Math 472: Homework 02

Due Friday, Jan 30

Problem 1. Prove **Proposition 2** in the lecture notes ("Basic properties of probability measures").
If you get stuck, refer to my old math 372 lecture notes (Week 2, Class 4, available on the course website).

Problem 2. In lecture, we noted that the variance of a sample mean decreases as the size of the sample increases. In this problem, we'll visualize this phenomenon for dice rolls like those of Example 7.1 in the textbook.

(a) For any positive integer n, the `R` code

```
mean(sample(1:6, n, replace=TRUE))
```

simulates rolling a 6-sided dice n times. This is our sample. Write code to compute the sample mean of this sample.

(b) For each $k \in \{3, 10, 100, 1000\}$, use the functions `replicate()` and `hist()` to plot a histogram consisting of 10,000 sample means (obtained using your code from part (a)).

When plotting the histograms with the `hist()` function, add the optional argument `xlim=c(0,6)` and observe how the histograms get narrower and more concentrated around 3.5. That's the whole point of this problem: the variance of a sample mean tends to zero as $n \to \infty$.

(c) In part (b), we visualized the variance of the *sample mean* through simulations. In this part, we will consider the *sample variance*, which is a different quantity. For

$$n \in \{10, 30, 60, 100, 1000, 10000, 100000, 500000, 1000000, 10000000, 100000000\},$$

draw a sample of n dice rolls and compute the sample variance S^2. What quantity does this appear to be converging to?

Problem 3. An example of a random variable whose expected value does not exist is the ***Cauchy random variable***, that is, one with pdf

$$f(x) = \frac{1}{\pi}\frac{1}{1+x^2}, \ x \in \mathbb{R}.$$

(a) Show that f is a valid probability density function by showing that it is nonnegative and that $\int_{-\infty}^{\infty} f(x)dx = 1$.

(b) Show that $\mathbb{E}[X]$ is not defined by showing that $E[|X|] = \int_{-\infty}^{\infty} |x| f(x)dx = +\infty$.

Problem 4. A "median" of a distribution is a value m such that $\mathbb{P}[X \le m] \ge \frac{1}{2}$ and $\mathbb{P}[X \ge m] \ge \frac{1}{2}$. (If X is continuous, m satisfies $\int_{-\infty}^{m} f(x)dx = \int_{m}^{\infty} f(x)dx = \frac{1}{2}$, where f is the pdf of X.) Find the median of the following distributions

(a) $f(x) = 3x^2, \ 0 < x < 1$

(b) $f(x) = \frac{1}{\pi}\frac{1}{1+x^2}, \ x \in \mathbb{R}$.

Problem 5 (Exercise 2.18 in textbook)**.** Suppose two fair coins are tossed and the upper faces are observed.

(a) List the sample points for this experiment.

(b) Assign a reasonable probability to each sample point. (Are the sample points equally likely?)

(c) Let A denote the event that exactly one head is observed and B the event that at least one head is observed. List the sample points in both A and B.

(d) From your answer to part (c), find $P(A)$, $P(B)$, $P(A \cap B)$, $P(A \cup B)$, and $P(A^c \cup B)$.

Problem 6 (Variance)**.** The ***variance*** of a random variable X is the quantity

$$\mathrm{Var}(X) = \mathbb{E}\left[(X-\mu)^2\right],$$

where $\mu = \mathbb{E}[X]$. The positive square root $\mathrm{Var}(X)$ is called the ***standard deviation*** of X.

In this problem, we'll prove three important facts about variance.

(a) Prove that
$$\mathrm{Var}(X) = \mathbb{E}\left[X^2\right] - \left(\mathbb{E}[X]\right)^2.$$

(b) Prove that if X is a random variable, then for any scalars a and b,
$$\mathrm{Var}(aX+b) = a^2\mathrm{Var}(X). \tag{1}$$

(c) If X and Y are independent random variables,
$$\mathrm{Var}(X+Y) = \mathrm{Var}(X) + \mathrm{Var}(Y). \tag{2}$$

Hint: Since X and Y are independent, $\mathbb{E}[(X-\mathbb{E}[X])(Y-\mathbb{E}[Y])] = 0$.

Problem 7. Let A and B be independent events. Show that the following pairs of events are independent.

(a) A and B^c

(b) A^c and B

(c) A^c and B^c.

Problem 8. Let X be a random variable with pdf f and cdf F. Assume that f is uniformly bounded, i.e., that there exists $M > 0$ such that $|f(t)| \leq M$ for all $t \in \mathbb{R}$.

(a) Prove that if $x, y \in \mathbb{R}$ then $|F(x) - F(y)| \leq M\,|x-y|$.

(b) Use part (a) to show that F is continuous on $\mathbb{R}$.

Problem 9 (Exercise 1.9 in the textbook)**.** Resting breathing rates for college-age students are approximately normally distributed with mean 12 and standard deviation 2.3 breaths per minute. What fraction of all college-age students have breathing rates in the following intervals?

(a) 9.7 to 14.3 breaths per minute

(b) 7.4 to 16.6 breaths per minute

(c) 9.7 to 16.6 breaths per minute

(d) Less than 5.1 or more than 18.9 breaths per minute