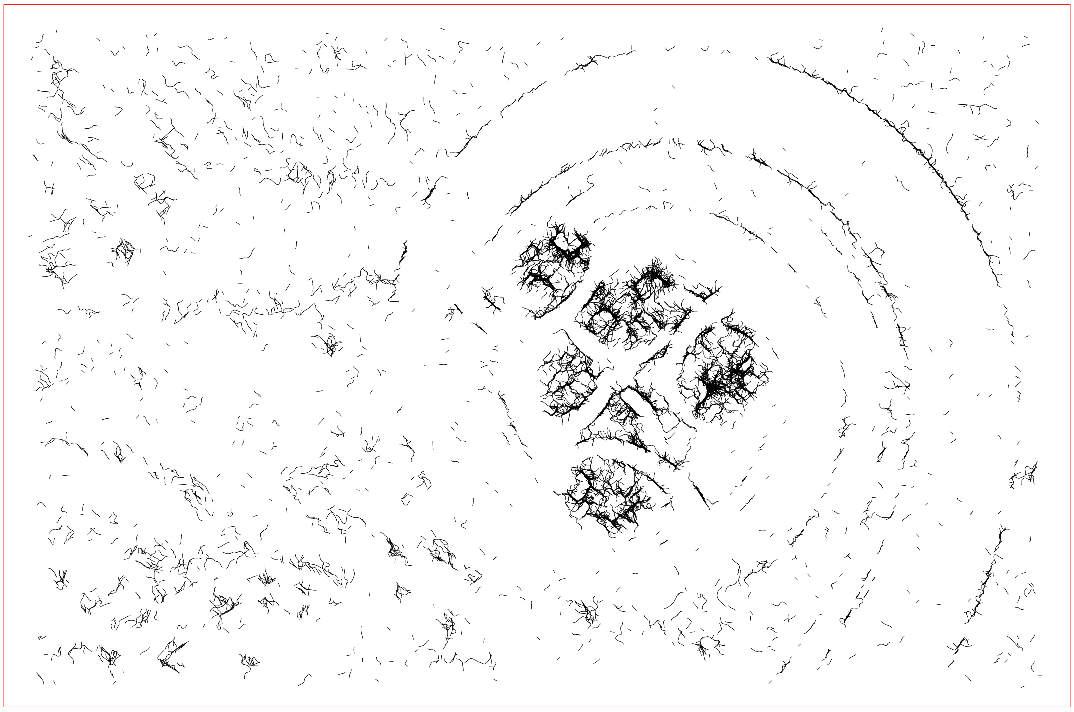
Linewise

Using thermodynamic simulations for art



Max Krummenacher

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Figure on cover is an example output of the model.

Abstract

This report describes two methods to decompose an image in to a collection of splines of cubic Bezier curves. For this the splines are interpreted as polymers moving in an environment which is derived from the image. The Hamiltonian of the system is used to define a Monte Carlo model. And the Lagrangian is used to find the equations of motion for this model. The models are compared.

Contents

I Introduction

I.1 Motivation

I.2 B-Splines

B-Splines are a type of parametric curve. In the following sections they are defined and equations for their paths and the derivative thereof is defined.

I.2.a Bezier Curves

A Bezier curve of order 1 is the linear interpolation of starting point $\vec{r}_0$ and an end point $\vec{r}_1$.

$$\vec{r}(s) = (1-s)\vec{r}_0 + s\vec{r}_1 \quad s \in [0,1] \tag{1}$$

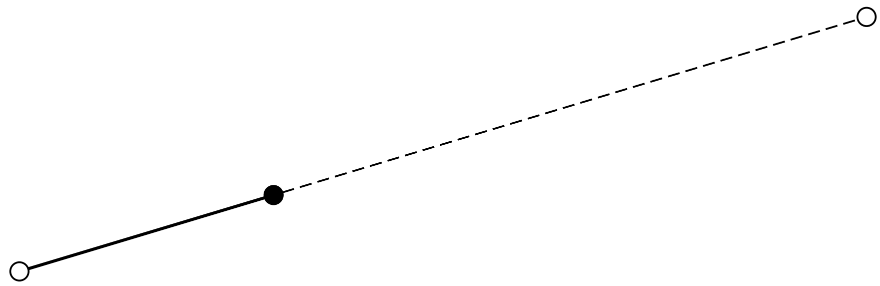


Figure 1: A Bezier curve of order 1 showing the interpolation of two points at $s = 0.3$

A Bezier curve of order 2 is the linear interpolation of two Bezier curves of order 1 with one ending at the start of the other bezier spline:

$$\begin{aligned} \vec{r}(s) &= (1-s)[(1-s)\vec{r}_0 + s\vec{r}_1] + s[(1-s)\vec{r}_1 + s\vec{r}_2] & s \in [0,1] \\ \vec{r}(s) &= (1-s)^2\vec{r}_0 + 2s(1-s)\vec{r}_1 + s^2\vec{r}_2 & s \in [0,1] \end{aligned} \tag{2}$$

