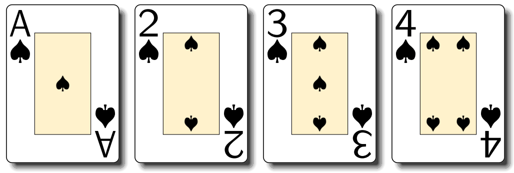
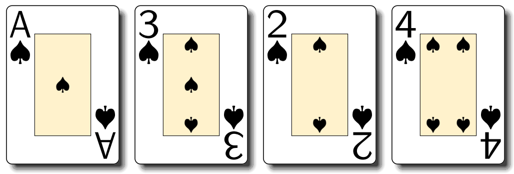
Worksheet 3: Random Permutations

Problem 1. Suppose some several distinct playing cards are laid out in a line like this:



A *random transposition* is where two different cards are selected at random and their positions swapped. For example, the transposition (23) swaps cards 2 and 3. If the cards started in the above order, then the effect of applying (23) would be to reorder the cards as follows:



Anna's nemesis Mark challenges her to a betting game. In the game, Mark lays out four cards in the order $1, 2, 3, 4$ and then performs k random transpositions, one after the other. As this happens, Anna is blindfolded, so she can't see which transpositions were performed—however, because she can hear Mark moving the cards, she does know the value of k. Anna then has to guess the final order of the cards. If she guesses correctly, Mark will pay her \$20; otherwise she has to pay Mark \$1.

Mark believes that as long as he performs at least 4 transpositions, then all $4! = 24$ permutations of the cards will be equally likely, so that Anna is expected to lose money when she plays the game. Moreover Mark believes that Anna is a sucker and he can make lots of money from her.

But Mark is wrong. Design a betting strategy for Anna to bleed Mark for all he's worth.

Hint: *First consider the simpler problem where the game is played with 3 cards. Write down a table which gives the probability mass function for each of the 6 permutations in that case:*

$$(1)(2)(3)$$
$$(1)(23)$$
$$(12)(3)$$
$$(13)(2)$$
$$(123)$$
$$(132)$$

Do this for each $k = 2, 3, 4, 5$. Once you understand the case with 3 cards, move to four.

Note that in the next two problems, we will use the notation AB to denote $A \cap B$.

Problem 2 (Inclusion-Exclusion)**.** In this problem we will generalize the inclusion-exclusion principle

$$\mathbb{P}[A_1 \cup A_2] = \mathbb{P}[A_1] + \mathbb{P}[A_2] - \mathbb{P}[A_1 \cap A_2] \tag{1}$$

to more than two sets. The formulas are a bit complicated, but we've broken it up into hopefully doable chunks.

(a) Let's start with some useful preliminaries. Show that $\mathbb{E}[\mathbf{1}_A] = \mathbb{P}[A]$ for any event A, and that $\mathbf{1}_A \mathbf{1}_B = \mathbf{1}_{AB}$ for any events A and B.

(b) The following polynomial equality holds for all $n \geq 1$:

$$(1-x_1)(1-x_2)\cdots(1-x_n) = 1 - \left(\sum_{1\leq i\leq n} x_i\right) + \left(\sum_{1\leq i<j\leq n} x_i x_j\right) - \left(\sum_{1\leq i<j<k\leq n} x_i x_j x_k\right) + \ldots + (-1)^n (x_1 x_2 \cdots x_n)$$

This formula can be proved by induction, but for now let's just convince ourselves that it holds by verifying the case when $n = 4$.

(c) Let $A_1, \ldots, A_n$ be events. Show that

$$\mathbf{1}_{A_1^c \cdots A_n^c} = (1 - \mathbf{1}_{A_1})(1 - \mathbf{1}_{A_2}) \cdots (1 - \mathbf{1}_{A_n}).$$

(d) Using parts (a), (b), and (c), prove the following formula:

$$\mathbb{P}[A_1^c \cdots A_n^c] = 1 - \left(\sum_{1\leq i\leq n} \mathbb{P}[A_i]\right) + \left(\sum_{1\leq i<j\leq n} \mathbb{P}[A_i A_j]\right) - \left(\sum_{1\leq i<j<k\leq n} \mathbb{P}[A_i A_j A_k]\right) + \ldots + (-1)^n \mathbb{P}[A_1 \cdots A_n]$$

(e) Deduce from the formula in (d) that

$$\mathbb{P}[A_1 \cup \cdots \cup A_n] = \left(\sum_{1\leq i\leq n} \mathbb{P}[A_i]\right) - \left(\sum_{1\leq i<j\leq n} \mathbb{P}[A_i A_j]\right) + \left(\sum_{1\leq i<j<k\leq n} \mathbb{P}[A_i A_j A_k]\right) + \ldots + (-1)^{n-1} \mathbb{P}[A_1 \cdots A_n].$$

This equation is called *the inclusion-exclusion principle.* It can be written more compactly in the following form (you don't have to show this):

$$\mathbb{P}[A_1 \cup \ldots \cup A_n] = \sum_{k=1}^{n} (-1)^{k-1} \left[\sum_{I\subseteq\{1,\ldots,n\}:|I|=k} \mathbb{P}[\cap_{i\in I} A_i]\right] \tag{2}$$

To explain this equation, note that for each k, the summation in the square brackets sums over all subsets I of $\{1, \ldots, n\}$ of size k. (Meaning it sums over $\binom{n}{k}$ terms.)

Problem 3. In this problem we consider the problem of a random permutation of n distinct elements, denoted $1, 2, \ldots, n$. Let N be the number of fixed points of the permutation. We will show that

$$\mathbb{P}[N=0] = \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \ldots + (-1)^n \frac{1}{n!}$$

which implies that $\mathbb{P}[N=0] \approx \frac{1}{e} \approx .37$ when n is large.

This problem uses the result of problem 2.

(a) Let A_i be the event that i is a fixed point. Using the ideas of conditional probability, show that $\mathbb{P}[A_i] = \frac{1}{n}$, $\quad \mathbb{P}[A_iA_j] = \frac{1}{n(n-1)}$, $\quad \mathbb{P}[A_iA_jA_k] = \frac{1}{n(n-1)(n-2)}$, and so forth, so that in general, we have

$$\mathbb{P}[\cap_{i \in I} A_i] = \frac{1}{n(n-1)(n-2)\cdots(n-k+2)}$$

whenever $I \subseteq \{1, 2 \ldots, n\}$ is a subset of size $k \geq 1$.

(b) Let k be a positive integer. How many terms are in the sum

$$\sum_{I \subseteq \{1,\ldots,n\} : |I| = k} \mathbb{P}[\cap_{i \in I} A_i]$$

(c) Using your answers to parts (a) and (b), show that

$$\sum_{I \subseteq \{1,\ldots,n\} : |I| = k} \mathbb{P}[\cap_{i \in I} A_i] = \frac{1}{k!}$$

for each $k = 1, 2, \ldots$.

(d) Plugging (c) into Equation (2), deduce that

$$\mathbb{P}[A_1 \cup \cdots \cup A_n] = 1 - \frac{1}{2!} + \frac{1}{3!} - \frac{1}{4!} + \ldots + (-1)^{n-1} \frac{1}{n!}$$

(e) Using the formula $e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}$, deduce that

$$\mathbb{P}[A_1 \cup \cdots \cup A_n] \approx 1 - \frac{1}{e}$$

and hence that

$$\mathbb{P}[A_1^c \cap A_2^c \cap \cdots \cap A_n^c] \approx \frac{1}{e}$$

(f) Interpret this result in words.

(g) How close is this approximation when $n = 4$? What about $n = 5$?