Shtuka cohomology and special values of Goss L-functions

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Introduction

Assuming everywhere good reduction we generalize the class number formula of Taelman [26] to Drinfeld modules over arbitrary coefficient rings. In order to prove this formula we develop a theory of shtukas and their cohomology.

1. A class number formula for Drinfeld modules

Fix a finite field $\mathbb{F}_q$. In the following all morphisms, fiber and tensor products will be over $\mathbb{F}_q$ unless indicated otherwise. Let C be a smooth projective connected curve over $\mathbb{F}_q$. Fix a closed point $\infty \in C$. The $\mathbb{F}_q$-algebra

$$A = \Gamma(C - \{\infty\}, \mathcal{O}_C)$$

will be called the *coefficient ring*. Fix an A-algebra R (the base ring). We denote $\iota\colon A \to R$ the natural map.

Consider the group scheme $\mathbb{G}_a$ over R. It is well-known that every $\mathbb{F}_q$-linear endomorphism of $\mathbb{G}_a$ can be uniquely written in the form of a *τ-polynomial*

$$r_0 + r_1\tau + \ldots + r_n\tau^n$$

where $r_0, \ldots, r_n \in R$, $r_n \neq 0$ and $\tau\colon \mathbb{G}_a \to \mathbb{G}_a$ denotes the q-Frobenius.

Recall that an A-module scheme is an abelian group scheme equipped with an action of A.

Definition. A Drinfeld module over R with coefficients in A is an A-module scheme E over R which has the following properties:

(1) The underlying additive group scheme of E is Zariski-locally isomorphic to $\mathbb{G}_a$.
(2) For every element $a \in A$ the induced endomorphism of the Lie algebra scheme Lie_E is the multiplication by $\iota(a)$.
(3) There exists an element $a \in A$ such that locally on $\operatorname{Spec} R$ the action of a on E is given by a τ-polynomial of positive degree and with top coefficient a unit.

Example. Let $A = \mathbb{F}_q[t]$, $R = \mathbb{F}_q[\theta]$ and let $\iota\colon A \to R$ be the isomorphism which sends t to θ. An example of a Drinfeld A-module over R is the *Carlitz module E*. Its underlying additive group scheme is $\mathbb{G}_a$. The action of $t \in A$ on E is given by the τ-polynomial

$$\theta + \tau.$$

The Frobenius τ induces the zero endomorphism on Lie_E. Hence t acts on Lie_E as the multiplication by $\theta = \iota(t)$. It follows that the condition (2) holds for E. The conditions (1) and (3) are clear.

From now on we assume that R is a domain and that it is finite flat over A. It follows that R is a Dedekind domain of finite type over $\mathbb{F}_q$. We denote K the fraction field of R. The generic fiber of a Drinfeld module over R is a Drinfeld module over K. However, not every Drinfeld module over K extends to a Drinfeld module over R. The ones which do are said to have *good reduction everywhere*.

Drinfeld modules behave in a way similar to elliptic curves. For the latter the role of the coefficient ring A is played by $\mathbb{Z}$. Given an elliptic curve E over a number field and a prime $(p) \subset \mathbb{Z}$ one can consider its p-adic Tate module

$$T_p E = \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Q}_p/\mathbb{Z}_p, E(\overline{\mathbb{Q}})).$$

Much in the same way for a Drinfeld module E over the function field K and a prime $\mathfrak{p} \subset A$ one has the $\mathfrak{p}$-adic Tate module

$$T_{\mathfrak{p}} E = \mathrm{Hom}_A(F_{\mathfrak{p}}/A_{\mathfrak{p}}, E(K^{\mathrm{sep}}))$$

where K^{sep} denotes a separable closure of K, $A_{\mathfrak{p}}$ is the completion of A at $\mathfrak{p}$ and $F_{\mathfrak{p}}$ is the field of fractions of $A_{\mathfrak{p}}$.

The Tate module $T_{\mathfrak{p}} E$ is a finitely generated free $A_{\mathfrak{p}}$-module. Its rank can be any integer greater than zero. This rank does not depend on $\mathfrak{p}$ and is called the *rank of E*. The Tate module $T_{\mathfrak{p}} E$ is naturally a continuous representation of the Galois group $G(K^{\mathrm{sep}}/K)$ unramified at almost all primes $\mathfrak{m} \subset R$.

Let us fix a Drinfeld module E over R. Let $\mathfrak{p} \subset A$ and $\mathfrak{m} \subset R$ be primes such that $\mathfrak{p} \neq \iota^{-1}(\mathfrak{m})$. Since E is defined over R the Tate module $T_{\mathfrak{p}} E$ is unramified at $\mathfrak{m}$. It thus makes sense to consider the inverse characteristic polynomial of the geometric Frobenius element at $\mathfrak{m}$ acting on $T_{\mathfrak{p}} E$. This polynomial has coefficients in the fraction field of A and is independent of the choice of $\mathfrak{p}$. We denote it $P_{\mathfrak{m}}(T)$.

Definition. Let F_∞ be the local field of the curve C at ∞. We define $L(E^*, 0) \in F_\infty$ by the formula

$$L(E^*, 0) = \prod_{\mathfrak{m}} \frac{1}{P_{\mathfrak{m}}(1)}$$

where the product ranges over all primes $\mathfrak{m} \subset R$.

It is not difficult to show that this product converges. The resulting element $L(E^*, 0) \in F_\infty$ is indeed a value of a certain function, the Goss L-function of the strictly compatible family of Galois representations given by the Tate modules $T_{\mathfrak{p}} E$. We use the notation $L(E^*, 0)$ instead of $L(E, 0)$ since the usual Goss L-function of E is given by the family of dual Tate modules $(T_{\mathfrak{p}} E)^*$. For the experts we remark that our $L(E^*, 0)$ coincides with the special value considered by Taelman in [26].

Example. Let us examine the Carlitz module E. For every prime $\mathfrak{m} \subset R = \mathbb{F}_q[\theta]$ there exists a unique monic irreducible polynomial $f \in \mathbb{F}_q[t]$ such that

$$\mathfrak{m} = \iota(f) R.$$

Let $g \in \mathbb{F}_q[t]$ be another monic irreducible polynomial, $g \neq f$. The Tate module $T_g E$ is free of rank 1. One can show that the *arithmetic* Frobenius at $\mathfrak{m}$ acts on $T_g E$ by multiplication by f. Therefore the inverse charateristic polynomial of the *geometric* Frobenius at $\mathfrak{m}$ is

$$P_{\mathfrak{m}}(T) = 1 - f^{-1} T.$$

We conclude that the special value $L(E^*, 0)$ is given by the Euler product

$$L(E^*, 0) = \prod_f \frac{1}{1 - \dfrac{1}{f}}$$

where $f \in \mathbb{F}_q[t]$ runs over monic irreducible polynomials. Expanding this product we get the formula

$$L(E^*, 0) = \sum_h \frac{1}{h}$$

where $h \in \mathbb{F}_q[t]$ runs over all monic polynomials.

Let $K_\infty = R \otimes_A F_\infty$. The exponential map of the Drinfeld module E is the unique map

$$\exp\colon \mathrm{Lie}_E(K_\infty) \to E(K_\infty)$$

satisfying the following conditions:

(1) exp is a homomorphism of A-modules,
(2) exp is an analytic function with derivative 1 at zero in the following sense. Fix an $\mathbb{F}_q$-linear isomorphism of group schemes $E \cong \mathbb{G}_a$ defined over K_∞. It identifies $E(K_\infty)$ with K_∞ while its differential identifies $\mathrm{Lie}_E(K_\infty)$ with K_∞. We demand that the resulting map $\exp\colon K_\infty \to K_\infty$ is given by an everywhere convergent power series
$$\exp(z) = z + a_1 z^q + a_2 z^{q^2} + \dots$$
with coefficients in K_∞.

Definition. The *complex of units* of E is the A-module complex

$$U_E = \left[\mathrm{Lie}_E(K_\infty) \xrightarrow{\exp} \frac{E(K_\infty)}{E(R)} \right]$$

where $\mathrm{Lie}_E(K_\infty)$ is placed in degree 0.

By construction

$$\mathrm{H}^0(U_E) = \exp^{-1} E(R),$$
$$\mathrm{H}^1(U_E) = \frac{E(K_\infty)}{\exp(K_\infty) + E(R)}.$$

The cohomology modules of U_E have individual names: $\mathrm{H}^0(U_E)$ is the module of units and $\mathrm{H}^1(U_E)$ is the class module.

Definition. The *regulator*

$$\rho\colon F_\infty \otimes_A U_E \to \mathrm{Lie}_E(K_\infty)[0]$$

is the F_∞-linear extension of the morphism $U_E \to \mathrm{Lie}_E(K_\infty)[0]$ given by the identity in degree zero.

Theorem 1.1 (Taelman [24]). *The A-module complex U_E is perfect and the regulator ρ is a quasi-isomorphism.*

This theorem is usually stated in a different form: $\mathrm{H}^1(U_E)$ is finite and $\mathrm{H}^0(U_E)$ is a lattice in $\mathrm{Lie}_E(K_\infty)$. Recall that an A-submodule Λ in a finite-dimensional F_∞-vector space V is called a *lattice* if one of the following equivalent conditions is satisfied:

- Λ is discrete and cocompact.
- The natural map $F_\infty \otimes_A \Lambda \to V$ is an isomorphism.

A lattice is automatically a finitely generated projective A-module.

One may interpret Taelman's theorem as saying that the complex U_E is a lattice in $\mathrm{Lie}_E(K_\infty)$ in a derived sense. $\mathrm{Lie}_E(K_\infty)$ contains one more natural lattice: the integral Lie algebra $\mathrm{Lie}_E(R)$. We would like to determine their relative position.

Since U_E is perfect the theory of Knudsen-Mumford determinants [17] provides us with an invertible A-module $\det_A U_E$ and an isomorphism of one-dimensional F_∞-vector spaces

$$\det_{F_\infty}(\rho)\colon F_\infty \otimes_A \det_A U_E \xrightarrow{\sim} \det_{F_\infty} \mathrm{Lie}_E(K_\infty).$$

The vector space $\det_{F_\infty} \mathrm{Lie}_E(K_\infty) = F_\infty \otimes_A \det_A \mathrm{Lie}_E(R)$ contains a canonical lattice $\det_A \mathrm{Lie}_E(R)$. Now we are ready to state our main result.

Theorem 1.2. *The image of* $\det_A U_E$ *under* $\det_{F_\infty}(\rho)$ *is*

$$L(E^*, 0) \cdot \det_A \mathrm{Lie}_E(R).$$

Remark. Theorem 1.2 implies that the A-modules $\det_A U_E$ and $\det_A \mathrm{Lie}_E(R)$ are isomorphic. This is by no means immediate if the class group of the coefficient ring A is not zero. In fact, it was not previously known apart from the trivial case Pic $A = 0$.

Remark. It is important to realize that Theorem 1.2 gives a formula for $L(E^*, 0)$ in terms of the lattices U_E and $\mathrm{Lie}_E(R)$. A priori the relation

$$\det_{F_\infty}(\rho)\Big(\det_A U_E\Big) = x \cdot \det_A \mathrm{Lie}_E(R)$$

determines $x \in F_\infty$ up to a unit of A. However one can prove that $L(E^*, 0)$ is a 1-unit in F_∞. The only unit of A which is also a 1-unit of F_∞ is the element 1. So the relation above determines a 1-unit x uniquely.

Remark. The statement of Theorem 1.2 goes back to the fundamental work of Taelman [26] where he established a formula for $L(E^*, 0)$ under assumption that the coefficient ring A is $\mathbb{F}_q[t]$. Unlike our Theorem 1.2 the result of Taelman applies to Drinfeld modules with arbitrary reduction type.

Fang [10] extended the result of Taelman to Anderson modules [1] which are a higher-dimensional generalization of Drinfeld modules. He also considered the coefficient ring $A = \mathbb{F}_q[t]$ only.

Debry [6] was the first to generalize this formula to coefficient rings A different from $\mathbb{F}_q[t]$. His result applies to coefficient rings with trivial class group. In our Theorem 1.2 the coefficient ring A can be arbitrary but the Drinfeld module E is assumed to have good reduction everywhere.

Remark. One can describe the image of $\det_A U_E$ under $\det_{F_\infty}(\rho)$ as follows. Theorem 1.1 implies that the A-submodule $\exp^{-1} E(R)$ of $\mathrm{Lie}_E(K_\infty)$ is a lattice so that its top exterior power $\det_A \exp^{-1} E(R)$ is an invertible A-submodule in the determinant of $\mathrm{Lie}_E(K_\infty)$. The image of $\det_A U_E$ is the A-submodule

$$I \cdot \det_A \exp^{-1} E(R)$$

where I is the 0-th Fitting ideal of the class module $\mathrm{H}^1(U_E)$. This ideal has the following explicit description. The A-module $\mathrm{H}^1(U_E)$ can be written as a finite direct sum $\bigoplus_n A/I_n$ where $I_n \subset A$ are ideals. The 0-th Fitting ideal of $\mathrm{H}^1(U_E)$ is

$$I = \prod_n I_n.$$

Example. Let us show how Theorem 1.2 works for the Carlitz module E. In this case $F_\infty = \mathbb{F}_q(\!(t^{-1})\!)$ and $K_\infty = \mathbb{F}_q(\!(\theta^{-1})\!)$. The exponential map $\exp\colon K_\infty \to K_\infty$ of the Carlitz module admits a local inverse around 0, the Carlitz logarithm map. It is given by the power series

$$\log z = z - \frac{z^q}{\theta^q - \theta} + \frac{z^{q^2}}{(\theta^q - \theta)(\theta^{q^2} - \theta)} - \dots$$

The series converges for z such that $|z| \leqslant q^{\frac{q}{q-1}}$. In particular the image of the exponential map contains the unit ball $\mathbb{F}_q[[\theta^{-1}]] \subset K_\infty$. As a consequence the class module

$$\mathrm{H}^1(U_E) = \frac{\mathbb{F}_q(\!(\theta^{-1})\!)}{\exp(K_\infty) + \mathbb{F}_q[\theta]}$$

is zero.

The F_∞-vector space $\mathrm{Lie}_E(K_\infty)$ is of dimension 1. Hence Theorem 1.1 implies that $\mathrm{H}^0(U_E)$ is a free A-module of rank 1. By construction the element

$$\widetilde{\pi} = \log(1)$$

belongs to $\mathrm{H}^0(U_E) = \exp^{-1} E(R)$. A priori it generates an A-submodule of finite index. However it is easy to show that a nonconstant element of A can not divide $1 \in E(R)$. Therefore

$$\mathrm{H}^0(U_E) = A \cdot \widetilde{\pi}.$$

The A-module $\mathrm{H}^0(U_E)$ coincides with its determinant since it is free of rank 1. Now $\mathrm{H}^1(U_E) = 0$ so Theorem 1.2 implies that

$$L(E^*, 0) = \alpha\iota^{-1}(\widetilde{\pi})$$

for some $\alpha \in A^\times$. Here $\iota\colon F_\infty \cong K_\infty$ is the natural isomorphism. As observed before, one can show that $L(E^*, 0)$ is a 1-unit of F_∞. Since $\widetilde{\pi}$ is a 1-unit by construction it follows that

$$L(E^*, 0) = \iota^{-1}(\widetilde{\pi}).$$

Expanding the definitions we obtain a formula

$$\sum_{\substack{h \in \mathbb{F}_q[t] \\ \text{monic}}} \frac{1}{h} = 1 - \frac{1}{t^q - t} + \frac{1}{(t^q - t)(t^{q^2} - t)} - \ldots$$

in $\mathbb{F}_q(\!(t^{-1})\!)$. Observe that the series on the right hand side converges much faster than the series on the left hand side. This formula was discovered by Carlitz [5] in the 1930-ies.

Remark. The complex of units U_E has an interesting analog in the context of number fields. For the moment, let K be a number field and let $\mathcal{O}_K \subset K$ be its ring of integers. Set $K_\infty = \mathbb{R} \otimes_{\mathbb{Q}} K$ and consider the complex

$$U_K = \Big[\mathrm{Lie}_{\mathbb{G}_\mathrm{m}}(K_\infty) \xrightarrow{\exp} \frac{\mathbb{G}_\mathrm{m}(K_\infty)}{\mathbb{G}_\mathrm{m}(\mathcal{O}_K)} \Big]$$

where exp is the exponential of the Lie group $\mathbb{G}_\mathrm{m}(K_\infty)$. In this setting the regulator

$$\rho\colon \mathbb{R} \otimes_{\mathbb{Z}} U_K \to \mathrm{Lie}_{\mathbb{G}_\mathrm{m}}(K_\infty)$$

is the $\mathbb{R}$-linear extension of the morphism $U_K \to \mathrm{Lie}_{\mathbb{G}_\mathrm{m}}(K_\infty)[0]$ given by the identity in degree zero. The Dirichlet's unit theorem for K is equivalent to the following statement:

Theorem 1.0 (Dirichlet). *The $\mathbb{Z}$-module complex U_K is perfect and the cone of ρ is quasi-isomorphic to $\mathbb{R}[0]$.*

2. Overview of the proof

To prepare the ground for the proof of Theorem 1.2 we develop a theory of shtukas and their cohomology. While retaining some features of the works of Taelman [26] and Fang [10], our approach differs from them in an essential way. Certain aspects of this approach were envisaged by Taelman in [25]. The central idea of using Anderson trace formula [2] to study special values of shtukas is due to V. Lafforgue [18]. In general, the ideas of Anderson [1, 2] play an important role in this text. Our cohomology theory for shtukas was heavily motivated by the works of Böckle-Pink [3] and V. Lafforgue [18]. The notion of a nilpotent τ-sheaf from [3] figures prominently in it. Our intellectual debt to Drinfeld [8, 9] is obvious.

Remark. To avoid confusion we should stress that the definitions in this section are simplified for expository purposes.

We begin with an overview of shtuka theory relevant to the proof of Theorem 1.2. Let us first describe the setting. The finite flat A-algebra R is a Dedekind domain of finite type over $\mathbb{F}_q$. To such an algebra R one can functorially associate a smooth connected projective curve X over $\mathbb{F}_q$ together with an open embedding $\operatorname{Spec} R \subset X$. Consider the scheme $\operatorname{Spec} A \times X$. Let $\tau\colon \operatorname{Spec} A \times X \to \operatorname{Spec} A \times X$ be the endomorphism which acts as the identity on A and as the q-Frobenius on X.

Definition. A *shtuka* $\mathcal{M}$ on $\operatorname{Spec} A \times X$ is given by a diagram

$$\mathcal{M}_0 \underset{j}{\overset{i}{\rightrightarrows}} \mathcal{M}_1$$

where $\mathcal{M}_0$, $\mathcal{M}_1$ are coherent sheaves on $\operatorname{Spec} A \times X$ and

$$i\colon \mathcal{M}_0 \to \mathcal{M}_1,$$
$$j\colon \mathcal{M}_0 \to \tau_*\mathcal{M}_1$$

are morphisms of coherent sheaves. Morphisms of shtukas are given by morphisms of underlying coherent sheaves which commute with i and j (cf. Definition 1.1.2).

An example of a shtuka is the *unit shtuka*

$$\mathbb{1} = \Big[\mathcal{O} \underset{\tau^\#}{\overset{1}{\rightrightarrows}} \mathcal{O}\Big]$$

where $\mathcal{O}$ is the structure sheaf of $\operatorname{Spec} A \times X$ and $\tau^\#\colon \mathcal{O} \to \tau_*\mathcal{O}$ is the map defined by the endomorphism τ. Shtukas on $\operatorname{Spec} A \times X$ form an abelian category.

Definition. Let $\mathcal{M}$ be a shtuka on $A \times X$. The *cohomology complex of* $\mathcal{M}$ is the A-module complex

$$\mathrm{R}\Gamma(\mathcal{M}) = \mathrm{RHom}(\mathbb{1}, \mathcal{M})$$

where RHom on the right hand side is computed in the derived category of shtukas.

Shtuka cohomology can be computed in terms of coherent cohomology: for every shtuka

$$\mathcal{M} = \Big[\mathcal{M}_0 \underset{j}{\overset{i}{\rightrightarrows}} \mathcal{M}_1\Big]$$

there exists a natural distinguished triangle

$$\mathrm{R}\Gamma(\mathcal{M}) \to \mathrm{R}\Gamma(\mathcal{M}_0) \xrightarrow{i-j} \mathrm{R}\Gamma(\mathcal{M}_1) \to [1]. \tag{0.1}$$

We write $\mathrm{R}\Gamma(-)$ instead of $\mathrm{R}\Gamma(\operatorname{Spec} A \times X, -)$ to improve legibility.

Definition. We define the *linearization functor* ∇ from the category of shtuka to itself in the following way:

$$\nabla\Big[\mathcal{M}_0 \underset{j}{\overset{i}{\rightrightarrows}} \mathcal{M}_1\Big] = \Big[\mathcal{M}_0 \underset{0}{\overset{i}{\rightrightarrows}} \mathcal{M}_1\Big].$$

The cohomology of $\nabla\mathcal{M}$ is often easier to compute than the cohomology of $\mathcal{M}$. Even though the complexes $\mathrm{R}\Gamma(\mathcal{M})$ and $\mathrm{R}\Gamma(\nabla\mathcal{M})$ are usually quite different, there is a subtle link between them.

The sheaves $\mathcal{M}_0$ and $\mathcal{M}_1$ are coherent by assumption. As a consequence the A-module complexes $\mathrm{R}\Gamma(\mathcal{M}_0)$ and $\mathrm{R}\Gamma(\mathcal{M}_1)$ are perfect. The distinguished triangle (0.1) now implies that $\mathrm{R}\Gamma(\mathcal{M})$ and $\mathrm{R}\Gamma(\nabla\mathcal{M})$ are perfect. So we can apply the theory of Knudsen-Mumford determinants to $\mathrm{R}\Gamma(\mathcal{M})$ and $\mathrm{R}\Gamma(\nabla\mathcal{M})$.

Definition. We define the *ζ-isomorphism*

$$\zeta_{\mathcal{M}}\colon \det_A \mathrm{R}\Gamma(\mathcal{M}) \xrightarrow{\sim} \det_A \mathrm{R}\Gamma(\nabla\mathcal{M})$$

as the composition of the isomorphisms

$$\det_A \mathrm{R}\Gamma(\mathcal{M}) \xrightarrow{\sim} \det_A \mathrm{R}\Gamma(\mathcal{M}_0) \otimes_A \det_A^{-1} \mathrm{R}\Gamma(\mathcal{M}_1) \xleftarrow{\sim} \det_A \mathrm{R}\Gamma(\nabla\mathcal{M})$$

induced by the natural distinguished triangles (0.1) for $\mathcal{M}$ and $\nabla\mathcal{M}$.

The ζ-isomorphisms are named by analogy with the ζ-elements of Kato [16]. Such isomorphisms first appeared in the article [18] of V. Lafforgue.

Now we are in position to describe the main steps in the proof of Theorem 1.2. To a Drinfeld module E over R we associate a certain shtuka called *a model of* E. Its precise definition is a bit technical and is not necessary to understand the following. The construction of a model proceeds roughly as follows. We take the Anderson motive of E and dualize it to obtain a shtuka on $\operatorname{Spec} A \otimes R$. We then extend it to $\operatorname{Spec} A \times X$ using the functor of extension by zero from the theory of Böckle-Pink [3].

Every Drinfeld module E admits many different shtuka models. However all the models share important properties. The underlying coherent sheaves of a model are locally free. Their rank coincides with the rank of E. The cohomology of a model captures important arithmetic invariants of E.

Theorem 2.1. *For every shtuka model $\mathcal{M}$ of E there are natural quasi-isomorphisms*

$$\mathrm{R}\Gamma(\mathcal{M}) \xrightarrow{\sim} U_E[-1], \quad \mathrm{R}\Gamma(\nabla\mathcal{M}) \xrightarrow{\sim} \mathrm{Lie}_E(R)[-1].$$

Recall that the complex of units U_E is defined in terms of an analytic map, the exponential of E. Theorem 2.1 provides an algebraic description of this complex. One important application of it is the following:

Corollary. $\det_A U_E \cong \det_A \mathrm{Lie}_E(R)$.

Proof. Indeed we have a ζ-isomorphism $\zeta_{\mathcal{M}}\colon \det_A \mathrm{R}\Gamma(\mathcal{M}) \cong \det_A \mathrm{R}\Gamma(\nabla\mathcal{M})$. □

Remark. The second quasi-isomorphism in Theorem 2.1 is easy to construct. In contrast there is no obvious natural map between the complexes $\mathrm{R}\Gamma(\mathcal{M})$ and $U_E[-1]$. The construction of the quasi-isomorphism $\mathrm{R}\Gamma(\mathcal{M}) \cong U_E[-1]$ is rather intricate. The proof of Theorem 2.1 uses Hochschild cohomology of A as the main computational tool.