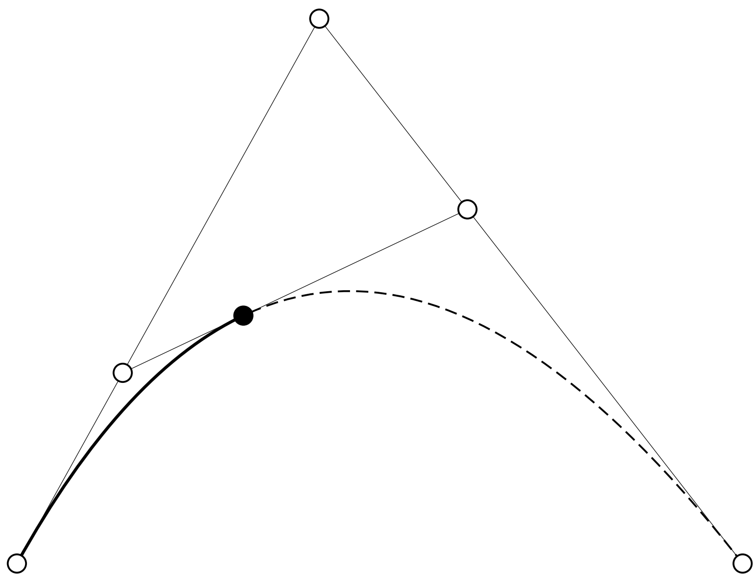


Figure 2: A Bezier curve of order 2 showing the construction of the point at $s = 0.35$

A Bezier curve of order 3 has four control points and is the interpolation of two order 2 Bezier curves:

$$\vec{r}(s) = (1-s)^3\vec{r}_0 + 3s(1-s)^2\vec{r}_1 + 3s^2(1-s)\vec{r}_2 + s^3\vec{r}_3 \quad s \in [0,1] \tag{3}$$

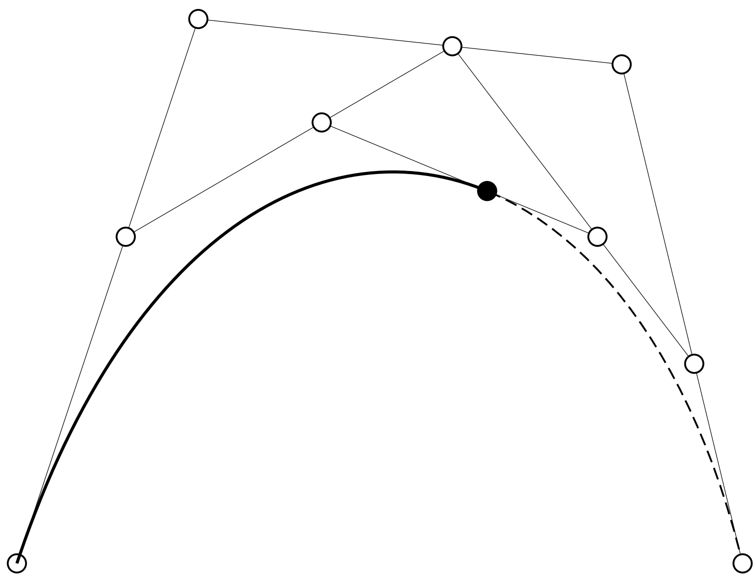


Figure 3: A Bezier curve of order 3 showing the contruction of the point at $s = 0.6$

In general a Bezier curve of order n is controled by $n+1$ points and the degree of the polynomial is n. The path is described by:

$$\vec{r}(s) = \sum_{i=0}^{n} \binom{n}{i} (1-s)^{n-i} s^i \vec{r}_i \tag{4}$$

The polynomials in Equation 4 are know as the Bernstein polynomials b_i^n where n is the degree of the polynomial.

$$b_i^n = \binom{n}{i} (1-s)^{n-i} s^i \tag{5}$$

Using the Bernstein polynomial for the path a Bezier curve of degree n can be written more compactly.

$$\vec{r}(s) = \sum_{i=0}^{n} \vec{r}_i b_i^n \tag{6}$$

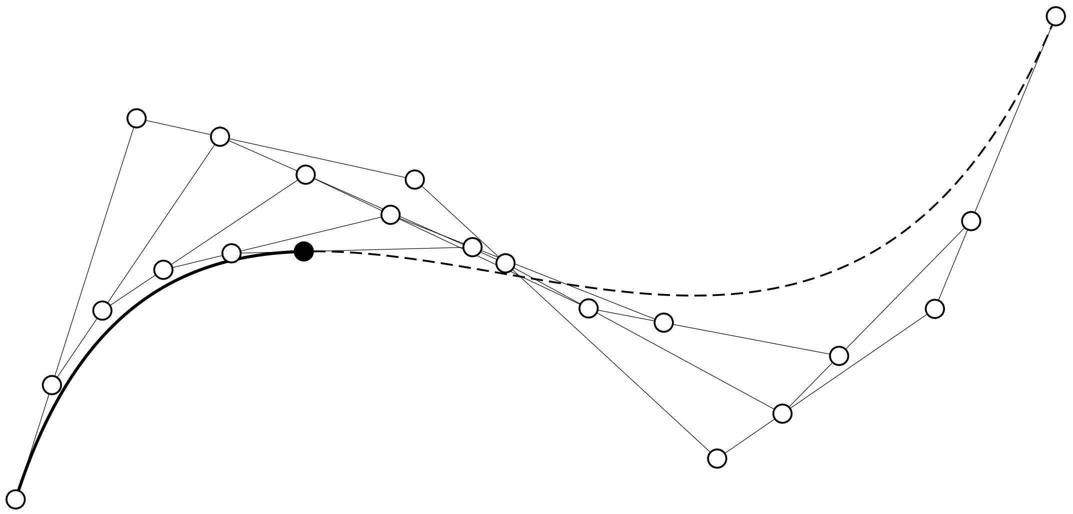


Figure 4: A Bezier curve of order 2 showing the construction of the point at $s = 0.35$

The derivative with respect to s

$$\vec{r}'(s) = \sum_{i=0}^{n-1} -(n-i)\binom{n}{i}(1-s)^{n-i-1}s^i\vec{r}_i + \sum_{i=1}^{n} i\binom{n}{i}(1-s)^{n-i}s^{i-1}\vec{r}_i \tag{7}$$

Note that the first sum only goes to $n-1$ because the n gives a exponent of 0 for $(1-s)$ which makes the first term of the product rule 0. The second sum start from 1 for a similar reason.

