A Monte Carlo Method for Image Decomposition into Collections of B-Splines

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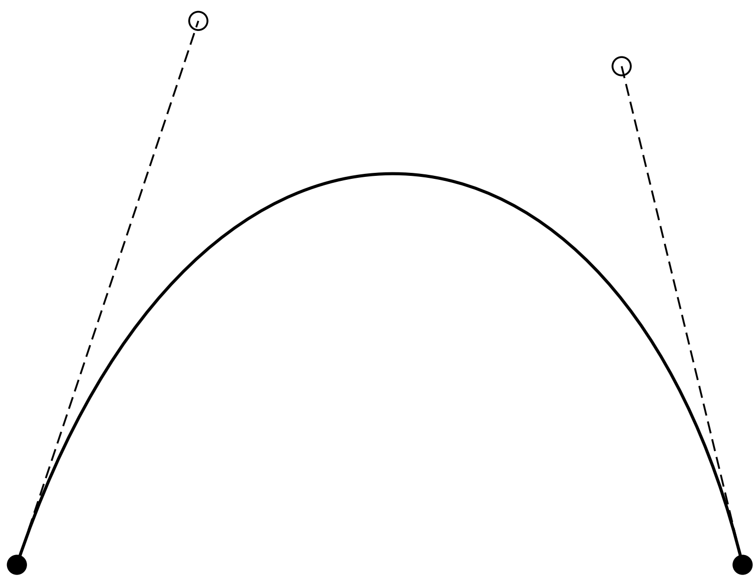
B-Splines

Cubic Bezier Curves



A

C_2 Continuous B-Splines of Cubic Bezier Curves

Thus a spline of n segments can be decribed in terms points $\vec{p}_k$ and vectors $\vec{v}_k$ with $0 \leq k \leq n$.

In the k-th segment the position $\vec{r}$ and its derivative with respect to s with $s \in [0, 1]$ are:

$$\vec{r} = (1-s)^3 \vec{p}_k + 3(1-s)^2 s(\vec{p}_k + \vec{v}_k) + 3(1-s)s^2(\vec{p}_{k+1} - \vec{v}_{k+1}) + s^3 \vec{p}_{k+1}$$

$$\frac{\partial}{\partial s}\vec{r} = 3(1-s)^2 \vec{v}_k + 6(1-s)s(\vec{p}_{k+1} - \vec{p}_k - \vec{v}_{k+1} - \vec{v}_k) + 3s^2 \vec{v}_{k+1}$$

$$\frac{\partial^2}{\partial s^2}\vec{r} = 6(1-s)(\vec{p}_{k+1} - \vec{p}_k - \vec{v}_{k+1} - 2\vec{v}_k) - 6s(\vec{p}_k - \vec{p}_{k+1} + \vec{v}_k + 2\vec{v}_{k+1})$$

By setting $k = \lfloor s \rfloor$ and $\hat{s} = \{s\}$ the fractional part of s this definition can be extended to whole range of s $[0, n]$

$$k = \frac{\dot{\vec{r}}_x \ddot{\vec{r}}_y - \ddot{\vec{r}}_x \dot{\vec{r}}_y}{\left\|\dot{\vec{r}}\right\|^3}$$

$$dl = \left\|\dot{\vec{r}}\right\| ds$$

Energies

Each energy term exist to promote a certain goal for the fitting of the splines. In this section all energy terms are derived and the reasoning behind them is explained.

Strain Energy

The strain energy promotes the splines to keep their length. As the name suggests it is inspired by the strain energy in elastic materials.

$$E_s = \boldsymbol{S} \frac{(L - L_0)^2}{L_0^2} L$$

Bending Energy

The bending energy promotes splines which are straight.

$$\begin{aligned} E_b &= \boldsymbol{B} \int_0^L k^2 dl \\ &= \boldsymbol{B} \int_0^n \frac{\left(\dot{\vec{r}}_x \ddot{\vec{r}}_y - \ddot{\vec{r}}_x \dot{\vec{r}}_y\right)^2}{\left\|\dot{\vec{r}}\right\|^5} ds \end{aligned}$$

Potential Energy

The potential energy promotes the correct density of the splines.

$$\begin{aligned} E_\Phi &= \boldsymbol{Q} \int_0^L \Phi(\vec{r}) dl \\ &= \boldsymbol{Q} \int_0^n \Phi(\vec{r}(t)) \left\|\dot{\vec{r}}\right\| ds \end{aligned}$$

Field Energy

The field energy promotes the alignement of the splines to a vector field.

$$\begin{aligned} E_{\vec{v}} &= \boldsymbol{P} \int_0^L -\frac{\left|\dot{\vec{r}} \cdot \vec{v}(\vec{r})\right|}{\left\|\dot{\vec{r}}\right\|} dl \\ &= -\boldsymbol{P} \int_0^n \left|\dot{\vec{r}} \cdot \vec{v}(\vec{r})\right| ds \end{aligned}$$