Remark. Taelman [25] established Theorem 2.1 for the Carlitz module E over an arbitrary finite flat A-algebra R, $A = \mathbb{F}_q[t]$. He constructed shtuka models of E in an ad hoc manner. His result was generalized by Fang [10] to Anderson modules with coefficients in $A = \mathbb{F}_q[t]$. The construction of shtuka models in [10] is also ad hoc.

Remark. It is necessary to mention that our proof of Theorem 2.1 extends without change to arbitrary Anderson A-modules, including the non-uniformizable ones. In this text we limit the exposition to Drinfeld modules since other important parts of the theory still depend on their special properties.

Remark. Our proof of Theorem 2.1 was inspired by the article [1] of Anderson. In [1, §2] he proves a vanishing statement for Ext^1 which in retrospect can be viewed as a statement on cohomology of certain shtukas related to Drinfeld modules.

As we mentioned above the cohomology complexes of a shtuka $\mathcal{M}$ and its linearization $\nabla\mathcal{M}$ are quite different in general. The ζ-isomorphism $\zeta_{\mathcal{M}}$ relates their determinants. A more direct link is given by the *regulator*

$$\rho_{\mathcal{M}}\colon F_\infty \otimes_A \mathrm{R}\Gamma(\mathcal{M}) \xrightarrow{\sim} F_\infty \otimes_A \mathrm{R}\Gamma(\nabla\mathcal{M}).$$

The regulator is a quasi-isomorphism and is natural in $\mathcal{M}$. It is defined for *elliptic shtukas*, a natural class of shtukas which generalize shtuka models of Drinfeld modules.

The central result about the regulator is the trace formula which expresses $\zeta_{\mathcal{M}}$ in terms of $\rho_{\mathcal{M}}$ and an explicit numerical invariant $L(\mathcal{M}) \in F_\infty$. This invariant is a product of local factors, one for each prime $\mathfrak{m} \subset R$. The local factor at $\mathfrak{m}$ depends only on the restriction of $\mathcal{M}$ to $A \otimes R/\mathfrak{m}$.

Theorem 2.2 (Trace formula) *If $\mathcal{M}$ is an elliptic shtuka satisfying certain technical condition*[1] *then*

$$\zeta_{\mathcal{M}} = L(\mathcal{M}) \cdot \det_{F_\infty}(\rho_{\mathcal{M}})$$

as maps from $F_\infty \otimes_A \det_A \mathrm{R}\Gamma(\mathcal{M})$ to $F_\infty \otimes_A \det_A \mathrm{R}\Gamma(\nabla\mathcal{M})$.

The following important lemma is very easy to prove:

Lemma. *If $\mathcal{M}$ is a shtuka model of E then $L(\mathcal{M}) = L(E^*, 0)$.*

Remark. Theorem 2.2 is basically the trace formula of Anderson [2] in disguise.

Remark. In general the invariant $L(\mathcal{M}) \in F_\infty$ is transcendental over $A \subset F_\infty$. Its inherent complexity reflects in the construction of the regulator making it rather involved. By contrast the definition of the regulator (Definition 5.14.1) is simple.

Now we have almost all the tools to prove Theorem 1.2. Thanks to Theorem 2.1 the shtuka-theoretic regulator $\rho_{\mathcal{M}}$ of a model $\mathcal{M}$ induces a quasi-isomorphism

$$F_\infty \otimes_A U_E \to \mathrm{Lie}_E(K_\infty)[0].$$

However there is no a priori reason for it to coincide with the arithmetic regulator

$$\rho_E \colon F_\infty \otimes_A U_E \to \mathrm{Lie}_E(K_\infty)[0]$$

which is defined purely in terms of the Drinfeld module E.

Theorem 2.3. *Let $\mathcal{M}$ be a shtuka model of E. The quasi-isomorphisms of Theorem 2.1 identify the shtuka-theoretic regulator $\rho_{\mathcal{M}}$ with the shifted arithmetic regulator $\rho_E[-1]$.*

Remark. The only proof of Theorem 2.3 which we have at the moment is rather technical, and is based on explicit computations in the case $A = \mathbb{F}_q[t]$.

As we observed above, shtuka models $\mathcal{M}$ of E exist and have the property that $L(\mathcal{M}) = L(E^*, 0)$. Hence Theorems 2.1, 2.2 and 2.3 imply Theorem 1.2 for E.

Remark. Our theory of shtukas is very sensitive to reduction properties of Drinfeld modules. Its extension to the bad reduction case is not at all straightforward and may be difficult. Such an extension is a subject of current research. We also work on an extension of our theory to Anderson modules [1].

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[1] At present we can only prove the theorem under a technical condition on the cohomology of sheaves underlying $\mathcal{M}$ (cf. Theorem 6.10.3). It is enough for the proof of the class number formula. We expect that the trace formula holds without this condition.

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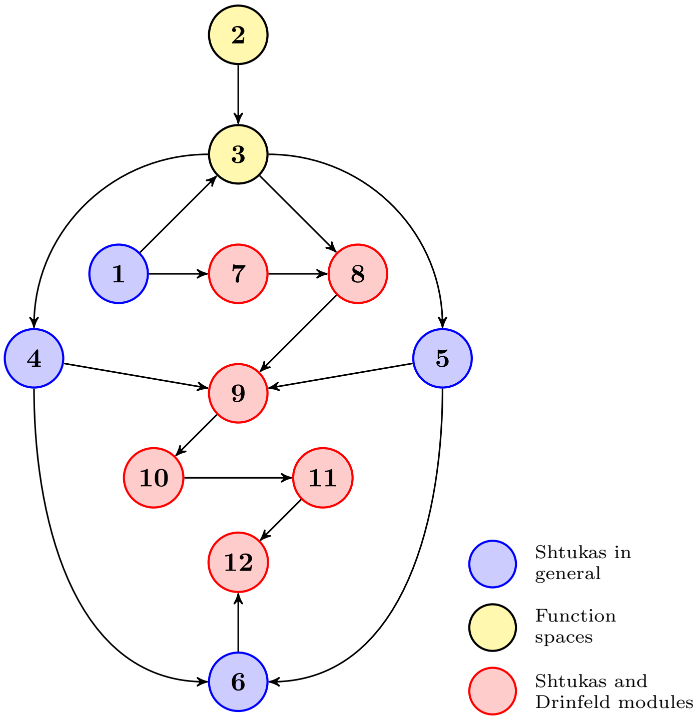
Leitfaden



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Notation and conventions

1. Stacks Project

We use the Stacks Project [27] as a reference for the theory of schemes, commutative and homological algebra. We follow the conventions, the terminology and the notation of the Stacks Project, with two amendments:

- We generally write distinguished triangles as $A \to B \to C \to [1]$ omitting the object A at the last place.
- We say that a complex A is *concentrated in degrees* $[a, b]$ if $H^n(A) = 0$ whenever $n \notin [a, b]$.

Since the order and numeration of items in the Stacks Project is subject to change we refer to them by tags as explained at the page `http://stacks.math.columbia.edu/tags`. A reference to a tag has the form $[wxyz]$ where "$wxyz$" is a combination of four letters and numbers. The corresponding item of the Stacks Project can be accessed by the URI `http://stacks.math.columbia.edu/tag/`$wxyz$.

2. Lattices

In Chapter 3 and Chapters 9–12 we will use the notion of a lattice in a module.

Definition 2.1. Let $R_0 \to R$ be a ring homomorphism, M an R-module and $M_0 \subset M$ an R_0-submodule. We say that M_0 is an R_0-*lattice* in M if the natural map $R \otimes_{R_0} M_0 \to M$ is an isomorphism.

3. Ground field

Throughout the text we fix a finite field $\mathbb{F}_q$. Correspondingly the letter q stands for its cardinality. Apart from Chapter 1 we work over $\mathbb{F}_q$. The tensor product $\otimes$ and the fiber product $\times$ without subscripts mean the products over $\mathbb{F}_q$.

4. Finite products of local fields

In our context a local field always means a local field containing $\mathbb{F}_q$. Let $F = \prod_{i=1}^n F_i$ be a finite product of local fields. It will be convenient for us to treat such products in a uniform way independent of n. To do that we set up some notation and terminology.

Observe that F is a locally compact $\mathbb{F}_q$-algebra. It has a compact open subalgebra $\mathcal{O}_F = \prod_{i=1}^n \mathcal{O}_{F_i}$ which we call the *ring of integers* of F. We call an element $\pi \in \mathcal{O}_F$ a *uniformizer* if its projection to every $\mathcal{O}_{F_i}$ is a uniformizer. Observe that $F = \mathcal{O}_F[\pi^{-1}]$. By a slight abuse of notation we denote $\mathfrak{m}_F \subset \mathcal{O}_F$ the Jacobson radical of $\mathcal{O}_F$. It is the cartesian product of maximal ideals $\mathfrak{m}_{F_i} \subset \mathcal{O}_{F_i}$. Every uniformizer generates $\mathfrak{m}_F$. If F is not a single local field then $\mathfrak{m}_F$ is not maximal.

5. Mapping fiber

Definition 5.1. Let $f\colon A \to B$ be a morphism in an abelian category. The *mapping fiber of* f is the complex

$$\big[A \xrightarrow{f} B\big]$$

where A is placed in degree 0 and B in degree 1. It coincides with $\operatorname{cone}(f)[-1]$ up to sign.

We extend this definition to a morphism $f\colon A \to B$ of complexes in an abelian category in the following way. The mapping fiber complex

$$\big[A \xrightarrow{f} B\big]$$

has the object $A^n \oplus B^{n-1}$ in degree n and the differential is given by the matrix

$$\begin{pmatrix} d_A & 0 \\ f & -d_B \end{pmatrix}$$

where d_A and d_B are differentials of A respectively B.

Alternatively one can describe the mapping fiber complex as the total complex of the double complex

$$\begin{tikzcd}
{} & {} \\
A^{n+1} \arrow[u] \arrow[r, "f"] & B^{n+1} \arrow[u] \\
A^n \arrow[u, "d_A"] \arrow[r, "f"] & B^n \arrow[u, "-d_B"'] \\
{} \arrow[u] & {} \arrow[u]
\end{tikzcd}$$

where A^n is placed in bidegree $(n, 0)$ and B^n in bidegree $(n, 1)$.

Denoting the mapping fiber complex C we get a natural distinguished triangle

$$C \xrightarrow{p} A \xrightarrow{f} B \xrightarrow{-i} C[1]$$

where p is the natural projection and i is the natural embedding. The sign change for i is necessary to make the triangle distinguished.

CHAPTER 1

Shtukas

In this chapter we present a theory of shtuka cohomology together with some supplementary constructions. By itself, shtuka cohomology is nothing new. It usually appears in the form of explicit complexes such as the one of Theorem 1.9.1 or the one of Theorem 1.15.4. By contrast the point of view we take in this chapter is rather abstract. Given a scheme X and an endomorphism τ we define an abelian category of shtukas on (X, τ), prove that it has enough injectives and define a shtuka cohomology functor as the right derived functor of a certain global sections functor. This theory is developed for an arbitrary scheme X over $\operatorname{Spec} \mathbb{Z}$ and an arbitrary endomorphism τ. Assumptions on X or τ are neither necessary nor will they make the theory simpler. Our treatment of shtuka cohomology was inspired by the article [18] of V. Lafforgue and the book [3] of G. Böckle and R. Pink.

The general theory of shtuka cohomology occupies the first nine sections of this chapter. Section 1.10 introduces the important notion of nilpotence borrowed from the theory of Böckle-Pink [3]. The construction of ζ-isomorphisms in Section 1.11 is due to V. Lafforgue [18]. The material of Section 1.12 is well-known. In Section 1.13 we study a Hom shtuka construction. Theorem 1.13.4 of that section relates the cohomology of the Hom shtuka to RHom in the category of left modules over a τ-polynomial ring. This result is of central importance to our computations of shtuka cohomology in the context of Drinfeld modules.

In reading this chapter a certain degree of familiarity with derived categories will be beneficial.

1.1. Basic definitions

Definition 1.1.1. A τ-ring is a pair (R, τ) consisting of a ring R and a ring endomorphism $\tau \colon R \to R$. A morphism of τ-rings $f \colon (R, \tau) \to (S, \sigma)$ is a ring homomorphism $f \colon R \to S$ such that $f\tau = \sigma f$.

A τ-scheme is a pair (X, τ) consisting of a scheme X and an endomorphism $\tau \colon X \to X$. A morphism of τ-schemes $f \colon (X, \tau) \to (Y, \sigma)$ is a morphism of schemes $f \colon X \to Y$ such that $f\tau = \sigma f$.

As we never work with more than one τ-ring structure on a given ring R we speak of a τ-ring R instead of (R, τ) and reserve the letter τ to denote the corresponding ring endomorphism. The same applies to τ-schemes.

A typical example of a τ-scheme appearing in this text is the following. Let $\mathbb{F}_q$ be a finite field with q elements, A an $\mathbb{F}_q$-algebra and X a smooth projective curve over $\mathbb{F}_q$. We equip the product $\operatorname{Spec} A \times_{\mathbb{F}_q} X$ with the τ-scheme structure given by the endomorphism which acts as the identity on $\operatorname{Spec} A$ and as the q-Frobenius on X.

Definition 1.1.2. Let X be a τ-scheme. An $\mathcal{O}_X$-module *shtuka* is a diagram

$$\mathcal{M}_0 \underset{j}{\overset{i}{\rightrightarrows}} \mathcal{M}_1$$

where $\mathcal{M}_0$, $\mathcal{M}_1$ are $\mathcal{O}_X$-modules and

$$i\colon \mathcal{M}_0 \to \mathcal{M}_1,$$
$$j\colon \mathcal{M}_0 \to \tau_*\mathcal{M}_1$$

are morphisms of $\mathcal{O}_X$-modules. A shtuka is called *quasi-coherent* if $\mathcal{M}_0$ and $\mathcal{M}_1$ are quasi-coherent $\mathcal{O}_X$-modules. It is called *locally free* if $\mathcal{M}_0$ and $\mathcal{M}_1$ are locally free $\mathcal{O}_X$-modules **of finite rank**.

Let $\mathcal{M}$ and $\mathcal{N}$ be $\mathcal{O}_X$-module shtukas given by diagrams

$$\mathcal{M} = \Big[\mathcal{M}_0 \overset{i_M}{\underset{j_M}{\rightrightarrows}} \mathcal{M}_1\Big], \quad \mathcal{N} = \Big[\mathcal{N}_0 \overset{i_N}{\underset{j_N}{\rightrightarrows}} \mathcal{N}_1\Big].$$

A morphism from $\mathcal{M}$ to $\mathcal{N}$ is a pair (f_0, f_1) where $f_n\colon \mathcal{M}_n \to \mathcal{N}_n$ are $\mathcal{O}_X$-module morphisms such that the diagrams

$$\begin{tikzcd} \mathcal{M}_0 \arrow[r, "f_0"] \arrow[d, "i_M"] & \mathcal{N}_0 \arrow[d, "i_N"] \\ \mathcal{M}_1 \arrow[r, "f_1"] & \mathcal{N}_1 \end{tikzcd} \qquad \begin{tikzcd} \mathcal{M}_0 \arrow[r, "f_0"] \arrow[d, "j_M"] & \mathcal{N}_0 \arrow[d, "j_N"] \\ \tau_*\mathcal{M}_1 \arrow[r, "\tau_*(f_1)"] & \tau_*\mathcal{N}_1 \end{tikzcd}$$

commute.

Our definition of a shtuka differs from the ones present in the literature in that we assume no restriction on $\mathcal{M}_0$, $\mathcal{M}_1$, i, j, X and even τ. This definition is the most convenient one for our purposes. We work with arbitrary $\mathcal{O}_X$-modules instead of just the quasi-coherent ones to make our definition of shtuka cohomology compatible with the cohomology of coherent sheaves. The latter relies on resolutions by injective $\mathcal{O}_X$-modules which are not quasi-coherent in general.

1.2. The category of shtukas

Let X be a τ-scheme. In the following we denote $\operatorname{Sht}\mathcal{O}_X$ the category of $\mathcal{O}_X$-module shtukas. Strictly speaking $\operatorname{Sht}\mathcal{O}_X$ depends not only on $\mathcal{O}_X$ but also on the endomorphism τ. We drop τ from the notation since we never work with more than one τ-structure on a given scheme X. In this section we establish basic properties of the category $\operatorname{Sht}\mathcal{O}_X$.

Lemma 1.2.1. *Let X be a τ-scheme. Let $\mathcal{M}$, $\mathcal{N}$ be $\mathcal{O}_X$-module shtukas defined by diagrams*

$$\mathcal{M} = \Big[\mathcal{M}_0 \overset{i_M}{\underset{j_M}{\rightrightarrows}} \mathcal{M}_1\Big], \quad \mathcal{N} = \Big[\mathcal{N}_0 \overset{i_N}{\underset{j_N}{\rightrightarrows}} \mathcal{N}_1\Big].$$

Denote $j_M^a\colon \tau^*\mathcal{M}_0 \to \mathcal{M}_1$, $j_N^a\colon \tau^*\mathcal{N}_0 \to \mathcal{N}_1$ the adjoints of $j_M\colon \mathcal{M}_0 \to \tau_*\mathcal{M}_1$, $j_N\colon \mathcal{N}_0 \to \tau_*\mathcal{N}_1$ respectively.

Let $f_0\colon \mathcal{M}_0 \to \mathcal{N}_0$ and $f_1\colon \mathcal{M}_1 \to \mathcal{N}_1$ be morphisms of $\mathcal{O}_X$-modules. The pair (f_0, f_1) is a morphism of shtukas if and only if the squares

$$\begin{tikzcd} \mathcal{M}_0 \arrow[r, "f_0"] \arrow[d, "i_M"'] & \mathcal{N}_0 \arrow[d, "i_N"] \\ \mathcal{M}_1 \arrow[r, "f_1"] & \mathcal{N}_1 \end{tikzcd} \qquad \begin{tikzcd} \tau^*\mathcal{M}_0 \arrow[r, "\tau^*(f_0)"] \arrow[d, "j_M^a"'] & \tau^*\mathcal{N}_0 \arrow[d, "j_N^a"] \\ \mathcal{M}_1 \arrow[r, "f_1"] & \mathcal{N}_1 \end{tikzcd}$$

are commutative. □

Definition 1.2.2. Let X be a τ-scheme. We define functors from $\operatorname{Sht}\mathcal{O}_X$ to $\mathcal{O}_X$-modules: $\alpha_*[\mathcal{M}_0 \rightrightarrows \mathcal{M}_1] = \mathcal{M}_0$ and $\beta_*[\mathcal{M}_0 \rightrightarrows \mathcal{M}_1] = \mathcal{M}_1$.

Proposition 1.2.3. *Let X be a τ-scheme.*

(1) $\operatorname{Sht}\mathcal{O}_X$ *is an abelian category.*
(2) *The functors α_* and β_* are exact.*
(3) *A sequence $\mathcal{M} \to \mathcal{M}' \to \mathcal{M}''$ of $\mathcal{O}_X$-module shtukas is exact if and only if the induced sequences*

$$\alpha_*\mathcal{M} \to \alpha_*\mathcal{M}' \to \alpha_*\mathcal{M}'', \quad \beta_*\mathcal{M} \to \beta_*\mathcal{M}' \to \beta_*\mathcal{M}''.$$

are exact. □

Proof. $\operatorname{Sht}\mathcal{O}_X$ is clearly an additive category. As the functor τ_* is left exact it is straightforward to show that kernels in $\operatorname{Sht}\mathcal{O}_X$ exist and commute with α_*, β_*. In a similar way Lemma 1.2.1 and the fact that τ^* is right exact imply that cokernels exist and commute with α_*, β_*. A morphism of shtukas $f\colon \mathcal{M} \to \mathcal{N}$ is an isomorphism if and only if $\alpha_*(f)$ and $\beta_*(f)$ are isomorphisms. Therefore $\operatorname{Sht}\mathcal{O}_X$ is an abelian category. (2) and (3) are clear. □

Definition 1.2.4. Let X be a τ-scheme. We define functors α^*, β^* from the category of $\mathcal{O}_X$-modules to $\operatorname{Sht}\mathcal{O}_X$:

$$\alpha^*\mathcal{F} = \Big[\mathcal{F} \underset{(0,\eta)}{\overset{(1,0)}{\rightrightarrows}} \mathcal{F} \oplus \tau^*\mathcal{F}\Big], \quad \beta^*\mathcal{F} = \Big[0 \rightrightarrows \mathcal{F}\Big].$$

Here $\eta\colon \mathcal{F} \to \tau_*\tau^*\mathcal{F}$ is the adjunction unit.

Lemma 1.2.5. *α^* is left adjoint to α_* and β^* is left adjoint to β_*.*

Proof. The first adjunction follows from Lemma 1.2.1. The second adjunction is clear. □

The following Theorem is of fundamental importance to our treatment of shtuka cohomology. Recall that an object U of an abelian category is called a generator if for every nonzero morphism $f\colon A \to B$ there is a morphism $g\colon U \to A$ such that the composition $f \circ g$ is nonzero.

Theorem 1.2.6. *Let X be a τ-scheme.*

(1) $\operatorname{Sht}\mathcal{O}_X$ *has all colimits and filtered colimits are exact.*
(2) $\operatorname{Sht}\mathcal{O}_X$ *admits a generator.*

It is a fundamental result of Grothendieck [12] that every abelian category satisfying (1) and (2) has enough injective objects.

Proof of Theorem 1.2.6. (1) Taking the direct sum of underlying $\mathcal{O}_X$-modules one concludes that $\operatorname{Sht}\mathcal{O}_X$ has arbitrary direct sums. As it is abelian it follows that it has all colimits. By construction the functors α_* and β_* commute with colimits. Applying α_* and β_* to a colimit of $\mathcal{O}_X$-module shtukas we deduce that filtered colimits are exact in $\operatorname{Sht}\mathcal{O}_X$ from the fact that they are exact in the category of $\mathcal{O}_X$-modules.

(2) Consider the $\mathcal{O}_X$-module

$$U = \bigoplus_{V \subset X} (\iota_V)_! \mathcal{O}_V$$

where $V \subset X$ runs over all open subsets and $\iota_V : V \hookrightarrow X$ denotes the corresponding open embedding. It is easy to see that U is a generator of the category of $\mathcal{O}_X$-modules.

We claim that $\alpha^*U \oplus \beta^*U$ is a generator of $\operatorname{Sht}\mathcal{O}_X$. Let $f\colon \mathcal{M} \to \mathcal{N}$ be a morphism of $\mathcal{O}_X$-module shtukas. If $f \neq 0$ then either $\alpha_* f$ or $\beta_* f$ is nonzero, say the first one. As U is a generator there exists a morphism $g\colon U \to \alpha_*\mathcal{M}$ such that $\alpha_* f \circ g \neq 0$. As a consequence the composition of the adjoint $g^a\colon \alpha^*U \to \mathcal{M}$ and f is nonzero. □

Our treatment of shtuka cohomology relies on the notion of a K-injective complex. Recall that a complex C of objects in an abelian category is called K-injective if every morphism from an acyclic complex to C is zero up to homotopy. A bounded below complex of injective objects is K-injective. In general K-injective objects play the role of injective resolutions for unbounded complexes. The reader who does not want to bother with unbounded complexes can safely replace K-injective complexes with bounded below complexes of injective objects in all the statements of this chapter. However unbounded complexes are used in some proofs.

Corollary 1.2.7. *Let X be a τ-scheme. The category $\operatorname{Sht}\mathcal{O}_X$ has enough injectives. Every complex of $\mathcal{O}_X$-module shtukas has a K-injective resolution.*

Proof. By [079I] it follows from Theorem 1.2.6. □

1.3. Injective shtukas

If $\mathcal{I}$ is an injective shtuka then, as we demonstrate below, $\beta_*\mathcal{I}$ is an injective sheaf of modules. On the contrary $\alpha_*\mathcal{I}$ need not be injective. Nevertheless we will show that it is good enough to compute derived pushforwards.

Lemma 1.3.1. *If $\mathcal{I}$ is a K-injective complex of $\mathcal{O}_X$-module shtukas over a τ-scheme X then $\beta_*\mathcal{I}$ is a K-injective complex of $\mathcal{O}_X$-modules.*

Proof. Immediate since β_* admits an exact left adjoint β^*. □

In the following $\mathrm{K}(\mathcal{O}_X)$ stands for the homotopy category of $\mathcal{O}_X$-module complexes and $\mathrm{D}(\mathcal{O}_X)$ for the derived category.