Swaping the sums and reindexing we achieve:

$$\begin{aligned}\vec{r}'(s) &= \sum_{i=0}^{n-1}(i+1)\binom{n}{i+1}(1-s)^{n-i-1}s^i\vec{r}_{i+1} - \sum_{i=0}^{n-1}(n-i)\binom{n}{i}(1-s)^{n-i-1}s^i\vec{r}_i \\ &= \sum_{i=0}^{n-1}\left((i+1)\binom{n}{i+1}\vec{r}_{i+1} - (n-i)\binom{n}{i}\vec{r}_i\right)(1-s)^{n-i-1}s^i\end{aligned} \tag{8}$$

Using the recurzion rules for the binomial coefficient:

$$\begin{aligned}\vec{r}'(s) &= \sum_{i=0}^{n-1}\left(n\binom{n-1}{i}\vec{r}_{i+1} - n\binom{n-1}{i}\vec{r}_i\right)(1-s)^{n-i-1}s^i \\ &= n\sum_{i=0}^{n-1}(\vec{r}_{i+1}-\vec{r}_i)\binom{n-1}{i}(1-s)^{(n-1)-i}s^i\end{aligned} \tag{9}$$

Thus the derivative of a Bezier curve of order n is itself a Bezier curve of order $n-1$

Defining $\Delta\vec{r}_i = \vec{r}_{i+1} - \vec{r}_i$ the forward difference operator allows us to write the derivative as follows:

$$\vec{r}'(s) = n\sum_{i=0}^{n-1}\Delta\vec{r}_i b_i^{n-1} \tag{10}$$

Furthermore now the mth derivative is:

$$\vec{r}^{(m)}(s) = \frac{n!}{(n-m)!} \sum_{i=0}^{n-m} \Delta^m \vec{r}_i b_i^{n-m} \tag{11}$$

I.2.b Splines of Bezier curves

Using Equation 6 we can define a Bezier curve with arbitrarily large degrees of freedom. However, Bezier curves of a high degree are hard to work with. The influence of a control becomes less local the higher the degree and the calculation of their path becomes more computationally intesive. This motivates the definition of splines of Bezier curves. As their name suggest the paths splines of Bezier curves are defined in terms of segments which themselves are Bezier curves. The most common type of splines are splines of cubic Bezier curves. They are used in many computer graphics applications. Cubic bezier curves are useful, because their control points can be conseptually understood as controling start and end point and the direction at start and end.

With $\vec{r}[m]$ being the Bezier curve for the mth segment of a spline of n segments and using $\{s\} = s - m \quad s \in [m, m+1)$ the fractional part of s, the equation for the path of such a spline is:

$$\vec{r}(s) = \begin{cases} \vec{r}[0](\{s\}) & s \in [0,1) \\ \vec{r}[1](\{s\}) & s \in [1,2) \\ \dots & \\ \vec{r}[n-1](\{s\}) & s \in [n-1, n) \end{cases} \tag{12}$$

Here the control points are $\vec{r}[m]_i$ where $m \in \{0, 1, ..., n-1\}$ and $i \in \{0, 1, 2, 3\}$.

I.2.c Continuty of Splines

In almost every application continuous B-splines are used. Since each segment on a spline is in itself C_∞-continuous only the points at

$$s \in \{1, 2, ..., n-1\} \tag{13}$$

For $s \to 1$ a Bezier curve approaches its last control point and $s \to 0$ a Bezier curve approaches its start point. Thus the condition for a continuous B-spline is:

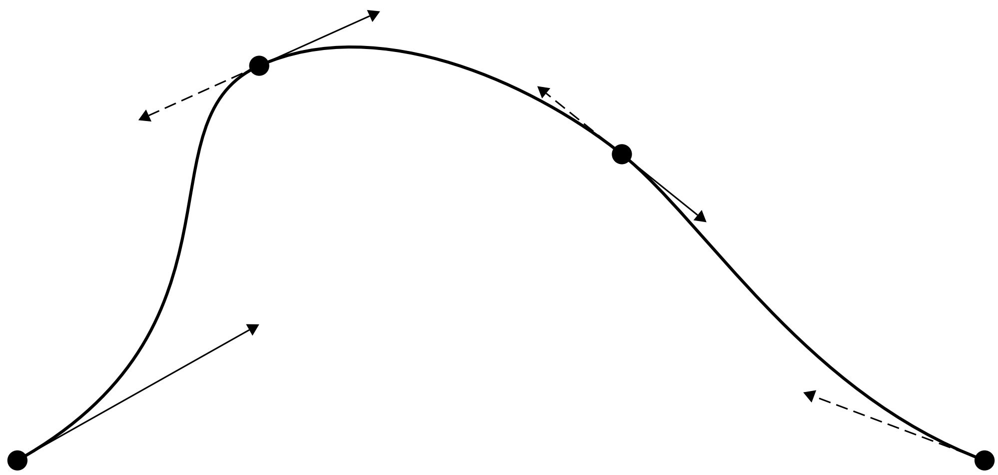
$$\vec{r}[m]_3 = \vec{r}[m+1]_0 \tag{14}$$

This trivialy means the end point of segment m must be the start point of segment $m+1$.

Simillarly for the first derivative:

$$\begin{gathered} \Delta\vec{r}[m]_2 = \Delta\vec{r}[m+1]_0 \\ \vec{r}[m]_3 - \vec{r}[m]_2 = \vec{r}[m+1]_1 - \vec{r}[m+1]_0 \end{gathered} \tag{15}$$

Renaming the common end/start point $\vec{r}[m]_3 = \vec{r}[m+1]_0 = \vec{p}[m+1]$ and the vector $\vec{r}[m+1]_1 - \vec{p}[m+1] = \vec{r}[m]_3 - \vec{r}[m]_2 = \vec{v}[m+1]$ we can describe all degrees of freedom of a C^2-continuous spline of cubic Besier curves using $\vec{p}[m]$ and $\vec{v}[m]$ with $m \in \{0, 1, ..., n\}$.



