16 2025-02-19 | Week 5 | Lecture 15

Definition 52 (Continuous random variable)**.** A random variable X is said to be *continuous* if it takes values in an interval I (or disjoint union of intervals) AND $\mathbb{P}[X = x] = 0$ for all $x \in I$.

To be precise, a random variable x is continuous if its cdf

$$F(x) = \mathbb{P}[X \leq x]$$

is a continuous function.

Definition 53 (pdf)**.** Let X be a continuous random variable. The *probability density function* (aka "pdf" or "density") of X is a function f_X such that for any real numbers a, b with $a \leq b$, we have

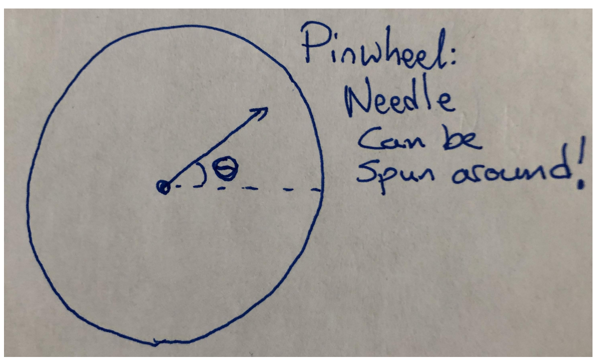
$$\mathbb{P}[X \in (a, b)] = \int_a^b f_X(x)dx$$

The graph of f_X is called the *density curve*.

Observations

- $f_X(x) \geq 0$ for all x
- $\int_{-\infty}^{\infty} f_X(x)dx = 1$
- $f_X(x)$ is not a probability!!! But you can think of it as "relative likeliood"

Example 54 (Pinwheel)**.** Suppose you have a pinwheel as follows.



Suppose we spin the needle. When the needle stops spinning, we measure its angle from the dotted line. Let X be the angle of the needle. If X is measured in degrees, then

$$X \in [0, 360)$$

but X could take *any* value, not just integer values. On the other hand, if we measure X in radians, then

$$X \in [0, 2\pi).$$

Moreover we have no reason to believe any one angle is more likely than any others – for our idealized pinwheel, all values between 0 and 2π should all be equally likely. This is an example of a uniform random variable, which we define next.

End of Example 54. □

Definition 55 (Uniform distribution)**.** Let I be an interval with endpoints a and b, such that $a < b$. A continuous random variable X has *uniform* distribution on I if

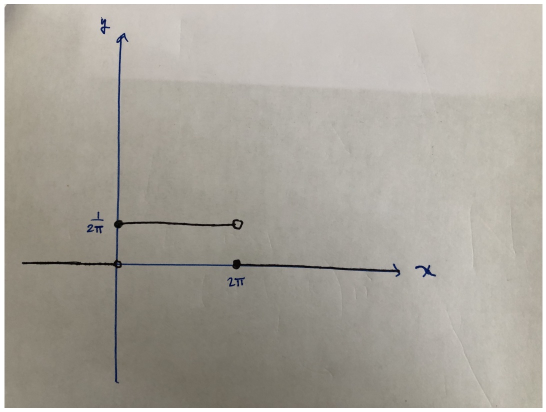
$$\begin{cases} \frac{1}{b-a} & : \quad x \in I \\ 0 & : \quad x \notin I \end{cases}$$

We note that $b - a$ is the length of the interval.

In the case of the pinwheel example, the pdf is

$$f_X(x) = \begin{cases} \frac{1}{2\pi} & : \quad 0 \le x < 2\pi \\ 0 & : \quad \text{else} \end{cases}$$

so the density curve is



Proposition 56. *The* cumulative density function *or cdf is the function*

$$F_X(x) = \mathbb{P}[X \le x] = \int_{-\infty}^{x} f_X(t)dt$$

Observations

- $\mathbb{P}[a \le X \le b] = F_X(b) - F_X(a)$