Recall that a complex $\mathcal{F}$ of $\mathcal{O}_X$-modules on a ringed space X is called K-flat if the functor $\mathcal{F}\otimes_{\mathcal{O}_X} -$ preserves quasi-isomorphisms. A bounded above complex of flat $\mathcal{O}_X$-modules is K-flat. Spaltenstein [21] proved that every complex of $\mathcal{O}_X$-modules has a K-flat resolution.

Lemma 1.3.2. *Let X be a τ-scheme. If $\mathcal{F}$ is a K-flat complex of $\mathcal{O}_X$-modules and $\mathcal{I}$ a K-injective complex of $\mathcal{O}_X$-module shtukas then $\operatorname{Hom}_{\mathrm{K}(\mathcal{O}_X)}(\mathcal{F},\alpha_*\mathcal{I}) = \operatorname{Hom}_{\mathrm{D}(\mathcal{O}_X)}(\mathcal{F},\alpha_*\mathcal{I})$.*

Proof. Assume that $\mathcal{F}$ is acyclic. By adjunction

$$\operatorname{Hom}_{\mathrm{K}(\mathcal{O}_X)}(\mathcal{F},\alpha_*\mathcal{I}) = \operatorname{Hom}_{\mathrm{K}(\operatorname{Sht}\mathcal{O}_X)}(\alpha^*\mathcal{F},\mathcal{I})$$

where $\mathrm{K}(\operatorname{Sht}\mathcal{O}_X)$ is the homotopy category of $\mathcal{O}_X$-module shtukas.

The complex $\tau^*\mathcal{F}$ is acyclic since $\mathcal{F}$ is K-flat. As a consequence $\alpha^*\mathcal{F}$ is acyclic and the Hom on the right side of the equation is zero. Now let $\mathcal{F}$ be an arbitrary K-flat complex and $f\colon \mathcal{F}'\to\mathcal{F}$ a quasi-isomorphism of K-flat complexes. The cone of f is K-flat and acyclic. Applying $\operatorname{Hom}_{\mathrm{K}(\mathcal{O}_X)}(-,\alpha_*\mathcal{I})$ to a distinguished triangle extending f we deduce that every map $g\colon \mathcal{F}'\to\alpha_*\mathcal{I}$ in $\mathrm{K}(\mathcal{O}_X)$ factors through $\mathcal{F}$. As every $\mathcal{O}_X$-module complex admits a K-flat resolution [06YF] we conclude that $\operatorname{Hom}_{\mathrm{K}(\mathcal{O}_X)}(\mathcal{F},\alpha_*\mathcal{I}) = \operatorname{Hom}_{\mathrm{D}(\mathcal{O}_X)}(\mathcal{F},\alpha_*\mathcal{I})$. □

Lemma 1.3.3. *Let X be a τ-scheme and $f\colon X\to Y$ a morphism of schemes. If $\mathcal{I}$ is a K-injective complex of $\mathcal{O}_X$-module shtukas then the natural map $f_*\alpha_*\mathcal{I}\to \mathrm{R}f_*\alpha_*\mathcal{I}$ is a quasi-isomorphism.*

Proof. Pick a K-injective resolution $\iota\colon \alpha_*\mathcal{I}\to\mathcal{J}$ and let C be the cone of ι so that we have a distinguished triangle

$$\alpha_*\mathcal{I}\xrightarrow{\iota}\mathcal{J}\to C\to[1]$$

in $\mathrm{K}(\mathcal{O}_X)$. We need to prove that $f_*(\iota)$ is a quasi-isomorphism or equivalently that f_*C is acyclic. Let $\mathcal{F}$ be a K-flat $\mathcal{O}_Y$-module complex. Applying the functors $\operatorname{Hom}_{\mathrm{K}(\mathcal{O}_X)}(f^*\mathcal{F},-)$ and $\operatorname{Hom}_{\mathrm{D}(\mathcal{O}_X)}(f^*\mathcal{F},-)$ to the triangle above we get a morphism of long exact sequences

$$\begin{tikzcd}
\mathrm{Hom}_{\mathrm{K}(\mathcal{O}_X)}(f^*\mathcal{F}, \alpha_*\mathcal{I}) \arrow[r] \arrow[d] & \mathrm{Hom}_{\mathrm{D}(\mathcal{O}_X)}(f^*\mathcal{F}, \alpha_*\mathcal{I}) \arrow[d] \\
\mathrm{Hom}_{\mathrm{K}(\mathcal{O}_X)}(f^*\mathcal{F}, \mathcal{J}) \arrow[r] \arrow[d] & \mathrm{Hom}_{\mathrm{D}(\mathcal{O}_X)}(f^*\mathcal{F}, \mathcal{J}) \arrow[d] \\
\mathrm{Hom}_{\mathrm{K}(\mathcal{O}_X)}(f^*\mathcal{F}, C) \arrow[r] \arrow[d] & \mathrm{Hom}_{\mathrm{D}(\mathcal{O}_X)}(f^*\mathcal{F}, C) \arrow[d] \\
\vdots & \vdots
\end{tikzcd}$$

The complex $f^*\mathcal{F}$ is K-flat so the top horizontal arrow in this diagram is an isomorphism by Lemma 1.3.2. The middle horizontal arrow is an isomorphism since $\mathcal{J}$ is K-injective. Thus the five lemma shows that the bottom horizontal arrow is an isomorphism. As C is acyclic we deduce that

$$0 = \mathrm{Hom}_{\mathrm{D}(\mathcal{O}_X)}(f^*\mathcal{F}, C) = \mathrm{Hom}_{\mathrm{K}(\mathcal{O}_X)}(f^*\mathcal{F}, C) = \mathrm{Hom}_{\mathrm{K}(\mathcal{O}_Y)}(\mathcal{F}, f_*C)$$

for an arbitrary K-flat complex $\mathcal{F}$. Since the complex f_*C admits a K-flat resolution $\mathcal{F} \to f_*C$ we conclude that f_*C is acyclic. $\square$

1.4. Cohomology of shtukas

We work over a fixed τ-scheme X.

Definition 1.4.1. The *ring of invariants* $\mathcal{O}_X(X)^{\tau=1}$ is $\{s \mid \tau(s) = s\} \subset \mathcal{O}_X(X)$.

The category of $\mathcal{O}_X$-module shtukas is $\mathcal{O}_X(X)^{\tau=1}$-linear by construction.

Definition 1.4.2. The *unit shtuka* $\mathbb{1}_X$ is defined by the diagram

$$\mathcal{O}_X \overset{1}{\underset{\tau^\sharp}{\rightrightarrows}} \mathcal{O}_X$$

where $\tau^\sharp \colon \mathcal{O}_X \to \tau_*\mathcal{O}_X$ is the homomorphism of sheaves of rings determined by τ.

Definition 1.4.3. We define the *cohomology functor* $\mathrm{R}\Gamma(X, -)$ from the derived category of $\mathrm{Sht}\,\mathcal{O}_X$ to the derived category of $\mathcal{O}_X(X)^{\tau=1}$-modules as follows:

$$\mathrm{R}\Gamma(X, \mathcal{M}) = \mathrm{RHom}(\mathbb{1}_X, \mathcal{M}).$$

We call $\mathrm{R}\Gamma(X, \mathcal{M})$ the *cohomology complex* of $\mathcal{M}$ or simply the cohomology of $\mathcal{M}$. The n-th cohomology module of $\mathrm{R}\Gamma(X, \mathcal{M})$ is denoted $\mathrm{H}^n(X, \mathcal{M})$. By construction $\mathrm{H}^n(X, \mathcal{M}) = \mathrm{Ext}^n(\mathbb{1}_X, \mathcal{M})$.

Let $\mathcal{M}$ be an $\mathcal{O}_X$-module shtuka given by a diagram

$$\mathcal{M}_0 \overset{i}{\underset{j}{\rightrightarrows}} \mathcal{M}_1$$

The arrows of $\mathcal{M}$ determine natural maps

$$i, j \colon \Gamma(X, \mathcal{M}_0) \to \Gamma(X, \mathcal{M}_1)$$

with the same source and target. In the case of j we identify $\Gamma(X, \tau_*\mathcal{M}_1)$ with $\Gamma(X, \mathcal{M}_1)$ using the fact that $\tau^{-1}X = X$. Observe that j is only $\mathcal{O}_X(X)^{\tau=1}$-linear since the natural identification $\Gamma(X, \mathcal{M}_1) = \Gamma(X, \tau_*\mathcal{M}_1)$ is τ-linear.

Proposition 1.4.4. $\mathrm{H}^0(X, \mathcal{M}) = \{\, s \in \Gamma(X, \mathcal{M}_0) \mid i(s) = j(s) \,\}$.

Proof. A morphism $f\colon \mathbb{1}_X \to \mathcal{M}$ is a pair of maps $f_0\colon \mathcal{O}_X \to \mathcal{M}_0$, $f_1\colon \mathcal{O}_X \to \mathcal{M}_1$ such that $i \circ f_0 = f_1$ and $j \circ f_0 = \tau_*(f_1) \circ \tau^\sharp$. The pair (f_0, f_1) is determined by the section $s = f_0(1)$ of $\Gamma(X, \mathcal{M}_0)$ which satisfies the equation $i(s) = j(s)$. □

1.5. Canonical triangle

The constructions of this section are due to V. Lafforgue [18, Section 4].

Definition 1.5.1. We define a morphism $\delta\colon \beta^*\mathcal{O}_X \to \alpha^*\mathcal{O}_X$ by the diagram

$$\begin{tikzcd} 0 \arrow[r] \arrow[d, shift right] \arrow[d, shift left] & \mathcal{O}_X \arrow[d, shift right, "{(1,0)}"'] \arrow[d, shift left, "{(0,\tau^\sharp)}"] \\ \mathcal{O}_X \arrow[r, "{(1,-1)}"] & \mathcal{O}_X \oplus \mathcal{O}_X \end{tikzcd}$$

Lemma 1.5.2. *For every $\mathcal{O}_X$-module shtuka $\mathcal{M}$ the diagram*

$$\begin{tikzcd} \mathrm{Hom}(\alpha^*\mathcal{O}_X, \mathcal{M}) \arrow[r, "\delta\circ-"] \arrow[d, equal] & \mathrm{Hom}(\beta^*\mathcal{O}_X, \mathcal{M}) \arrow[d, equal] \\ \Gamma(X,\, \alpha_*\mathcal{M}) \arrow[r, "i-j"] & \Gamma(X,\, \beta_*\mathcal{M}) \end{tikzcd}$$

is commutative. □

Lemma 1.5.3. *For every complex of $\mathcal{O}_X$-module shtukas the maps*

$$\mathrm{RHom}(\alpha^*\mathcal{O}_X, \mathcal{M}) \to \mathrm{R}\Gamma(X,\, \alpha_*\mathcal{M})$$
$$\mathrm{RHom}(\beta^*\mathcal{O}_X, \mathcal{M}) \to \mathrm{R}\Gamma(X,\, \beta_*\mathcal{M})$$

induced by the natural identifications

$$\mathrm{Hom}(\alpha^*\mathcal{O}_X, \mathcal{M}) = \Gamma(X,\, \alpha_*\mathcal{M})$$
$$\mathrm{Hom}(\beta^*\mathcal{O}_X, \mathcal{M}) = \Gamma(X,\, \beta_*\mathcal{M})$$

are quasi-isomorphisms.

Proof. Without loss of generality we assume that $\mathcal{M}$ is K-injective. The result for β is then immediate since $\beta_*\mathcal{M}$ is K-injective by Lemma 1.3.1. Applying Lemma 1.3.3 to the map $X \to \operatorname{Spec}\mathbb{Z}$ we conclude that $\Gamma(X,\, \alpha_*\mathcal{M}) = \mathrm{R}\Gamma(X,\, \alpha_*\mathcal{M})$. □

Let $\mathcal{M}$ be a complex of $\mathcal{O}_X$-module shtukas. The arrows of the shtukas of $\mathcal{M}$ determine natural maps

$$i, j\colon\ \mathrm{R}\Gamma(X, \alpha_*\mathcal{M}) \to \mathrm{R}\Gamma(X, \beta_*\mathcal{M}),$$

in the following way. The first map is induced by the i-arrows. The j-arrows induce a map $\mathrm{R}\Gamma(X, \alpha_*\mathcal{M}) \to \mathrm{R}\Gamma(X, \tau_*\beta_*\mathcal{M})$. Taking its composition with the natural map $\mathrm{R}\Gamma(X, \tau_*\beta_*\mathcal{M}) \to \mathrm{R}\Gamma(X, \mathrm{R}\tau_*\beta_*\mathcal{M})$ and using the identity $\mathrm{R}\Gamma(X, \beta_*\mathcal{M}) = \mathrm{R}\Gamma(X, \mathrm{R}\tau_*\beta_*\mathcal{M})$ we get a map of the desired form.

Lemma 1.5.4. *For every complex of $\mathcal{O}_X$-module shtukas $\mathcal{M}$ the square*

$$\begin{tikzcd} \mathrm{RHom}(\alpha^*\mathcal{O}_X,\, \mathcal{M}) \arrow[r, "\delta\circ-"] \arrow[d, "\wr"'] & \mathrm{RHom}(\beta^*\mathcal{O}_X,\, \mathcal{M}) \arrow[d, "\wr"] \\ \mathrm{R}\Gamma(X,\, \alpha_*\mathcal{M}) \arrow[r, "i-j"] & \mathrm{R}\Gamma(X,\, \beta_*\mathcal{M}) \end{tikzcd}$$

is commutative.

Proof. Without loss of generality we may assume that $\mathcal{M}$ is K-injective. The result then follows from Lemmas 1.5.3 and 1.5.2. □

Definition 1.5.5. We define a morphism $\rho\colon \alpha^*\mathcal{O}_X \to \mathbb{1}_X$ by the diagram

$$\begin{tikzcd}
\mathcal{O}_X \arrow[rr, "1"] \arrow[d, shift right, "(1{,}0)"'] \arrow[d, shift left, "(0{,}\tau^\sharp)"] & & \mathcal{O}_X \arrow[d, shift right, "1"'] \arrow[d, shift left, "\tau^\sharp"] \\
\mathcal{O}_X \oplus \mathcal{O}_X \arrow[rr, "\Sigma"] & & \mathcal{O}_X
\end{tikzcd}$$

Definition 1.5.6. We denote $\mathbb{U}_X = \operatorname{cone}\big(\beta^*\mathcal{O}_X \xrightarrow{-\delta} \alpha^*\mathcal{O}_X\big)$. We define a map $\pi\colon \mathbb{U}_X \to \beta^*\mathcal{O}_X[1]$ by the identity in degree -1 and a map $\widetilde{\rho}\colon \mathbb{U}_X \to \mathbb{1}_X[0]$ by ρ in degree 0.

Lemma 1.5.7. *$\widetilde{\rho}$ is a quasi-isomorphism.* □

Proposition 1.5.8. *The triangle*

$$\mathbb{1}_X[-1] \xrightarrow{-\pi\widetilde{\rho}^{-1}[-1]} \beta^*\mathcal{O}_X[0] \xrightarrow{\delta} \alpha^*\mathcal{O}_X[0] \xrightarrow{\rho} \mathbb{1}_X[0] \tag{1.1}$$

in $\mathrm{D}(\mathrm{Sht}\,\mathcal{O}_X)$ *is distinguished.*

The following definition is central to our theory:

Definition 1.5.9. Let $\mathcal{M}$ be a complex of $\mathcal{O}_X$-module shtukas. We define the *canonical distinguished triangle*

$$\mathrm{R}\Gamma(X,\,\mathcal{M}) \longrightarrow \mathrm{R}\Gamma(X,\,\alpha_*\mathcal{M}) \xrightarrow{i-j} \mathrm{R}\Gamma(X,\,\beta_*\mathcal{M}) \longrightarrow [1]$$

by applying $\mathrm{RHom}(-,\mathcal{M})$ to the triangle (1.1) and using the identifications of Lemma 1.5.4.

Proof of Proposition 1.5.8. Let $\iota\colon \mathbb{U}_X \to \alpha^*\mathcal{O}_X[0]$ be the map given by the identity in degree 0. By definition of $\mathbb{U}_X$ the triangle

$$\beta^*\mathcal{O}_X[0] \xrightarrow{-\delta} \alpha^*\mathcal{O}_X[0] \xrightarrow{\iota} \mathbb{U}_X \xrightarrow{-\pi} \beta^*\mathcal{O}_X[1]$$

is distinguished [014P]. Hence so is the isomorphic triangle

$$\beta^*\mathcal{O}_X[0] \xrightarrow{-\delta} \alpha^*\mathcal{O}_X[0] \xrightarrow{\rho} \mathbb{1}_X[0] \xrightarrow{-\pi\widetilde{\rho}^{-1}} \beta^*\mathcal{O}_X[1]$$

Rotating it we obtain a triangle

$$\mathbb{1}_X[-1] \xrightarrow{\pi\widetilde{\rho}^{-1}[-1]} \beta^*\mathcal{O}_X[0] \xrightarrow{-\delta} \alpha^*\mathcal{O}_X[0] \xrightarrow{\rho} \mathbb{1}_X[0]$$

which is isomorphic to (1.1). □

1.6. Associated complex

It will often be convenient for us to view the functor $\mathrm{R}\Gamma$ on shtukas not as the derived functor of $\mathrm{Hom}(\mathbb{1}_X,-)$ but as the derived functor of the so-called *associated complex* functor which we now introduce

Definition 1.6.1. Let $\mathcal{M}$ be a complex of $\mathcal{O}_X$-module shtukas. We define

$$\Gamma_{\mathrm{a}}(X,\,\mathcal{M}) = \mathrm{Hom}^\bullet(\mathbb{U}_X,\,\mathcal{M})$$

and call $\Gamma_{\mathrm{a}}(X,\,\mathcal{M})$ the *associated complex* of $\mathcal{M}$. Here $\mathrm{Hom}^\bullet$ is the Hom complex [09L9]. We define a natural morphism $\Gamma_{\mathrm{a}}(X,\,\mathcal{M}) \to \mathrm{R}\Gamma(X,\,\mathcal{M})$ as the composition

$$\mathrm{Hom}^\bullet(\mathbb{U}_X,\,\mathcal{M}) \longrightarrow \mathrm{RHom}(\mathbb{U}_X,\,\mathcal{M}) \xrightarrow{\widetilde{\rho}^{-1}\circ -} \mathrm{RHom}(\mathbb{1}_X,\,\mathcal{M})$$

where $\widetilde{\rho}$ is the quasi-isomorphism of Definition 1.5.6.

Example. Let an $\mathcal{O}_X$-module shtuka $\mathcal{M}$ be given by a diagram

$$\mathcal{M}_0 \underset{j}{\overset{i}{\rightrightarrows}} \mathcal{M}_1.$$

Regarding $\mathcal{M}$ as a complex of shtukas concentrated in degree 0 we have

$$\Gamma_{\mathrm{a}}(X,\,\mathcal{M}) = \Big[\Gamma(X,\,\mathcal{M}_0) \xrightarrow{i-j} \Gamma(X,\,\mathcal{M}_1)\Big].$$

The square brackets denote the mapping fiber complex of Chapter "Notation and conventions".

Lemma 1.6.2. *Let $\mathcal{M}$ be a complex of $\mathcal{O}_X$-module shtukas. Denote $\mathcal{M}^n$ the shtuka in degree n. Then $\Gamma_{\mathrm{a}}(X,\,\mathcal{M})$ coincides with the total complex of the double complex*

$$\begin{tikzcd}
{} & {} \\
\Gamma(X,\,\alpha_*\mathcal{M}^{n+1}) \arrow[u] \arrow[r, "(-1)^{n+1}(i-j)"] & \Gamma(X,\,\beta_*\mathcal{M}^{n+1}) \arrow[u] \\
\Gamma(X,\,\alpha_*\mathcal{M}^{n}) \arrow[u] \arrow[r, "(-1)^{n}(i-j)"] & \Gamma(X,\,\beta_*\mathcal{M}^{n}) \arrow[u] \\
{} \arrow[u] & {} \arrow[u]
\end{tikzcd}$$

The object $\Gamma(X,\,\alpha_*\mathcal{M}^n)$ is placed in the bidegree $(n,0)$ while $\Gamma(X,\,\beta_*\mathcal{M}^n)$ is in the bidegree $(n,1)$. The vertical maps are the differentials of $\Gamma(X,\,\alpha_*\mathcal{M})$ respectively $\Gamma(X,\,\beta_*\mathcal{M})$. The maps i and j are induced by the arrows of the shtukas $\mathcal{M}^n$.

Proof. In view of Lemma 1.5.2 it follows from the definition of $\mathrm{Hom}^\bullet$ [09L9]. $\square$

Definition 1.6.3. Let $\mathcal{M}$ be a complex of $\mathcal{O}_X$-module shtukas. We define a triangle

$$\Gamma_{\mathrm{a}}(X,\,\mathcal{M}) \xrightarrow{p} \Gamma(X,\,\alpha_*\mathcal{M}) \xrightarrow{i-j} \Gamma(X,\,\beta_*\mathcal{M}) \xrightarrow{-q} \Gamma_{\mathrm{a}}(X,\,\mathcal{M})[1] \tag{1.2}$$

as follows. Denote $\mathcal{M}^n$ the shtuka in degree n. According to Lemma 1.6.2 the object of $\Gamma_{\mathrm{a}}(X,\mathcal{M})$ in degree n is

$$\Gamma(X,\alpha_*\mathcal{M}^n) \oplus \Gamma(X,\beta_*\mathcal{M}^{n-1}).$$

The morphism p is the natural projection. The morphism q is defined by the formula $b \mapsto (0,(-1)^n b)$ in degree n. The maps i and j are induced by the arrows of the shtukas $\mathcal{M}^n$.

Proposition 1.6.4. *For every complex of $\mathcal{O}_X$-module shtukas $\mathcal{M}$ the following holds:*

(1) *The triangle* (1.2) *is distinguished.*
(2) *The natural diagram*

$$\begin{tikzcd}
\Gamma_{\mathrm{a}}(X,\,\mathcal{M}) \arrow[r, "p"] \arrow[d] & \Gamma(X,\,\alpha_*\mathcal{M}) \arrow[r, "i-j"] \arrow[d] & \Gamma(X,\,\beta_*\mathcal{M}) \arrow[r, "-q"] \arrow[d] & {[1]} \\
\mathrm{R}\Gamma(X,\,\mathcal{M}) \arrow[r] & \mathrm{R}\Gamma(X,\alpha_*\mathcal{M}) \arrow[r, "i-j"] & \mathrm{R}\Gamma(X,\,\beta_*\mathcal{M}) \arrow[r] & {[1]}
\end{tikzcd}$$

is a morphism of triangles. Here the bottom row is the canonical triangle of Definition 1.5.9, the first vertical map is the one of Definition 1.6.1, the second and the third are the natural ones.
(3) *If $\mathcal{M}$ is K-injective then the diagram* (2) *is an isomorphism.*

In particular the triangle (1.2) gives an explicit description of the canonical triangle of Definition 1.5.9 in the case when $\mathcal{M}$ is K-injective.

Proof of Proposition 1.6.4. Consider a commutative diagram

$$\begin{tikzcd} \mathbb{U}_X[-1] \arrow[r, "-\pi"] \arrow[d, "\widetilde{\rho}[-1]"', "\wr"] & \beta^*\mathcal{O}_X[0] \arrow[r, "\delta"] \arrow[d, equal] & \alpha^*\mathcal{O}_X[0] \arrow[r, "\iota"] \arrow[d, equal] & \mathbb{U}_X \arrow[d, "\widetilde{\rho}", "\wr"'] \\ \mathbb{1}_X[-1] \arrow[r, "-\pi\widetilde{\rho}^{-1}[-1]"] & \beta^*\mathcal{O}_X[0] \arrow[r, "\delta"] & \alpha^*\mathcal{O}_X[0] \arrow[r, "\rho"] & \mathbb{1}_X[0] \end{tikzcd} \tag{1.3}$$

The map ι is given by the identity in degree 0. The bottom triangle is distinguished by Proposition 1.5.8. Hence so is the top one.

We next apply $\mathrm{Hom}^\bullet(-, \mathcal{M})$ to the top row. We use the map given by $(-1)^n$ in degree n to identify $\mathrm{Hom}^\bullet(-[-1], \mathcal{M})$ with $\mathrm{Hom}^\bullet(-, \mathcal{M})[1]$. A straightforward computation shows that the resulting triangle is (1.2). In particular this triangle is distinguished.