To calculate the controlpoints the following equations can be used.

$$
\begin{aligned}
\vec{r}[m]_0 &= \vec{p}[m] \\
\vec{r}[m]_1 &= \vec{p}[m] + \vec{v}[m] \\
\vec{r}[m]_2 &= \vec{p}[m+1] - \vec{v}[m+1] \\
\vec{r}[m]_3 &= \vec{p}[m+1]
\end{aligned}
\tag{16}
$$

Curves of this type are used for the implementation of this simulation and will from here on out be referred to simply as *splines*.

This choice was made for two reasons. Making sure the result of the simulation is visually apealing requires that the splines don't have any kinks. I.e. the splines should not change their direction suddenly. This does not require C^1-continuity as between two segments in a spline the derivative of the path would still be allowed to change in magnitude without changing its direction.

This however makes the splines harder to represent efficiently. For this reason the choice was made to enforce C^1-continuity.

I.2.d Representiation of the Splines

Sorting all points $\vec{p}[m]$ and vectors $\vec{v}[m]$ in to a list Q:

$$
Q = \begin{array}{c}
\begin{matrix} \vec{p}[0] & \vec{v}[0] & \vec{p}[1] & \vec{v}[1] & \vec{p}[2] & \vec{v}[2] & \dots & \vec{p}[n] & \vec{v}[n] \end{matrix} \\
\left[\begin{pmatrix}\cdot\\\cdot\end{pmatrix} \begin{pmatrix}\cdot\\\cdot\end{pmatrix} \begin{pmatrix}\cdot\\\cdot\end{pmatrix} \begin{pmatrix}\cdot\\\cdot\end{pmatrix} \begin{pmatrix}\cdot\\\cdot\end{pmatrix} \begin{pmatrix}\cdot\\\cdot\end{pmatrix} \dots \begin{pmatrix}\cdot\\\cdot\end{pmatrix} \begin{pmatrix}\cdot\\\cdot\end{pmatrix} \right]
\end{array}
\tag{17}
$$

We can define a special indexing operation to get the coordiantes controlling the shape of segment $m \in \{0, 1, 2, ..., n-1\}$ as a 2×4 matrix:

$$
Q[m] = \begin{array}{c}
\begin{matrix} \vec{p}[m] & \vec{v}[m] & \vec{p}[m+1] & \vec{v}[m+1] \end{matrix} \\
\left[\begin{pmatrix}\cdot\\\cdot\end{pmatrix} \quad \begin{pmatrix}\cdot\\\cdot\end{pmatrix} \quad \begin{pmatrix}\cdot\\\cdot\end{pmatrix} \qquad \begin{pmatrix}\cdot\\\cdot\end{pmatrix} \right]
\end{array}
\tag{18}
$$

Note:

$$Q[m]_{i,j+2} = Q[m+1]_{i,j} \tag{19}$$

Using $s - m = \{s\} \quad s \in [m, m+1)$ – the fractional part of s – and Einstein sum notation, we can write the path of the spline and its derivatives in the following way:

$$\begin{aligned}
r(s)_i &= \ \ Q[m]_{ij} A_{jk} b^3_k(\{s\}) \quad s \in [m, m+1) \\
r'(s)_i &= 3Q[m]_{ij} B_{jk} b^2_k(\{s\}) \quad s \in [m, m+1) \\
r''(s)_i &= 6\,Q[m]_{ij} C_{jk} b^1_k(\{s\}) \quad s \in [m, m+1)
\end{aligned} \tag{20}$$

with

$$A = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix} \tag{21}$$

$$B = \begin{pmatrix} 0 & -1 & 0 \\ 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix} \tag{22}$$

$$C = \begin{pmatrix} -1 & 1 \\ -2 & 1 \\ 1 & -1 \\ -1 & 2 \end{pmatrix} \tag{23}$$

and

$$\vec{b}^3(s) = \begin{pmatrix} (1-s)^3 \\ 3(1-s)^2 s \\ 3(1-s)s^2 \\ s^3 \end{pmatrix} \tag{24}$$

$$\vec{b}^2(s) = \begin{pmatrix} (1-s)^2 \\ 2(1-s)s \\ s^2 \end{pmatrix} \tag{25}$$

$$\vec{b}^1(s) = \begin{pmatrix} 1-s \\ s \end{pmatrix} \tag{26}$$

II Model

Each energy term exist to promote a certain goal for the fitting of the splines. In this section all energy terms are derived and the reasoning behind them is explained.

II.1 Intrinsic Terms

The intrinsic terms are what control the local shape of the spline.

II.1.a Strain Energy

The strain energy promotes the splines to keep their length. As the name suggests it is inspired by the strain energy in elastic materials.

$$E_s = \boldsymbol{S}\frac{\left(L - L_0\right)^2}{L_0^2}L \tag{27}$$

The length is definded by the path integral:

$$L = \int_\gamma dl = \int_0^n \|\vec{r}'\| ds \tag{28}$$

For a consistant look the strain energy is calculated per segment. The control points stay equally spaced through the spline.

$$E_s = \boldsymbol{S}\sum_m \frac{\left(L[m] - L[m]_0\right)^2}{L[m]_0^2}L[m] \tag{29}$$

with:

$$L[m] = \int_m^{m+1} \|\vec{r}'\| ds \tag{30}$$

II.1.b Bending Energy

The bending energy promotes splines which are straight.

