Describing a Moving Source With the Advection-Diffusion Equation

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1. Introduction

This is an attempted solution to diffusion from a uniform moving source. Assume that a species A is diffusing vertically downward (e.g. in the y-direction) into a moving fluid B with velocity v_x. Assume no chemical reactions occur within the mass transfer zone. This 2-D system is governed by the following partial differential equation

$$\frac{\partial C_A}{\partial t} + v_x \frac{\partial C_A}{\partial x} = D_{AB}\left(\frac{\partial^2 C_A}{\partial y^2}\right), \qquad 0 < x < \infty, \qquad 0 < y < \infty, \qquad t > 0$$

Now consider a system in which a mass transfer source is moving with a velocity v_s in the x-direction on top of a surface. If the reference frame is the source of mass transfer, then the system can be treated like an advection-diffusion problem with v being the relative velocity $v = v_x - v_s$ which captures the velocity between the moving source (v_s) and moving fluid (v_x). At steady state

$$v \frac{\partial C_A}{\partial x} = D_{AB}\left(\frac{\partial^2 C_A}{\partial y^2}\right), \qquad 0 < x < \infty, \qquad 0 < y < \infty \tag{1}$$

where v is relative velocity, $v = v_x - v_s$

2. Similarity Solution

Because the domain is not bounded, separation of variables cannot be used. Consider a similarity solution to reduce the solution as a function of x and y to just one variable, η,

$$C_A(x, y) = C_{As} f(\eta), \quad \eta = \frac{y}{g(x)} \tag{2}$$

The two boundary conditions in y become,

$C_A(x, 0) = C_{As} \implies f(0) = 1$
$C_A(x, \infty) = 0 \implies f(\infty) = 0$

From physical intuition based on equation (1), $g(x)$ grows by $\sqrt{x}$ with y. Physical constants D_{AB} and v will be added to η later to make it dimensionless. Plugging $C_A(x, y) = f(\eta) = f(\frac{y}{g(x)})$ into equation (1),

$$\frac{\partial C_A}{\partial x} = \Big(f'(\eta)\Big)\Big(-\frac{yg'(x)}{(g(x))^2}\Big)$$

$$\frac{\partial^2 C_A}{\partial y^2} = \Big(f''(\eta)\Big)\Big(\frac{1}{(g(x))^2}\Big)$$

$$v\frac{\partial C_A}{\partial x} = D_{AB}\Big(\frac{\partial^2 C_A}{\partial y^2}\Big) \implies v\Big(f'(\eta)\Big(-\frac{yg'(x)}{(g(x))^2}\Big)\Big) = D_{AB}\Big(f''(\eta)\Big(\frac{1}{(g(x))^2}\Big)\Big) \implies$$

$$v\Big(f'(\eta)\Big(-yg'(x)\Big)\Big) = D_{AB}\Big(f''(\eta)\Big)$$

Note from equation (2), $y = \eta g(x)$ so,

$$v\Big(f'(\eta)\Big(-\eta g(x)g'(x)\Big)\Big) = D_{AB}\Big(f''(\eta)\Big) \tag{3}$$

Because the differential equation above needs to be only in terms of η, $g(x)g'(x)$ must be independent of x. That is,

$$g(x)\frac{dg(x)}{dx} = C$$

Where C is some constant.

$$\int g(x)\frac{dg(x)}{\cancel{dx}}\ \cancel{dx} = \int C\ \ dx \implies \frac{(g(x))^2}{2} = Cx \implies g(x) = \sqrt{2Cx}$$

Plugging back into equation (2),

$$\eta = \frac{y}{\sqrt{2Cx}}$$

Let $C = 2\Big(\frac{D_{AB}x}{v}\Big)$. Note that the added factor of 2 in front results allows for simplifications in solving a differential equation later on. Thus,

$$\eta = \frac{y}{2\sqrt{\frac{D_{AB}x}{v}}}$$

Now η accounts for scaling between y and $\sqrt{x}$ as well as other parameters that make η dimensionless. These results agree with other dimensionless variables (η, etc.) in partial differential equations similar to that of equation (1) such as the transient heat conduction equation in 1-D [1].

Plugging back $g(x)\frac{dg(x)}{dx} = C = 2\Big(\frac{D_{AB}x}{v}\Big)$ into equation (3),

$$\cancel{v}\Big(f'(\eta)\Big(-\eta\Big(2\frac{\cancel{D_{AB}}x}{\cancel{v}}\Big)\Big)\Big) = \cancel{D_{AB}}\Big(f''(\eta)\Big) \implies -2\eta f'(\eta) = f''(\eta) \implies \frac{f''(\eta)}{f'(\eta)} = -2\eta$$

Let $h(\eta) = f'(\eta)$. Then,

$$\frac{h'(\eta)}{h(\eta)} = -2\eta \implies \int \frac{h'(\eta)}{h(\eta)}\ \ d\eta = \int -2\eta\ \ d\eta \implies h(\eta) = C_1 e^{-\eta^2}$$

Plugging back in $f'(\eta)$ for $h(\eta)$,

$$f'(\eta) = C_1 e^{-\eta^2} \implies \int f'(\eta)\ d\eta = \int_0^\eta C_1 e^{-s^2}\ ds \implies f(\eta) = C_1 \frac{\sqrt{\pi}}{2} \text{erf}(\eta) + C_2$$

From boundary conditions,

$f(0) = 1 = C_1 \frac{\sqrt{\pi}}{2} \text{erf}(0) + C_2 \implies C_2 = 1$
$f(\infty) = 0 = C_1 \frac{\sqrt{\pi}}{2} \text{erf}(\infty) + 1 \implies C_1 = -\frac{2}{\sqrt{\pi}}$

Therefore,

$$f(\eta) = 1 - \text{erf}(\eta) \implies C_A(x, y) = C_{As}(1 - \text{erf}(\eta)) \implies \boxed{C_A(x, y) = C_{As} \left(1 - \text{erf} \left(\frac{y}{2\sqrt{\frac{D_{AB}x}{v}}} \right) \right)}$$

Where $C_A(x, y)$ is concentration in amount (e.g. mol, gram, etc.)/length^3, C_{As} is the surface concentration of the moving source in amount/length^3, D_{AB} is the diffusivity of A in B in length^2/time, v is relative velocity ($v = v_x - v_s$) where v_s is the velocity of the source and v_x is the velocity of the fluid, and y and x are position, both with units length.

Note that this solution agrees with the solutions of other partial differential equations with similar semi-infinite Dirichlet boundary conditions and the same order and number of independent and dependent variables as this problem has (1 dependent variable, C_A, 2 independent variables, x and y). This follows closely to solution 5. of transient diffusion cases provided by professor Mohraz [2].

This work would not be possible without the aid of [3, 4] for clarity, coding in LaTeX, and solutions to similar partial differential equations and steps involved throughout this document. A special thanks to Professor Mohraz for fruitful discussions.

3. References

[1] Leif Helge Hjertaker. *TKT4140 - Partial Differential Equations*. NTNU Teaching Website. Accessed: 2024. 2020. URL: `https://leifh.folk.ntnu.no/teaching/tkt4140/._main034.html`.

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[3] OpenAI. *ChatGPT*. AI language model. Version GPT-4. 2023. URL: `https://chat.openai.com`.

[4] DeepSeek. *DeepSeek Chat*. AI language model. Version DeepSeek-V3. 2024. URL: `https://www.deepseek.com`.