Applying $\mathrm{RHom}(-, \mathcal{M})$ to (1.3) and using Lemma 1.5.4 we obtain an isomorphism from the canonical distinguished triangle of $\mathcal{M}$ to a distinguished triangle

$$\mathrm{RHom}(\mathbb{U}_X, \mathcal{M}) \longrightarrow \mathrm{R\Gamma}(X, \alpha_*\mathcal{M}) \xrightarrow{i-j} \mathrm{R\Gamma}(X, \beta_*\mathcal{M}) \longrightarrow [1].$$

Composing the inverse of this isomorphism with the map of triangles induced by the natural morphism $\mathrm{Hom}^\bullet(-, \mathcal{M}) \to \mathrm{RHom}(-, \mathcal{M})$ we get the diagram (2). It is therefore a morphism of triangles.

If $\mathcal{M}$ is K-injective then the third vertical arrow in (2) is a quasi-isomorphism by Lemma 1.3.1. Applying Lemma 1.3.3 to the natural map $X \to \operatorname{Spec} \mathbb{Z}$ we conclude that the second vertical arrow of (2) is a quasi-isomorphism. Hence so is the first one, and the diagram (2) is an isomorphism of triangles. $\square$

1.7. Pushforward

We work with a fixed morphism of τ-schemes $f\colon X \to Y$.

Definition 1.7.1. Let $\mathcal{M}$ be an $\mathcal{O}_X$-module shtuka given by a diagram

$$\mathcal{M}_0 \underset{j}{\overset{i}{\rightrightarrows}} \mathcal{M}_1.$$

Define

$$f_*\mathcal{M} = \Big[f_*\mathcal{M}_0 \underset{f_*j}{\overset{f_*i}{\rightrightarrows}} f_*\mathcal{M}_1 \Big].$$

Here we use the natural isomorphism $f_*\tau_*\mathcal{M}_1 = \tau_* f_*\mathcal{M}_1$ to interpret f_*j as a map to $\tau_* f_*\mathcal{M}_1$.

Definition 1.7.2. The functor f_* on the category of $\mathcal{O}_X$-module shtukas is left exact. We define $\mathrm{R}f_*$ as its right derived functor.

Lemma 1.7.3. *The natural maps $\alpha_*\mathrm{R}f_* \to \mathrm{R}f_*\alpha_*$ and $\beta_*\mathrm{R}f_* \to \mathrm{R}f_*\beta_*$ are quasi-isomorphisms.*

Proof. Let $\mathcal{M}$ be a K-injective complex of $\mathcal{O}_X$-module shtukas. Lemma 1.3.3 shows that the natural map $f_*\alpha_*\mathcal{M} \to \mathrm{R}f_*\alpha_*\mathcal{M}$ is a quasi-isomorphism. However $f_*\alpha_*\mathcal{M} = \alpha_* f_*\mathcal{M} = \alpha_*\mathrm{R}f_*\mathcal{M}$ and we get the result for α_*. The result for β_* follows in a similar way since the complex $\beta_*\mathcal{I}$ is K-injective by Lemma 1.3.1. $\square$

Let $\mathcal{M}$ be a complex of $\mathcal{O}_X$-module shtukas. We have a natural isomorphism $\mathrm{Hom}(\mathbb{1}_X, \mathcal{M}) = \mathrm{Hom}(\mathbb{1}_Y, f_*\mathcal{M})$. It induces a map $\mathrm{R\Gamma}(X, \mathcal{M}) \to \mathrm{R\Gamma}(Y, \mathrm{R}f_*\mathcal{M})$. Furthermore we have a natural quasi-isomorphism

$$\mathrm{R\Gamma}(X, \alpha_*\mathcal{M}) \xrightarrow{\sim} \mathrm{R\Gamma}(Y, \mathrm{R}f_*\alpha_*\mathcal{M}) \xrightarrow{\sim} \mathrm{R\Gamma}(Y, \alpha_*\mathrm{R}f_*\mathcal{M})$$

where the second arrow is the quasi-isomorphism of Lemma 1.7.3. In a similar way we have a natural quasi-isomorphism $\mathrm{R}\Gamma(X,\, \beta_*\mathcal{M}) \xrightarrow{\sim} \mathrm{R}\Gamma(Y,\, \beta_*\mathrm{R}f_*\mathcal{M})$.

Proposition 1.7.4. *For every complex $\mathcal{M}$ of $\mathcal{O}_X$-module shtukas the natural map $\mathrm{R}\Gamma(X,\, \mathcal{M}) \to \mathrm{R}\Gamma(Y,\, \mathrm{R}f_*\mathcal{M})$ is a quasi-isomorphism. Moreover the natural diagram*

$$\begin{tikzcd} \mathrm{R}\Gamma(X,\, \mathcal{M}) \arrow[r] \arrow[d] & \mathrm{R}\Gamma(X,\, \alpha_*\mathcal{M}) \arrow[r, "i-j"] \arrow[d] & \mathrm{R}\Gamma(X,\, \beta_*\mathcal{M}) \arrow[r] \arrow[d] & {[1]} \\ \mathrm{R}\Gamma(Y,\, \mathrm{R}f_*\mathcal{M}) \arrow[r] & \mathrm{R}\Gamma(Y,\, \alpha_*\mathrm{R}f_*\mathcal{M}) \arrow[r, "i-j"] & \mathrm{R}\Gamma(Y,\, \beta_*\mathrm{R}f_*\mathcal{M}) \arrow[r] & {[1]} \end{tikzcd} \tag{1.4}$$

is an isomorphism of distinguished triangles.

Proof. Without loss of generality we assume that $\mathcal{M}$ is K-injective so that $f_*\mathcal{M} = \mathrm{R}f_*\mathcal{M}$. Let $\mathcal{I}$ be a K-injective resolution of $f_*\mathcal{M}$. The map $f_*\mathcal{M} \to \mathcal{I}$ induces a morphism of triangles (1.2):

$$\begin{tikzcd} \Gamma_{\mathrm{a}}(Y,\, f_*\mathcal{M}) \arrow[r] \arrow[d] & \Gamma(Y,\, \alpha_*f_*\mathcal{M}) \arrow[r, "i-j"] \arrow[d] & \Gamma(Y,\, \beta_*f_*\mathcal{M}) \arrow[r] \arrow[d] & {[1]} \\ \Gamma_{\mathrm{a}}(Y,\, \mathcal{I}) \arrow[r] & \Gamma(Y,\, \alpha_*\mathcal{I}) \arrow[r, "i-j"] & \Gamma(Y,\, \beta_*\mathcal{I}) \arrow[r] & {[1]} \end{tikzcd} \tag{1.5}$$

The top triangle coincides with the triangle (1.2) for $\Gamma_{\mathrm{a}}(X,\, \mathcal{M})$. Proposition 1.6.4 then identifies the diagram (1.5) with the diagram (1.4). Whence the result. $\square$

1.8. Pullback

Let $f\colon X \to Y$ be a morphism of τ-schemes and let

$$\mathcal{M} = \Big[\mathcal{M}_0 \overset{i}{\underset{j}{\rightrightarrows}} \mathcal{M}_1\Big]$$

be an $\mathcal{O}_Y$-module shtuka.

Definition 1.8.1. We set

$$f^*\mathcal{M} = \Big[f^*\mathcal{M}_0 \overset{f^*i}{\underset{\mu\circ f^*j}{\rightrightarrows}} f^*\mathcal{M}_1\Big]$$

where μ is the base change map $f^*\tau_*\mathcal{M}_1 \to \tau_*f^*\mathcal{M}_1$ arising from the commutative square

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\tau"] & Y \arrow[d, "\tau"] \\ X \arrow[r, "f"] & Y. \end{tikzcd}$$

We will use the following notation:

- For an affine τ-scheme $X = \operatorname{Spec} R$ we denote $\mathcal{M}(R)$ the R-module shtuka
$$\Gamma(X, f^*\mathcal{M}_0) \overset{f^*i}{\underset{\mu\circ f^*j}{\rightrightarrows}} \Gamma(X, f^*\mathcal{M}_1)$$
arising by pullback of $\mathcal{M}$ along $f\colon X \to Y$.
- To make the expressions more legible we will generally write $\mathrm{R}\Gamma(X, \mathcal{M})$ in place of $\mathrm{R}\Gamma(X, f^*\mathcal{M})$.

Lemma 1.8.2. *There exists a unique adjunction*

$$\mathrm{Hom}_{\mathrm{Sht}\,\mathcal{O}_X}(f^*-,-) \cong \mathrm{Hom}_{\mathrm{Sht}\,\mathcal{O}_Y}(-,f_*-)$$

which is compatible with the adjunction

$$\mathrm{Hom}_{\mathcal{O}_X}(f^*-,-) \cong \mathrm{Hom}_{\mathcal{O}_Y}(-,f_*-)$$

through the natural maps given by functors α_* and β_*. ☐

Definition 1.8.3. We define the *pullback map*

$$\mathrm{R}\Gamma(Y,\mathcal{M}) \xrightarrow{f^*} \mathrm{R}\Gamma(X,f^*\mathcal{M})$$

in the following way. Let $\eta\colon \mathcal{M} \to f_*f^*\mathcal{M}$ be the adjunction unit. Taking its composition with the natural map $f_*f^*\mathcal{M} \to \mathrm{R}f_*f^*\mathcal{M}$ and applying $\mathrm{R}\Gamma(Y,-)$ we obtain a map from $\mathrm{R}\Gamma(Y,\mathcal{M})$ to $\mathrm{R}\Gamma(Y,\mathrm{R}f_*f^*\mathcal{M})$. Proposition 1.7.4 identifies $\mathrm{R}\Gamma(Y,\mathrm{R}f_*f^*\mathcal{M})$ with $\mathrm{R}\Gamma(X,f^*\mathcal{M})$. The resulting map from $\mathrm{R}\Gamma(Y,\mathcal{M})$ to $\mathrm{R}\Gamma(X,f^*\mathcal{M})$ is the pullback map.

Observe that $\alpha_*f^*\mathcal{M} = f^*\alpha_*\mathcal{M}$ and $\beta_*f^*\mathcal{M} = f^*\beta_*\mathcal{M}$ by construction. So we have natural pullback maps $\mathrm{R}\Gamma(Y,\alpha_*\mathcal{M}) \to \mathrm{R}\Gamma(X,\alpha_*f^*\mathcal{M})$ and $\mathrm{R}\Gamma(Y,\beta_*\mathcal{M}) \to \mathrm{R}\Gamma(X,\beta_*f^*\mathcal{M})$.

Proposition 1.8.4. *For every $\mathcal{O}_Y$-module shtuka $\mathcal{M}$ the natural diagram*

$$\begin{tikzcd}
\mathrm{R}\Gamma(Y,\,\mathcal{M}) \arrow[r] \arrow[d] & \mathrm{R}\Gamma(Y,\,\alpha_*\mathcal{M}) \arrow[r, "i-j"] \arrow[d] & \mathrm{R}\Gamma(Y,\,\beta_*\mathcal{M}) \arrow[r] \arrow[d] & {[1]} \\
\mathrm{R}\Gamma(X,\,f^*\mathcal{M}) \arrow[r] & \mathrm{R}\Gamma(X,\,\alpha_*f^*\mathcal{M}) \arrow[r, "i-j"] & \mathrm{R}\Gamma(X,\,\beta_*f^*\mathcal{M}) \arrow[r] & {[1]}
\end{tikzcd} \tag{1.6}$$

is a morphism of distinguished triangles.

Proof. The natural map $\mathcal{M} \to \mathrm{R}f_*f^*\mathcal{M}$ induces a morphism of distinguished triangles

$$\begin{tikzcd}
\mathrm{R}\Gamma(Y,\,\mathcal{M}) \arrow[r] \arrow[d] & \mathrm{R}\Gamma(Y,\,\alpha_*\mathcal{M}) \arrow[r, "i-j"] \arrow[d] & \mathrm{R}\Gamma(Y,\,\beta_*\mathcal{M}) \arrow[r] \arrow[d] & {[1]} \\
\mathrm{R}\Gamma(Y,\,\mathrm{R}f_*f^*\mathcal{M}) \arrow[r] & \mathrm{R}\Gamma(Y,\,\alpha_*\mathrm{R}f_*f^*\mathcal{M}) \arrow[r, "i-j"] & \mathrm{R}\Gamma(Y,\,\beta_*\mathrm{R}f_*f^*\mathcal{M}) \arrow[r] & {[1]}
\end{tikzcd} \tag{1.7}$$

At the same time Proposition 1.7.4 states that the natural diagram

$$\begin{tikzcd}
\mathrm{R}\Gamma(X,\,f^*\mathcal{M}) \arrow[r] \arrow[d, "\wr"] & \mathrm{R}\Gamma(X,\,\alpha_*f^*\mathcal{M}) \arrow[r, "i-j"] \arrow[d, "\wr"] & \mathrm{R}\Gamma(X,\,\beta_*f^*\mathcal{M}) \arrow[r] \arrow[d, "\wr"] & {[1]} \\
\mathrm{R}\Gamma(Y,\,\mathrm{R}f_*f^*\mathcal{M}) \arrow[r] & \mathrm{R}\Gamma(Y,\,\alpha_*\mathrm{R}f_*f^*\mathcal{M}) \arrow[r, "i-j"] & \mathrm{R}\Gamma(Y,\,\beta_*\mathrm{R}f_*f^*\mathcal{M}) \arrow[r] & {[1]}
\end{tikzcd} \tag{1.8}$$

is an isomorphism of distinguished triangles. A quick inspection shows that the composition of (1.7) and the inverse of (1.8) gives the diagram (1.6). ☐

1.9. Shtukas over affine schemes

It follows from Definition 1.1.2 that a quasi-coherent shtuka $\mathcal{M}$ on an affine τ-scheme $X = \operatorname{Spec} R$ is given by a diagram

$$M_0 \overset{i}{\underset{j}{\rightrightarrows}} M_1$$

where M_0, M_1 are R-modules, $i\colon M_0 \to M_1$ an R-module homomorphism and $j\colon M_0 \to M_1$ a τ-linear R-module homomorphism: for all $r \in R$ and $m \in M_0$ one has $j(rm) = \tau(r)j(m)$. The associated complex of $\mathcal{M}$ is

$$\Gamma_{\mathrm{a}}(X, \mathcal{M}) = \Big[M_0 \xrightarrow{i-j} M_1\Big].$$

We will show that this complex computes the cohomology of $\mathcal{M}$.

Theorem 1.9.1. *If $\mathcal{M}$ is a quasi-coherent shtuka over an affine τ-scheme X then the natural map $\Gamma_{\mathrm{a}}(X, \mathcal{M}) \to \mathrm{R}\Gamma(X, \mathcal{M})$ is a quasi-isomorphism.*

Proof. By Proposition 1.6.4 the natural map in question extends to a morphism of distinguished triangles

$$\begin{tikzcd} \Gamma_{\mathrm{a}}(X,\, \mathcal{M}) \arrow[r] \arrow[d] & \Gamma(X,\, \mathcal{M}_0) \arrow[r, "i-j"] \arrow[d] & \Gamma(X,\, \mathcal{M}_1) \arrow[r] \arrow[d] & {[1]} \\ \mathrm{R}\Gamma(X,\, \mathcal{M}) \arrow[r] & \mathrm{R}\Gamma(X,\, \mathcal{M}_0) \arrow[r, "i-j"] & \mathrm{R}\Gamma(X,\, \mathcal{M}_1) \arrow[r] & {[1]} \end{tikzcd}$$

As $\mathcal{M}_0$ and $\mathcal{M}_1$ are quasi-coherent $\mathcal{O}_X$-modules over an affine scheme X the complexes $\mathrm{R}\Gamma(X,\, \mathcal{M}_0)$ and $\mathrm{R}\Gamma(X,\, \mathcal{M}_1)$ are concentrated in degree 0 [01XB]. Hence the second and third vertical maps in the diagram above are quasi-isomorphism. It follows that so is the first map. $\square$

To make the expressions more legible we will often write $\mathrm{R}\Gamma(R,\, \mathcal{M})$ instead of $\mathrm{R}\Gamma(\operatorname{Spec} R,\, \mathcal{M})$. If there is no ambiguity in the choice of R then we further shorten it to $\mathrm{R}\Gamma(\mathcal{M})$. For a quasi-coherent shtuka $\mathcal{M}$ we identify $\mathrm{R}\Gamma(\mathcal{M})$ with $\Gamma_{\mathrm{a}}(X,\, \mathcal{M})$ using the Theorem above.

1.10. Nilpotence

The notion of nilpotence for shtukas is crucial to this work. It first appeared in the book of Böckle-Pink [3] in the context of τ-sheaves.

Definition 1.10.1. Let X be a τ-scheme. An $\mathcal{O}_X$-module shtuka

$$\mathcal{M}_0 \overset{i}{\underset{j}{\rightrightarrows}} \mathcal{M}_1$$

is called *nilpotent* if i is an isomorphism and the composition

$$\mathcal{M}_0 \xrightarrow{\mu} \tau_*\mathcal{M}_0 \xrightarrow{\tau_*\mu} \tau_*^2\mathcal{M}_0 \to \ldots \to \tau_*^n\mathcal{M}_0, \quad \mu = \tau_*(i^{-1}) \circ j, \tag{1.9}$$

is zero for some $n \geqslant 1$.

Proposition 1.10.2. *Let $f\colon X \to Y$ be a morphism of τ-schemes and let $\mathcal{M}$ be an $\mathcal{O}_Y$-module shtuka. If $\mathcal{M}$ is nilpotent then $f^*\mathcal{M}$ is nilpotent.*

Proof. Without loss of generality we assume that

$$\mathcal{M} = \Big[\mathcal{M}_0 \overset{1}{\underset{j}{\rightrightarrows}} \mathcal{M}_0\Big].$$

Let $j^a\colon \tau^*\mathcal{M}_0 \to \mathcal{M}_0$ be the adjoint of j. Using the naturality of the adjunction $\tau^* \leftrightarrow \tau_*$ it is easy to show that (1.9) is zero if and only if the composition

$$\tau^{*n}(\mathcal{M}_0) \xrightarrow{\tau^{*(n-1)}(j^a)} \tau^{*(n-1)}(\mathcal{M}_0) \to \ldots \to \tau^*(\mathcal{M}_0) \xrightarrow{j^a} \mathcal{M}_0$$

is zero. Taking the pullback by f and using the equality $\tau \circ f = f \circ \tau$ we get the result. $\square$

Proposition 1.10.3. *Let $\mathcal{M}$ be an $\mathcal{O}_X$-module shtuka over a τ-scheme X. If $\mathcal{M}$ is nilpotent then $\mathrm{R}\Gamma(X,\, \mathcal{M}) = 0$.*

Proof. Without loss of generality we assume that $\mathcal{M}$ is given by a diagram

$$\mathcal{M}_0 \overset{1}{\underset{j}{\rightrightarrows}} \mathcal{M}_0.$$

As $\mathcal{M}$ is nilpotent one easily deduces that the endomorphism of $\mathrm{R}\Gamma(\mathcal{M}_0)$ induced by j is nilpotent. As a consequence the endomorphism $1-j$ is a quasi-isomorphism. Now the canonical trinangle of $\mathrm{R}\Gamma(\mathcal{M})$ shows that $\mathrm{R}\Gamma(\mathcal{M})$ is the mapping fiber of $1-j$ so the result follows. □

The following proposition is our main tool to deduce vanishing of cohomology.

Proposition 1.10.4. *Let R be a Noetherian τ-ring complete with respect to an ideal $I \subset R$. Assume that $\tau(I) \subset I$ so that τ descends to the quotient R/I. Let $\mathcal{M}$ be a locally free R-module shtuka. If $\mathcal{M}(R/I)$ is nilpotent then the following holds:*

(1) $\mathrm{R}\Gamma(\mathcal{M}) = 0$.
(2) *For every $n > 0$ the shtuka $\mathcal{M}(R/I^n)$ is nilpotent.*

Proof. Suppose that $\mathcal{M}$ is given by a diagram

$$M_0 \overset{i}{\underset{j}{\rightrightarrows}} M_1.$$

The ideal I is in the Jacobson radical of R so Nakayama's lemma implies that i is surjective. Now i splits since M_1 is projective and one more application of Nakayama's lemma shows that i is an isomorphism.

The endomorphism $i^{-1}j$ of M_0 preserves the filtration by powers of I. Furthermore $(i^{-1}j)^m M_0 \subset IM_0$ for some $m \geqslant 0$ since $\mathcal{M}(R/I)$ is nilpotent. As a consequence $(i^{-1}j)^{mn} M_0 \subset I^n M_0$ and we get (2). Moreover (2) implies that $1 - i^{-1}j$ is an isomorphism modulo every power of I. Since M_0 is I-adically complete we deduce that $1 - i^{-1}j$ is an isomorphism. As i is an isomorphism the claim (1) now follows from Theorem 1.9.1. □

1.11. The linearization functor and ζ-isomorphisms

The constructions of this section are due to V. Lafforgue [18] but the terminology and the notation is our own. The notion of a ζ-isomorphism is at the heart of our approach to the class number formula.

Definition 1.11.1. Let X be a τ-scheme. We define the *linearization functor* ∇ from $\operatorname{Sht}\mathcal{O}_X$ to $\operatorname{Sht}\mathcal{O}_X$ as follows:

$$\nabla\Big[\mathcal{M}_0 \overset{i}{\underset{j}{\rightrightarrows}} \mathcal{M}_1\Big] = \Big[\mathcal{M}_0 \overset{i}{\underset{0}{\rightrightarrows}} \mathcal{M}_1\Big].$$

We say that an $\mathcal{O}_X$-module shtuka $[\mathcal{M}_0 \overset{i}{\underset{j}{\rightrightarrows}} \mathcal{M}_1]$ is *linear* if $j = 0$.

The complex $\mathrm{R}\Gamma(X, \nabla\mathcal{M})$ is often easier to compute than $\mathrm{R}\Gamma(X, \mathcal{M})$. Even though the complexes $\mathrm{R}\Gamma(X, \mathcal{M})$ and $\mathrm{R}\Gamma(X, \nabla\mathcal{M})$ are very different in general, a link beween them exists under some natural assumptions on $\mathcal{M}$.

Fix a subring $A \subset \mathcal{O}_X(X)^{\tau=1}$ and assume that $\mathrm{R}\Gamma(X, \mathcal{M})$ is a perfect complex of A-modules. The theory of Knudsen-Mumford [17] associates to $\mathrm{R}\Gamma(X, \mathcal{M})$ an invertible A-module[1] $\det_A \mathrm{R}\Gamma(X, \mathcal{M})$. This determinant is functorial in quasi-isomorphisms. If the A-module complexes $\mathrm{R}\Gamma(X, \mathcal{M})$, $\mathrm{R}\Gamma(X, \mathcal{M}_0)$ and $\mathrm{R}\Gamma(X, \mathcal{M}_1)$

[1]Strictly speaking the determinant is a pair (L, α) consisting of an invertible A-module L and a continuous function $\alpha\colon \operatorname{Spec} A \to \mathbb{Z}$. This function is not important for the following discussion so we ignore it.

are bounded and their cohomology modules are perfect then the canonical distinguished triangle

$$\mathrm{R}\Gamma(X, \mathcal{M}) \to \mathrm{R}\Gamma(X, \mathcal{M}_0) \xrightarrow{i-j} \mathrm{R}\Gamma(X, \mathcal{M}_1) \to [1].$$

determines a natural A-module isomorphism

$$\det_A \mathrm{R}\Gamma(X, \mathcal{M}) \xrightarrow{\sim} \det_A \mathrm{R}\Gamma(X, \mathcal{M}_0) \otimes_A \det_A^{-1} \mathrm{R}\Gamma(X, \mathcal{M}_1)$$

[17, Corollary 2 after Theorem 2].

Definition 1.11.2. Let X be a τ-scheme and let $A \subset \mathcal{O}_X(X)^{\tau=1}$ be a subring. Let $\mathcal{M}$ be an $\mathcal{O}_X$-module shtuka given by a diagram

$$\mathcal{M}_0 \underset{j}{\overset{i}{\rightrightarrows}} \mathcal{M}_1.$$

We say that the *ζ-isomorphism is defined for $\mathcal{M}$* if the complexes of A-modules $\mathrm{R}\Gamma(X, \mathcal{M})$, $\mathrm{R}\Gamma(X, \nabla\mathcal{M})$, $\mathrm{R}\Gamma(X, \mathcal{M}_0)$ and $\mathrm{R}\Gamma(X, \mathcal{M}_1)$ are bounded and their cohomology modules are perfect. Under this assumption we define the *ζ-isomorphism*

$$\zeta_{\mathcal{M}} \colon \det_A \mathrm{R}\Gamma(X, \mathcal{M}) \xrightarrow{\sim} \det_A \mathrm{R}\Gamma(X, \nabla\mathcal{M})$$

as the composition

$$\det_A \mathrm{R}\Gamma(X, \mathcal{M}) \xrightarrow{\sim} \det_A \mathrm{R}\Gamma(X, \mathcal{M}_0) \otimes_A \det_A^{-1} \mathrm{R}\Gamma(X, \mathcal{M}_1) \xleftarrow{\sim} \det_A \mathrm{R}\Gamma(X, \nabla\mathcal{M})$$

of isomorphisms determined by the canonical triangles

$$\mathrm{R}\Gamma(X, \mathcal{M}) \to \mathrm{R}\Gamma(X, \mathcal{M}_0) \xrightarrow{i-j} \mathrm{R}\Gamma(X, \mathcal{M}_1) \to [1],$$
$$\mathrm{R}\Gamma(X, \nabla\mathcal{M}) \to \mathrm{R}\Gamma(X, \mathcal{M}_0) \xrightarrow{i} \mathrm{R}\Gamma(X, \mathcal{M}_1) \to [1].$$

1.12. τ-polynomials

In this section we work with a fixed τ-ring R.