$$\begin{aligned} E_b &= \boldsymbol{B}\int_\gamma k^2 dl \\ &= \boldsymbol{B}\int_0^n \frac{\left(\vec{r}_0'\vec{r}_1'' - \vec{r}_0''\vec{r}_1'\right)^2}{\|\vec{r}'\|^5} ds \end{aligned} \tag{31}$$

II.2 Environment Terms

The environment terms are what are used to control the resulting image macroscopically. The dertermination of the potential Φ and vector field $\vec{v}$ are treated seperatly in Section VI

II.2.a Potential Energy

The potential energy promotes the correct density of the splines.

$$\begin{aligned} E_\Phi &= \boldsymbol{Q}\int_\gamma \Phi(\vec{r}) dl \\ &= \boldsymbol{Q}\int_0^n \Phi(\vec{r}(t))\|\vec{r}'\| ds \end{aligned} \tag{32}$$

II.2.b Field Energy

The field energy promotes the alignement of the splines to a vector field.

$$\begin{aligned} E_{\vec{v}} &= \boldsymbol{P} \int_\gamma \frac{\vec{r}' \cdot \vec{v}(\vec{r})}{\|\vec{r}'\|} dl \\ &= \boldsymbol{P} \int_0^n \vec{r}' \cdot \vec{v}(\vec{r}) ds \end{aligned} \tag{33}$$

II.2.c Boundary Energy

The boundary energy exist so the splines stay within the boundaries during the simulation. Let $d(\vec{r})$ be the signed distance function to the boundary, defined such that $d(\vec{r}) < 0$ if $\vec{r}$ is with in the simulation boundaries.

Let $f(x) = \begin{cases} \infty \text{ if } x > 0 \\ \frac{1}{x^2} \text{ else} \end{cases}$

$$\begin{aligned} E_r &= \boldsymbol{R} \int_\gamma f(d(\vec{r})) dl \\ &= \boldsymbol{R} \int_0^n f(d(\vec{r})) \|\vec{r}'\| ds \end{aligned} \tag{34}$$

II.3 Pair Interaction

The pair interaction energy creates a repulsive force between the splines. In contrast to all other energies this energy depends on two splines.

$$\begin{aligned} E_g &= \boldsymbol{C} \int_{\gamma_0} \int_{\gamma_1} \frac{1}{\|\vec{r}_0 - \vec{r}_1\|} dl_1 dl_0 \\ &= \boldsymbol{C} \int_0^{n_0} \int_0^{n_1} \frac{\|\vec{r}_0'\| \|\vec{r}_1'\|}{\|\vec{r}_0 - \vec{r}_1\|} ds_1 ds_0 \end{aligned} \tag{35}$$

II.4 Kinetic Energy

The kinetic energy is only used in the Lagrangian model.

$$E_{\text{kin}} = \int_\gamma \rho \dot{\vec{r}}^2 dl \tag{36}$$

$$\begin{aligned} E_{\text{kin}} &= \sum_{m=0}^{n-1} \int_m^{m+1} \rho \dot{\vec{r}}^2 \|\vec{r}'\| ds \\ &= \boldsymbol{M} \sum_{m=0}^{n-1} \frac{1}{L[m]} \int_m^{m+1} \dot{\vec{r}}^2 \|\vec{r}'\| ds \end{aligned} \tag{37}$$

II.5 Summary

In Table 1 the formulas for the energies are summerized some constants are renamed and the parameters are given.

Energy	Formula
strain energy	$$E_s = \boldsymbol{S}\frac{(L - L_0)^2}{L_0^2}L \tag{38}$$
bending energy	$$E_b = \boldsymbol{B}\int_0^n \frac{(\vec{r}_0'\vec{r}_1'' - \vec{r}_0''\vec{r}_1')^2}{\|\vec{r}'\|^5}ds \tag{39}$$
potential energy	$$E_\Phi = \boldsymbol{Q}\int_0^n \Phi(\vec{r}(s))\|\vec{r}'\|ds \tag{40}$$
field energy	$$E_{\vec{v}} = \boldsymbol{P}\int_0^n \vec{r}' \cdot \vec{v}(\vec{r})ds \tag{41}$$
boundary energy	$$E_r = \boldsymbol{R}\int_0^n f(s(\vec{r}))\|\vec{r}'\|ds \tag{42}$$
pair interaction energy	$$E_g = \boldsymbol{C}\int_0^{n_0}\int_0^{n_1} \frac{\|\vec{r}_0'\|\|\vec{r}_1'\|}{\|\vec{r}_0 - \vec{r}_1\|}ds_1 ds_0 \tag{43}$$

Table 1: All energy terms

III Dimensional Analysis

This Model has a huge amount of paramaters to control, which need to be fine tuned.

Parameter	Dimension
bold(S)	$\frac{E}{L}$
bold(B)	LE
bold(Q)	

IV Monte Carlo Model

IV.1 Overview

IV.2 Variations of the Polymers

IV.3 Updating the Transition Parameters

IV.4 Generation of the initial state

V Lagrangian Model

For the Lagranian view two new matrices $\dot{Q}$ and $\ddot{Q}$ are defined, refering to the first and second derivative of the coordinates with respect to time.

The generalized cooridnates are $\vec{p}_{ij}$ and $\vec{v}_{ij}$ with $1 \leq i \leq N$ and $0 \leq j \leq n_i$ where N is the total number of particles and n_i is the number of segments of particle i.

$$E_{\text{kin}} = \int_0^L \frac{1}{2} \vec{v}^2 \rho dl = \frac{1}{2} \rho \int_0^n \dot{\vec{r}}^2 \|\vec{r}'\| ds \tag{44}$$

Setting V to E_{tot} and T to the sum of kinetic energies $\sum_i E_{\text{kin},i}$

Pluging $L = T + V$ into Lagrange's equation and assuming no external forces we get:

For any q in our coordinates:

$$\frac{d}{dt}\left(\frac{\partial}{\partial \dot{q}} T\right) - \frac{\partial}{\partial q} T + \frac{\partial}{\partial q} V = 0 \tag{45}$$

Consider each term seprately.

