9 2025-01-31 | Week 3 | Lecture 8

9.1 The Multiplication Rule for Conditional Probability

Recall, the multiplication rule for conditional probability is

$$\mathbb{P}[AB] = \mathbb{P}[A \mid B]\,\mathbb{P}[B]$$

where A and B are any events.

Example 29 (Urn - example of multiplication rule)**.** An urn contains 6 white balls and 9 black balls. If 4 balls are drawn at random, what is the probability that the first 2 are white and the last 2 are black?
Solution: Let W be the event that the first two balls are white, and B the event that the last two balls are black. Then the desired probability is

$$\mathbb{P}[W \cap B] = \mathbb{P}[B \mid W]\,\mathbb{P}[W] \tag{2}$$

Now,

$$\mathbb{P}[W] = \frac{\binom{6}{2}}{\binom{15}{2}} = \frac{1}{7}.$$

Moreover, given that W occurs, then 4 white balls and 9 black balls remain. So

$$\mathbb{P}[B \mid W] = \frac{\binom{9}{2}}{\binom{13}{2}} = \frac{6}{13}.$$

Therefore by Eq. (2),

$$\mathbb{P}[W \cap B] = \frac{1}{7} \cdot \frac{6}{13} = \frac{6}{91} \approx 0.07$$

End of Example 29. □

Example 30 (Application of multiplication rule (c.f. Example 2.27 in textbook))**.** A vampire goes to the blood bank looking to find some type O+ blood (the most delicious type). He finds four unlabeled bags of blood. Only one of the bags is O+, but he doesn't know which one. The vampire resorts to taste-testing to find the O+ bag.

(a) What is the probability that he must test at least 3 bags to find the desired type?

Solution: Let X be the number of tested bags. We want to find $\mathbb{P}[X \geq 3]$.

Let $A =$ [first bag isn't O+]. Let $B =$ [second bag isn't O+]

$$\begin{aligned}\mathbb{P}[X \geq 3] &= \mathbb{P}[A \cap B] \\ &= \mathbb{P}[A]\,\mathbb{P}[B \mid A] \\ &= \frac{3}{4} \cdot \frac{2}{3} \\ &= \frac{1}{2}.\end{aligned}$$

(b) What's the probability that the third bag he tests is the one containing the O+ blood?

Solution: We want $\mathbb{P}[X = 3] = \mathbb{P}[\text{third bag is O+}]$.

$$\begin{aligned}\mathbb{P}[\text{third bag is O+}] &= \mathbb{P}[\text{third bag is} \mid \text{first isn't} \cap \text{second isn't}] \times \mathbb{P}[\text{second isn't} \mid \text{first isn't}] \cdot \mathbb{P}[\text{first isn't}] \\ &= \frac{1}{2} \cdot \frac{2}{3} \cdot \frac{3}{4} \\ &= \frac{1}{4}\end{aligned}$$

End of Example 30. □

9.2 Law of Total Probability

Theorem 31 (The Law of Total Probability)**.** *Let $A_1, \ldots, A_n$ be events which partition the sample space (i.e., they are mutually exclusive and $A_1 \cup \cdots \cup A_n = S$). Then*

$$\mathbb{P}[B] = \sum_{i=1}^{n} \mathbb{P}[B \mid A_i]\, \mathbb{P}[A_i]$$

Proof. Proof by picture. $\square$

Example 32 (Similar to example 2.29 from text)**.** A large group of adventurers are given the choice of one of three quests: slay a dragon (Q_1), defeat the lich king (Q_2), or retrieve the long-lost scepter of fire in the underground kingdom of Avernus (Q_3). Out of this group, 50% of the adventurers undertake Q_1, 30% undertake Q_2, and the remaining 20% choose Q_3.

It is known that within the first year, 25% of adventurers who set out to slay a dragon become dragon food instead. Moreover, 20% of adventurers who attempt to stop the lich king end up joining the ranks of his growing undead army, and 10% of adventurers who descend into Avernus to retrieve the scepter of fire get slain by underground lizardmen.

(a) What is the probability that a randomly-selected adventurer undertakes quest (1) and gets eaten by a dragon?

Solution: In other words, we want to compute $\mathbb{P}\left[☠ \cap Q_1\right]$. Using the multiplication rule, we have

$$\mathbb{P}\left[☠ \cap Q_1\right] = \mathbb{P}\left[☠ \mid Q_1\right] \cdot \mathbb{P}[Q_1] = \frac{1}{4} \cdot \frac{1}{2} = \frac{1}{8}$$

(b) What is the probability that a randomly-selected adventurer meets an untimely demise?