Definition 1.12.1. We define the ring $R\{\tau\}$ as follows. Its elements are formal polynomials $r_0 + r_1\tau + r_2\tau^2 + \ldots r_n\tau^n$, $r_0, \ldots r_n \in R$, $n \geqslant 0$. The multiplication in $R\{\tau\}$ is subject to the following identity: for every $r \in R$ we have

$$\tau \cdot r = r^\tau \cdot \tau$$

where $r^\tau = \tau(r)$ is the image of r under $\tau \colon R \to R$.

Unlike all the other rings in this text the ring $R\{\tau\}$ is not commutative in general. Still it is associative and has the multiplicative unit 1. Left $R\{\tau\}$-modules are directly related to R-module shtukas.

Definition 1.12.2. Let M be a left $R\{\tau\}$-module. The *R-module shtuka associated to M* is

$$M \underset{\tau}{\overset{1}{\rightrightarrows}} M.$$

Here $\tau \colon M \to M$ is the τ-multiplication map. It is tautologically τ-linear so that the diagram above indeed defines a shtuka.

Let M be a left $R\{\tau\}$-module. In this section we denote $a \colon R\{\tau\} \otimes_R M \to M$ the map which sends a tensor $\varphi \otimes m$ to $\varphi \cdot m$. The letter a stands for "action".

Lemma 1.12.3. *If M is a left $R\{\tau\}$-module then the sequence of left $R\{\tau\}$-modules*

$$0 \to R\{\tau\}\tau \otimes_R M \xrightarrow{d} R\{\tau\} \otimes_R M \xrightarrow{a} M \to 0$$

is exact. Here $d(\varphi\tau \otimes m) = \varphi \otimes \tau \cdot m - \varphi\tau \otimes m$.

Proof. It is clear that $a \circ d = 0$ and a is surjective. Let us verify the injectivity of d. The modules $R\{\tau\}$, $R\{\tau\}\tau$ carry filtrations by degree of τ-polynomials. The map d is compatible with the induced filtrations on $R\{\tau\}\tau \otimes_R M$ and $R\{\tau\} \otimes_R M$ and is injective on subquotients. It is therefore injecitve.

Let us verify the exactness of the sequence at $R\{\tau\} \otimes_R M$. Consider the quotient of $R\{\tau\} \otimes_R M$ by the image of d. In this quotient we have the identity $r\tau^{n+1} \otimes m \equiv r\tau^n \otimes \tau m$ for all $r \in R$, $m \in M$, $n \geqslant 0$. As a consequence $\varphi \otimes m \equiv \varphi \cdot m$ for every $\varphi \in R\{\tau\}$. Hence every element $y \in R\{\tau\} \otimes_R M$ is equivalent to $1 \otimes a(y)$. If $a(y) = 0$ then $y \equiv 0$ or in other words y is in the image of d. □

Remark 1.12.4. Let M be an R-module. We denote τ^*M the R-module $R^\tau \otimes_R M$ where R^τ is R with the R-algebra structure given by the homomorphism $\tau \colon R \to R$. We write the elements of τ^*M as sums of pure tensors $r \otimes m$, $r \in R^\tau$, $m \in M$. If $r, r_1 \in R$ and $m \in M$ then

$$r \otimes r_1 m = r\tau(r_1) \otimes m.$$

The ring R acts on $\tau^*M = R^\tau \otimes_R M$ via the factor $R^\tau = R$. If $r, r_1 \in R$ and $m \in M$ then

$$r_1 \cdot (r \otimes m) = r_1 r \otimes m.$$

Lemma 1.12.5. *Let M be an R-module. The maps*

$$R\{\tau\}\tau \otimes_R M \to R\{\tau\} \otimes_R \tau^*M, \quad \varphi\tau \otimes m \mapsto \varphi \otimes (1 \otimes m)$$

and

$$R\{\tau\} \otimes_R \tau^*M \to R\{\tau\}\tau \otimes_R M, \quad \varphi \otimes (r \otimes m) \mapsto \varphi r\tau \otimes m$$

are mutually inverse isomorphisms of left $R\{\tau\}$-modules. □

Proposition 1.12.6. *If M is a left $R\{\tau\}$-module then the sequence of left $R\{\tau\}$-modules*

$$0 \to R\{\tau\} \otimes_R \tau^*M \xrightarrow{1\otimes\tau^a - \eta} R\{\tau\} \otimes_R M \xrightarrow{a} M \to 0$$

is exact. Here $\tau^a \colon \tau^*M \to M$ is the adjoint of the τ-multiplication map $M \to \tau_*M$ and η is the map given by the formula

$$\eta \colon \varphi \otimes (r \otimes m) \mapsto \varphi r\tau \otimes m.$$

Proof. Using the isomorphism $R\{\tau\}\tau \otimes_R M \cong R\{\tau\} \otimes_R \tau^*M$ of Lemma 1.12.5 we rewrite the short sequence in question as

$$0 \to R\{\tau\}\tau \otimes_R M \xrightarrow{d} R\{\tau\} \otimes_R M \xrightarrow{a} M \to 0.$$

An easy computation shows that

$$d(\varphi\tau \otimes m) = \varphi \otimes \tau \cdot m - \varphi\tau \otimes m.$$

The result thus follows from Lemma 1.12.3. □

1.13. The Hom shtuka in an affine setting

Let R be a τ-ring and let M and N be R-module shtukas. In this section we introduce the Hom shtuka $\mathcal{H}\mathrm{om}_R(M, N)$. To some extent it behaves like an internal Hom in the category of R-module shtukas. It is literally the internal Hom for shtukas which come from left $R\{\tau\}$-modules. Even if both M and N are left $R\{\tau\}$-modules, $\mathcal{H}\mathrm{om}_R(M, N)$ is in general a genuine shtuka which does not come from a left $R\{\tau\}$-module. Apart from the Drinfeld construction of Chapter 7 the $\mathcal{H}\mathrm{om}$ construction is the main source of nontrivial shtukas in the present work.

Definition 1.13.1. Let M and N be R-module shtukas given by diagrams

$$M = \Big[M_0 \overset{i_M}{\underset{j_M}{\rightrightarrows}} M_1\Big], \quad N = \Big[N_0 \overset{i_N}{\underset{j_N}{\rightrightarrows}} N_1\Big].$$

The *Hom shtuka* $\mathcal{H}\mathrm{om}_R(M, N)$ is defined by the diagram

$$\mathrm{Hom}_R(M_1, N_0) \overset{i}{\underset{j}{\rightrightarrows}} \mathrm{Hom}_R(\tau^* M_0, N_1)$$

where

$$i(f) = i_N \circ f \circ j_M^a,$$
$$j(f) = (j_N \circ f)^a \circ \tau^*(i_M),$$

j_M^a is the τ-adjoint of j_M and $(-)^a$ denotes the adjunction $\mathrm{Hom}_R(M_1, \tau_* N_1) = \mathrm{Hom}_R(\tau^* M_1, N_1)$.

The adjunction $\mathrm{Hom}_R(M_1, \tau_* N_1) = \mathrm{Hom}_R(\tau^* M_1, N_1)$ is τ-linear so the diagram above indeed defines a shtuka.

If M and N are left $R\{\tau\}$-modules then $\mathcal{H}\mathrm{om}_R(M, N)$ means $\mathcal{H}\mathrm{om}_R$ applied to the R-module shtukas associated to M and N as in Definition 1.12.2. The Hom shtukas we work with are typically of this sort. We will also need $\mathcal{H}\mathrm{om}_R(M, N)$ in the case when M is an R-module shtuka which does not come from a left $R\{\tau\}$-module (cf. Section 9.5).

In the rest of this section we use the notation of Remark 1.12.4 for the elements of $\tau^* M$.

Lemma 1.13.2. *If M and N are left $R\{\tau\}$-modules then $\mathcal{H}\mathrm{om}_R(M, N)$ is represented by the diagram*

$$\mathrm{Hom}_R(M, N) \overset{i}{\underset{j}{\rightrightarrows}} \mathrm{Hom}_R(\tau^* M, N),$$
$$i(f) = f \circ \tau_M^a, \quad j(f) = \tau_N^a \circ \tau^*(f)$$

where τ_M^a and τ_N^a are the adjoints of the τ-multiplication maps. In other words

$$i(f)\colon r \otimes m \mapsto f(r\tau \cdot m), \quad j(f)\colon r \otimes m \mapsto r\tau \cdot f(m). \quad \square$$

Our goal is to describe the cohomology of the shtuka $\mathcal{H}\mathrm{om}_R(M, N)$ in the case when M and N are left $R\{\tau\}$-modules.

Proposition 1.13.3. *If M and N are left $R\{\tau\}$-modules then*

$$\mathrm{Hom}_{R\{\tau\}}(M, N) = \mathrm{H}^0(\mathcal{H}\mathrm{om}_R(M, N))$$

as abelian subgroups of $\mathrm{Hom}_R(M, N)$.

Proof. Let i and j be the arrows of $\mathcal{H}\mathrm{om}_R(M, N)$. Let $f \in \mathrm{Hom}_R(M, N)$. Lemma 1.13.2 implies that $i(f) = j(f)$ if and only if f commutes with τ. $\square$

If M is projective as an R-module then the functor $N \mapsto \mathcal{H}\mathrm{om}_R(M, N)$ on the category of left $R\{\tau\}$-modules is exact. So the isomorphism of Proposition 1.13.3 induces a natural map $\mathrm{RHom}_{R\{\tau\}}(M, N) \to \mathrm{R\Gamma}(\mathcal{H}\mathrm{om}_R(M, N))$.

Theorem 1.13.4. *Let M and N be left $R\{\tau\}$-modules. If M is projective as an R-module then the natural map* $\mathrm{RHom}_{R\{\tau\}}(M, N) \to \mathrm{R\Gamma}(\mathcal{H}\mathrm{om}_R(M, N))$ *is a quasi-isomorphism*

Proof. By Proposition 1.12.6 we have a short exact sequence

$$0 \to R\{\tau\} \otimes_R \tau^* M \xrightarrow{1\otimes\tau_M^a - \eta} R\{\tau\} \otimes_R M \xrightarrow{a} M \to 0. \tag{1.10}$$

If M is a projective R-module then so is $\tau^* M$. As a consequence $R\{\tau\} \otimes_R M$ and $R\{\tau\} \otimes_R \tau^* M$ are projective left $R\{\tau\}$-modules. Thus (1.10) is a projective resolution of M as a left $R\{\tau\}$-module. Applying $\mathrm{Hom}_{R\{\tau\}}(-, N)$ to (3) we conclude that

$$\mathrm{RHom}_{R\{\tau\}}(M, N) = \Big[\mathrm{Hom}_R(M, N) \xrightarrow{(1\otimes\tau_M^a)^* - \eta^*} \mathrm{Hom}_R(\tau^* M, N)\Big]. \tag{1.11}$$

where $*$ indicates the induced maps. Consider the shtuka

$$\mathcal{H}\mathrm{om}_R(M, N) = \Big[\mathrm{Hom}_R(M, N) \underset{j}{\overset{i}{\rightrightarrows}} \mathrm{Hom}_R(\tau^* M, N)\Big].$$

According to Lemma 1.13.2 the maps i and $(1 \otimes \tau_M^a)^*$ coincide. A straightforward computation shows that $\eta^* = j$. Therefore (1.11) computes $\mathrm{R}\Gamma(\mathcal{H}\mathrm{om}_R(M, N))$ by Theorem 1.9.1. $\square$

1.14. A global variant of the Hom sthuka

Let X be a τ-scheme and let $\mathcal{M}$ and $\mathcal{N}$ be $\mathcal{O}_X$-module shtukas. In this section define the $\mathcal{O}_X$-module Hom shtuka $\underline{\mathcal{H}\mathrm{om}}_X(\mathcal{M}, \mathcal{N})$, a global variant of the construction of the previous section. This construction is important to Chapter 12.

Given $\mathcal{O}_X$-modules $\mathcal{E}$ and $\mathcal{F}$ we denote $\underline{\mathcal{H}\mathrm{om}}_X(\mathcal{E}, \mathcal{F})$ the sheaf Hom [01CM].

Definition 1.14.1. Let $\mathcal{M}$ and $\mathcal{N}$ be $\mathcal{O}_X$-module shtukas given by diagrams

$$\mathcal{M} = \Big[\mathcal{M}_0 \underset{j_{\mathcal{M}}}{\overset{i_{\mathcal{M}}}{\rightrightarrows}} \mathcal{M}_1\Big], \quad \mathcal{N} = \Big[\mathcal{N}_0 \underset{j_{\mathcal{N}}}{\overset{i_{\mathcal{N}}}{\rightrightarrows}} \mathcal{N}_1\Big].$$

The *Hom shtuka* $\underline{\mathcal{H}\mathrm{om}}_X(\mathcal{M}, \mathcal{N})$ is defined by the diagram

$$\underline{\mathcal{H}\mathrm{om}}_X(\mathcal{M}_1, \mathcal{N}_0) \underset{j}{\overset{i}{\rightrightarrows}} \underline{\mathcal{H}\mathrm{om}}_X(\tau^* \mathcal{M}_0, \mathcal{N}_1).$$

The arrow i is the compostion

$$\underline{\mathcal{H}\mathrm{om}}_X(\mathcal{M}_1, \mathcal{N}_0) \xrightarrow{i_{\mathcal{N}} \circ -} \underline{\mathcal{H}\mathrm{om}}_X(\mathcal{M}_1, \mathcal{N}_1) \xrightarrow{- \circ j_{\mathcal{M}}^a} \underline{\mathcal{H}\mathrm{om}}_X(\tau^* \mathcal{M}_0, \mathcal{N}_1)$$

where $j_{\mathcal{M}}^a$ is the τ-adjoint of $j_{\mathcal{M}}$. The arrow j is defined by the diagram

$$\begin{tikzcd}
\underline{\mathcal{H}\mathrm{om}}_X(\mathcal{M}_1, \mathcal{N}_0) \arrow[r, "j"] \arrow[d, "j_{\mathcal{N}} \circ -"'] & \tau_* \underline{\mathcal{H}\mathrm{om}}_X(\tau^* \mathcal{M}_0, \mathcal{N}_1) \\
\underline{\mathcal{H}\mathrm{om}}_X(\mathcal{M}_1, \tau_* \mathcal{N}_1) \arrow[r, "\sim"] & \tau_* \underline{\mathcal{H}\mathrm{om}}_X(\tau^* \mathcal{M}_1, \mathcal{N}_1), \arrow[u, "\tau_*(h)"']
\end{tikzcd}$$

the unlabelled arrow is the natural adjunction isomorphism and h is given by composition with $\tau^*(i_{\mathcal{M}}) \colon \tau^* \mathcal{M}_0 \to \tau^* \mathcal{M}_1$.

Lemma 1.14.2. *Let $X = \operatorname{Spec} R$ be an affine τ-scheme and let M and N be R-module shtukas. Denote $\mathcal{M}$ and $\mathcal{N}$ the $\mathcal{O}_X$-module shtukas corresponding to M and N respectively. If the underlying R-modules of M are of finite type then $\underline{\mathcal{H}\mathrm{om}}_X(\mathcal{M}, \mathcal{N})$ is the $\mathcal{O}_X$-module shtuka associated to $\mathcal{H}\mathrm{om}_R(M, N)$.* $\square$

1.15. Special τ-structure

The last section of this chapter is devoted to a technical result which will only be used in Section 6.4. Let X be a τ-scheme. We assume that τ acts as the identity on the underlying topological space. The most important case for our applications is $X = \operatorname{Spec}\Lambda \times_{\mathbb{F}_q} X_0$ where X_0 is an $\mathbb{F}_q$-scheme, Λ a finite artinian $\mathbb{F}_q$-algebra and $\tau\colon X \to X$ the endomorphism which acts as the identity on Λ and as the q-Frobenius on X_0.

Let $\mathcal{F}$ be an $\mathcal{O}_X$-module. Since τ acts as the identity on the underlying topological space we can identify $\mathcal{F}$ and $\tau_*\mathcal{F}$ as sheaves of abelian groups. We can thus make the following construction. Let $\mathcal{M}$ be a shtuka on X given by a diagram

$$\mathcal{M}_0 \underset{j}{\overset{i}{\rightrightarrows}} \mathcal{M}_1.$$

We associate to it a complex of $\mathcal{O}_X(X)^{\tau=1}$-module sheaves

$$(*) \qquad \big[\mathcal{M}_0 \xrightarrow{i-j} \mathcal{M}_1\big]$$

on the underlying topological space of X. We identify $\tau_*\mathcal{M}_1$ and $\mathcal{M}_1$ as sheaves and view $j\colon \mathcal{M}_0 \to \tau_*\mathcal{M}_1$ as a morphism from $\mathcal{M}_0$ to $\mathcal{M}_1$.

Our goal is to show that the sheaf cohomology of $(*)$ coincides with the shtuka cohomology of $\mathcal{M}$. It is useful to first extend the definition of $(*)$ to complexes of shtukas.

Definition 1.15.1. Let $\mathcal{M}$ be a complex of $\mathcal{O}_X$-module shtukas. We define a complex of $\mathcal{O}_X(X)^{\tau=1}$-module sheaves $\mathcal{G}_a(\mathcal{M})$ as the total complex of the double complex

$$\begin{tikzcd}
 & \\
\alpha_*\mathcal{M}^{n+1} \arrow[u] \arrow[r, "(-1)^{n+1}(i-j)"] & \beta_*\mathcal{M}^{n+1} \arrow[u] \\
\alpha_*\mathcal{M}^{n} \arrow[u] \arrow[r, "(-1)^{n}(i-j)"] & \beta_*\mathcal{M}^{n} \arrow[u] \\
 \arrow[u] & \arrow[u]
\end{tikzcd}$$

The vertical maps are the differentials of $\alpha_*\mathcal{M}$ respectively $\beta_*\mathcal{M}$. The object $\alpha_*\mathcal{M}^n$ is placed in the bidegree $(n,0)$ while $\beta_*\mathcal{M}^n$ is in the bidegree $(n,1)$.

Definition 1.15.2. We define a natural triangle

$$\mathcal{G}_a(\mathcal{M}) \xrightarrow{p} \alpha_*\mathcal{M} \xrightarrow{i-j} \beta_*\mathcal{M} \xrightarrow{-q} [1]$$

as follows. The object of $\mathcal{G}_a(\mathcal{M})$ in degree n is $\alpha_*\mathcal{M}^n \oplus \beta_*\mathcal{M}^{n-1}$. The morphism p is the projection to the first factor. The morphism q is the injection to the second factor multiplied by $(-1)^n$.

Example. If $\mathcal{M}$ is a single shtuka placed in degree 0 then the triangle above is the canonical distinguished triangle

$$\big[\mathcal{M}_0 \xrightarrow{i-j} \mathcal{M}_1\big] \longrightarrow \mathcal{M}_0 \xrightarrow{i-j} \mathcal{M}_1 \longrightarrow [1]$$

of the mapping fiber complex.

Lemma 1.15.3. *The triangle of Definition 1.15.2 is distinguished.*

Proof. The sequence

$$0 \to \beta_*\mathcal{M}[-1] \xrightarrow{q[-1]} \mathcal{G}_a(\mathcal{M}) \xrightarrow{p} \alpha_*\mathcal{M} \to 0$$

is exact and is termwise split. Such a sequence determines a distinguished triangle in the following way. Let r be the splitting of $q[-1]$ given by the formula $(a, b) \mapsto (-1)^n b$ in degree $n+1$ and let s be the splitting of p given by the formula $a \mapsto (a, 0)$. Let $f = r \circ d \circ s$ where d is the differential of $\mathcal{G}_a(\mathcal{M})$. The triangle

$$\beta_*\mathcal{M}[-1] \xrightarrow{q[-1]} \mathcal{G}_a(\mathcal{M}) \xrightarrow{p} \alpha_*\mathcal{M} \xrightarrow{f} \beta_*\mathcal{M}$$

is distinguished [014Q]. An easy computation reveals that $f = i - j$. Rotating this triangle we conclude that (1.2) is distinguished. □

Lemma 1.6.2 implies that $\Gamma_{\mathrm{a}}(X, \mathcal{M}) = \Gamma(X, \mathcal{G}_a(\mathcal{M}))$. Moreover the functor $\mathcal{G}_a$ is exact on the level of homotopy categories. So taking the derived functors we obtain a canonical morphism $\mathrm{R}\Gamma(X, \mathcal{M}) \to \mathrm{R}\Gamma(X, \mathcal{G}_a(\mathcal{M}))$.

Theorem 1.15.4. *Assume that $\tau\colon X \to X$ acts as the identity on the underlying topological space. For every complex of $\mathcal{O}_X$-module shtukas $\mathcal{M}$ the natural map $\mathrm{R}\Gamma(X, \mathcal{M}) \to \mathrm{R}\Gamma(X, \mathcal{G}_a(\mathcal{M}))$ is a quasi-isomorphism. Furthermore it extends to an isomorphism of distinguished triangles*

$$\begin{tikzcd}
\mathrm{R}\Gamma(X, \mathcal{M}) \arrow[r] \arrow[d, "\wr"] & \mathrm{R}\Gamma(X, \alpha_*\mathcal{M}) \arrow[r, "i-j"] \arrow[d, equal] & \mathrm{R}\Gamma(X, \beta_*\mathcal{M}) \arrow[r] \arrow[d, equal] & {[1]} \\
\mathrm{R}\Gamma(X, \mathcal{G}_a(\mathcal{M})) \arrow[r] & \mathrm{R}\Gamma(X, \alpha_*\mathcal{M}) \arrow[r, "i-j"] & \mathrm{R}\Gamma(X, \beta_*\mathcal{M}) \arrow[r] & {[1]}
\end{tikzcd}$$

Here the top row is the canonical triangle and the bottom row is obtained by applying $\mathrm{R}\Gamma(X, -)$ *to the triangle of Definition 1.15.2.*

In a way this result justifies the use of notation $\mathrm{R}\Gamma$ for shtuka cohomology. With some caution one may think of a shtuka as a two term complex (∗). Shtuka cohomology is then the usual sheaf cohomology of this complex.

Proof of Theorem 1.15.4. Without loss of generality we can assume that the complex $\mathcal{M}$ is K-injective. Consider the diagram

$$\begin{tikzcd}
\Gamma(X, \mathcal{G}_a(\mathcal{M})) \arrow[r] \arrow[d] & \Gamma(X, \alpha_*\mathcal{M}) \arrow[r, "i-j"] \arrow[d] & \Gamma(X, \beta_*\mathcal{M}) \arrow[r] \arrow[d] & {[1]} \\
\mathrm{R}\Gamma(X, \mathcal{G}_a(\mathcal{M})) \arrow[r] & \mathrm{R}\Gamma(X, \alpha_*\mathcal{M}) \arrow[r, "i-j"] & \mathrm{R}\Gamma(X, \beta_*\mathcal{M}) \arrow[r] & {[1]}
\end{tikzcd}$$

where the rows are obtained by applying $\Gamma(X, -)$ respectively $\mathrm{R}\Gamma(X, -)$ to the distinguished triangle of Definition 1.15.2 and the vertical arrows are the natural morphisms $\Gamma(X, -) \to \mathrm{R}\Gamma(X, -)$. By construction the top row coincides with the distinguished triangle

$$\Gamma_{\mathrm{a}}(X, \mathcal{M}) \to \Gamma(X, \alpha_*\mathcal{M}) \xrightarrow{i-j} \Gamma(X, \beta_*\mathcal{M}) \to [1]$$

of Definition 1.6.3. The diagram above is a morphism of distinguished triangles by naturality. Now $\mathcal{M}$ is K-injective so Proposition 1.6.4 identifies the top row with the canonical distinguished triangle

$$\mathrm{R}\Gamma(X, \mathcal{M}) \to \mathrm{R}\Gamma(X, \alpha_*\mathcal{M}) \xrightarrow{i-j} \mathrm{R}\Gamma(X, \beta_*\mathcal{M}) \to [1].$$

and the second and third vertical arrows with the identity morphisms. It follows that the first vertical arrow is a quasi-isomorphism. □

CHAPTER 2

Topological vector spaces over finite fields

In this chapter we present some results on topological vector spaces over finite fields. The base field $\mathbb{F}_q$ is fixed throughout the chapter. It is assumed to carry the discrete topology. In the following a subspace of a vector space always means an $\mathbb{F}_q$-vector subspace, not an arbitrary topological subspace. As is usual in the theory of topological groups all our locally compact topological vector spaces are assumed to be Hausdorff. A topological vector space is said to be *linearly topologized* if every open neighbourhood of zero contains an open subspace. We mainly study linearly topologized Hausdorff vector spaces. Throughout this chapter we abbreviate "linearly topologized Hausdorff" as "*lth*".