The Lagranian equation results in a system of equations, one linear equation for every coordinate. Observe that the equation that most coordiantes can be referred to in two ways:

$$Q[m]_{i,j+2} = Q[m+1]_{i,j} \tag{46}$$

To refer to each coordinate once m has to be in $\{0, 1, 2, \ldots n\}$ and j from 1 to 2 for a polymer with n segments.

$$\frac{\partial r_{\tilde{i}}(s)}{\partial Q[m]_{ij}} = \frac{\partial Q[\tilde{m}]_{\tilde{i}\tilde{j}}}{\partial Q[m]_{ij}} A_{\tilde{j}k} b_k^3(\{s\}) \quad s \in [\tilde{m}, \tilde{m}+1) \tag{47}$$

$$\frac{\partial Q[\tilde{m}]_{\tilde{i}\tilde{j}}}{\partial Q[m]_{ij}} = \delta_{i\tilde{i}}\left(\delta_{j\tilde{j}}\delta_{m\tilde{m}} + \delta_{m,\tilde{m}-1}\delta_{j,\tilde{j}+2}\right) \tag{48}$$

$$\frac{\partial r_{\tilde{i}}(s)}{\partial Q[m]_{ij}} = \begin{cases} \delta_{i\tilde{i}} A_{j+2,k} b_k^3 & s \in [m-1, m) \\ \delta_{i\tilde{i}} A_{jk} b_k^3 & s \in [m, m+1) \\ 0 & \text{else} \end{cases} \tag{49}$$

Similarly for the derivatives with respect to s

$$\frac{\partial r'_{\tilde{i}}(s)}{\partial Q[m]_{ij}} = \begin{cases} 3\delta_{i\tilde{i}} B_{j+2,k} b_k^2 & s \in [m-1, m) \\ 3\delta_{i\tilde{i}} B_{jk} b_k^2 & s \in [m, m+1) \\ 0 & \text{else} \end{cases} \tag{50}$$

$$\frac{\partial r''_{\tilde{i}}(s)}{\partial Q[m]_{ij}} = \begin{cases} 6\delta_{i\tilde{i}} C_{j+2,k} b_k^1 & s \in [m-1, m) \\ 6\delta_{i\tilde{i}} C_{jk} b_k^1 & s \in [m, m+1) \\ 0 & \text{else} \end{cases} \tag{51}$$

$$\frac{\partial}{\partial Q[m]_{ij}}\|\vec{r}'\| = \cancel{2}\left(\frac{\partial}{\partial Q[m]_{ij}}\vec{r}'\right)\cdot\vec{r}'\frac{\cancel{1}}{\cancel{2}}\|\vec{r}'\|^{-1}$$

$$= \begin{cases} 3\delta_{i\tilde{\imath}}B_{j+2,k}b_k^2 r'_{\tilde{\imath}}\|\vec{r}'\|^{-1} & s\in[m-1,m] \\ 3\delta_{i\tilde{\imath}}B_{jk}b_k^2 r'_{\tilde{\imath}}\|\vec{r}'\|^{-1} & s\in[m,m+1] \\ 0 & \text{else} \end{cases} \tag{52}$$

$$= \begin{cases} 3B_{j+2,k}b_k^2 r'_i\|\vec{r}'\|^{-1} & s\in[m-1,m] \\ 3B_{jk}b_k^2 r'_i\|\vec{r}'\|^{-1} & s\in[m,m+1] \\ 0 & \text{else} \end{cases}$$

V.1 Term 1

For any coordinate q used to describe polymer p the derivate is zero for all terms in T except for the contribution of the polymer p itself.

The first term becomes:

$$\begin{aligned} \frac{d}{dt}\left(\frac{\partial}{\partial\dot{q}}T\right) &= \frac{d}{dt}\left(\frac{\partial}{\partial\dot{q}}\frac{1}{2}\rho\int_0^n \dot{\vec{r}}^2\|\vec{r}'\|ds\right) \\ &= \rho\frac{d}{dt}\int_0^n\left(\dot{\vec{r}}\cdot\frac{\partial\dot{\vec{r}}}{\partial\dot{q}}\right)\|\vec{r}'\|ds \\ &= \rho\int_0^n\left(\ddot{\vec{r}}\cdot\frac{\partial\dot{\vec{r}}}{\partial\dot{q}}\right)\|\vec{r}'\| + \left(\dot{\vec{r}}\cdot\frac{\partial\dot{\vec{r}}}{\partial\dot{q}}\right)\left\|\dot{\vec{r}}'\right\|ds \end{aligned} \tag{53}$$

Note that $\frac{\partial\dot{\vec{r}}}{\partial\dot{q}}$ is independent of t since $\dot{\vec{r}}$ depends linearly on $\dot{q}$. Thus, its derivative with respect to t is 0.

