21 2025-03-03 | Week 8 | Lecture 20

Sections 5.1, 5.2 Recall that two events are independent if knowledge that one has occurred gives no information about whether the other has occurred. We can extend this idea to random variables:

Definition 68 (Independence of rvs)**.** Two random variables X and Y are *independent* if for every pair of (x, y)-values,

$$p(x, y) = p_X(x) \cdot p_Y(y) \quad \text{when } X \text{ and } Y \text{ are discrete}$$

or

$$f(x, y) = f_X(x) \cdot f_Y(y) \quad \text{when } X \text{ and } Y \text{ are continuous}$$

If this condition isn't satisfied for all (x, y), then X and Y are said to be *dependent*.

[For clarity in the above: $p(x, y)$ is the joint pmf of (X, Y). And $f(x, y)$ is the joint pdf of (X, Y).]

The intuition is this: two random variables are independent if observing one of them doesn't give you any information about the other.

Example 69. In the car factory example from last lecture, we had two random variables, X and Y, with joint pmf given by

		Y			
		0	1	2	3
X	0	.840	.030	.020	.010
	1	.060	.010	.008	.002
	2	.010	.005	.004	.001

Are X and Y independent? To determine this, we need to first obtain the pmfs:

x	$p_X(x)$
0	.9
1	.08
2	.02

and

y	$p_Y(y)$
0	.91
1	.045
2	.032
3	.013

Having computed the pmfs, we can verify that X and Y are **not independent**. This is because

$$p(0, 0) = .84 \neq (.9)(.91) = .819 = p_X(0)p_Y(0).$$

End of Example 69. □

Example 70. Let X and Y be the lifetimes of two lightbulbs. Assume that $X \sim \exp(\alpha)$ and $Y \sim \exp(\beta)$, where α and β are positive numbers. It's reasonable to assume of these as independent. Then the joint pdf is

$$\begin{aligned} f(x, y) &= f_X(x)f_Y(y) \\ &= \left(\alpha e^{-\alpha x}\mathbf{1}_{[x>0]}\right)\left(\beta e^{-\beta y}\mathbf{1}_{[y>0]}\right) \\ &= \alpha\beta e^{-\alpha x - \beta y}\mathbf{1}_{[x,y>0]} \\ &= \begin{cases} \alpha\beta e^{-\alpha x-\beta y} & : \quad x, y > 0 \\ 0 & : \quad \text{else} \end{cases} \end{aligned}$$

Suppose $\alpha = 1/1000$ and $\beta = 1/1200$. Then the expected lifetimes of the bulbs are 1000 hours and 1200 hours, respectively.

Question: What is the probability that both lightbulbs last at least 900 hours?

Want:

$$
\begin{aligned}
\mathbb{P}\left[X \geq 900, Y \geq 900\right] &= \int_{900}^{\infty}\int_{900}^{\infty}\left(\alpha e^{-\alpha x}\right)\left(\beta e^{-\beta y}\right) dx dy \\
&= \int_{900}^{\infty}\left[\int_{900}^{\infty}\alpha e^{-\alpha x} dx\right]\left(\beta e^{-\beta y}\right) dy \\
&= \left(\int_{900}^{\infty}\alpha e^{-\alpha x} dx\right)\left(\int_{900}^{\infty}\beta e^{-\beta y} dy\right) \\
&= \left(\left[e^{-\alpha x}\right]_{x\to\infty}^{x=900}\right)\left(\left[e^{-\beta y}\right]_{y\to\infty}^{y=900}\right) \\
&= \left(e^{-900\alpha} - 0\right)\left(e^{-900\beta} - 0\right) \\
&= e^{-900(\alpha+\beta)} \\
&= e^{-900\left(\frac{1}{1000}+\frac{1}{1200}\right)} \\
&= e^{-1.65} \\
&\approx .2
\end{aligned}
$$

So with probability about one fifth, both ligthbulbs will last at least 900 hours.

A faster way to do this. Observe that the events $[X \geq 900]$ and $[Y \geq 900]$ are independent, because the random variables are. Then

$$
\begin{aligned}
\mathbb{P}\left[X \geq 900, Y \geq 900\right] &= \mathbb{P}\left[X \geq 900\right]\mathbb{P}\left[Y \geq 900\right] \\
&= e^{-900\alpha}e^{-900\beta} \\
&= e^{-1.65}
\end{aligned}
$$

End of Example 70. $\square$