2.1. Overview

In our computations of shtuka cohomology we will extensively use various spaces of continuous $\mathbb{F}_q$-linear functions and germs of such functions. This chapter is devoted to their study.

Let V and W be locally compact topological $\mathbb{F}_q$-vector spaces. We consider the following function spaces:

- the space $c(V, W)$ of continuous $\mathbb{F}_q$-linear maps from V to W,
- the space $b(V, W)$ of bounded continuous $\mathbb{F}_q$-linear maps, i.e. the maps which have image in a compact subspace,
- the space $a(V, W)$ of locally constant bounded $\mathbb{F}_q$-linear maps.

The function spaces are equipped with topologies which make them into complete vector spaces. These topologies are carefully chosen to suit the applications. The space $c(V, W)$ carries the compact-open topology. The topologies on its dense subspaces $a(V, W)$ and $b(V, W)$ are finer than the induced ones.

An important object related to the function spaces is the space of germs $g(V, W)$. Its elements are equivalence classes of continuous $\mathbb{F}_q$-linear maps from V to W. Two such maps are equivalent if they restrict to the same map on an open subspace. The main property of $g(V, W)$ is invariance under local isomorphisms on the source V and the target W. This property will be indispensable for cohomological computations of Chapter 8 among others.

To describe the structure of the function spaces we employ two topological tensor products:

- the completed tensor product $V \mathbin{\widehat{\otimes}} W = \lim_{U,Y} V/U \otimes W/Y$.
- the ind-complete tensor product $V \mathbin{\check{\otimes}} W = \lim_{U,Y} (V \otimes W)/(U \otimes Y)$.

Here $U \subset V$ and $Y \subset W$ run over all open subspaces. The tensor products $\widehat{\otimes}$ and $\check{\otimes}$ are closely related: we will show that the natural commutative square

$$\begin{tikzcd}
V \mathbin{\check{\otimes}} W \arrow[r] \arrow[d] & V \mathbin{\widehat{\otimes}} W^{\#} \arrow[d] \\
V^{\#} \mathbin{\widehat{\otimes}} W \arrow[r] & V \mathbin{\widehat{\otimes}} W
\end{tikzcd}$$

is cartesian in the category of topological vector spaces (Proposition 2.7.6). Here $(-)^{\#}$ denotes the same vector space taken with the discrete topology.

Using the tensor products above we will construct natural topological isomorphisms

$$\begin{aligned} c(V,W) &\cong V^* \mathbin{\widehat{\otimes}} W && \text{(Proposition 2.8.5)},\\ b(V,W) &\cong (V^*)^{\#} \mathbin{\widehat{\otimes}} W && \text{(Proposition 2.9.4)},\\ a(V,W) &\cong V^* \mathbin{\check{\otimes}} W && \text{(Proposition 2.10.3)} \end{aligned}$$

where V^* is the continuous $\mathbb{F}_q$-linear dual of V.

Function spaces, germ spaces and the topological tensor products $\widehat{\otimes}$, $\check{\otimes}$ figure prominently in this text. Past this chapter some degree of familiarity with them will be assumed.

Much of the material in this chapter is largely well-known. However a reservation should be made about the ind-complete tensor product $V \mathbin{\check{\otimes}} W$, the function spaces $a(V,W)$, $b(V,W)$ and the germ space $g(V,W)$. While these constructions are very natural and should have certainly appeared before, we are not aware of a reference for them in the literature.

Last but not least we should acknowledge our intellectual debt to G. W. Anderson. The inspiration for this chapter comes from his article [1], specifically from §2 of that text where he uses function spaces to compute what in retrospect is the cohomology of certain shtukas associated to Drinfeld modules.

2.2. Examples

To lend this discussion more substance let us give some examples. We begin with a few examples of locally compact vector spaces:

- $A = \mathbb{F}_q[t]$ with the discrete topology,
- $F = \mathbb{F}_q((t^{-1}))$ with the locally compact topology,
- the compact open subspace $\mathcal{O}_F = \mathbb{F}_q[[t^{-1}]]$ in F.

The quotient F/A is naturally a compact $\mathbb{F}_q$-vector space.

Let K be a local field containing $\mathbb{F}_q$. An example of a function space which is particularly relevant to our study is $c(F/A, K)$, the space of continuous $\mathbb{F}_q$-linear maps from F/A to K. Proposition 2.8.5 provides us with a topological isomorphism

$$(F/A)^* \mathbin{\widehat{\otimes}} K \cong c(F/A, K).$$

Let us show that this isomorphism gives a rather hands-on description of $c(F/A,K)$. Let $\Omega = \mathbb{F}_q[t]\,dt$ be the module of Kähler differentials of A over $\mathbb{F}_q$ equipped with the discrete topology and let

$$\operatorname{res}\colon \Omega \otimes_A F \to \mathbb{F}_q, \quad \sum_n a_n t^n \cdot dt \mapsto -a_{-1}$$

be the residue map at infinity. The map

$$\Omega \to (F/A)^*, \quad \omega \mapsto [x \mapsto \operatorname{res}(x\omega)]$$

is easily shown to be an isomorphism of topological vector spaces. As a consequence the isomorphism $\Omega \mathbin{\widehat{\otimes}} K \cong c(F/A,K)$ of Proposition 2.8.5 identifies $c(F/A,K)$ with the topological vector space of formal series à la Tate:

$$K\langle t\rangle\,dt = \Big\{ \sum_{n\geqslant 0} \alpha_n t^n dt \in K[[t]]dt \;\Big|\; \lim_{n\to\infty} \alpha_n = 0 \Big\}.$$

The topology on this space is induced by the norm

$$\Big| \sum_{n\geqslant 0} \alpha_n t^n \cdot dt \Big| = \sup_{n\geqslant 0} |\alpha_n|.$$

A series $\sum_{n\geqslant 0}\alpha_n t^n dt$ corresponds to the continuous function

$$F/A \to K, \quad x \mapsto \sum_{n\geqslant 0} \alpha_n \operatorname{res}(xt^n dt).$$

This function maps t^{-n} to $-\alpha_{n-1}$.

From this discussion one easily deduces that $g(F/A,K)$, the space of germs of continuous functions from F/A to K, is isomorphic to the quotient

$$\frac{\Omega \mathbin{\widehat{\otimes}} K}{\Omega \otimes K}.$$

Such quotients arise naturally in the cohomological computations of the subsequent chapters. The utility of $g(F/A,K)$ is that it gives them an accessible interpretation.

2.3. Basic properties

Lemma 2.3.1. *Every open embedding of topological vector spaces is continuously split.*

Proof. Let $j\colon U \hookrightarrow V$ be an open embedding. The quotient topology on V/U is discrete. So every $\mathbb{F}_q$-linear splitting $i\colon V/U \to V$ of the quotient map is continuous and the map $j \oplus i\colon U \oplus V/U \to V$ is a continuous bijection. If $U' \subset U$ is an open subset and $Y' \subset V/U$ any subset then the image of $U' \oplus Y'$ in V is a union of translates of U' whence open. Thus $j \oplus i$ is a topological isomorphism. □

Corollary 2.3.2. *Let V,W be topological vector spaces, $U \subset V$ an open subspace. Every continuous $\mathbb{F}_q$-linear map $f\colon U \to W$ admits an extension to V with image $f(U)$.*

Proof. Take a splitting $V = U \oplus Y$ and extend f to Y by zero. □

Lemma 2.3.3. *A topological vector space is Hausdorff if and only if its zero point is closed.* □

Lemma 2.3.4. *Let V be an lth vector space, $V' \subset V$ a closed subspace. The quotient topology on V/V' is lth.* □

Lemma 2.3.5. *The category of lth vector spaces and continuous $\mathbb{F}_q$-linear maps is additive and has arbitrary limits. The limits commute with forgetful functors to $\mathbb{F}_q$-vector spaces (without topology) and to topological abelian groups.* □

2.4. Locally compact vector spaces

Proposition 2.4.1. *Every locally compact vector space is lth and contains a compact open subspace.*

Proof. Let V be a locally compact vector space. We first assume that $\mathbb{F}_q = \mathbb{F}_p$ for a prime p. In effect we work with a locally compact p-torsion abelian group V. If V is connected then its Pontrjagin dual is trivial since the p-torsion subgroup of $\mathbb{C}^\times$ is disconnected. Hence V is itself trivial. The connected component of 0 for a general V is a connected locally compact subgroup so it is trivial. Translating by elements of V we conclude that V is totally disconnected. Now a theorem of van Dantzig [15, Ch. II, Theorem 7.7, p. 62] states that every open neighbourhood of $0 \in V$ contains a compact open subgroup.

Next let $\mathbb{F}_q$ be an arbitrary finite field. If $U \subset V$ is an open subgroup then the subgroup

$$U' = \bigcap_{\alpha \in \mathbb{F}_q{}^\times} \alpha U$$

is open and stable under the $\mathbb{F}_q$-action. □

Lemma 2.4.2. *Let V be a locally compact vector space. For every compact subset $K \subset V$ there exists a compact open subspace of V containing K.*

Proof. Let $U \subset V$ be a compact open subspace which exists by Proposition 2.4.1. The quotient V/U is discrete so the image of K in it is finite. Let $K' \subset U/V$ be a finite $\mathbb{F}_q$-vector subspace containing the image of K. The preimage of K' in V is compact open and contains K by construction. $\square$

Definition 2.4.3. For a locally compact vector space V we define V^* to be the space of continuous $\mathbb{F}_q$-linear functions $V \to \mathbb{F}_q$ equipped with the compact-open topology.

In our context the duality theorem of Pontrjagin takes the following form:

Theorem 2.4.4. *Let V be a locally compact vector space.*

1. *V^* is locally compact. Moreover:*
   (a) *V is discrete if and only if V^* is compact.*
   (b) *V is compact if and only if V^* is discrete.*
2. *The natural map $V \to V^{**}$ is a topological isomorphism.*

Proof. Let $\mathbb{F}_p \subset \mathbb{F}_q$ be the prime subfield. Using the trace map $\mathrm{tr}\colon \mathbb{F}_q \to \mathbb{F}_p$ one easily reduces the problem to the case $\mathbb{F}_q = \mathbb{F}_p$. In this case we can invoke the usual Pontrjagin duality as follows. A choice of a primitive p-th root of unity determines a topological isomorphism of $(\mathbb{F}_p, +)$ and the p-torsion subgroup $\mu_p(\mathbb{C}) \subset \mathbb{C}^\times$. Every character of V as a locally compact p-torsion abelian group has the image in $\mu_p(\mathbb{C})$. Thus the chosen isomorphism $(\mathbb{F}_p, +) \cong \mu_p(\mathbb{C})$ identifies V^* with the Pontrjagin dual of V. The result is now clear. $\square$

2.5. Completion

Let V be an lth vector space. The completion of V is the lth vector space

$$\widehat{V} = \lim_{U} V/U$$

where $U \subset V$ runs over all open subspaces. It is enough to take the limit over a fundamental system of open subspaces. V is called complete if the natural map $V \to \widehat{V}$ is a topological isomorphism. Every continuous $\mathbb{F}_q$-linear map from V to a complete lth vector space factors uniquely over $\widehat{V}$.

Lemma 2.5.1. *If U is an open subspace in an lth vector space V then the natural sequence $0 \to \widehat{U} \to \widehat{V} \to V/U \to 0$ is exact. In particular the natural map $\widehat{U} \to \widehat{V}$ is an open embedding.* $\square$

Lemma 2.5.2. *Let V be an lth vector space.*

1. *$\widehat{V}$ is complete.*
2. *The natural map $V \to \widehat{V}$ is injective with dense image.*
3. *If $V \to \widehat{V}$ is bijective then it is a topological isomorphism.* $\square$

Lemma 2.5.3. *A locally compact vector space is complete.*

Proof. Let V be a compact vector space. The natural map $V \to \widehat{V}$ is injective with dense image so a topological isomorphism. Thus a compact space is complete. If a space V admits a complete open subspace then it is complete. In particular every locally compact space is complete. $\square$

Proposition 2.5.4. *Let $\mathcal{U} = \{U_i\}$ be a covering of an lth vector space V by open subspaces. If for every $U_i, U_j \in \mathcal{U}$ there exists $U_k \in \mathcal{U}$ such that $U_i + U_j \subset U_k$ then $\{\widehat{U_i}\}$ covers $\widehat{V}$.*

Remark. The draft version of this proposition was wrong. Many thanks to Hendrik Lenstra for the correction.

Proof of Proposition 2.5.4. According to Lemma 2.5.1 the natural maps $\widehat{U}_i \to \widehat{V}$ are open embeddings. Let W be the union of $\widehat{U}_i$ inside $\widehat{V}$. The assumption on $\mathcal{U}$ implies that W is an $\mathbb{F}_q$-vector subspace. By construction W contains V and so is dense. Being an open $\mathbb{F}_q$-vector subspace it is automatically closed hence coincides with $\widehat{V}$. □

2.6. Completed tensor product

Recall that according to our convention a tensor product $\otimes$ without subscript means a tensor product over $\mathbb{F}_q$.

Definition 2.6.1. Let V, W be lth vector spaces. We define the *tensor product topology* on $V \otimes W$ by the fundamental system of subspaces $U \otimes W + V \otimes Y$ where $U \subset V, Y \subset W$ run over all open subspaces. The $\mathbb{F}_q$-vector space $V \otimes W$ equipped with this topology is denoted $V \otimes_{\mathrm{c}} W$. We reserve the tensor product $V \otimes W$ without decorations to indicate the corresponding vector space without topology.

In general a continuous bilinear map $U \times V \to W$ does not induce a continuous map $U \otimes_{\mathrm{c}} V \to W$.

Lemma 2.6.2. *If V and W are lth vector spaces then $V \otimes_{\mathrm{c}} W$ is lth.*

Proof. We need to prove that $V \otimes_{\mathrm{c}} W$ is Hausdorff. According to Lemma 2.3.3 it suffices to prove that $0 \in V \otimes_{\mathrm{c}} W$ is closed. Assume W is discrete. If $U \subset V$ is an open subspace then $U \otimes W \subset V \otimes_{\mathrm{c}} W$ is open and hence closed. Letting U run over all open subspaces of V we conclude that $0 = \bigcap_U U \otimes W$ is closed. Now let W be arbitrary. Fix an open subspace $Y \subset W$. The natural map $V \otimes_{\mathrm{c}} W \to V \otimes_{\mathrm{c}} W/Y$ is continuous. As the latter space is Hausdorff it follows that $V \otimes Y$ is closed. The intersection $\bigcap_Y V \otimes Y = 0$ is thus closed. □

Definition 2.6.3. Let V, W be lth vector spaces. We define the *completed tensor product* $V \mathbin{\widehat{\otimes}} W$ as the completion of $V \otimes_{\mathrm{c}} W$. In other words

$$V \mathbin{\widehat{\otimes}} W = \varprojlim_{U,Y} V/U \otimes W/Y$$

where $U \subset V$, $Y \subset W$ run over all open subspaces and the tensor products in the limit diagram are equipped with the discrete topology. If $f\colon V_1 \to V_2$ and $g\colon W_1 \to W_2$ are continuous $\mathbb{F}_q$-linear maps then $f \mathbin{\widehat{\otimes}} g\colon V_1 \mathbin{\widehat{\otimes}} W_1 \to V_2 \mathbin{\widehat{\otimes}} W_2$ is defined as the completion of $f \otimes g$.

A completed tensor product of two compact spaces is compact. However a completed tensor product of an infinite discrete and an infinite compact space is never locally compact.

Definition 2.6.4. For a vector space V we define $V^{\#}$ to be this space equipped with the discrete topology.

Proposition 2.6.5. *Let V, W be lth vector spaces. Consider the natural map*

$$\iota\colon V^{\#} \mathbin{\widehat{\otimes}} W \to V \mathbin{\widehat{\otimes}} W.$$

(1) *The map ι is injective with dense image.*
(2) *If V is complete and W compact then ι is a bijection.*

Proof. For every open subspace $Y \subset W$ let $\iota_Y : V \otimes_{\mathrm{c}} W/Y \to \lim_U (V/U \otimes_{\mathrm{c}} W/Y)$ be the completion map. At the level of $\mathbb{F}_q$-vector spaces without topology the map ι is the limit of ι_Y over all open subspaces $Y \subset W$.

(1) The space $V \otimes_{\mathrm{c}} W/Y$ is Hausdorff by Lemma 2.6.2. Hence ι_Y is injective by Lemma 2.5.2. It follows that ι is injective. The density statement is clear.

(2) The space W/Y is finite since W is compact. Hence the natural map

$$\lim_U (V/U \otimes W/Y) \to \lim_U (V/U) \otimes W/Y$$

is an isomorphism of $\mathbb{F}_q$-vector spaces without topology. Since V is complete we conclude that ι_Y is bijective. As a consequence ι is a bijection. □

2.7. Ind-complete tensor product

In this section we introduce a different topology on $V \otimes W$ which is better for some purposes than the usual tensor product topology.

Definition 2.7.1. Let V, W be lth vector spaces. We define the *ind-tensor product topology* on $V \otimes W$ by the fundamental system of open subspaces $U \otimes Y$ where $U \subset V$, $Y \subset W$ run over all open subspaces. We denote $V \otimes_{\mathrm{ic}} W$ the tensor product $V \otimes W$ equipped with this topology.

One can prove that a continuous bilinear map $U \times V \to W$ induces a continuous map $U \otimes_{\mathrm{ic}} V \to W$. On the downside the ind-tensor product topology has some counterintuitive properties. For example $V \otimes_{\mathrm{ic}} \mathbb{F}_q = V^{\#}$ so $\mathbb{F}_q$ is not a tensor unit for $\otimes_{\mathrm{ic}}$.

Lemma 2.7.2. *If V, W are lth vector spaces then the natural map $V \otimes_{\mathrm{ic}} W \to V^{\#} \otimes_{\mathrm{c}} W$ is continuous.*

Proof. Indeed if $Y \subset W$ is an open subspace then $V \otimes Y$ is open both in $V \otimes_{\mathrm{ic}} W$ and in $V^{\#} \otimes_{\mathrm{c}} W$. The result follows since the subspaces $V \otimes Y$ form a fundamental system in $V^{\#} \otimes_{\mathrm{c}} W$. □

Lemma 2.7.3. *If V, W are lth vector spaces then $V \otimes_{\mathrm{ic}} W$ is lth.*

Proof. The space $V^{\#} \otimes_{\mathrm{c}} W$ is Hausdorff by Lemma 2.6.2. As the natural bijection $V \otimes_{\mathrm{ic}} W \to V^{\#} \otimes_{\mathrm{c}} W$ is continuous the point $0 \in V \otimes_{\mathrm{ic}} W$ is closed. So $V \otimes_{\mathrm{ic}} W$ is Hausdorff by Lemma 2.3.3. □

Lemma 2.7.4. *If $f : V_1 \to V_2$ and $g : W_1 \to W_2$ are continuous $\mathbb{F}_q$-linear maps of lth vector spaces then the map $f \otimes g : V_1 \otimes_{\mathrm{ic}} W_1 \to V_2 \otimes_{\mathrm{ic}} W_2$ is continuous.* □

Definition 2.7.5. Let V, W be lth vector spaces. We define the *ind-complete tensor product* $V \check{\otimes} W$ as the completion of $V \otimes_{\mathrm{ic}} W$. If $f : V_1 \to V_2$ and $g : W_1 \to W_2$ are continuous $\mathbb{F}_q$-linear maps of lth vector spaces then $f \check{\otimes} g : V_1 \check{\otimes} W_1 \to V_2 \check{\otimes} W_2$ is defined as the completion of $f \otimes g$.

Lemma 2.7.2 equips us with natural maps $V \check{\otimes} W \to V^{\#} \widehat{\otimes} W$ and $V \check{\otimes} W \to V \widehat{\otimes} W^{\#}$.

Proposition 2.7.6. *If V, W are lth vector spaces then the natural square*

$$\begin{tikzcd}
V \check{\otimes} W \arrow[r] \arrow[d] & V \widehat{\otimes} W^{\#} \arrow[d] \\
V^{\#} \widehat{\otimes} W \arrow[r] & V \widehat{\otimes} W
\end{tikzcd}$$

is cartesian in the category of topological vector spaces.

Proof. The proposition claims that the map

$$f\colon V \mathbin{\check{\otimes}} W \to (V^{\#} \mathbin{\widehat{\otimes}} W) \times_{V \widehat{\otimes} W} (V \mathbin{\widehat{\otimes}} W^{\#})$$

defined by the diagram above is a topological isomorphism.

Given open subspaces $U \subset V$, $Y \subset W$ let us temporarily denote

$$[U, Y] = V/U \otimes_{\mathrm{c}} W/Y,$$
$$\langle U, Y\rangle = (V^{\#} \otimes_{\mathrm{c}} W/Y) \times_{[U,Y]} (V/U \otimes_{\mathrm{c}} W^{\#}).$$

As limits commute with limits the target of the map f is $\lim_{U,Y} \langle U, Y\rangle$ where U, Y range over all open subspaces. Hence f is defined by the natural projections

$$f_{U,Y} \colon V \otimes_{\mathrm{ic}} W \to \langle U, Y\rangle.$$

A choice of splittings $V \cong U \oplus V/U$, $W \cong Y \oplus W/Y$ induces isomorphisms

$$\langle U, Y\rangle \cong (U^{\#} \otimes_{\mathrm{c}} W/Y) \times (V/U \otimes_{\mathrm{c}} Y^{\#}) \times [U, Y],$$
$$V \otimes_{\mathrm{ic}} W \cong (U \otimes_{\mathrm{ic}} Y) \times (U \otimes_{\mathrm{ic}} W/Y) \times (V/U \otimes_{\mathrm{ic}} W) \times [U, Y]$$

which identify $f_{U,Y}$ with the projection to the last three factors. Hence $f_{U,Y}$ is onto with the kernel $U \otimes_{\mathrm{ic}} Y$. The resulting continuous bijection

$$(V \otimes_{\mathrm{ic}} W)/(U \otimes_{\mathrm{ic}} Y) \to \langle U, Y\rangle$$

is a topological isomorphism since its target and source are both discrete. Taking the limit over all U, Y we deduce the desired result. ☐

Corollary 2.7.7. *If V, W are lth vector spaces then the natural maps $V \mathbin{\check{\otimes}} W \to V^{\#} \mathbin{\widehat{\otimes}} W$ and $V \mathbin{\check{\otimes}} W \to V \mathbin{\widehat{\otimes}} W^{\#}$ are injective.*

Proof. According to Proposition 2.6.5 both natural maps $V^{\#} \mathbin{\widehat{\otimes}} W \to V \mathbin{\widehat{\otimes}} W$ and $V \mathbin{\widehat{\otimes}} W^{\#} \to V \mathbin{\widehat{\otimes}} W$ are injective. So the result follows from Proposition 2.7.6. ☐

Corollary 2.7.8. *If V is a compact vector space and W a complete lth vector space then the natural map $V \mathbin{\check{\otimes}} W \to V^{\#} \mathbin{\widehat{\otimes}} W$ is a continuous bijection.*

Proof. Indeed $V \mathbin{\widehat{\otimes}} W^{\#} \to V \mathbin{\widehat{\otimes}} W$ is a bijection by Proposition 2.6.5 (2) whence the result follows from Proposition 2.7.6. ☐

2.8. Continuous functions

Definition 2.8.1. Let V, W be topological vector spaces. We define $c(V, W)$ to be the space of continuous $\mathbb{F}_q$-linear maps $V \to W$ equipped with the compact-open topology.