$$\begin{aligned} \frac{d}{dt}\left(\frac{\partial T}{\partial\dot{Q}[m]_{ij}}\right) &= \rho\int_0^n\frac{\partial\dot{r}_{\tilde{\imath}}}{\partial\dot{Q}[m]_{ij}}\left(\ddot{r}_{\tilde{\imath}}\|\vec{r}'\| + \dot{r}_{\tilde{\imath}}\left\|\dot{\vec{r}}'\right\|\right)ds \\ &= \rho\int_{m-1}^m \delta_{i\tilde{\imath}}A_{j+2,k}b_k^3(\{s\})\left(\ddot{r}_{\tilde{\imath}}\|\vec{r}'\| + \dot{r}_{\tilde{\imath}}\left\|\dot{\vec{r}}'\right\|\right)ds \\ &\quad+\rho\int_m^{m+1}\delta_{i\tilde{\imath}}A_{jk}b_k^3(\{s\})\left(\ddot{r}_{\tilde{\imath}}\|\vec{r}'\| + \dot{r}_{\tilde{\imath}}\left\|\dot{\vec{r}}'\right\|\right)ds \\ &= \rho\int_{m-1}^m A_{j+2,k}b_k^3(\{s\})\left(\ddot{r}_i\|\vec{r}'\| + \dot{r}_i\left\|\dot{\vec{r}}'\right\|\right)ds \\ &\quad+\rho\int_m^{m+1}A_{jk}b_k^3(\{s\})\left(\ddot{r}_i\|\vec{r}'\| + \dot{r}_i\left\|\dot{\vec{r}}'\right\|\right)ds \end{aligned} \tag{54}$$

the first integral

$$
\begin{aligned}
&\rho \int_{m-1}^{m} A_{jk} b_k^3(\{s\}) \left(\ddot{r}_i \|\vec{r}'\| + \dot{r}_i \left\|\dot{\vec{r}}'\right\|\right) ds \\
&= \rho \int_{m-1}^{m} A_{jk} b_k^3(\{s\}) \ddot{Q}[p, m]_{il} A_{ln} b_n^3(\{s\}) \|\vec{r}'\| ds \\
&+ \rho \int_{m-1}^{m} A_{jk} b_k^3(\{s\}) \dot{Q}[m]_{il} A_{ln} b_n^3(\{s\}) \left\|\dot{\vec{r}}'\right\| ds \\
&= \ddot{Q}[m]_{il} \rho \int_{m-1}^{m} A_{jk} b_k^3(\{s\}) A_{ln} b_n^3(\{s\}) \|\vec{r}'\| ds \\
&+ \dot{Q}[m]_{il} \rho \int_{m-1}^{m} A_{jk} b_k^3(\{s\}) A_{ln} b_n^3(\{s\}) \left\|\dot{\vec{r}}'\right\| ds
\end{aligned}
\tag{55}
$$

the second integral

$$
\begin{aligned}
&\rho \int_{m}^{m+1} A_{j-2,k} b_k^3(\{s\}) \left(\ddot{r}_i \|\vec{r}'\| + \dot{r}_i \left\|\dot{\vec{r}}'\right\|\right) ds \\
&= \rho \int_{m}^{m+1} A_{j-2,k} b_k^3(\{s\}) \ddot{Q}[m+1]_{il} A_{ln} b_n^3(\{s\}) \|\vec{r}'\| ds \\
&+ \rho \int_{m}^{m+1} A_{j-2,k} b_k^3(\{s\}) \dot{Q}[m+1]_{il} A_{ln} b_n^3(\{s\}) \left\|\dot{\vec{r}}'\right\| ds
\end{aligned}
\tag{56}
$$

V.2 Term 2

The second term becomes:

$$
\begin{aligned}
\frac{\partial}{\partial q} T &= \frac{\partial}{\partial q} \frac{1}{2} \rho \int_0^n \dot{\vec{r}}^2 \|\vec{r}'\| ds \\
&= \frac{1}{2} \rho \int_0^n \dot{\vec{r}}^2 \frac{\cancel{1}}{\cancel{2}} \frac{1}{\|\vec{r}'\|} \left(\cancel{2} \vec{r}' \cdot \frac{\partial}{\partial q} \vec{r}'\right) ds \\
&= \frac{1}{2} \rho \int_0^n \dot{\vec{r}}^2 \left(\vec{r}' \cdot \frac{\partial}{\partial q} \vec{r}'\right) \|\vec{r}'\|^{-1} ds
\end{aligned}
\tag{57}
$$

This integral again splits up into two parts

$$
\begin{aligned}
&\frac{1}{2} \rho \int_{m-1}^{m} \dot{\vec{r}}^2 3 \delta_{i\tilde{i}} B_{jk} b_k^2(\{s\}) 3 Q[m]_{\tilde{i}l} B_{ln} b_n^2(\{s\}) \|\vec{r}'\|^{-1} ds \\
&+ \frac{1}{2} \rho \int_{m}^{m+1} \dot{\vec{r}}^2 3 \delta_{i\tilde{i}} B_{j-2,k} b_k^2(\{s\}) 3 Q[m+1]_{\tilde{i}l} B_{ln} b_n^2(\{s\}) \|\vec{r}'\|^{-1} ds \\
&= \frac{9}{2} \rho Q[m]_{il} \int_{m-1}^{m} \dot{\vec{r}}^2 B_{jk} b_k^2(\{s\}) B_{ln} b_n^2(\{s\}) \|\vec{r}'\|^{-1} ds \\
&+ \frac{9}{2} \rho Q[m+1]_{il} \int_{m}^{m+1} \dot{\vec{r}}^2 B_{j-2,k} b_k^2(\{s\}) B_{ln} b_n^2(\{s\}) \|\vec{r}'\|^{-1} ds
\end{aligned}
\tag{58}
$$

V.3 Term 3

The third term can be split into the contributions of the different potentials in the system.

