# 2 2025-01-14 | Week 01 | Lecture 02

> ***The nexus question of this lecture:*** *What is the general framework for probability?*
> ***Reading assignment:*** *chapters 2.1 - 2.5*

## 2.1 A general framework for probability

We begin with a general framework and some terminology to formalize the notions of probability.

- An ***experiment*** is an activity or process whose outcome is subject to uncertainty, and about which an observation is made.

  Examples include flipping a coin, rolling a dice, measuring the size of a wave, or the amount of rainfall, conducting a poll, performing a diagnostic test, opening a pack of Pokemon cards, etc.

- The ***sample space*** $S$ of an experiment is the set of all possible outcomes. The elements of the sample space are called ***sample points***.

  We think of each sample point as representing a unique outcome of the experiment. In the case of rolling a dice, the sample points are $1, 2, 3, 4, 5$ and $6$, and the sample space is $S = \{1, 2, 3, 4, 5, 6\}$.

- We use the term ***event*** to refer to a collection of outcomes. So we think of an event as being a *subset* of the sample space:

  $$\text{an event} = \text{a set of outcomes} = \text{a subset of } S.$$

  Example: if our experiment is rolling a 6-sided dice, here are some events

  $$\begin{aligned} A &= [\text{observe an odd number}] & E_2 &= [\text{observe a } 2] \\ B &= [\text{observe an even number}] & E_3 &= [\text{observe a } 3] \\ C &= [\text{observe a number less than } 5] & E_4 &= [\text{observe a } 4] \\ D &= [\text{observe a } 2 \text{ or a } 3] & E_5 &= [\text{observe a } 5] \\ E_1 &= [\text{observe a } 1] & E_6 &= [\text{observe a } 6] \end{aligned}$$

- There are two types of events: ***compound events***, which can be decomposed into other events, and ***simple events***, which cannot.

  In the above example, the events $A, B, C$ and $D$ are compound events. $E_1, \ldots, E_6$ are simple events.

  $$\begin{aligned} A &= [\text{observe an odd number}] = \{1, 3, 5\} & E_2 &= [\text{observe a } 2] = \{2\} \\ B &= [\text{observe an even number}] = \{2, 4, 6\} & E_3 &= [\text{observe a } 3] = \{3\} \\ C &= [\text{observe a number less than } 5] = \{1, 2, 3, 4\} & E_4 &= [\text{observe a } 4] = \{4\} \\ D &= [\text{observe a } 2 \text{ or a } 3] = \{2, 3\} & E_5 &= [\text{observe a } 5] = \{5\} \\ E_1 &= [\text{observe a } 1] = \{1\} & E_6 &= [\text{observe a } 6] = \{6\} \end{aligned}$$

**Example 1** (Examples of sample spaces)**.**

- If I roll dice, the sample space is $S = \{1, 2, 3, 4, 5, 6\}$
- If I flip a coin, the sample space is $S = \{T, H\}$
- The amount of rainfall in a day is $S = \{x \in \mathbb{R} : x \geq 0\}$

  An example of an event for this sample space is:

  $$[\text{between 1 and 2 inches of rain}] = \{x \in \mathbb{R} : 1 \leq x \leq 2\}.$$

- You've got an urn filled with 300,000,000 balls. Exactly one ball is made of gold. Draw a ball out at random. If it's not the gold ball, put it back and keep repeating until you get the gold ball. Once you get the gold ball, you are done. The output of this experiment is *the number of times you drew a ball from the urn.* The sample space is

$$S = \{1, 2, 3, 4, \ldots\}.$$

The last two examples show that the sample space need not be finite.

End of Example 1. □

## 2.2 The enumeration principle

**Example 2** (Rolling two dice)**.** When rolling a red and a blue dice, the sample space consists of 36 possible outcomes:

| | | | | | |
|---|---|---|---|---|---|
| 1 1 | 1 2 | 1 3 | 1 4 | 1 5 | 1 6 |
| 2 1 | 2 2 | 2 3 | 2 4 | 2 5 | 2 6 |
| 3 1 | 3 2 | 3 3 | 3 4 | 3 5 | 3 6 |
| 4 1 | 4 2 | 4 3 | 4 4 | 4 5 | 4 6 |
| 5 1 | 5 2 | 5 3 | 5 4 | 5 5 | 5 6 |
| 6 1 | 6 2 | 6 3 | 6 4 | 6 5 | 6 6 |

so we can write the sample space as:

$$\begin{aligned} S &= \{(x, y) : x, y \in \{1, 2, 3, 4, 5, 6\}\} \\ &= \{(1, 1), (1, 2), \ldots, (6, 6)\} \end{aligned}$$

The event that the dice are equal is

$$[\text{dice are equal}] = \{(1, 1), (2, 2), (3, 3), (4, 4), (5, 5), (6, 6)\}$$

We could have other events too, like that the dice sum to 3:

$$[\text{dice sum to } 4] = \{(1, 3), (2, 2), (3, 1)\}$$

All of the outcome pairs are equally likely, so we can compute the probabilities of by counting entires in the table. Let

$$Z = \text{the sum of the red dice and the blue dice}$$

By counting entries in our table, we see that

$$\mathbb{P}[Z = 4] = \frac{3}{36}.$$

Similarly,

$$\mathbb{P}[Z = 7] = \frac{6}{36} = \frac{1}{6}$$

and

$$\mathbb{P}[Z \leq 5] = \frac{10}{36}.$$

End of Example 2. □

The previous example illustrates the critically imporant idea of **enumeration:**

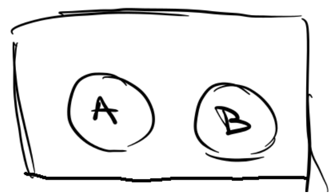
> If your sample space is finite and consists of *equally likely outcomes*, then you can compute lots of probabilities easily by listing outcomes and counting them. To be prcecise, for any event $A$,
>
> $$\mathbb{P}[A] = \frac{|A|}{|S|} = \frac{\#\text{ of outcomes in } A}{\#\text{ of outcomes in } S}$$

I already had you do a bunch of these problems on Monday.

### 2.3 Events

Some observations about events:

- Two events $E$ and $F$ are ***mutually exclusive*** if $E \cap F = \emptyset$. This means that $E$ and $F$ cannot both happen at the same time.

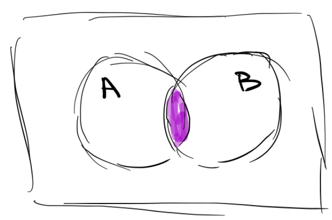
  
  

  In the dice example, the events $A$ and $B$ are mutually exclusive, since the dice roll cannot be both even and odd. But $A$ and $C$ are not mutually exclusive because $A \cap C = \{1, 3\} \neq \emptyset$. If a 1 or a 3 is rolled, then both $A$ and $C$ occur.

- The sample points are *elements* of $S$. The simple events are *singleton subsets* of $S$. In the dice example, we have:
  - Sample points: 1,2,3,4,5,6.
  - Simple events: $\{1\}, \{2\}, \{3\}, \{4\}, \{5\}, \{6\}$.

- The empty set $\emptyset$ and the whole sample space $S$ are always both events: $\emptyset$ is the event "nothing happens" and $S$ is the event "something happens".

- If $E$ and $F$ are events, then $E \cap F$ is the event that both $E$ **and** $F$ occur:

  Here, $E \cap F$ is the ***intersection*** of $E$ and $F$; that is, $E \cap F$ the set of sample points contained in both $E$ and $F$.