Lemma 2.8.2. *Let V be a topological vector space. If W is an lth vector space then so is $c(V, W)$.* ☐

Lemma 2.8.3. *If an $\mathbb{F}_q$-linear map $V \to W$ is continuous then so are the induced natural transformations $c(W, -) \to c(V, -)$ and $c(-, V) \to c(-, W)$.* ☐

The natural map $c(U \otimes_{\mathrm{c}} V, W) \to c(U, c(V, W))$ is not surjective in general and so does not define an adjunction of $- \otimes_{\mathrm{c}} V$ and $c(V, -)$.

Definition 2.8.4. Let V, W be topological vector spaces. We denote

$$\sigma_{V,W} \colon V^{*} \otimes W \to c(V, W)$$

the map which sends a tensor $f \otimes w$ to the function $v \mapsto f(v)w$.

Proposition 2.8.5. *For every locally compact vector space V and a complete lth vector space W there exists a unique topological isomorphism*

$$V^{*} \mathbin{\widehat{\otimes}} W \xrightarrow{\sim} c(V, W)$$

extending $\sigma_{V,W}$ on $V^{*} \otimes W$.

Proof. We split the proof in two steps.

Step 1. *V is compact and W is discrete.*
The space V^* is discrete by Theorem 2.4.4 whence $V^* \mathbin{\widehat{\otimes}} W = V^* \otimes_{\mathrm{c}} W$ is discrete. As a consequence

$$V^* \mathbin{\widehat{\otimes}} W = \bigcup_{W' \subset W} V^* \otimes_{\mathrm{c}} W'$$

where $W' \subset W$ ranges over all finite subspaces.

The space $c(V, W)$ is discrete since $V \subset V$ is compact and $\{0\} \subset W$ is open. As V is compact and W is discrete the image of every continuous $\mathbb{F}_q$-linear map $V \to W$ is finite. Therefore

$$c(V, W) = \bigcup_{W' \subset W} c(V, W')$$

where $W' \subset W$ again ranges over all finite subspaces. Hence it is enough to consider the case when W is finite. This case instantly reduces to the case $W \cong \mathbb{F}_q$ which is clear.

Step 2. *V is locally compact and W is complete lth.*
For an open subspace $Y \subset W$ let $q_Y : W \to W/Y$ be the quotient map. For a compact open subspace $U \subset V$ let $\rho_{U,Y} : c(V, W) \to c(U, W/Y)$ be the map given by restriction to U and composition with q_Y. The subspace $\ker \rho_{U,Y} \subset c(V, W)$ consists of functions which send the compact subset U to the open subset Y. As a consequence it is open. Lemma 2.4.2 implies that the collection of all the subspaces $\ker \rho_{U,Y}$ is a fundamental system of open subspaces in $c(V, W)$. Every $\rho_{U,Y}$ is surjective by Corollary 2.3.2. Therefore the limit map

$$\rho \colon c(V, W) \to \lim_{U,Y} c(U, W/Y).$$

defined by the $\rho_{U,Y}$ is the completion map $c(V, W) \to c(V, W)\widehat{\ }$. The map ρ is bijective since V is the union of its compact open subspaces and $W = \lim_Y W/Y$. As ρ is the completion map of $c(V, W)$ it is in fact a topological isomorphism.

Let $\rho_U : V^* \to U^*$ be the restriction map. Arguing as in the case of $\rho_{U,Y}$ above we conclude that the collection of subspaces $\ker \rho_U$ is a fundamental system in V^* and that every map ρ_U is surjective. As a consequence the limit map

$$\psi \colon V^* \otimes_{\mathrm{c}} W \to \lim_{U,Y} (U^* \otimes W/Y).$$

defined by the $\rho_U \otimes q_Y$ is the completion map of $V^* \otimes_{\mathrm{c}} W$. Altogether we obtain a commutative diagram

$$\begin{tikzcd}
V^* \otimes_{\mathrm{c}} W \arrow[r, "\sigma_{V,W}"] \arrow[d, "\psi"] & c(V, W) \arrow[d, "\rho"] \\
\lim_{U,Y} (U^* \otimes_{\mathrm{c}} W/Y) \arrow[r, "\lim \sigma_{U,W/Y}"] & \lim_{U,Y} c(U, W/Y).
\end{tikzcd}$$

The bottom arrow is a topological isomorphism as a limit of topological isomorphisms $\sigma_{U,W/Y}$. Hence $\sigma_{V,W}$ factors through a topological isomorphism $V^* \mathbin{\widehat{\otimes}} W \to c(V, W)$. As $V^* \otimes W$ is dense in $V^* \mathbin{\widehat{\otimes}} W$ this is the topological isomorphism σ we need to construct. $\square$

2.9. Bounded functions

Definition 2.9.1. Let V, W be locally compact vector spaces. A continuous $\mathbb{F}_q$-linear map $f\colon V \to W$ is said to be bounded if its image is contained in a compact subspace. We define $b(V,W) \subset c(V,W)$ to be the space of all bounded maps. The topology on $b(V,W)$ is given by the fundamental system of subspaces $c(V,Y)$ where $Y \subset W$ ranges over all compact open subspaces.

Lemma 2.9.2. *The inclusion $b(V,W) \subset c(V,W)$ is continuous.* □

Lemma 2.9.3. *If an $\mathbb{F}_q$-linear map $V \to W$ is continuous then so are the induced natural transformations $b(W,-) \to b(V,-)$ and $b(-,V) \to b(-,W)$.* □

Observe that the map $\sigma_{V,W}\colon V^* \otimes W \to c(V,W)$ of Definition 2.8.4 has image in $b(V,W)$.

Proposition 2.9.4. *For every pair V, W of locally compact vector spaces there exists a unique topological isomorphism*

$$(V^*)^\# \mathbin{\widehat{\otimes}} W \xrightarrow{\sim} b(V,W)$$

extending $\sigma_{V,W}$ on $V^* \otimes W$.

Proof. The composition

$$(V^*)^\# \mathbin{\widehat{\otimes}} W \xrightarrow{\iota} V^* \mathbin{\widehat{\otimes}} W \xrightarrow{\sigma} c(V,W)$$

of the natural inclusion ι and the topological isomorphism σ of Proposition 2.8.5 is continuous and restricts to $\sigma_{V,W}$ on $V^* \otimes W$. Hence our claim follows if $\sigma\iota$ is a homeomorphism onto $b(V,W)$.

Let $\mathcal{U}$ be the family of all compact open subspaces of W. $\mathcal{U}$ is a fundamental system in W by Proposition 2.4.1. It covers W by Lemma 2.4.2. Since $(V^*)^\#$ is discrete it follows that the family

$$\{(V^*)^\# \otimes Y \mid Y \in \mathcal{U}\}$$

is a fundamental system which covers $(V^*)^\# \otimes_{\mathrm{c}} W$. As a consequence the family

$$\{(V^*)^\# \mathbin{\widehat{\otimes}} Y \mid Y \in \mathcal{U}\}$$

is a fundamental system in $(V^*)^\# \mathbin{\widehat{\otimes}} W$. It covers $(V^*)^\# \mathbin{\widehat{\otimes}} W$ by Proposition 2.5.4.

As $Y \in \mathcal{U}$ is compact the map ι identifies $(V^*)^\# \mathbin{\widehat{\otimes}} Y$ with $(V^*) \mathbin{\widehat{\otimes}} Y$ by Proposition 2.6.5 (2). The map σ sends the latter subspace isomorphically onto $c(V,Y) \subset b(V,W)$. The subspaces $c(V,Y)$ form a fundamental system in $b(V,W)$. This system covers $b(V,W)$ as a consequence of Lemma 2.4.2. We conclude that $\iota\sigma$ is a homeomorphism onto $b(V,W)$. □

Corollary 2.9.5. *Let V and W be locally compact vector spaces. If V is compact then the inclusion $b(V,W) \hookrightarrow c(V,W)$ is a topological isomorphism.*

Proof. Indeed V^* is discrete by Theorem 2.4.4 so the natural map $(V^*)^\# \mathbin{\widehat{\otimes}} W \to V^* \mathbin{\widehat{\otimes}} W$ is a topological isomorphism. □

2.10. Bounded locally constant functions

Definition 2.10.1. Let V, W be locally compact vector spaces. A continuous $\mathbb{F}_q$-linear map $f\colon V \to W$ is called bounded locally constant if it is bounded and its kernel is open. We define $a(V,W) \subset b(V,W)$ to be the space of all bounded locally constant maps. The space $a(V,W)$ is equipped with the minimal topology such that the inclusions $a(V,W) \subset b(V,W)$ and $a(V,W) \subset c(V,W^\#)$ are continuous.

One can describe $a(V,W)$ set-theoretically as the intersection

$$a(V,W) = b(V,W) \cap c(V,W^{\#}) \subset c(V,W).$$

By construction the topology on $a(V,W)$ is that of the fibre product

$$b(V,W) \times_{c(V,W)} c(V,W^{\#}).$$

Lemma 2.10.2. *If an $\mathbb{F}_q$-linear map $V \to W$ is continuous then so are the induced natural transformations $a(W,-) \to a(V,-)$ and $a(-,V) \to a(-,W)$.* □

Observe that the map $\sigma_{V,W} \colon V^* \otimes W \to c(V,W)$ of Definition 2.8.4 has image in $a(V,W)$.

Proposition 2.10.3. *For every pair of locally compact vector spaces V, W there exists a unique topological isomorphism*

$$V^* \mathbin{\check{\otimes}} W \xrightarrow{\sim} a(V,W)$$

extending $\sigma_{V,W}$ on $V^* \otimes W$.

Proof. Consider the commutative diagram

$$(*) \qquad \begin{array}{ccccc} (V^*)^{\#} \mathbin{\widehat{\otimes}} W & \longrightarrow & V^* \mathbin{\widehat{\otimes}} W & \longleftarrow & V^* \mathbin{\widehat{\otimes}} W^{\#} \\ \downarrow & & \downarrow & & \downarrow \\ b(V,W) & \longrightarrow & c(V,W) & \longleftarrow & c(V,W^{\#}) \end{array}$$

where the horizontal arrows are the natural maps, the left vertical arrow is the topological isomorphism of Proposition 2.9.4 and the other two vertical arrows are the topological isomorphisms provided by Propositon 2.8.5. According to Proposition 2.7.6

$$V^* \mathbin{\check{\otimes}} W = ((V^*)^{\#} \mathbin{\widehat{\otimes}} W) \times_{V^* \widehat{\otimes} W} (V^* \mathbin{\widehat{\otimes}} W^{\#}).$$

At the same time

$$a(V,W) = b(V,W) \times_{c(V,W)} c(V,W^{\#}).$$

Thus $(*)$ defines a topological isomorphism $V^* \mathbin{\check{\otimes}} W \to a(V,W)$. It extends $\sigma_{V,W}$ since the vertical maps in $(*)$ do so. □

Corollary 2.10.4. *Let V and W be locally compact vector spaces. If W is discrete then the inclusion $a(V,W) \hookrightarrow b(V,W)$ is a topological isomorphism.*

Proof. The map $V^* \widehat{\otimes} W^{\#} \to V^* \widehat{\otimes} W$ is a topological isomorphism. Proposition 2.7.6 implies that the map $V^* \mathbin{\check{\otimes}} W \to (V^*)^{\#} \mathbin{\widehat{\otimes}} W$ is a topological isomorphism. Whence the result. □

2.11. Germ spaces

Definition 2.11.1. Let V, W be lth vector spaces. The $\mathbb{F}_q$-vector *space of germs* $g(V,W)$ is

$$g(V,W) = \operatorname*{colim}_{U} c(U,W)$$

where $U \subset V$ runs over all open subspaces and the transition maps are restrictions. We do not equip $g(V,W)$ with a topology. The image of $f \in c(U,W)$ under the natural map $c(U,W) \to g(V,W)$ is called the germ of f.

An element of $g(V,W)$ can be represented by a pair (U,f) where $U \subset V$ is an open subspace and $f \colon U \to W$ a continuous $\mathbb{F}_q$-linear map. Two such pairs (U_1,f_1), (U_2,f_2) represent the same element of $g(V,W)$ if there exists an open subspace $U \subset U_1 \cap U_2$ such that $f_1|_U = f_2|_U$.

Proposition 2.11.2. *The natural sequence*

$$0 \to a(V,W) \to b(V,W) \to g(V,W) \to 0$$

is exact for all locally compact vector spaces V, W.

Proof. The sequence is clearly left exact. We need to prove that $b(V,W) \to g(V,W)$ is surjective. Let $U \subset V$ be an open subspace. As V is locally compact there exists a compact open subspace $U' \subset U$. According to Corollary 2.3.2 the restriction map $c(U,W) \to c(U',W)$ is onto. Furthermore $c(U',W) = b(U',W)$ since U' is compact. The map $b(V,W) \to b(U',W)$ is surjective by Corollary 2.3.2 again. Hence the map $b(V,W) \to g(V,W)$ is surjective. □

Definition 2.11.3. Let V, W be lth vector spaces. A continuous $\mathbb{F}_q$-linear map $f\colon V \to W$ is called a *local isomorphism* if there exists an open subspace $U \subset V$ such that $f(U) \subset W$ is open and the restriction $f|_U\colon U \to f(U)$ is a topological isomorphism.

Proposition 2.11.4. *If $f\colon V \to W$ is a local isomorphism of lth vector spaces then the induced natural transformations $g(W,-) \to g(V,-)$ and $g(-,V) \to g(-,W)$ are isomorphisms.* □

Lemma 2.11.5. *For every pair of locally compact vector spaces V, W the natural map $V \mathbin{\check{\otimes}} W \to V^{\#} \mathbin{\widehat{\otimes}} W$ extends to a natural short exact sequence*

$$0 \to V \mathbin{\check{\otimes}} W \to V^{\#} \mathbin{\widehat{\otimes}} W \to g(V^*,W) \to 0.$$

Proof. Consider the short exact sequence

$$0 \to a(V^*,W) \to b(V^*,W) \to g(V^*,W) \to 0$$

of Proposition 2.11.2. The isomorphisms

$$b(V^*,W) \cong (V^{**})^{\#} \mathbin{\widehat{\otimes}} W,$$
$$a(V^*,W) \cong V^{**} \mathbin{\check{\otimes}} W$$

of Propositions 2.9.4, 2.10.3 and Pontrjagin duality of Theorem 2.4.4 transform it to

$$0 \to V \mathbin{\check{\otimes}} W \to V^{\#} \mathbin{\widehat{\otimes}} W \to g(V^*,W) \to 0$$

and the result follows. □

Proposition 2.11.6. *Let $f_V : V_1 \to V_2$ and $f_W : W_1 \to W_2$ be continuous $\mathbb{F}_q$-linear maps of locally compact vector spaces. If f_V^* and f_W are local isomorphisms then the induced map*

$$\frac{V_1^{\#} \mathbin{\widehat{\otimes}} W_1}{V_1 \mathbin{\check{\otimes}} W_1} \to \frac{V_2^{\#} \mathbin{\widehat{\otimes}} W_2}{V_2 \mathbin{\check{\otimes}} W_2}$$

is an isomorphism.

Proof. The induced map $g(V_1^*,W_1) \to g(V_2^*,W_2)$ is an isomorphism by Proposition 2.11.4. Hence the result follows from Lemma 2.11.5. □

CHAPTER 3

Topological rings and modules

In this chapter we use the tensor product and function space constructions of Chapter 2 to produce and study rings and modules over them.

We keep the conventions of Chapter 2. In particular we continue using the acronym "lth" and assume all locally compact vector spaces to be Hausdorff. We work with topological algebras over the fixed field $\mathbb{F}_q$ and with modules over such algebras. In this chapter an algebra (without further qualifications) means an $\mathbb{F}_q$-algebra.

A topological $\mathbb{F}_q$-algebra is a topological $\mathbb{F}_q$-vector space A equipped with a continuous multiplication map $A \times A \to A$ which makes A into a commutative associative unital $\mathbb{F}_q$-algebra. A topological module M over a topological algebra A is a topological vector space M equipped with a continuous A-action map $A \times M \to M$ which makes M into an A-module.

3.1. Overview

Let A, B be locally compact $\mathbb{F}_q$-algebras. Typical examples of such algebras relevant to our applications are

- the discrete algebra $\mathbb{F}_q[t]$,
- the locally compact algebra $\mathbb{F}_q(\!(t^{-1})\!)$,
- the compact algebra $\mathbb{F}_q[[t^{-1}]]$.

The first goal of this chapter is to equip the tensor products $A \mathbin{\check{\otimes}} B$ and $A \mathbin{\widehat{\otimes}} B$ with topological $\mathbb{F}_q$-algebra structures compatible with the dense subalgebra $A \otimes B$. In the case of $A \mathbin{\widehat{\otimes}} B$ it can be done only under certain assumptions on A, B (cf. Example 3.3.1). To handle this difficulty we work out some preliminaries in Section 3.2. The rings $A \mathbin{\check{\otimes}} B$ and $A \mathbin{\widehat{\otimes}} B$ play a prominent role in this work. Some degree of familiarity with them will be assumed in the subsequent chapters. We discuss examples of such rings in Section 3.4.

Let M be a locally compact A-module and N a locally compact B-module. Another important goal of this chapter is to equip the function spaces $a(M, N)$, $b(M, N)$, $c(M, N)$ and the germ space $g(M, N)$ with natural actions of tensor product rings:

- an $A \mathbin{\check{\otimes}} B$-module structure on $a(M, N)$ and $g(M, N)$,
- an $A^{\#} \mathbin{\widehat{\otimes}} B$-module structure on $b(M, N)$,
- an $A \mathbin{\widehat{\otimes}} B$-module structure on $c(M, N)$, under certain assumptions.

In Section 3.8 we fix τ-ring structures on $A \mathbin{\check{\otimes}} B$ and $A \mathbin{\widehat{\otimes}} B$ to facilitate applications in the context of shtukas. We also fix the structures of left modules over suitable τ-polynomial rings on the function spaces and the germ spaces. As a result one can use them as arguments for the Hom shtuka construction of Section 1.13. A Hom shtuka with a function space or a germ space argument is one of the main constructions of this text.

In Section 3.9 we study $a(M, N)$, $b(M, N)$ and $c(M, N)$ as modules in one particular case which is central to our applications. Let C be a smooth projective connected curve over $\mathbb{F}_q$. Fix a closed point $\infty \in C$ and set $A = \Gamma(C - \{\infty\}, \mathcal{O}_C)$. The local field F of C at ∞ is a locally compact $\mathbb{F}_q$-algebra. Its ring of integers $\mathcal{O}_F \subset F$ is a compact open subalgebra while $A \subset F$ is a discrete cocompact subalgebra. Let ω_A be the module of Kähler differentials of A over $\mathbb{F}_q$ and let $\omega_F = \omega_A \otimes_A F$. Serre duality for C implies important results for module structures on $a(M, N)$, $b(M, N), c(M, N)$ where M is one of the spaces

$$F, \quad F/A, \quad F/\mathcal{O}_F.$$

For example $a(F, B)$ is topologically isomorphic to $\omega_F \mathbin{\check{\otimes}} B$, a free $F \mathbin{\check{\otimes}} B$-module of rank 1, while $b(F/A, B) \cong \omega_A \mathbin{\widehat{\otimes}} B$ is a locally free $A \mathbin{\widehat{\otimes}} B$-module of rank 1.

The content of Section 3.2 is well-known. The same applies to the rest of the chapter modulo the reservations we made on the tensor product $\check{\otimes}$, the function spaces $a(M, N)$, $b(M, N)$ and the germ space $g(M, N)$ in Chapter 2.

As is the case for Chapter 2, this chapter was inspired by and owes much to Anderson's work [1], especially to §2 of that text where Anderson uses function spaces to compute certain Ext's for modules over τ-polynomial rings.

3.2. Preliminaries

Lemma 3.2.1. *Let U, V and W be topological vector spaces. A bilinear map $\mu\colon U \times V \to W$ is continuous if and only if*

(1) *For every $u \in U$ the map $\mu(u, -)\colon V \to W$ is continuous.*
(2) *For every $v \in V$ the map $\mu(-, v)\colon U \to W$ is continuous.*
(3) *The map μ is continuous at $(0, 0)$.*

Proof. Assume (1), (2), (3). Let $(u, v) \in U \times V$ and let $u' \in U$, $v' \in V$. By bilinearity of μ we have

$$\mu(u + u', v + v') = \mu(u, v) + \mu(u, v') + \mu(u', v) + \mu(u', v').$$

From (1), (2), (3) it follows that the map $(u', v') \mapsto \mu(u + u', v + v')$ is continuous at $(0, 0)$. Hence μ is continuous. □

Lemma 3.2.2. *Let U, V and W be lth vector spaces, $\mu\colon U \times V \to W$ a bilinear map. If μ is continuous then there exists a unique bilinear continuous map $\widehat{\mu}\colon \widehat{U} \times \widehat{V} \to \widehat{W}$ extending μ.*

Proof. [4, III, p.50, Theorem 1]. □

Lemma 3.2.3. *Let A be an lth vector space equipped with an $\mathbb{F}_q$-algebra structure.*

(1) *$\widehat{A}$ admits at most one structure of a topological $\mathbb{F}_q$-algebra such that the natural map $A \to \widehat{A}$ is a homomorphism.*
(2) *Such a structure exists if and only if the multiplication map $\mu\colon A \times A \to A$ is continuous.*

Proof. (1) follows from the fact that $A \times A$ is dense in $\widehat{A} \times \widehat{A}$. (2) If μ is continuous then we get a continuous multiplication $\widehat{\mu}\colon \widehat{A} \times \widehat{A} \to \widehat{A}$ by completion. Conversely if there exists a $\widehat{\mu}$ such that the natural map $A \to \widehat{A}$ is a homomorphism then μ is continuous since every open subset of A is a preimage of an open subset in $\widehat{A}$. □

Lemma 3.2.4. *Let M be an lth module of finite type over an lth algebra A. If M is a topological direct summand of $A^{\oplus n}$ for some n then the natural map $M \otimes_A \widehat{A} \to \widehat{M}$ is an $\widehat{A}$-module isomorphism.*

Proof. The claim reduces to the case when $M = A^{\oplus n}$ and this case is clear. □

We will need the notion of boundedness from topological ring theory.

Definition 3.2.5. Let A be a topological algebra and M a topological A-module.

(1) If $V \subset A$, $K \subset M$ are subsets then $V \cdot K$ denotes the set of products
$$\{a \cdot m \mid a \in V, m \in K\} \subset M.$$

(2) A subset $K \subset M$ is called *bounded* if for every open subspace $U \subset M$ there exists an open subspace $V \subset A$ such that $V \cdot K \subset U$.

Lemma 3.2.6. *Let M be an lth module over an lth algebra A. Every compact subset of M is bounded.*

Proof. Let K be a compact subset and $m \in K$ a point. The preimage of U in $A \times M$ under the multiplication map contains the point $(0, m)$. As this preimage is open there exist open subspaces $V_m \subset A$, $Y_m \subset M$ such that $V_m \cdot (m + Y_m) \subset U$. The translates $m + Y_m$ cover K. As K is compact we can choose finitely many such translates. Let V be the intersection of the corresponding V_m. By construction $V \cdot K \subset U$. □

Lemma 3.2.7. *Let A be a compact algebra and M an lth A-module. Every compact open subspace $U \subset M$ contains an open submodule.*

Proof. Given $a \in A$ let $\mu_a \colon M \to M$ denote the multiplication by a map. As U is compact and open Lemma 3.2.6 provides us with an open subspace $V \subset A$ such that $V \cdot U \subset U$. Consider the intersection
$$U' = U \cap \bigcap_a \mu_a^{-1} U$$
where a runs over a set of representatives of the classes in A/V. The subspace U' is open since A/V is finite. By construction $A \cdot U' \subset U$ hence the submodule generated by U' is contained in U. □

3.3. Tensor products

Let A, B be locally compact algebras. Even though $A \otimes_{\mathrm{c}} B$ is both an $\mathbb{F}_q$-algebra and an lth vector space it is not necessarily an lth algebra since its multiplication need not be continuous in the tensor product topology.

Example 3.3.1. Let us demonstrate that the multiplication on $\mathbb{F}_q(\!(z)\!) \otimes_{\mathrm{c}} \mathbb{F}_q(\!(z)\!)$ is not continuous. Denote $F = \mathbb{F}_q(\!(z)\!)$ temporarily. We have
$$\lim_{n\to\infty} z^{-n} \otimes z^n = 0 = \lim_{n\to\infty} z^n \otimes z^{-n}$$
in $F \otimes_{\mathrm{c}} F$. As a consequence
$$\lim_{n\to\infty} (z^{-n} \otimes z^n, z^n \otimes z^{-n}) = 0$$
in $(F \otimes_{\mathrm{c}} F) \times (F \otimes_{\mathrm{c}} F)$. The multiplication maps this sequence to the constant sequence $1 \otimes 1$. Hence it is not continuous.

