17 2025-02-21 | Week 5 | Lecture 16

Recall the example of the pinwheel Example 54, with the angle X measured in radians, so that the state space of X is $[0, 2\pi)$, with all values equally likely. Recall that we computed the pdf as

$$f_X(x) = \frac{1}{2\pi} \cdot \mathbf{1}_{[0,2\pi)}(x) = \begin{cases} \frac{1}{2\pi} & : \quad x \in [0, 2\pi) \\ 0 & : \quad x \notin [0, 2\pi) \end{cases}$$

Recall also that the cdf is defined as

$$F_X(x) := \int_{-\infty}^{x} f_X(t)dt.$$

So that for the pinwheel angle X, we have

$$\begin{aligned} F_X(x) &= \int_{-\infty}^{x} \frac{1}{2\pi} \cdot \mathbf{1}_{[0,2\pi)}(t)dt \\ &= \frac{1}{2\pi} \int_{-\infty}^{x} \mathbf{1}_{[0,2\pi)}(t)dt. \end{aligned} \tag{12}$$

To evaluate an integral of this form, we have to consider three cases:

- $x < 0$ (i.e. when x is to the left of the interval)
- $x \geq 2\pi$ (i.e. when x is to the right of the interval)
- $0 \leq x < 2\pi$ (i.e. when x is in the interval)

Let's think through these cases:

- When $x < 0$, the integral on the right-hand side of Eq. (12) is zero.
- When $x \geq 2\pi$, the integral equals

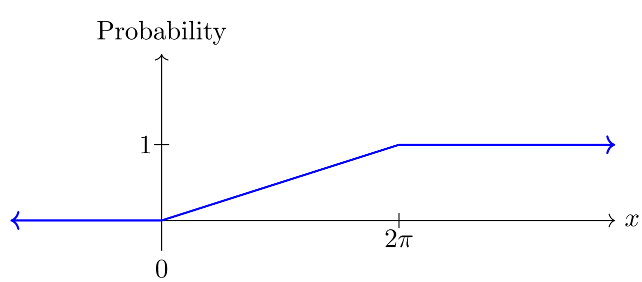
$$\int_{-\infty}^{0} 0dt + \frac{1}{2\pi}\int_{0}^{2\pi} dt + \int_{2\pi}^{x} 0dt = 0 + 1 + 0 = 1$$

- When $0 \leq x < 2\pi$, the integral equals

$$\int_{-\infty}^{0} 0dt + \frac{1}{2\pi}\int_{0}^{x} dt = 0 + \frac{x}{2\pi} = \frac{x}{2\pi}$$

From this analysis and Eq. (12), we can conclude that

$$F_X(x) = \begin{cases} 0 & : \quad x < 0 \\ \frac{x}{2\pi} & : \quad 0 \leq x < 2\pi \\ 1 & : \quad x \geq 2\pi \end{cases}$$



The *median* of a continuous random variable is the point x^* where $F(x^*) = \frac{1}{2}$. Interpretation: A random variable is equally likely to be greater or less than than its median.

The *expected value* of a continuous random variable X is

$$\mathbb{E}[X] := \int_{-\infty}^{\infty} x f_X(x) dx$$

where $f_X(x)$ is the pdf of X. (Provided that the integral above converges absolutely).

Similarly, for any function $h : \mathbb{R} \to \mathbb{R}$ we have

$$\mathbb{E}[h(X)] = \int_{-\infty}^{\infty} h(x) f_X(x) dx$$

This is the continuous analogue of the very important theorem Theorem 40 which was for discrete random variables.

We also have variance defined like it was for discrete random variables:

$$\mathrm{Var}(X) := \mathbb{E}\left[(X - \mu)^2\right],$$

where $\mu = \mathbb{E}[X]$, and as before we have the formula

$$\mathrm{Var}(X) = \mathbb{E}\left[X^2\right] - (\mathbb{E}[X])^2$$

Example 57 (Waiting times)**.** Suppose you station yourself at spot on a road and watch as cars pass by. Let T be the time in minutes until the first car passes by. This random variable is called the *waiting time*.

Under certain circumstances, it is reasonable for T to have pdf

$$f(x) = \lambda e^{-\lambda x} \mathbf{1}_{[x \geq 0]} \tag{13}$$

for a certain (fixed) value of $\lambda > 0$. In this expression, the term $\mathbf{1}_{[x \geq 0]}$ is an "indicator function" which takes value 1 when $x \geq 0$ and 0 if $x < 0$. A random variable with pdf like in Eq. (13) is called an *exponential random variable with rate λ*, and these are some of the most important random variables.

For concreteness, let's assume that $\lambda = 3$, so

$$f(x) = 3e^{-3x} \mathbf{1}_{[x \geq 0]}$$

Then the cdf is

$$\begin{aligned} F(t) &= \int_{-\infty}^{t} f_T(x) dx \\ &= \int_{0}^{t} 3e^{-3x} dx \\ &= 3\left[\frac{-e^{-4x}}{3}\right]_0^t \\ &= 1 - e^{-3t}. \end{aligned}$$

Equivalently,

$$\mathbb{P}[T > t] = e^{-3t}.$$

For example the probability that you have to wait more than $t = 30$ seconds for a car to pass by is

$$\mathbb{P}[T > 5] = e^{-3 \cdot 0.5} \approx .22$$

What about

$$\mathbb{E}[T]?$$

If the cars come at a rate of $\lambda = 3$ cars per minute, then on average how long would you have to wait for a car to arrive? This should be 20 seconds.

End of Example 57. □