$$
\frac{\partial}{\partial q} V \tag{59}
$$

V.3.a Strain Energy

$$
\begin{aligned}
\frac{\partial}{\partial q} E_s &= \boldsymbol{S} \frac{\partial}{\partial q} \left(\frac{(L - L_0)^2}{L_0^2} L \right) \\
&= \boldsymbol{S} \left(2 \frac{L - L_0}{L_0^2} \frac{\partial L}{\partial q} + \frac{(L - L_0)^2}{L_0^2} \frac{\partial L}{\partial q} \right) \\
&= \boldsymbol{S} (2 + L - L_0) \frac{L - L_0}{L_0^2} \frac{\partial L}{\partial q}
\end{aligned}
\tag{60}
$$

$$
\begin{aligned}
\frac{\partial L}{\partial Q[m]_{ij}} &= \frac{\partial}{\partial Q[m]_{ij}} \int_0^n \|\vec{r}'\| ds \\
&= \int_0^n \cancel{2} \left(\frac{\partial}{\partial Q[m]_{ij}} \vec{r}' \right) \cdot \vec{r}' \frac{\cancel{1}}{\cancel{2}} \|\vec{r}'\|^{-1} ds \\
&= \int_{m-1}^{m} 3 \delta_{i\tilde{i}} B_{j+2,k} b_k^2 r_{\tilde{i}}' \|\vec{r}'\|^{-1} ds \\
&\quad + \int_{m}^{m+1} 3 \delta_{i\tilde{i}} B_{jk} b_k^2 r_{\tilde{i}}' \|\vec{r}'\|^{-1} ds \\
&= \int_{m-1}^{m} 3 B_{j+2,k} b_k^2 r_i' \|\vec{r}'\|^{-1} ds \\
&\quad + \int_{m}^{m+1} 3 B_{jk} b_k^2 r_i' \|\vec{r}'\|^{-1} ds
\end{aligned}
\tag{61}
$$

V.3.b Bending Energy

$$
E_b = \boldsymbol{B} \int_0^n \frac{(\vec{r}_0' \vec{r}_1'' - \vec{r}_0'' \vec{r}_1')^2}{\|\vec{r}'\|^5} ds
\tag{62}
$$

V.3.c Potential Energy

$$
\begin{aligned}
\frac{\partial}{\partial q}E_\Phi &= \boldsymbol{Q}\frac{\partial}{\partial q}\int_0^n \Phi(\vec{r}(s))\|\vec{r}'\|ds \\
&= \boldsymbol{Q}\int_0^n \Phi'(\vec{r}(s))\frac{\partial}{\partial q}\vec{r}\|\vec{r}'\|ds + \boldsymbol{Q}\int_0^n \Phi(\vec{r}(s))\left(\frac{\partial}{\partial q}\vec{r}'\right)\cdot\vec{r}'\|\vec{r}'\|^{-1}ds
\end{aligned}
\tag{63}
$$

The first integral:

$$
\begin{aligned}
\int_0^n \Phi'(\vec{r}(s))\frac{\partial}{\partial q}\vec{r}\|\vec{r}'\|ds = &\int_{m-1}^{m} \Phi'(\vec{r}(s))\delta_{i\tilde{\imath}}A_{j+2,k}b_k^3\|\vec{r}'\|ds \\
&+ \int_{m}^{m+1} \Phi'(\vec{r}(s))\delta_{i\tilde{\imath}}A_{jk}b_k^3\|\vec{r}'\|ds
\end{aligned}
\tag{64}
$$

The second integral:

$$
\begin{aligned}
\int_0^n \Phi'(\vec{r}(s))\frac{\partial}{\partial q}\vec{r}\|\vec{r}'\|ds = &\int_{m-1}^{m} \Phi(\vec{r}(s))3B_{j+2,k}b_k^2r_i'\|\vec{r}'\|^{-1}ds \\
&+ \int_{m}^{m+1} \Phi(\vec{r}(s))3B_{jk}b_k^2r_i'\|\vec{r}'\|^{-1}ds
\end{aligned}
\tag{65}
$$

$$
\frac{\partial}{\partial Q[m]_{ij}}E_\Phi = \boldsymbol{Q}
\tag{66}
$$

$$
\begin{aligned}
\frac{\partial r_{\tilde{\imath}}(s)}{\partial Q[m]_{ij}} &= \begin{cases} \delta_{i\tilde{\imath}}A_{j+2,k}b_k^3 & s\in[m-1,m) \\ \delta_{i\tilde{\imath}}A_{jk}b_k^3 & s\in[m,m+1) \\ 0 & \text{else} \end{cases} \\
\frac{\partial r_{\tilde{\imath}}'(s)}{\partial Q[m]_{ij}} &= \begin{cases} 3\delta_{i\tilde{\imath}}B_{j+2,k}b_k^2 & s\in[m-1,m) \\ 3\delta_{i\tilde{\imath}}B_{jk}b_k^2 & s\in[m,m+1) \\ 0 & \text{else} \end{cases} \\
\frac{\partial r_{\tilde{\imath}}''(s)}{\partial Q[m]_{ij}} &= \begin{cases} 6\delta_{i\tilde{\imath}}C_{j+2,k}b_k^1 & s\in[m-1,m) \\ 6\delta_{i\tilde{\imath}}C_{jk}b_k^1 & s\in[m,m+1) \\ 0 & \text{else} \end{cases}
\end{aligned}
\tag{67}
$$

V.3.d Field Energy

$$
E_{\vec{v}} = \boldsymbol{P}\int_0^n \vec{r}'\cdot\vec{v}(\vec{r})ds
\tag{68}
$$

V.3.e Pair Interaction Energy

$$
E_g = \boldsymbol{C}\int_0^{n_0}\int_0^{n_1}\frac{\|\vec{r}_0'\|\|\vec{r}_1'\|}{\|\vec{r}_0-\vec{r}_1\|}ds_1ds_0
\tag{69}
$$

V.3.f Boundary Energy

$$
E_r = \boldsymbol{R}\int_0^n f(s(\vec{r}))\|\vec{r}'\|ds
\tag{70}
$$

VI Image Processing

VII Comparison of the Model

VIII Conclusion

IX Appendix