Pair Interaction Energy

The pair interaction energy creates a repulsive force between the splines. In contrast to all other energies this energy depends on two splines.

$$E_g = \boldsymbol{C} \int_0^{L_0} \int_0^{L_1} \frac{1}{\|\vec{r}_0 - \vec{r}_1\|} dl_1 dl_0$$
$$= \boldsymbol{C} \int_0^{n_0} \int_0^{n_1} \frac{\|\dot{\vec{r}}_0\| \|\dot{\vec{r}}_1\|}{\|\vec{r}_0 - \vec{r}_1\|} ds_1 ds_0$$

Boundary Energy

The boundary energy exist so the splines stay within the boundaries during the simulation. Let $d(\vec{r})$ be the signed distance function to the boundary, defined such that $d(\vec{r}) < 0$ if $\vec{r}$ is with in the simulation boundaries.

Let $f(x) = \begin{cases} \infty \text{ if } x > 0 \\ \frac{1}{x^2} \text{ else} \end{cases}$

$$E_r = \boldsymbol{R} \int_0^{L} f(d(\vec{r})) dl$$
$$= \boldsymbol{R} \int_0^{n} f(d(\vec{r})) \|\dot{\vec{r}}\| ds$$

Total Energy

In Table 1 the formulas for the energies are summerized some constants are renamed and the parameters are given.

Energy	Formula
strain energy	$E_{s,i} = \boldsymbol{S} \frac{(L_i - L_{i,0})^2}{L_{i,0}^2} L_i$
bending energy	$E_{b,i} = \boldsymbol{B} \int_0^{n_i} \frac{\left(\dot{\vec{r}}_{i,x} \ddot{\vec{r}}_{i,y} - \ddot{\vec{r}}_{i,x} \dot{\vec{r}}_{i,y}\right)^2}{\left\|\dot{\vec{r}}_i\right\|^5} dt$
potential energy	$E_{\Phi,i} = \boldsymbol{Q} \int_0^{n_i} \Phi(\vec{r}_i(t)) \left\|\dot{\vec{r}}_i\right\| dt$
field energy	$E_{\vec{v},i} = \boldsymbol{P} \int_0^{n_i} \dot{\vec{r}}_i \cdot \vec{v}(\vec{r}_i) dt$
pair interaction energy	$E_{g,ij} = \boldsymbol{C} \int_0^{n_i} \int_0^{n_j} \frac{\left\|\dot{\vec{r}}_i\right\| \left\|\dot{\vec{r}}_j\right\|}{\left\|\vec{r}_i - \vec{r}_j\right\|} dt_j dt_i$
boundary energy	$E_{r,i} = \boldsymbol{R} \int_0^{n_i} f(s(\vec{r}_i)) \left\|\dot{\vec{r}}_i\right\| dt$

Table 1: All energy terms

The total energy E_{tot} is simply the sum of all energies.

$$E_{\text{tot}} = \sum_i \left(E_{s,i} + E_{b,i} + E_{\Phi,i} + E_{\vec{v},i} + E_{r,i}\right) + \sum_i \sum_{j, j>i} E_{g,ij}$$

or

$$E_{\text{tot}} = \sum_i (E_{s,i} + E_{b,i} + E_{\Phi,i} + E_{\vec{v},i} + E_{r,i}) + \frac{1}{2}\sum_i \sum_{j,j\neq i} E_{g,ij}$$

Lagrangian View

The generalized cooridnates are $\vec{p}_{ij}$ and $\vec{v}_{ij}$ with $1 \leq i \leq N$ and $0 \leq j \leq n_i$ where N is the total number of particles and n_i is the number of segments of particle i.

$$E_{\text{kin}} = \int_0^L \frac{1}{2}\vec{v}^2 \rho dl = \frac{1}{2}\rho \int_0^n \left(\dot{\vec{r}}\right)^2 \left\| \frac{\partial \vec{r}}{\partial s} \right\| ds$$

Setting V to E_{tot} and T to the sum of kinetic energies $\sum_i E_{\text{kin},i}$

Pluging $L = T + V$ into Lagrange's equation and assuming no external forces we get:

For any q in our coordinates:

$$\frac{d}{dt}\left(\frac{\partial}{\partial \dot{q}} T\right) - \frac{\partial}{\partial q} T + \frac{\partial}{\partial q} V = 0$$

With fixed i and $q \in \left\{\vec{p}_{ij}\right\} \cup \left\{\vec{v}_{ij}\right\}$ T can be rewritten as $E_{\text{kin},i}$ since this is the only term in T which depends on any cooridate with index i.

