5 2026-01-23 | Week 02 | Lecture 05

5.1 St. Petersburg Paradox

The game is the following: repeatedly flip a coin until you get heads. You win 2^n dollars, where n is the number of coin flips. How much would you be willing to pay to play this game????

Let X be your (random) payoff. The sample space of X is

$$S = \{2, 4, 8, 16, 32, \ldots\}.$$

What is the expected value of X?

$$\begin{aligned}
\mathbb{E}[X] &= \sum_x x\mathbb{P}[X = x] \\
&= 2\mathbb{P}[X = 2] + 4\mathbb{P}[X = 4] + 8\mathbb{P}[X = 8] + 16\mathbb{P}[X = 16] + \ldots
\end{aligned}$$

Next, we observe that $\mathbb{P}\left[X = 2^k\right] = \frac{1}{2^k}$ for all $k = 1, 2, \ldots$. Plugging these probabilities in, we get

$$\mathbb{E}[X] = 1 + 1 + 1 + 1 + 1 + \ldots = +\infty.$$

The expected value of playing this game is positive infinity!!!!! N matter how much you pay to play this game, you will eventually make money if you play enough times.

5.2 What is probability, continued

Definition 9 (Probability Measure)**.** A ***probability measure*** $\mathbb{P}$ is a function which assigns to each event a probability. We denote the probability of an event E by

$$\mathbb{P}[E] \quad \text{or} \quad \mathbb{P}(E).$$

To be a ***probability measure***, $\mathbb{P}$ must satisfy the following three axioms:

A.1 (Nonnegativity) For every event E, we have

$$\mathbb{P}[E] \geq 0.$$

A.2 (Sum-to-one) If S is the whole sample space, then $\mathbb{P}[S] = 1$.

A.3 (Countable additivity) Let $E_1, E_2, \ldots$ be an infinite sequence of events. If the sequence is pairwise disjoint, then

$$\mathbb{P}[E_1 \cup E_2 \cup \ldots] = \mathbb{P}[E_1] + \mathbb{P}[E_2] + \ldots.$$

Proposition 10 (Basic properties of probability measure)**.**

(i.) *(The null event has probability zero)* $\mathbb{P}[\emptyset] = 0$

(ii.) *(Finite additivity) If $E_1, \ldots, E_n$ are pairwise disjoint, then*

$$\mathbb{P}[E_1 \cup E_2 \cup \ldots \cup E_n] = \mathbb{P}[E_1] + \mathbb{P}[E_2] + \ldots + \mathbb{P}[E_n]$$

(iii.) *("With probability one, an event E either does occur or doesn't")* $\mathbb{P}[E^c] = 1 - \mathbb{P}[E]$

(iv.) *(Excision Property) If A, B are events and $A \subseteq B$, then*

$$\mathbb{P}[B \backslash A] = \mathbb{P}[B] - \mathbb{P}[A].$$

(v.) *("The particular is less likely than the general") If A, B are events and $A \subseteq B$, then* $\mathbb{P}[A] \leq \mathbb{P}[B]$

(vi.) *("Probabilities are between 0 and 1") For any event E, $\mathbb{P}[E] \in [0,1]$*

Proposition 11 (De Morgan's Laws)**.** *The following equalities hold:*

$$(A \cup B)^c = A^c \cap B^c \quad \textit{and} \quad (A \cap B)^c = A^c \cap B^c.$$

Proof. Draw a picture. $\square$