11 2025-02-05 | Week 4 | Lecture 10

11.1 Application of the Law of Total Probability: The probability of winning at craps

Recall the rules of craps:

1. You roll two dice and add them up.
   - If the sum is $2, 3$, or 12, you lose ("craps")
   - If you roll 7 or 11, then you win ("natural")
   - Otherwise you establish "point", which is whatever number you got.
2. If you established point, then you must keep rolling until one of two events occurs:
   - You roll a 7: you lose
   - You roll your "point" number: you win.

Let's introduce some useful notation: Let

$$W = [\text{You win}]\,,$$

and for each $k = 2, 3, 4, \ldots, 12$, define

$$F_k = [\text{Your first roll is } k]\,.$$

Question: What is $\mathbb{P}[W]$?
Solution: By the *Law of Total Probability*, we have

$$\mathbb{P}[W] = \sum_{k=2}^{12} \mathbb{P}[W \mid F_k] \times \mathbb{P}[F_k] \tag{6}$$

In words,

$$\mathbb{P}[W \mid F_k] = \text{ probability of winning, given you rolled } n \text{ on the first roll}$$

k	$P[W \mid F_i]$	$\mathbb{P}[F_k]$
2	0	1/36
3	0	2/36
4	1/3	3/36
5	4/10	4/36
6	5/11	5/36
7	1	6/36
8	5/11	5/36
9	4/10	4/36
10	1/3	3/36
11	1	2/36
12	0	1/36

How do we get, say, $\mathbb{P}[W \mid F_4]$?
(Method 1: The long way) Let's enumerate the ways we can win:

4	win on the first roll after establishing point
$*\,4$	win on second roll
$*\,*4$	win on third roll
$*\,*\,*\,4$	etc
$*\,*\,*\,*4$	
$\ldots$	

where $*$ denotes any roll other than a 4 or a 7.

These have probabitilies

$$\mathbb{P}[4] = \frac{3}{36}$$
$$\mathbb{P}[*4] = \frac{27}{36}\frac{3}{36}$$
$$\mathbb{P}[**4] = \left(\frac{27}{36}\right)^2\frac{3}{36}$$
$$\mathbb{P}[***4] = \left(\frac{27}{36}\right)^3\frac{3}{36}$$

$$\mathbb{P}[****4] = \left(\frac{27}{36}\right)^4\frac{3}{36}$$

$$\ldots$$

So

$$\begin{aligned}
\mathbb{P}[W \mid F_4] &= \frac{3}{36} + \frac{27}{36}\frac{3}{36} + \left(\frac{27}{36}\right)^2\frac{3}{36} \\
&= \frac{3}{36}\left(1 + \frac{27}{36} + \left(\frac{27}{36}\right)^2 + \ldots\right) \\
&= \frac{3}{36}\cdot\frac{1}{1-\frac{27}{36}} \\
&= \frac{3}{9} \\
&= \frac{1}{3}.
\end{aligned}$$

(Method 2: The sort way). Given F_4, you know you're gonna keep rolling until you get a 4 or a 7, which is eventually going to happen. And whether you win or lose is determined by what you roll on that last roll. Your probability of winning is the probability of rolling a 4 on your last roll, given that your last roll is a 4 or a 7. Thus,

$$\begin{aligned}
\mathbb{P}[W \mid F_4] &= \mathbb{P}[\text{roll 4} \mid \text{roll 4 or 7}] \\
&= \frac{\mathbb{P}[(\text{roll 4}) \text{ and } (\text{roll 4 or 7})]}{\mathbb{P}[\text{roll 4 or 7}]} && \text{by def of conditional prob.} \\
&= \frac{\mathbb{P}[\text{roll 4}]}{\mathbb{P}[\text{roll 4 or 7}]} \\
&= \frac{3/36}{\frac{3}{36}+6/36} \\
&= \frac{1}{3}.
\end{aligned}$$

Continue in this manner to get all the values of the table. Then plugging the values from the table into Eq. (6) gives

$$\mathbb{P}[W] = 0\cdot\frac{1}{36} + 0\cdot\frac{2}{36} + \frac{1}{3}\cdot\frac{3}{36} + \frac{4}{10}\cdot\frac{4}{36} + \frac{5}{11}\cdot\frac{5}{36} + 1\cdot\frac{6}{36} + \ldots + 0\cdot\frac{1}{36} = 0.492999...$$

Your chance of winning is about 49.3%.

11.2 Independence

Two events A and B are said to be *indpendent* if and only if

$$\mathbb{P}[A \cap B] = \mathbb{P}[A] \times \mathbb{P}[B].$$

We denote this by writing $A \perp\!\!\!\perp B$.

Otherwise, we say that A and B are *dependent*.

Remarks:

- *Intuition:* Independence means that if I know that A occurred or didn't occur, that gives me no information about whether B occurred or not. And vis-versa. Then

$$\begin{aligned} \mathbb{P}[A \mid B] &= \frac{\mathbb{P}[A \cap B]}{\mathbb{P}[B]} \\ &= \frac{\mathbb{P}[A\cap]\,\mathbb{P}[B]}{\mathbb{P}[B]} && \text{by independence} \\ &= \mathbb{P}[A] \end{aligned}$$

- Physical indpendence always implies mathematical independence. For example, when you roll two dice, the results are independent because they are physically independent.
- Mutually exclusive events are never independent.[1]. That's because if you know that one happened, then you know that the other didn't!

12 Random Variables

Based on sections 3.1 and 3.2 of the textbook

A *random variable* is a variable whose value depends on the outcome of a random process or phenomenon.

Technically, a random variable X is a function which assigns to each outcome $\omega \in S$ in the sample space a real number $X(\omega)$:

$$X : S \to \mathbb{R}.$$

But in this class, we usually won't think of a random variable in this way.

It is customary to denote random variables by capital letters like X, Y etc., and the values they take by x, y, etc.

The *state space* of a random variable X is the set of values it can take. We say that a random variable is *discrete* if the state space is "countable" (either finitely many values, or, for example integers or rational numbers).

Given a discrete random variable X, the *probability mass function (pmf)* is the function

$$p(x) := \mathbb{P}[X = x]$$

To avoid ambiguity, sometimes we name the pmf of X $p_X(x)$ instead of just $p(x)$.

Here are some standard random variables to be familiar with:

Example 34 (Bernoulli random variable)**.** Any random varible whose only values are 0 or 1. For any fixed value $0 < \alpha < 1$, we say that X *is a Bernoulli random variable with success parameter σ* if

$$X = \begin{cases} 1 & \text{with probability } \alpha \\ 0 & \text{with probability } 1 - \alpha \end{cases}$$

In this case, we write $X \sim \mathrm{Bern}(\alpha)$. The pmf of X is the function taking the following values:

$$p(1) = \alpha, \quad p(0) = 1 - \alpha, \quad \text{and} \quad p(x) = 0 \text{ for all other values of } x$$

A Bernoulli random variable is nothing other than a coin flip (with a coin that lands 'head' with probability α.)

End of Example 34. □

[1]As long as the events are of positive probability