12 2025-02-07 | Week 4 | Lecture 11

- please read sections 3.1-3.5 in the textbook
- there is a webassign homework due next Friday
- handwritten assignment too

12.1 Discrete random variables (continued)

We often associate numbers with outcomes of experiments. This is what we mean by random variables.

Example 35 (Random variables and pmfs)**.** Say we flip a fair coin three times

	HHH	HHT	HTH	THH	TTH	THT	HTT	TTT
$X = \#$ heads	3	2	2	2	1	1	1	0
$Y = \#$ tails prior to first heads	0	0	0	1	2	1	0	3
$Z =$ (more tails than heads?)	1	1	1	1	0	0	0	0

Here $Z = \begin{cases} 1 & : \text{ more tails than heads} \\ 0 & : \text{ otherwise} \end{cases}$ X, Y, Z are examples of random variables. This is an example of what's called an "indicator" random variable. To be precise random variables are functions $X : S \to \mathbb{R}$ and $Y : S \to \mathbb{R}$.

In this example, the sample space is

$$S = \{\text{HHH, HHT, HTH, THH, TTH, THT, HTT, TTT}\}$$

$X = 2$ is an event:

$$[X = 2] = \{s \in S : X(s) = 2\} = \{\text{HHT, HTH, THH}\}$$

$X \leq 2$ is also an event:

$$[X \leq 2] = \{s \in S : X(s) \leq 2\} = \{\text{HHT, HTH, THH, TTH, THT, HTT, TTT}\}$$

We can specify X, Y, Z by the following tables:

x	$\mathbb{P}[X = x]$
0	1/8
1	3/8
2	3/8
3	1/8

y	$\mathbb{P}[Y = y]$
0	4/8
1	2/8
2	1/8
3	1/8

z	$\mathbb{P}[Z = z]$
0	4/8
1	4/8

Recall that given a discrete random variable X, the *probability mass function (pmf)* is the function

$$p(x) := \mathbb{P}[X = x]$$

For example, the pmf of X is the function p which take the following values

$$p(0) = 1/8 \quad p(1) = 3/8 \quad p(2) = 3/8, \quad \text{and} \quad p(3) = 1/8$$

It's just a different way of writing the above table!

General fact: For any discrete r.v., we have $p(x) \geq 0$ for all x, and that

$$\sum_x p(x) = 1,$$

where the sum ranges over all values that X can take.

End of Example 35. □

Let X be a discrete random varible. The *cumulative distribution function (cdf)* of X is the function

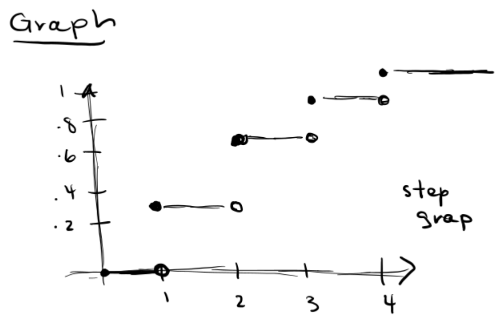
$$F_X(x) := \mathbb{P}[X \leq x]$$

Example 36 (cdf of a dice roll)**.** What is the pmf and cdf a dice roll?

The pmf is:

x	1	2	3	4	5	6
$p(x)$	1/6	1/6	1/6	1/6	1/6	1/6

The cdf is



End of Example 36. □

Proposition 37 (cdf properties)**.** *We note that* $F_X : \mathbb{R} \to [0,1]$ *and that*

$$F_X(x) = \sum_{y \leq x} p_X(y)$$

From this equation, we can deduce that

$$F_X(x_1) \leq F_X(x_2) \text{ whenever } x_1 \leq x_2.$$

In other words, F_X *is an increasing function (or maybe more appropriately, "nondecreasing"). Also,*

$$\mathbb{P}[a < X \leq b] = F_X(b) - F_X(a)$$

Example 38 (Geometric distribution)**.** Throw a basketbball. Then

$$1 = \text{success} = \text{made basket} \tag{7}$$
$$0 = \text{failed} = \text{missed basket} \tag{8}$$

Fix a parameter $\alpha \in (0,1)$. Say that success has probability α and failure has probability $1-\alpha$.

Let X be the number of throws until I make a basket. Let $p(x)$ the pmf of X. Then

x	$p(x)$
1	α
2	$(1-\alpha)\alpha$
3	$(1-\alpha)^2\alpha$
4	$(1-\alpha)^3\alpha$
$\vdots$	$\vdots$

In other words, the pmf is

$$p(x) = \begin{cases} (1-\alpha)^{x-1}\,\alpha & \text{if } x = 1, 2, \ldots \\ 0 & \text{otherwise} \end{cases}$$

A random variable with this pmf is called a *geometric random variable with success parameter* α*.*

End of Example 38. □