18 2025-02-24 | Week 7 | Lecture 17

Topic: moment generating functions – this isn't in the textbook

Definition 58 (Equal in distribution)**.** Let X and Y be two random variables. We say that X and Y *have the same distribution*, denoted $X \overset{d}{=} Y$ if they have the same cdf (i.e., if $F_X = F_Y$).

- If X and Y are discrete and have pmfs p_X and p_Y, then this is equivalent to $p_X = p_Y$.
- If X and Y are continuous with pdfs f_X and f_Y, then this is equivalent to $f_X = f_Y$.

Example 59. Warning: equality in distribution does not imply that $X = Y$. For example, if I roll a red dice X and a blue dice Y, both with 6 sides, the chances are pretty good that $X \neq Y$. But of course these are equal in distribution, since

$$p_X(k) = p_Y(k) = \frac{1}{6} \quad \text{for all } k = 1, 2, \ldots, 6.$$

End of Example 59. □

Definition 60 (Standard deviation)**.** The *standard deviation* of X is the square root of $\mathrm{Var}(X)$:

$$\mathrm{SD}(X) = \sqrt{\mathrm{Var}(X)}.$$

Standard deviation is often denoted σ.

Definition 61 (Moment generating function)**.** The *moment generating function* or $m.g.f.$ of a random variable X is the function

$$M(t) := \mathbb{E}\left[e^{tX}\right].$$

- For a discrete random variable with pmf $p(x)$, we have

$$M(t) = \sum_x e^{tx} p(x) dx.$$

- For a continuous r.v., with pdf $f(x)$, we have

$$M(t) = \int_{-\infty}^{\infty} e^{tx} f(x) dx.$$

The moment generating function is important because if two random variables X and Y have the same mgf, then they have the same distribution.

Question: But what are "moments" and why is this called a "moment-generating" function?!

Definition 62. Let X be a r.v., and let k be a nonnegative integer. The *k-th moment of X* is the quantity

$$\mathbb{E}\left[X^k\right].$$

So, the 0-th moment is always 1. The first moment is just the expectation of the r.v. The second moment shows up in the variance fomrula:

$$\mathrm{Var}(X) = \mathbb{E}\left[X^2\right] - \left(\mathbb{E}\left[X\right]\right)^2 = \mathbb{E}\left[X^2\right].$$

To the best of my knowledge, moments don't have a straightforward interpretation that holds generally. But they are important for two reasons (1) they often show up in things we care about, like the variance formula, as well as in the assumptions and proofs of important theorems, like the central limit theorem, and (2) when taken together, all the moments of a random variable $\left\{\mathbb{E}\left[X^k\right] : k = 0, 1, 2, \ldots\right\}$ will usually completely determine the distribution of the random variable.[2]

[2] It is necessary that the sequence $\mathbb{E}\left[X\right], \mathbb{E}\left[X^2\right], \mathbb{E}\left[X^3\right], \ldots$ does not grow too quickly, but we can ignore this technical issue for now.

As the name suggests, the mgf of X has a close connection to the moments of X. To see this, observe that

$$e^{tX} = \sum_{k=0}^{\infty} \frac{(tX)^k}{k!} = \sum_{k=0}^{\infty} \frac{t^k X^k}{k!}.$$

So

$$\mathbb{E}\left[e^{tX}\right] = \mathbb{E}\left[\sum_{k=0}^{\infty} \frac{t^k X^k}{k!}\right] = \sum_{k=0}^{\infty} \frac{t^k}{k!}\mathbb{E}\left[X^k\right] \tag{14}$$

Example 63 (Using a mgf to compute moments)**.** Let $X \sim \exp(\lambda)$. That is, X is an exponential random variable with rate λ.

Question #1: What is the moment generating function of X?

First, recall that

- The pdf of X is $f_X(x) = \lambda e^{-\lambda x}$, for all $x > 0$.
- For any function $h : \mathbb{R} \to \mathbb{R}$, we have

$$\mathbb{E}\left[h(X)\right] = \int_{-\infty}^{\infty} h(x) f_X(x) dx$$

Using these two facts, we have:

$$\begin{aligned}
M(t) &= \mathbb{E}\left[e^{tX}\right] \\
&= \int_0^{\infty} e^{tx} \lambda e^{-\lambda x} dx \\
&= \lambda \int_0^{\infty} e^{(t-\lambda)x} dx \\
&= \frac{\lambda}{t-\lambda}\left[e^{(t-\lambda)x}\right]_{x=0}^{x=\infty}. \\
&= \frac{\lambda}{t-\lambda}\left[\left(\lim_{x\to\infty} e^{(t-\lambda)x}\right) - 1\right]
\end{aligned}$$

When we try to evaluate the limit as $x \to \infty$, something goes wrong if $t > \lambda$. Indeed, we have:

$$\lim_{x\to\infty} e^{(t-\lambda)x} = \begin{cases} +\infty & : \quad t > \lambda \\ 0 & : \quad t < \lambda \end{cases}$$

Therefore we have

$$M(t) = \frac{\lambda}{t-\lambda}(-1) = \frac{\lambda}{\lambda - t}$$

but this function is defined only for $t < \lambda$.

Question #2: For concreteness, let's suppose that $\lambda = 9$. What is the 2-nd moment of X?

Solution 1. We can compute the second moment using the formula

$$\mathbb{E}\left[X^2\right] = \int_0^{\infty} x^2 f_X(x) dx = \int_0^{\infty} x^2 9 e^{-9x} dx$$

This would work. But this would require integrating by parts twice, which we may wish to avoid. So instead, I'll present a different approach which utilizes the moment generating function.

Solution 2. From our earlier calculations, we have

$$M(t) = \frac{9}{9-t}, \quad \text{defined for } t < 9$$

Differentiating M with respect to t gives

$$M'(t) = \frac{9}{(9-t)^2}.$$

Differentiating again, we have:

$$M''(t) = \frac{9}{2(9-t)^3} \tag{15}$$

But recall by Eq. (14), we have

$$M(t) = \sum_{k=0}^{\infty} \frac{t^k}{k!}\mathbb{E}\left[X^k\right] = 1 + t\mathbb{E}\left[X\right] + \frac{t^2}{2!}\mathbb{E}\left[X^2\right] + \frac{t^3}{3!}\mathbb{E}\left[X^3\right] + \frac{t^4}{4!}\mathbb{E}\left[X^4\right] + \ldots,$$

and differentiating *this* equation twice gives

$$M''(t) = \mathbb{E}\left[X^2\right] + t\mathbb{E}\left[X^3\right] + \frac{t^2}{2!}\mathbb{E}\left[X^4\right] + \ldots. \tag{16}$$

Combining Eqs. (15) and (16), it follows that

$$\mathbb{E}\left[X^2 e^{tX}\right] = \frac{9}{2(9-t)^3}.$$

The above equation holds for all t such that both sides are defined. In particular, it holds when $t = 0$. Setting $t = 0$, something almost magical happens:

$$\mathbb{E}\left[X^2 e^0\right] = \frac{9}{2(9)^3}$$

which simplifies to

$$\mathbb{E}\left[X^2\right] = \frac{2}{81}.$$

We've computed the second moment (and all we had to do was differentiate the mgf twice and then plug in zero!).

End of Example 63. □

This example illustrates the following general property of a moment generating function:

$$\mathbb{E}\left[X^k\right] = \left[\frac{d^k}{dx^k}\left[M(t)\right]\right]_{t=0}$$

And this is how the moment generating function got its name.