33 2025-04-23 | Week 14 | Lecture 33

33.1 Ingredient #2: The Chi-Squared Distribution

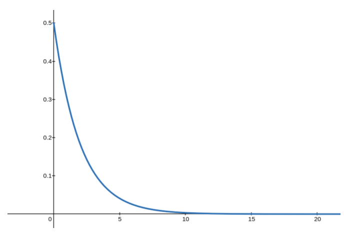
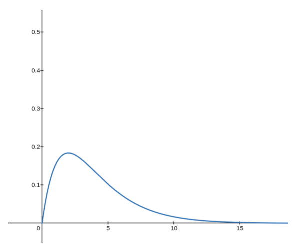
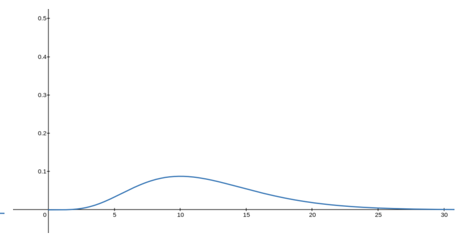
A special kind of gamma distribution:

Definition 94 (Chi-Squared Distribution)**.** Let k be a positive integer. A nonnegative random variable X is said to have *chi-squared distribution with parameter d* if it has pdf

$$f(x) = \begin{cases} \frac{1}{2^{d/2}\Gamma(d/2)} x^{\frac{d}{2}-1} e^{-x/2} & : \quad x \geq 0 \\ 0 & : \quad x < 0 \end{cases}$$

In other words, if X is a Gamma distributed random variable with parameters $\alpha = d/2$ and $\beta = 2$. The parameter d is usually called the *degrees of freedom* of X.

From left-to-right, here's what this looks like for $d = 2$, $d = 4$, and $d = 12$:

Using properties of gamma distribution, we have:

Proposition 95. *A χ^2 random variable X with d degrees of freedom has*

$$\mathbb{E}[X] = \alpha\beta = d \quad \text{and} \quad \mathrm{Var}(X) = \alpha\beta^2 = 2d$$

The chi-squared distribution plays a central role in statistical inference. It is completely determined by its degrees of freedom.

Theorem 96. *Suppose $X = (X_1, \ldots, X_k)$ is a multinomial random variable with parameters $p_1, \ldots, p_k$. Define a new random variable χ^2, called the chi-squared statistic by*

$$\chi^2 := \sum_{i=1}^{k} \frac{(X_i - np_i)^2}{np_i}.$$

Then χ^2 has approximately chi-squared distribution with $k-1$ degrees of freedom (as long as $np_i \geq 5$ for all $i = 1, \ldots, m$).

33.2 The goodness-of-fit test

With ingredients #1 and #2, we can formulate what is called a "goodness of fit test". This isn't quite what we set out to do (i.e., we want to do *tests of association*), but it's a good first step. Goodness-of-fit tests are described in detail in chatper 14.1 of the textbook.

In Example 93 (one armed bandit), we considered a game with three outcomes, 1,2, and 3 with unknown probabilities p_1, p_2, and p_3. Our null hypothesis was that

$$H_0 : (p_1, p_2, p_3) = (0.5, 0.3, 0.2)$$

and the alternative hypothesis is

$$H_1 : (p_1, p_2, p_3) \neq (0.5, 0.3, 0.2).$$

We played the game 100 times. Our data is summarized in the following table:

Category	1	2	3
Observed	43	35	22
Expected	50	30	20

In this table, the expected values were calculated assuming the null hypothesis that was true. The chi-squred statistic is

$$\begin{aligned}\chi^2 &:= \frac{(43-50)^2}{50} + \frac{(35-30)^2}{30} + \frac{(22-20)^2}{20} \\ &= \frac{49}{50} + \frac{25}{30} + \frac{4}{20} \\ &\approx 2.01\end{aligned}$$

Recall that we reject the null hypothesis if χ^2 is big, and we fail to reject the null hypothesis if χ^2 is small.

Question: So is 2.01 big enough to reject H_0?

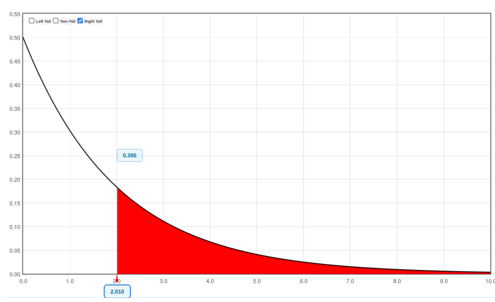
Answer: Let's compute a p-value, which if you recall is the probability of observing results at least as extreme as you observed, assuming the null hypothesis is true.

By Theorem 96, we know that the statistic has a chi-squared distribution with 2 degrees of freedom.

Therefore our p-value is

$$\begin{aligned}p\text{-value} &= \mathbb{P}\left[\chi^2 \geq 2.01\right] \\ &= \int_2^\infty \frac{1}{2} e^{-x/2} && \text{by Definition 94} \\ &= 0.366\end{aligned}$$

In other words, if the null hypothesis were true, we would expect to see our chi-squared statistic be 2.01 or greater about 37% of the time. That's pretty common, so we **fail to reject** the null hypothesis.



The test that we just did is called a goodness of fit test. These are described in chapter 14.1 in the textbook.