19 2025-02-26 | Week 7 | Lecture 18

Please read chapter 4.3

Announcment: I would like to pushing back the midterm until March 14.

Example 64. Let $X \sim \exp(3/2)$. What is the probability is that X is greater than twice its standard deviation σ?

Solution: Recall that for an exponential r.v. Y with rate λ, we have

$$\mathbb{E}[Y] = \frac{1}{\lambda}. \tag{18}$$

and

$$F_Y(t) := \mathbb{P}[Y \le t] = 1 - e^{-\lambda t}.$$

This implies that

$$\mathbb{P}[Y > t] = e^{-\lambda t}. \tag{19}$$

In our case, we have $\lambda = 3/2$. Let σ be the standard deviation of X. We want to compute

$$\mathbb{P}[X > 2\sigma].$$

By Eq. (19), with $\lambda = \frac{3}{2}$ and $t = 2\sigma$, we have

$$\mathbb{P}[X > 2\sigma] = e^{-\frac{3}{2}(2\sigma)} = e^{-3\sigma}. \tag{20}$$

So it remains to find σ. We have

$$\mathrm{Var}(X) = \mathbb{E}\left[X^2\right] - (\mathbb{E}[X])^2.$$

From Eq. (18), have $\mathbb{E}[X] = \frac{2}{3}$, so

$$\mathrm{Var}(X) = \mathbb{E}\left[X^2\right] - \frac{4}{9} \tag{21}$$

Next, let's compute $\mathbb{E}\left[X^2\right]$. Recall from Example 63 that

$$M(t) = \begin{cases} \frac{\lambda}{\lambda - t} & : \quad t < \lambda \\ 0 & : \quad t \ge \lambda \end{cases}$$

which for our setting is

$$M(t) = \frac{\frac{3}{2}}{\frac{3}{2} - t} \quad \text{for } t < \frac{3}{2}.$$

From Eq. (17), we have $M''(0) = \mathbb{E}\left[X^2\right]$.

Indeed, $M''(t) = 3\left(\frac{3}{2} - t\right)^{-3}$. Then

$$\mathbb{E}\left[X^2\right] = M''(0) = 3\left(\frac{3}{2}\right)^{-3} = \frac{8}{9}$$

Plugging this into Eq. (21), we get

$$\mathrm{Var}(X) = \frac{8}{9} - \frac{4}{9} = \frac{4}{9}.$$

And hence

$$\sigma = \sqrt{\mathrm{Var}(X)} = \sqrt{\frac{4}{9}} = \frac{2}{3}.$$

Therefore by Eq. (20), we have

$$\mathbb{P}[X > 2\sigma] = e^{-3\sigma} = e^{-2} \approx 0.14.$$

End of Example 64. □

19.1 Normal random variables

The most important continuous r.v. is the following,

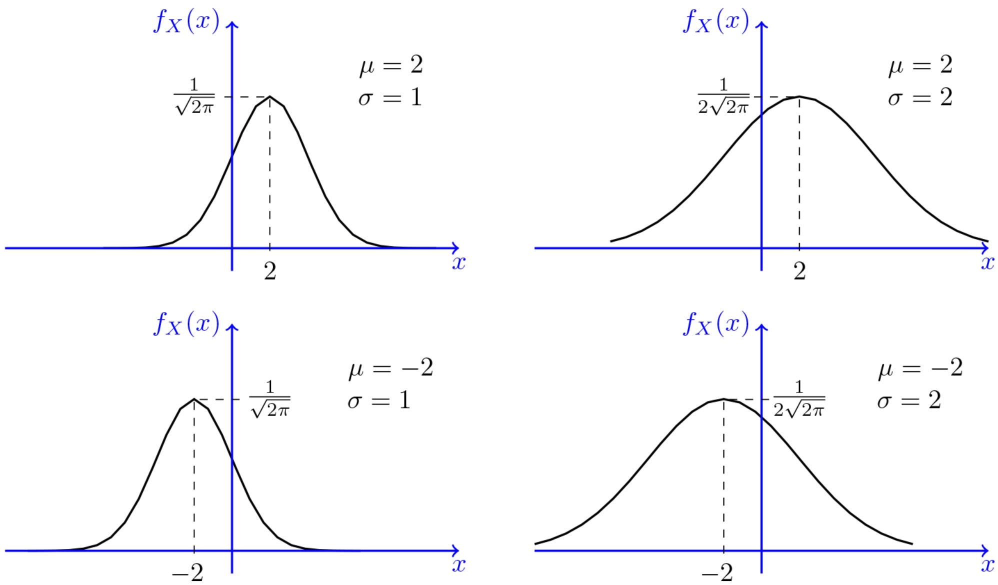
Example 65 (Gaussian random variable)**.** We say that X is a *Gaussian random variable with mean μ and variance σ^2* if X is a continuous r.v. with density given by the formula

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}, \quad -\infty < x < \infty$$

In this case, we write $X \sim \mathcal{N}(\mu, \sigma^2)$. Gaussian r.v.'s are also known as *Normal* random variables.

End of Example 65. □

Normal random variables have a "bell-curve" distribution. Here are some example density curves for various values of μ and σ. Note that μ determines where the peak of the bell curve is located, and σ determines how wide or narrow it is:



Normal random variables arise naturally when we have a random quantity that is the result of adding up many small random quantities. A classic example of this is the height of a random person, as height is influenced by the effects of thousands of genes and environmental factors.

Most important special case: The most important special case is where $\mu = 0$ and $\sigma^2 = 1$, in which case the density is

$$\phi(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \quad -\infty < x < \infty$$

A random variable with this density is called a *standard normal* and is usually denoted with the letter Z.

Fact: if $X \sim \mathcal{N}(\mu, \sigma^2)$, then the random variable

$$Z = \frac{X - \mu}{\sigma} \tag{22}$$

is a standard normal random variable.

This transformation

$$X \mapsto \frac{X - \mu}{\sigma}$$

is called *standardization.* Because we can standardize any normal random variable, that means we can focus our attention on standard normal random variables.

But first, we should really check that ϕ is really a density. Namely, we need to check that

$$\int_{-\infty}^{\infty} \phi(x) dx = 1.$$

We showed how to do this by using Fubini's theorem and converting to polar coordinates.

The CDF of a standard normal: The cdf of a standard normal is usually denoted with the symbol Φ:

$$\Phi(x) = \int_{-\infty}^{x} \phi(u) du = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{u^2}{2}} du.$$

There is no elementary form for the antiderivative Φ, so generally we compute the values with a computer.