The first term becomes:

$$\begin{aligned}
\frac{d}{dt}\left(\frac{\partial}{\partial \dot{q}} T\right) &= \frac{d}{dt}\left(\frac{\partial}{\partial \dot{q}} \frac{1}{2}\rho \int_0^{n_i} \left(\dot{\vec{r}_i}\right)^2 \left\| \frac{\partial \vec{r}_i}{\partial s} \right\| ds\right) \\
&= \rho \frac{d}{dt} \int_0^{n_i} \left(\dot{\vec{r}_i} \cdot \frac{\partial \dot{\vec{r}_i}}{\partial \dot{q}}\right) \left\| \frac{\partial \vec{r}_i}{\partial s} \right\| ds \\
&= \rho \int_0^{n_i} \left(\left(\ddot{\vec{r}_i} \cdot \frac{\partial \dot{\vec{r}_i}}{\partial \dot{q}}\right) + \left(\dot{\vec{r}_i} \cdot \frac{d}{dt}\frac{\partial \dot{\vec{r}_i}}{\partial \dot{q}}\right)\right) \left\| \frac{\partial \vec{r}_i}{\partial s} \right\| ds
\end{aligned}$$

Note that in out case $\frac{\partial \dot{\vec{r}_i}}{\partial \dot{q}}$ is independent of t since $\dot{\vec{r}}$ depends linearly on q. Thus, its derivative with respect to t is 0.

$$\frac{d}{dt}\left(\frac{\partial}{\partial \dot{q}} T\right) = \rho \int_0^{n_i} \ddot{\vec{r}_i} \cdot \frac{\partial \dot{\vec{r}_i}}{\partial \dot{q}} \left\| \frac{\partial \vec{r}_i}{\partial s} \right\| ds$$

Furthermore note that $\vec{r}_i$ only depends on the coordinates with subscript j for $s \in (j-1, j)$ and $s \in (j, j+1)$. For $j = 0$ and $j = n$ the first or the second interval respectively is not included. For $\vec{p}_{ij}$ we get

$$\rho \int_{j-1}^{j} \ddot{\vec{r}_i} \frac{\partial \dot{\vec{r}_i}}{\partial \dot{\vec{p}}_{ij}} \left\| \frac{\partial \vec{r}_i}{\partial s} \right\| ds = \rho \int_0^1 \ddot{\vec{r}}_{ij} (3s^2(1-s) + s^3) \left\| \frac{\partial \vec{r}_{ij}}{\partial s} \right\| ds$$

$$\rho \int_{j}^{j+1} \ddot{\vec{r}_i} \frac{\partial \dot{\vec{r}_i}}{\partial \dot{\vec{p}}_{ij}} \left\| \frac{\partial \vec{r}_i}{\partial s} \right\| ds = \rho \int_0^1 \ddot{\vec{r}}_{ij+1} ((1-s)^3 + 3s(1-s)^2) \left\| \frac{\partial \vec{r}_{ij+1}}{\partial s} \right\| ds$$

Similarly for $\vec{v}_{ij}$

$$\rho \int_{j-1}^{j} \ddot{\vec{r}}_i \frac{\partial \dot{\vec{r}}_i}{\partial \dot{\vec{v}}_{ij}} \left\| \frac{\partial \vec{r}_i}{\partial s} \right\| ds = \rho \int_0^1 \ddot{\vec{r}}_{ij}(-3s^2(1-s)) \left\| \frac{\partial \vec{r}_{ij}}{\partial s} \right\| ds$$

$$\rho \int_{j}^{j+1} \ddot{\vec{r}}_i \frac{\partial \dot{\vec{r}}_i}{\partial \dot{\vec{v}}_{ij}} \left\| \frac{\partial \vec{r}_i}{\partial s} \right\| ds = \rho \int_0^1 \ddot{\vec{r}}_{ij+1}(3s(1-s)^2) \left\| \frac{\partial \vec{r}_{ij+1}}{\partial s} \right\| ds$$

The second term becomes:

$$\begin{aligned}
\frac{\partial}{\partial q} T &= \frac{\partial}{\partial q} \frac{1}{2} \rho \int_0^{n_i} \left(\dot{\vec{r}}_i\right)^2 \left\| \frac{\partial \vec{r}_i}{\partial s} \right\| ds \\
&= \frac{1}{2} \rho \int_0^{n_i} \left(\dot{\vec{r}}_i\right)^2 \frac{\cancel{1}}{\cancel{2}} \frac{1}{\left\| \frac{\partial \vec{r}_i}{\partial s} \right\|} \left(\cancel{2} \frac{\partial \vec{r}_i}{\partial s} \cdot \frac{\partial}{\partial q} \frac{\partial \vec{r}_i}{\partial s} \right) ds \\
&= \frac{1}{2} \rho \int_0^{n_i} \left(\dot{\vec{r}}_i\right)^2 \frac{\frac{\partial \vec{r}_i}{\partial s} \cdot \frac{\partial}{\partial q} \frac{\partial \vec{r}_i}{\partial s}}{\left\| \frac{\partial \vec{r}_i}{\partial s} \right\|} ds
\end{aligned}$$