A Monte Carlo Method for Image Decomposition into Collections of B-Splines

Max Krummenacher

B-Splines

Cubic Bezier Curves

A

C_2 Continuous B-Splines of Cubic Bezier Curves

$$\hat{t} = t - \lfloor t \rfloor$$

$$\vec{r}(t) = \left(1 - \hat{t}\right)^3 \vec{r}_{\lfloor t \rfloor} \left(1 - \hat{t}\right)...$$

$$\dot{\vec{r}}(t) =$$

$$\ddot{\vec{r}}(t) =$$

$$k = \frac{\dot{\vec{r}}_x \ddot{\vec{r}}_y - \ddot{\vec{r}}_x \dot{\vec{r}}_y}{\left\|\dot{\vec{r}}\right\|^3}$$

$$dl = \left\|\dot{\vec{r}}\right\| dt$$

Energies

Each energy term exist to promote a certain goal for the fitting of the splines. In this section all energy terms are derived and the reasoning behind them is explained.

Strain Energy

The strain energy promotes the splines to keep their length.

$$E_s = \int_0^L \frac{\sigma^2}{2Yw} dl$$

$$Y = \frac{\sigma}{\varepsilon} \Rightarrow \sigma = Y\varepsilon$$

$$\begin{aligned} E_s &= \int_0^L w\varepsilon^2 \frac{Y}{2} dl \\ &= \frac{1}{2} wY\varepsilon^2 \int_0^n \left\|\dot{\vec{r}}\right\| dt \\ &= \frac{1}{2} wY\varepsilon^2 L \\ &= \frac{1}{2} wY \frac{\left(L - L_0\right)^2}{L_0^2} L \end{aligned}$$

Bending Energy

The bending energy promotes splines which are straight.

$$E_b = \int_0^L \frac{M^2}{2YI} dl$$

$$YI = \frac{M}{k} \Rightarrow M = YIk$$

$$I = \frac{w^3}{12}$$

$$\begin{aligned} E_b &= \int_0^n \frac{YI}{2} k(t)^2 \left\| \dot{\vec{r}} \right\| dt \\ &= \frac{1}{24} Y w^3 \int_0^n \frac{\left(\dot{\vec{r}}_x \ddot{\vec{r}}_y - \ddot{\vec{r}}_x \dot{\vec{r}}_y \right)^2}{\left\| \dot{\vec{r}} \right\|^5} dt \end{aligned}$$

Potential Energy

The potential energy promotes the correct density of the splines.

$$\begin{aligned} E_\Phi &= \int_0^L q\Phi(\vec{r}) dl \\ &= q \int_0^n \Phi(\vec{r}(t)) \left\| \dot{\vec{r}} \right\| dt \end{aligned}$$

Field Energy

The field energy promotes the alignement of the splines to a vector field.

$$\begin{aligned} E_{\vec{v}} &= \int_0^L p \frac{\dot{\vec{r}} \cdot \vec{v}(\vec{r})}{\left\| \dot{\vec{r}} \right\|} dl \\ &= p \int_0^n \dot{\vec{r}} \cdot \vec{v}(\vec{r}) dt \end{aligned}$$

Pair Interaction Energy

The pair interaction energy creates a repulsive force between the splines. In contrast to all other energies this energy depends on two splines γ_0 and γ_1.

$$\begin{aligned} E_g(\gamma_0, \gamma_1) &= \int_0^{L_0} \int_0^{L_1} \frac{\rho}{\|\vec{r}_0 - \vec{r}_1\|} dl_1 dl_0 \\ &= \rho \int_0^{n_0} \int_0^{n_1} \frac{\left\| \dot{\vec{r}}_0 \right\| \left\| \dot{\vec{r}}_1 \right\|}{\|\vec{r}_0 - \vec{r}_1\|} dt_1 dt_0 \end{aligned}$$

Boundary Energy

The border energy exist so the splines stay within the boundaries during the simulation. Let $s(\vec{r})$ be the signed distance function to the boundary. Let $f(x) = \begin{cases} \infty \text{ if } x>0 \\ -\frac{1}{x} \text{ else} \end{cases}$

$$
\begin{aligned}
E_r &= \int_0^L a f(s(\vec{r}))dl \\
&= a \int_0^n f(s(\vec{r}))\left\|\dot{\vec{r}}\right\|dt
\end{aligned}
$$

Total Energy

In Table 1 the formulas for the energies are summerized some constants are renamed and the parameters are given.

Energy	Formula	Parameter
strain energy	$E_{s,i} = S\frac{\left(L_i - L_{i,0}\right)^2}{L_{i,0}^2} L_i$	S
bending energy	$E_{b,i} = B\int_0^{n_i} \frac{\left(\dot{\vec{r}}_{i,x}\ddot{\vec{r}}_{i,y} - \ddot{\vec{r}}_{i,x}\dot{\vec{r}}_{i,y}\right)^2}{\left\|\dot{\vec{r}}_i\right\|^5} dt$	B
potential energy	$E_{\Phi,i} = Q\int_0^{n_i} \Phi(\vec{r}_i(t))\left\|\dot{\vec{r}}_i\right\|dt$	Q
field energy	$E_{\vec{v},i} = P\int_0^{n_i} \dot{\vec{r}}_i \cdot \vec{v}(\vec{r}_i)dt$	P
pair interaction energy	$E_{g,ij} = C\int_0^{n_i}\int_0^{n_j} \frac{\left\|\dot{\vec{r}}_i\right\|\left\|\dot{\vec{r}}_j\right\|}{\left\|\vec{r}_i - \vec{r}_j\right\|} dt_j dt_i$	C
boundary energy	$E_{r,i} = A\int_0^{n_i} f(s(\vec{r}_i))\left\|\dot{\vec{r}}_i\right\|dt$	A

Table 1: All energy terms

The total energy E_{tot} is simply the sum of all energies.

$$
E_{\text{tot}} = \sum_i \left(E_{s,i} + E_{b,i} + E_{\Phi,i} + E_{\vec{v},i} + E_{r,i}\right) + \sum_i \sum_{j,j>i} E_{g,ij}
$$