

PIRATES AND PROBABILITIES

Part 1: Coins

Devon, a pirate, loves to gamble when he is not scrubbing the decks or plundering booty. But lately he has been on an unlucky streak, so he decides to study probability theory so that he will have an advantage the next time he tries his luck in the gambling dens ashore.

He starts by thinking about his favorite thing: gold coins. If Devon flips a coin 3 times, what is the probability that he gets 3 heads in a row? When all outcomes are equally likely, the **probability** of a particular result can be calculated with the formula

$$\text{Probability of a particular result} = \frac{\text{Number of ways the result can happen}}{\text{Number of possible results that can happen}}.$$

When Devon flips a coin 3 times, there are 8 possible results that can happen:

HHH, HHT, HTH, HTT, THH, THT, TTH, and TTT.

Out of these possible results, only one has 3 heads, so the probability that Devon gets 3 heads in 3 flips is $\frac{1}{8}$.

Problem 1 (Coin Flipping). Suppose Devon flips a coin three times in a row.

- (a) What is the probability that Devon get exactly 2 heads?

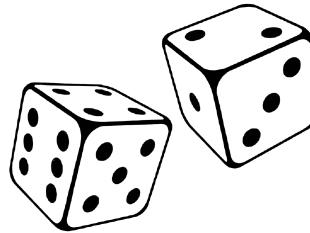
- (b) What is the probability that Devon gets at least 2 heads?

Part 2: Standard Dice Problems

Devon decides he's had enough of flipping coins and wants to start rolling some dice. For the next three problems, assume the dice are standard 6-sided dice with faces labeled 1,2,3,4,5, and 6, like this: 

Problem 2 (Standard Dice: Joint Probabilities). Devon has two dice and he rolls both of them. He wants to know the probability that both of them land on 1. He draws the following table to represent all the possible outcomes:

		Dice 2					
		1	2	3	4	5	6
Dice 1	1						
	2						
	3						
	4						
	5						
	6						



- (a) What is the probability that both dice land on 1? (Hint: use the table above by drawing an \times on the box(s) representing the outcome in which both dice land on 1.)

- (b) What is the probability that the dice land on the same number?

- (c) What is the probability that both dice are even?

- (d) When rolling two dice, what is the probability that at least one dice does not land on a 1?

Problem 3 (Standard Dice: Summing Problems). Now suppose Devon rolls two dice and then adds the numbers together. The possible outcomes are

$$2, 3, 4, 5, 6, 7, 8, 9, 10, 11, \text{ and } 12$$

but these numbers are not all equally likely. To help figure out the probabilities of the possible outcomes, Devon draws the following table on the back of one of his treasure maps:

		Dice 2					
		1	2	3	4	5	6
Dice 1		1	2	3	4	5	6
		2	3	4	5	6	7
		3	4	5	6	7	8
		4	5	6	7	8	9
		5	6	7	8	9	10
		6	7	8	9	10	11
			7	8	9	10	11
							12

- (a) What is the probability that the two dice add up to 3?

- (b) Which of the numbers 2 through 12 is the most likely to be obtained?

- (c) What is the probability of getting the most likely number?

- (d) What is the probability that the dice add up to an odd number?

Problem 4 (Standard Dice: Expected Values). The **expected value** of a dice roll is the value one would expect to get **on average** if the dice roll were repeated many times. The expected value can be calculated by adding up each possible outcome times that outcome's probability. For example, the expected value for a roll of a single 6-sided dice (with faces 1,2,3,4,5, and 6) is:

$$\text{Expected value} = 1 \times \frac{1}{6} + 2 \times \frac{1}{6} + 3 \times \frac{1}{6} + 4 \times \frac{1}{6} + 5 \times \frac{1}{6} + 6 \times \frac{1}{6} = 3.5.$$

- (a) Suppose you roll 1 dice and square the result (meaning you multiply the result by itself). What is the expected value?

- (b) If you roll 2 dice and add them up, what is the expected value?

- (c) If you roll 120 dice and add them up, what is the expected value?

Part 3: Treasure Dice Problems

When Devon's pirate ship sinks in a storm, he washes up on a mysterious island and takes shelter in a cave. Upon exploring the cave, he finds an ancient treasure chest containing three dice: one made of amethyst (A), one made of black obsidian (B), and one made of an unknown crystal (C). The dice are unlike any dice he has ever seen. They each have 6 sides, but they are not labeled 1 through 6. Instead, their faces are labeled as follows:

$$A : \quad 2, 2, 4, 4, 9, 9$$

$$B : \quad 1, 1, 6, 6, 8, 8$$

$$C : \quad 3, 3, 5, 5, 7, 7$$

Problem 5. Devon wonders what are the probabilities for these new mysterious dice.

- (a) If Devon rolls dice A, what is the probability that it lands on a 2?

- (b) If Devon rolls dice A and dice B and then adds them together, how many different sums can he possibly get?

- (c) If Devon rolls dice A and dice B, what is the probability that they add up to 10?

- (d) What is the expected value of each of the dice A, B, C ?

Expected value of A =

Expected value of B =

Expected value of C =

- (e) Suppose Devon rolls dice A and B and then adds up the two numbers. What is the expected value?

- (f) Devon puts all three dice in a bag. Then he picks one out at random and rolls it. What is the expected value in this case?

Problem 6. After the storm passes, Devon is found in the cave by rival pirate Udani. Devon challenges her to a game of dice, using the dice that Devon just found (the same dice A, B, and C from the previous problem). In the game, Devon and Udani both choose one dice to roll and whoever gets the higher number wins.

- (a) If Devon rolls dice A and Udani rolls dice B, what is the probability that Devon wins?

- (b) If Devon rolls dice B and Udani rolls dice C, what is the probability that Devon wins?

- (c) If Devon rolls dice C and Udani rolls dice A, what is the probability that Devon wins?

- (d) If Devon decides to play this game with dice A, what dice should Udani choose to maximize her probability of winning?

Part 4: Extreme Dice Problems

Problem 7 (Extreme dice). After losing several games in a row, Udani suggests they play with a different set of dice that she found in a different part of the cave. These dice are made of diamond (D), emerald (E), and fieldstone (F), and their faces are labeled as follows:

$$D : \quad 2, 2, 2, 5, 5, 5$$

$$E : \quad 1, 4, 4, 4, 4, 4$$

$$F : \quad 3, 3, 3, 3, 3, 6$$

- (a) Suppose Udani rolls dice E. What is the probability that she rolls a 4?

- (b) Suppose Udani rolls dice D and F together. What is the probability that she rolls a 2 with dice D and a 3 with dice F?

Problem 8. Udani agrees to give Devon passage back to the seaport, but only if he can beat her in a game involving these dice. Each player will choose one of the three dice, and the winner is the player who rolls the higher number.

- (a) In making a decision on which dice to use, Devon's first thought is to compute the expected value of each dice. What is the expected value of a roll of dice D? What about E and F?

Expected value of D =

Expected value of E =

Expected value of F =

- (b) If Devon chooses D and Udani chooses E, what is the probability that Devon wins?

- (c) If Devon chooses D and Udani chooses F, what is the probability that Devon wins?

- (d) If Devon chooses E and Udani chooses F, what is the probability that Devon wins?