$$

Then the following sets are defined as follows:

$$A^c = [\text{I didn't bring a black marker to class}]$$

and

$$B^c = [\text{I didn't bring a blue marker to class}]$$

and

$$A \cup B = [\text{I brought a black marker OR a blue marker, possibly both}]$$

and

$$A \cap B = [\text{I brought both a blue and a black marker}]$$

and we also have *De Morgan's Laws:*

$$(A \cup B)^c = A^c \cap B^c = [\text{I brought neither a blue nor a black marker}]$$

$$(A \cap B)^c = A^c \cup B^c = [\text{I didn't bring } \textit{both} \text{ a blue marker and a black marker (but maybe I brought one)}]$$

End of Example 22. □

8.2 Conditional Probability

For any two events A and B, conditional probability describes the probability that A happens given that we know B happened.

Definition 23 (Conditional Probability)**.** Let A, B be events, and assume that $\mathbb{P}[A] > 0$. Then the *conditional probability of B, given A*, denoted $\mathbb{P}[B \mid A]$, is given by the formula

$$\mathbb{P}[B \mid A] := \frac{\mathbb{P}[A \cap B]}{\mathbb{P}[A]}.$$

Example 24 (Conditional Probability)**.** I roll two dice and add them up. Call this number X. Then $\mathbb{P}[X = 7] = \frac{1}{6}$, which we can see by counting up the possibilies. Now, what about if I tell you that I rolled the dice and got a sum greater than 4? Now that you have a little additional information, you can exclude the possibilities that $X = 1, 2, 3$ or $X = 4$. So with this new information, you should evaluate the probability that $X = 7$ as greater than before. That is, now we want

$$\begin{aligned}
\mathbb{P}[X = 7 | X > 4] &= \frac{\mathbb{P}[X = 7 \text{ and } X > 4]}{\mathbb{P}[X > 4]} \\
&= \frac{\mathbb{P}[X = 7]}{\mathbb{P}[X > 4]} \\
&= \frac{1/6}{30/36} \\
&= \frac{1}{5}.
\end{aligned}$$

This answer makes sense because $\frac{1}{5} > \frac{1}{6}$, consistent with our intution.

End of Example 24. □

Example 25 (Conditional probability)**.** Suppose you flip a coin 10 times and get more than 6 heads. What's the probability that you get less than 9 heads?
Solution: Let X be the number of heads. Let $A = [X < 9]$, and let $B = [X > 6]$. Then

$$\mathbb{P}[B] = \frac{\binom{10}{7} + \binom{10}{8} + \binom{10}{9} + \binom{10}{10}}{2^{10}}$$

Observe that $A \cap B = [X \in \{7, 8\}]$. Therefore

$$\mathbb{P}[A \cap B] = \frac{\binom{10}{7} + \binom{10}{8}}{2^{10}}$$

Therefore

$$\mathbb{P}[A \mid B] = \frac{\mathbb{P}[AB]}{\mathbb{P}[B]} = \frac{\binom{10}{7} + \binom{10}{8}}{\binom{10}{7} + \binom{10}{8} + \binom{10}{9} + \binom{10}{10}} = \frac{15}{16}.$$

End of Example 25. □

Notation 26 (Set intersections)**.** If A, B are events, then we will sometimes write AB to denote the set $A \cap B$.

Proposition 27 (Multiplication Rule)**.** *For two events A, B,*

$$\mathbb{P}[AB] = \mathbb{P}[A]\,\mathbb{P}[B \mid A]$$

For three events A, B, C,

$$\mathbb{P}[ABC] = \mathbb{P}[A]\,\mathbb{P}[B \mid A]\,\mathbb{P}[C \mid AB]$$

This rule is useful for analyzing experiments which proceed in stages, like the following problem:

Example 28 (Urn - example of multiplication rule)**.** An urn contains 6 white balls and 9 black balls. If 4 balls are drawn at random, what is the probability that the first 2 are white and the last 2 are black?
Solution: Next time.

End of Example 28. □