4 2026-01-21 | Week 02 | Lecture 04

4.1 Repeated coin flipping

Example 5 (Repeated coin flipping)**.** Consider an experiment in which we flip a coin repeatedly until we get a heads. The **output** of this experiment is *the number of times we flipped the coin.*

The sample space of the experiment is

$$S = \{1, 2, 3, 4, \dots\}.$$

Let $X =$ (number of coin flips). Here, X is a random quantity which will vary between experiments. Then

$$\mathbb{P}[X = 1] = \mathbb{P}[\text{first coin flip was heads}] = \frac{1}{2}$$

and

$$\mathbb{P}[X = 2] = \mathbb{P}[\text{first flip was tails and second was heads}] = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}.$$

Similarly, for any $n = 1, 2, 3, 4, \dots$, we have

$$\mathbb{P}[X = n] = \left(\frac{1}{2}\right)^n \tag{1}$$

Question 1: How many times do we expect to flip the coin? In other words, what is $\mathbb{E}[X]$?

We use the formula

$$\mathbb{E}[X] = \sum_{n=1}^{\infty} n\mathbb{P}[X = n]$$

which gives

$$\begin{aligned}
\mathbb{E}[X] &= \sum_{n=1}^{\infty} n \left(\frac{1}{2}\right)^n \\
&= \frac{1}{2} + 2\left(\frac{1}{2}\right)^2 + 3\left(\frac{1}{2}\right)^3 + 4\left(\frac{1}{2}\right)^4 + \dots \\
&= \frac{1}{2} + 2 \cdot \frac{1}{4} + 3 \cdot \frac{1}{8} + 4 \cdot \frac{1}{16} + \dots \\
&= \frac{1}{2} + \left(\frac{1}{4} + \frac{1}{4}\right) + \left(\frac{1}{8} + \frac{1}{8} + \frac{1}{8}\right) + \left(\frac{1}{16} + \frac{1}{16} + \frac{1}{16} + \frac{1}{16}\right) + \dots \\
&= \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \dots \\
&\qquad + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \dots \\
&\qquad\qquad + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \dots \\
&\qquad\qquad\qquad + \frac{1}{16} + \frac{1}{32} + \dots \\
&\qquad\qquad\qquad\qquad + \frac{1}{32} + \dots
\end{aligned}$$

The first row adds up to 1 by the geometric series formula (which says that $\sum_{n=1}^{\infty} r^n = \frac{r}{1-r}$). The second row adds up to 1/2 (because it's 1/2 less than the first row). The third row adds up to 1/4. The fourth adds up to 1/8. etc. So

$$\mathbb{E}[X] = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \dots$$

and applying the geometric series formula again, we see that

$$\mathbb{E}[X] = 2.$$

In other words, we expect to flip the coin twice.

End of Example 5. $\square$

4.2 The three probability axioms

Sections 2.1-2.2 in the textbook.

Definition 6 (Probability Measure)**.** A ***probability measure*** $\mathbb{P}$ is a function which assigns to each event a probability. We denote the probability of an event E by

$$\mathbb{P}[E] \quad \text{or} \quad \mathbb{P}(E).$$

To be a ***probability measure***, $\mathbb{P}$ must satisfy the following three axioms:

A.1 (Nonnegativity) For every event E, we have

$$\mathbb{P}[E] \geq 0.$$

A.2 (Sum-to-one) If S is the whole sample space, then $\mathbb{P}[S] = 1$.

A.3 (Countable additivity) Let $E_1, E_2, \ldots$ be an infinite sequence of events. If the sequence is pairwise disjoint, then

$$\mathbb{P}[E_1 \cup E_2 \cup \ldots] = \mathbb{P}[E_1] + \mathbb{P}[E_2] + \ldots.$$

Notice that Definition 6 doesn't say how to assign probabilities. It merely tells us the conditions that probabilities must satisfy. The first two, **A.1** and **A.2** seem almost self-evident. The last one, **A.3**, is more technical and harder to deciper. The next example gives involves an application of axiom **A.3**.

Example 7 (Continuation of Example 5.)**.** Assume the same experiment and notation as Example 5.

Question 2: What is the probability that we flip the coin an even number of times?

For each $n = 1, 2, \ldots$, define the event E_n as

$$E_n = [X = n].$$

We want to know the probability that X is even. Observe that

$$[X \text{ is even}] = E_2 \cup E_4 \cup E_6 \cup E_8 \cup \ldots$$

We note that the sequence $E_2, E_4, E_6 \ldots$ is pairwise disjoint. So we can use **A.3** to compute the probability of this event:

$$\begin{aligned}
\mathbb{P}[X \text{ is even}] &= \mathbb{P}[E_2 \cup E_4 \cup E_6 \cup E_8 \cup \ldots] \\
&= \mathbb{P}[E_2] + \mathbb{P}[E_4] + \mathbb{P}[E_6] + \mathbb{P}[E_8] + \ldots && \text{(by } \textbf{A.3}\text{)} \\
&= \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^4 + \left(\frac{1}{2}\right)^6 + \left(\frac{1}{2}\right)^8 + \ldots && \text{(by Eq. (1))} \\
&= \frac{1}{4} + \left(\frac{1}{4}\right)^2 + \left(\frac{1}{4}\right)^3 + \left(\frac{1}{4}\right)^4 + \ldots \\
&= \frac{\frac{1}{4}}{1 - \frac{1}{4}} && \text{by geometric series formula } \sum_{n=1}^{\infty} r^n = \frac{r}{1-r} \\
&= \frac{1}{3}.
\end{aligned}$$

So the probability that you flip the coin an *even* number of times is only $1/3$.

End of Example 7. $\square$

4.3 Inclusion-exclusion principle

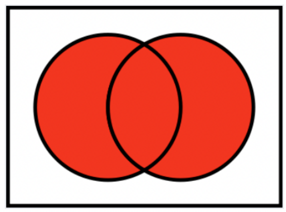
Theorem 8 (Inclusion-Exclusion Prinicple)**.** *If A and B are events, then*

$$\mathbb{P}[A \cup B] = \mathbb{P}[A] + \mathbb{P}[B] - \mathbb{P}[A \cap B]$$

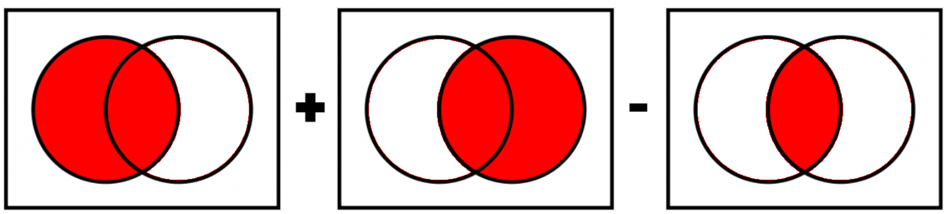
In particular, if A and B are distjoint, then

$$\mathbb{P}[A \cup B] = \mathbb{P}[A] + \mathbb{P}[B].$$

Idea of proof. $\mathbb{P}[A \cup B]$ is the area of the red-shaded region:



Compare with $\mathbb{P}[A] + \mathbb{P}[B] - \mathbb{P}[A \cap B]$:



$\square$