5 2025-01-24 | Week 2 | Lecture 5

5.1 St. Petersburg Paradox

The game is the following: repeatedly flip a coin until you get heads. You win 2^n dollars, where n is the number of coin flips. How much would you be willing to pay to play this game????

Let X be your (random) payoff. The sample space of X is

$$S = \{2, 4, 8, 16, 32, \dots\}.$$

What is the expected value of X?

$$\begin{aligned}
\mathbb{E}[X] &= \sum_{n=1}^{\infty} n\mathbb{P}[X = n] \\
&= \mathbb{P}[X = 1] + 2\mathbb{P}[X = 2] + 3\mathbb{P}[X = 3] + 4\mathbb{P}[X = 4] + 5\mathbb{P}[X = 5] + 6\mathbb{P}[X = 6] + 7\mathbb{P}[X = 7] \\
&\quad + 8\mathbb{P}[X = 8] + 9\mathbb{P}[X = 9] + \dots \\
&= 2\mathbb{P}[X = 2] + 4\mathbb{P}[X = 4] + 8\mathbb{P}[X = 8] + 16\mathbb{P}[X = 16] + \\
&= \sum_{k=1}^{\infty} 2^k \mathbb{P}\left[X = 2^k\right]
\end{aligned}$$

where the third equality follow because $\mathbb{P}[X = i] = 0$ whenever $i \notin S$.

Next, we observe that $\mathbb{P}\left[X = 2^k\right] = \frac{1}{2^k}$ (check this for, say, $k = 1, 2, 3$).

Plugging these probabilities in, we get

$$\mathbb{E}[X] = \sum_{k=1}^{\infty} 2^k \cdot \frac{1}{2^k} = \sum_{k=1}^{\infty} 1 = 1 + 1 + 1 + 1 + 1 + \dots = +\infty.$$

The expected value of playing this game is positive infinity!!!!!

5.2 Counting Techniques

There are three main counting principles we'll look at today:

- multiplication principle
- permutation principle
- combination principle

Recall that a k-tuple is an ordered list of things (usually but not always numbers). A k-tuple can have repeats!

Proposition 12 (Multiplication Principle). *Suppose S consist of k-tuples and there are n_1 choices for the first element, n_2 choices for the second, etc. Then there are*

$$n_1 \times n_2 \times n_k$$

possible k-tuples.

Example 13. I roll my lucky dice, flip a gold coin, then flip a silver coin. How many possible outcomes are there?

[draw a tree to illustrate the product rule]

$$6 \times 2 \times 2 = 24$$

End of Example 13. □

Example 14. How many ways are there to order the 3 letters A, B, C?

Use the multiplication principle

$$3 \times 2 \times 1$$

End of Example 14. □

Proposition 15 (Counting permutations). *The total number of ways to order n distinct objects is*

$$n! = n \times (n-1) \times (n-2) \times \cdots \times 2 \times 1.$$

Each ordering is called a permutation. (Note that we define $0! = 1$.)

(Permutations don't have repeats).

Definition 16 (set). A *set* is an *unordered* collection of elements, all of which are *distinct.*

- $\{1, 2, 4, 5, 3\}$ is a set
- $\{1, 1, 2, 4, 5, 3\}$ is not a set

Proposition 17 (Counting subsets – this is the "combination principle"). *Let S be a set with n elements. The number of (unordered) subsets of size k is*

$$\binom{n}{k} = \frac{n!}{k!\,(n-k)!}$$

Example 18. Let $S = \{1, 2, 3, 4, 5, 6\}$. How many subsets of size 2 are there?

$$\binom{6}{2} = \frac{6!}{2! \cdot 4!} = \frac{6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{(2 \cdot 1) \cdot (4 \cdot 3 \cdot 2 \cdot 1)} = \frac{6 \cdot 5}{2} = 15.$$

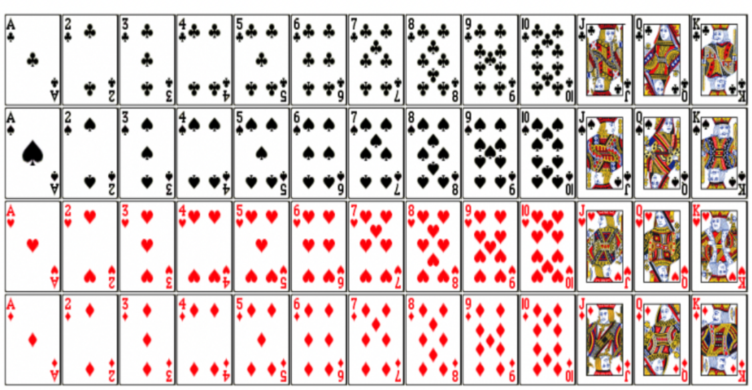
These are

$$\begin{gathered} \{1,2\}, \{1,3\}, \{1,4\}, \{1,5\}, \{1,6\} \\ \{2,3\}, \{2,4\}, \{2,5\}, \{2,6\} \\ \{3,4\}, \{3,5\}, \{3,6\} \\ \{4,5\}, \{4,6\} \\ \{5,6\} \end{gathered}$$

note that the order doesn't matter, i.e., $\{2, 1\} = \{1, 2\}$ so this just counts as one, not two.

End of Example 18. □

Example 19 (Poker). A poker hand consist of 5 randomly chosen cards from a standard 52-card deck.



Some questions about poker hands:

1. How many distinct poker hands are there?

$$\binom{52}{5} = \frac{52!}{5! \cdot 47!} = 2,598,960$$

2. How many ways are there to get a flush (i.e., all the same suit)?

$$4 \times \binom{13}{5} = 4 \cdot \frac{13!}{5!\,8!} = 4 \cdot \frac{13 \cdot 12 \cdot 11 \cdot 10 \cdot 9}{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} = 5148$$

   The 4 is to choose a suit. Then we choose 5 out of 13 cards in that suit.

3. What's the probability that we are dealt a hand in which this occurs? (i.e., in which all cards are of the same suit?)

   To answer this, we use the enumeration principle:

$$\frac{\binom{4}{1} \times \binom{13}{5}}{\binom{52}{5}} = \frac{5,148}{2,598,960} \approx 0.002$$

   In other words, the probability of a flush is pretty low: about 0.2%.

End of Example 19. □