We would like to give a sufficient condition for $A \mathbin{\widehat{\otimes}} B$ to carry a natural topological algebra structure. To do it we need some preparation.

Lemma 3.3.2. *Let M be an lth module over an lth algebra A. If M is bounded and admits a fundamental system of open submodules then the map $\mu \colon A \otimes_{\mathrm{c}} M \to M$ induced by the A-multiplication on M is continuous.*

Proof. Let $U \subset M$ be an open submodule. As M is bounded there exists an open subspace $V \subset A$ such that $V \cdot M \subset U$. Therefore $\mu(V \otimes M + A \otimes U) \subset U$. □

Lemma 3.3.3. *Let A, B be lth algebras, M an lth A-module and N an lth B-module. If the map $A \otimes_{\mathrm{c}} M \to M$ induced by the A-multiplication on M is continuous then the $A \otimes_{\mathrm{c}} B$-multiplication on $M \otimes_{\mathrm{c}} N$ is continuous.*

Proof. The $A \otimes_{\mathrm{c}} B$-multiplication on $M \otimes_{\mathrm{c}} N$ is a bilinear map which satisfies the conditions (1) and (2) of Lemma 3.2.1. It remains to show that the condition (3) holds. Let $U_M \subset M$, $U_N \subset N$ be open subspaces. As the map $A \otimes_{\mathrm{c}} M \to M$ is continuous there are open subspaces $U_A \subset A$ and $V_M \subset M$ such that $U_A \cdot M \subset U_M$ and $A \cdot V_M \subset U_M$. By continuity of B-multiplication on N there exist open subspaces $U_B \subset B$ and $V_N \subset N$ such that $U_B \cdot V_N \subset U_N$. We then have

$$(U_A \otimes B + A \otimes U_B) \cdot (V_M \otimes N + M \otimes V_N) \subset$$
$$(U_A \cdot V_M) \otimes N + (U_A \cdot M) \otimes N + (A \cdot V_M) \otimes N + M \otimes (U_B \cdot V_N) \subset$$
$$U_M \otimes N + M \otimes U_N$$

and the result follows. □

Proposition 3.3.4. *Let A, B be lth algebras. If A is compact or discrete then $A \mathbin{\widehat{\otimes}} B$ admits a unique structure of a topological algebra such that the natural map $A \otimes B \to A \mathbin{\widehat{\otimes}} B$ is a homomorphism.*

Proof. We first prove that the multiplication map $A \otimes_{\mathrm{c}} A \to A$ is continuous. This is clear if A is discrete. If A is compact then it is bounded by Lemma 3.2.6 and admits a fundamental system of open A-submodules by Lemma 3.2.7. Hence the map $A \otimes_{\mathrm{c}} A \to A$ is continuous by Lemma 3.3.2. Now Lemma 3.3.3 implies that the multiplication map $(A \otimes_{\mathrm{c}} B) \times (A \otimes_{\mathrm{c}} B) \to (A \otimes_{\mathrm{c}} B)$ is continuous. Whence the result follows from Lemma 3.2.3. □

Let A, B be lth algebras, M an lth A-module and N an lth B-module. The topological vector space $M \mathbin{\widehat{\otimes}} N$ comes equipped with a natural action of $A \otimes B$ by functoriality of $\widehat{\otimes}$. Recall that according to our convention the ring $A \otimes B$ carries no topology so its action is not supposed to be continuous. Nonetheless we can prove the following result.

Proposition 3.3.5. *Let A, B be lth algebras, M an lth A-module and N an lth B-module. Assume that one of the following conditions hold:*

(1) *A and M are discrete,*
(2) *A and M are compact.*

Then the $A \otimes B$-module structure on $M \mathbin{\widehat{\otimes}} N$ extends to a unique topological $A \mathbin{\widehat{\otimes}} B$-module structure.

Proof. The unicity is clear. If either (1) or (2) holds then the same argument as in the proof of Proposition 3.3.4 shows that the multiplication map $(A \otimes_{\mathrm{c}} B) \times (M \otimes_{\mathrm{c}} N) \to (M \otimes_{\mathrm{c}} N)$ is continuous. Taking its completion we get a topological $A \mathbin{\widehat{\otimes}} B$-module structure on $M \mathbin{\widehat{\otimes}} N$. It is compatible with the natural $A \otimes B$-module structure on the dense subspace $M \otimes N$. Whence the result. □

As we observed above the completed tensor product $A \mathbin{\widehat{\otimes}} B$ of locally compact algebras A and B need not carry a natural lth algebra structure. The ind-complete tensor product $A \mathbin{\check{\otimes}} B$ does not suffer from such a problem.

Lemma 3.3.6. *If A, B are lth algebras, M an lth A-module and N an lth B-module then the multiplication map $(A \otimes_{\mathrm{ic}} B) \times (M \otimes_{\mathrm{ic}} N) \to M \otimes_{\mathrm{ic}} N$ is continuous.*

Proof. The conditions (1) and (2) of Lemma 3.2.1 are clear and the condition (3) follows instantly from the definition of the ind-tensor product topology (Definition 2.7.1). □

Proposition 3.3.7. *If A, B are locally compact algebras then there exists a unique lth algebra structure on $A \mathbin{\check{\otimes}} B$ such that the natural map $A \otimes B \to A \mathbin{\check{\otimes}} B$ is a homomorphism.* □

Proposition 3.3.8. *If A, B are locally compact algebras, M a locally compact A-module and N a locally compact B-module then the natural $A \otimes B$-module structure on $M \overset{\smile}{\otimes} N$ extends to a unique topological $A \overset{\smile}{\otimes} B$-module structure.* □

Next we study some natural maps of tensor product algebras.

Proposition 3.3.9. *Let A, B be lth algebras. Assume that either A or B is compact or discrete.*

(1) *The natural map $\iota\colon A \widehat{\otimes} B^{\#} \to A \widehat{\otimes} B$ is an injective $\mathbb{F}_q$-algebra homomorphism.*
(2) *If A is compact then ι is an $\mathbb{F}_q$-algebra isomorphism.*

Proof. The map ι is injective by Proposition 2.6.5 (1). It is a homomorphism of $\mathbb{F}_q$-algebras since it is the completion of the continuous homomorphism $A \otimes_{\mathrm{c}} B^{\#} \to A \otimes_{\mathrm{c}} B$. If A is compact then ι is bijective by Proposition 2.6.5 (2). □

Proposition 3.3.10. *Let A, B be locally compact algebras.*

(1) *The natural map $\iota\colon A \overset{\smile}{\otimes} B \to A^{\#} \widehat{\otimes} B$ is an injective $\mathbb{F}_q$-algebra homomorphism.*
(2) *If A is compact then ι is an isomorphism.*

Proof. Follows from Corollaries 2.7.7 and 2.7.8. □

Proposition 3.3.11. *If A, B are compact algebras then the natural map $A \overset{\smile}{\otimes} B \to A \widehat{\otimes} B$ is a $\mathbb{F}_q$-algebra isomorphism.*

Proof. The map in question is the composition of continuous homomorphisms $A \overset{\smile}{\otimes} B \to A \widehat{\otimes} B^{\#} \to A \widehat{\otimes} B$. The first one is an isomorphism by Proposition 3.3.10 while the second one is an isomorphism by Proposition 3.3.9 (2). □

The natural maps of Propositions 3.3.9 and 3.3.10 are always continuous. However their inverses, if they exist, are not continuous in general. Similarly the natural map of Proposition 3.3.11 is a continuous bijection whose inverse is not continuous in general.

3.4. Examples of tensor product algebras

We are now in position to discuss some examples of tensor product algebras. Consider the locally compact algebras

$$F = \mathbb{F}_q(\!(z)\!), \quad \mathcal{O}_F = \mathbb{F}_q[[z]]$$
$$K = \mathbb{F}_q(\!(\zeta)\!), \quad \mathcal{O}_K = \mathbb{F}_q[[\zeta]].$$

We have

$$\mathcal{O}_F \overset{\smile}{\otimes} \mathcal{O}_K = \mathbb{F}_q[[z, \zeta]],$$
$$\mathcal{O}_F \widehat{\otimes} \mathcal{O}_K = \mathbb{F}_q[[z, \zeta]]$$

as abstract $\mathbb{F}_q$-algebras. However the topologies on $\mathcal{O}_F \overset{\smile}{\otimes} \mathcal{O}_K$ and $\mathcal{O}_F \widehat{\otimes} \mathcal{O}_K$ are different. The topology on $\mathcal{O}_F \overset{\smile}{\otimes} \mathcal{O}_K$ is given by the powers of the ideal $(z\zeta)$ while the topology on $\mathcal{O}_F \widehat{\otimes} \mathcal{O}_K$ is given by the powers of the ideal (z, ζ). In particular $\mathcal{O}_F \widehat{\otimes} \mathcal{O}_K$ is compact while $\mathcal{O}_F \overset{\smile}{\otimes} \mathcal{O}_K$ is not even locally compact.

The completed tensor products with a discrete factor look as follows:

$$\mathcal{O}_F^{\#} \widehat{\otimes} \mathcal{O}_K = \mathbb{F}_q[[z, \zeta]],$$
$$\mathcal{O}_F^{\#} \widehat{\otimes} K = \mathbb{F}_q[[z]](\!(\zeta)\!),$$
$$F^{\#} \widehat{\otimes} \mathcal{O}_K = \mathbb{F}_q(\!(z)\!)[[\zeta]],$$
$$F^{\#} \widehat{\otimes} K = \mathbb{F}_q(\!(z)\!)(\!(\zeta)\!).$$

The topologies on $\mathcal{O}_F^{\#} \widehat{\otimes} \mathcal{O}_K$ and $F^{\#} \widehat{\otimes} \mathcal{O}_K$ are given by powers of the ideals (ζ). The topologies on $\mathcal{O}_F^{\#} \widehat{\otimes} K$ and $F^{\#} \widehat{\otimes} K$ are determined by open subalgebras $\mathcal{O}_F^{\#} \widehat{\otimes} \mathcal{O}_K$ and $F^{\#} \widehat{\otimes} \mathcal{O}_K$ respectively.

The ind-complete tensor products with a compact factor has the following form:

$$F \breve{\otimes} \mathcal{O}_K = \mathbb{F}_q[[\zeta]]((z)).$$

Its topology is defined by the open subalgebra $\mathcal{O}_F \breve{\otimes} \mathcal{O}_K$. The ind-complete tensor product of F and K is

$$F \breve{\otimes} K = \mathbb{F}_q[[z, \zeta]][(z\zeta)^{-1}]$$

with the topology given by the open subalgebra $\mathcal{O}_F \breve{\otimes} \mathcal{O}_K$. As we demonstrated in Example 3.3.1 the tensor product $F \widehat{\otimes} K$ makes no sense as an lth algebra.

Another important example is $F \widehat{\otimes} \mathbb{F}_q[x]$. It is topologically isomorphic to the algebra of Tate series

$$F\langle x\rangle = \Big\{ \sum_{n \geqslant 0} \alpha_n x^n \in F[[x]] \;\Big|\; \lim_{n \to \infty} \alpha_n = 0 \Big\}.$$

For the sake of completeness let us describe the algebra $\mathcal{O}_F \widehat{\otimes} K$. It can be identified with the algebra of power series

$$\sum_{n \in \mathbb{Z}} \alpha_n \zeta^n, \;\; \alpha_n \in \mathcal{O}_F, \;\; \lim_{n \to -\infty} \alpha_n = 0.$$

For every nonzero ideal $I \subset \mathcal{O}_F$ and every integer $m \in \mathbb{Z}$ the subspace

$$\Big\{ \sum_{n \in \mathbb{Z}} \alpha_n \zeta^n \mid \alpha_n \in I \text{ for all } n \leqslant m \Big\} \subset \mathcal{O}_F \widehat{\otimes} K$$

is open. Such subspaces form a fundamental system.

3.5. Algebraic properties

In this section we study the properties of $A \breve{\otimes} B$, $A \widehat{\otimes} B$ as commutative rings without topology. We are primarily interested in the case when A and B are finite products of local fields or the rings of integers in such finite products. We begin with some localization properties.

Proposition 3.5.1. *If A is an $\mathbb{F}_q$-algebra, K a finite product of local fields and $\zeta \in \mathcal{O}_K$ a uniformizer then $(A^{\#} \widehat{\otimes} \mathcal{O}_K)[\zeta^{-1}] = A^{\#} \widehat{\otimes} K$.*

Proof. The family of open subspaces $\{A^{\#} \otimes_{\mathrm{c}} \zeta^{-n}\mathcal{O}_K\}_{n \geqslant 0}$ covers $A^{\#} \otimes_{\mathrm{c}} K$. Proposition 2.5.4 implies that the family $\{A^{\#} \widehat{\otimes} \zeta^{-n}\mathcal{O}_K\}_{n \geqslant 0}$ covers $A^{\#} \widehat{\otimes} K$. Multiplication by ζ^n maps $A^{\#} \widehat{\otimes} \zeta^{-n}\mathcal{O}_K$ bijectively onto $A^{\#} \widehat{\otimes} \mathcal{O}_K$ since the same is true with $\otimes_{\mathrm{c}}$ in place of $\widehat{\otimes}$. The claim is now clear. □

Proposition 3.5.2. *If A is a compact $\mathbb{F}_q$-algebra, K a finite product of local fields and $\zeta \in K$ a uniformizer then $(A \breve{\otimes} \mathcal{O}_K)[\zeta^{-1}] = A \breve{\otimes} K$.*

Proof. As A is compact the natural maps $A \breve{\otimes} \mathcal{O}_K \to A^{\#} \widehat{\otimes} \mathcal{O}_K$ and $A \breve{\otimes} K \to A^{\#} \widehat{\otimes} K$ are isomorphisms by Proposition 3.3.10. Hence the claim follows from Proposition 3.5.1. □

Proposition 3.5.3. *If F, K are finite products of local fields with uniformizers $z \in \mathcal{O}_F$, $\zeta \in \mathcal{O}_K$ then $(\mathcal{O}_F \breve{\otimes} \mathcal{O}_K)[(z\zeta)^{-1}] = F \breve{\otimes} K$.*

Proof. The family of open subspaces $\{z^{-n}\mathcal{O}_F \otimes_{\mathrm{ic}} \zeta^{-n}\mathcal{O}_K\}_{n \geqslant 0}$ covers $F \otimes_{\mathrm{ic}} K$. Its completion $\{z^{-n}\mathcal{O}_F \breve{\otimes} \zeta^{-n}\mathcal{O}_K\}_{n \geqslant 1}$ covers $F \breve{\otimes} K$ by Proposition 2.5.4. Multiplication by $(z\zeta)^n$ maps $z^{-n}\mathcal{O}_F \breve{\otimes} \zeta^{-n}\mathcal{O}_K$ bijectively onto $\mathcal{O}_F \breve{\otimes} \mathcal{O}_K$ since the same is true with $\otimes_{\mathrm{ic}}$ in place of $\breve{\otimes}$. The claim is now clear. □

Next we study quotients of tensor product algebras. Observe that an ideal $I \subset \mathcal{O}_K$ is open if and only if it projects to nonzero ideals in all factors of $\mathcal{O}_K = \prod_{i=1}^n \mathcal{O}_{K_i}$. If A is an $\mathbb{F}_q$-algebra and $I \subset \mathcal{O}_K$ an open ideal then we have a natural map $A^\# \mathbin{\widehat{\otimes}} \mathcal{O}_K \to A \otimes \mathcal{O}_K/I$.

Proposition 3.5.4. *Let A be an $\mathbb{F}_q$-algebra and K a finite product of local fields. If $I \subset \mathcal{O}_K$ is an open ideal then the following holds:*

(1) *The sequence $0 \to A^\# \mathbin{\widehat{\otimes}} I \to A^\# \mathbin{\widehat{\otimes}} \mathcal{O}_K \to A \otimes \mathcal{O}_K/I \to 0$ is exact.*
(2) *The natural map $(A^\# \mathbin{\widehat{\otimes}} \mathcal{O}_K) \otimes_{\mathcal{O}_K} I \to A^\# \mathbin{\widehat{\otimes}} \mathcal{O}_K$ is injective with image $A^\# \mathbin{\widehat{\otimes}} I$.*

Proof. (1) Indeed the sequence $0 \to A^\# \otimes_{\mathrm{c}} I \to A^\# \otimes_{\mathrm{c}} \mathcal{O}_K \to A^\# \otimes_{\mathrm{c}} \mathcal{O}_K/I \to 0$ is clearly exact and the first map in it is an open embedding. Hence the result follows from Lemma 2.5.1. (2) Observe that I is a free $\mathcal{O}_K$-module of rank 1. Let $x \in I$ be a generator. Multiplication by x identifies $A^\# \otimes_{\mathrm{c}} \mathcal{O}_K$ with $A^\# \otimes_{\mathrm{c}} I$. Taking completion we get the result. □

If $I \subset \mathcal{O}_K$ is an open ideal then the quotient $\mathcal{O}_K/I$ is discrete. As a consequence $A \otimes_{\mathrm{ic}} \mathcal{O}_K/I$ is discrete. Taking the completion of $A \otimes_{\mathrm{ic}} \mathcal{O}_K \to A \otimes_{\mathrm{ic}} \mathcal{O}_K/I$ we get a natural map $A \mathbin{\breve{\otimes}} \mathcal{O}_K \to A \otimes \mathcal{O}_K/I$.

Proposition 3.5.5. *Let A be an lth $\mathbb{F}_q$-algebra, K a finite product of local fields. If $I \subset \mathcal{O}_K$ is an open ideal then then the following holds:*

(1) *The sequence $0 \to A \mathbin{\breve{\otimes}} I \to A \mathbin{\breve{\otimes}} \mathcal{O}_K \to A \otimes \mathcal{O}_K/I \to 0$ is exact.*
(2) *The natural map $(A \mathbin{\breve{\otimes}} \mathcal{O}_K) \otimes_{\mathcal{O}_K} I \to A \mathbin{\breve{\otimes}} \mathcal{O}_K$ is injective with image $A \mathbin{\breve{\otimes}} I$.*

Proof. Follows by the same argument as Proposition 3.5.4. □

Finally we discuss some structural properties of tensor product algebras.

Proposition 3.5.6. *Let A be a noetherian $\mathbb{F}_q$-algebra and K a finite product of local fields.*

(1) *$A^\# \mathbin{\widehat{\otimes}} \mathcal{O}_K$ is noetherian and complete with respect to the ideal $A^\# \mathbin{\widehat{\otimes}} \mathfrak{m}_K$.*
(2) *$A^\# \mathbin{\widehat{\otimes}} K$ is noetherian.*

Proof. Without loss of generality we assume that K is a local field. In this case $K \cong k(\!(\zeta)\!)$ for some finite field extension k of $\mathbb{F}_q$. (1) By definition of the completed tensor product

$$A^\# \mathbin{\widehat{\otimes}} \mathcal{O}_K = \lim_{n \geqslant 1} A^\# \otimes \mathcal{O}_K/\mathfrak{m}_K^n.$$

Therefore $A^\# \mathbin{\widehat{\otimes}} \mathcal{O}_K$ is the completion of the ring $(A \otimes k)[\zeta]$ at the ideal (ζ). The ring $A \otimes k$ is of finite type over A and so is noetherian. Thus $(A \otimes k)[\zeta]$ is noetherian and so is its completion $A^\# \mathbin{\widehat{\otimes}} \mathcal{O}_K$. The completion of the ideal $(\zeta) \subset (A \otimes k)[\zeta]$ is clearly $A^\# \mathbin{\widehat{\otimes}} \mathfrak{m}_K$ so $A^\# \mathbin{\widehat{\otimes}} \mathcal{O}_K$ is complete with respect to $A^\# \mathbin{\widehat{\otimes}} \mathfrak{m}_K$. (2) follows from (1) in view of Proposition 3.5.1. □

Proposition 3.5.7. *Let F, K be finite products of local fields, $z \in \mathcal{O}_F$ and $\zeta \in \mathcal{O}_K$ uniformizers.*

(1) *$\mathcal{O}_F \mathbin{\widehat{\otimes}} \mathcal{O}_K$ is a finite product of complete regular 2-dimensional local rings.*
(2) *The maximal ideals of $\mathcal{O}_F \mathbin{\widehat{\otimes}} \mathcal{O}_K$ are precisely the prime ideals containing z and ζ.*

Proof. It is enough to assume that F and K are local fields. In this case $F \cong k_1[\![z]\!]$ and $K \cong k_2[\![\zeta]\!]$ for some finite field extensions k_1 and k_2 of $\mathbb{F}_q$. Therefore

$$\mathcal{O}_F \mathbin{\widehat{\otimes}} \mathcal{O}_K = \lim_{n,m \geqslant 0} (k_1 \otimes k_2)[z, \zeta]/(z^n, \zeta^m) = (k_1 \otimes k_2)[\![z, \zeta]\!].$$

Observe that $k_1 \otimes k_2$ is a finite product of finite fields. (1) and (2) are now clear. □

3.6. Function spaces as modules

Let A, B be locally compact algebras, M a locally compact A-module and N a locally compact B-module. The function spaces $a(M,N)$, $b(M,N)$ and $c(M,N)$ carry an action of A on the right and B on the left by functoriality. Since A is commutative we get a natural $A \otimes B$-module structure. In this section we show that it extends canonically to a structure of an $A \check{\otimes} B$-module on $a(M,N)$ and an $A^{\#} \widehat{\otimes} B$-module on $b(M,N)$. Under certain assumptions on A and M we show that it extends to a structure of an $A \widehat{\otimes} B$-module on $c(M,N)$.

Definition 3.6.1. Let A be an lth algebra and M an lth A-module. We equip M^*, the continuous $\mathbb{F}_q$-linear dual of M, with the A-module structure given by the action of A on M.

Lemma 3.6.2. *If M is locally compact then the A-action map $A \times M^* \to M^*$ is continuous.*

Proof. We will deduce the result from Lemma 3.2.1. To do it we need to check the following conditions:

1. For every $a \in A$ the induced map $M^* \to M^*$, $f \mapsto fa$ is continuous.
2. For every $f \in M^*$ the induced map $A \to M^*$, $a \mapsto fa$ is continuous.
3. The A-action map $A \times M^* \to M^*$ is continuous at $(0,0)$.

The condition (1) holds by functoriality of M^*. Let us check (3). For a compact open subset $U \subset M$ let $[U] \subset M^*$ be the subspace of functions which vanish on U. By Lemma 3.2.6 there exists an open $\mathbb{F}_q$-vector subspace $V \subset A$ such that $V \cdot U \subset U$. As a consequence $V \cdot [U] \subset [U]$ so the condition (3) holds. Given $f \in M^*$ there exists a compact open $U \subset M$ such that f vanishes on U. As before we can find an open $V \subset A$ with the property that $V \cdot U \subset U$. Hence $V \cdot f \subset [U]$ and the condition (2) holds as well. $\square$

Proposition 3.6.3. *Let A, B be locally compact algebras, M a locally compact A-module and N a locally compact B-module.*

1. *The $A \otimes B$-module structure on $a(M,N)$ extends uniquely to a topological $A \check{\otimes} B$-module structure.*
2. *The map $M^* \otimes N \to a(M,N)$, $f \otimes n \mapsto (m \mapsto f(m)n)$ extends uniquely to a topological isomorphism $M^* \check{\otimes} N \cong a(M,N)$ of $A \check{\otimes} B$-modules.*

Proof. The map $M^* \otimes N \to a(M,N)$ above extends to a topological isomorphism $M^* \check{\otimes} N \cong a(M,N)$ of $\mathbb{F}_q$-vector spaces by Proposition 2.10.3. It is $A \otimes B$-linear by naturality. The $A \otimes B$-module structure on $M^* \check{\otimes} N$ extends canonically to a topological $A \check{\otimes} B$-module structure by Proposition 3.3.8 and we get the result. $\square$

Lemma 3.6.4. *Let A, B be locally compact algebras and M a locally compact A-module. If M^* is a topological direct summand of a free A-module of finite rank then the natural map $M^* \otimes_A (A \check{\otimes} B) \to a(M,B)$ is an $A \check{\otimes} B$-module isomorphism.*

Proof. By Proposition 3.6.3 we can identify $a(M,B)$ with the completion of an lth $A \otimes_{\mathrm{ic}} B$-module $M^* \otimes_{\mathrm{ic}} B$. Due to our assumption on M^* the module $M^* \otimes_{\mathrm{ic}} B$ is a topological direct summand of $(A \otimes_{\mathrm{ic}} B)^{\oplus n}$ for some n. So the result is a consequence of Lemma 3.2.4. $\square$

Proposition 3.6.5. *Let A, B be locally compact algebras, M a locally compact A-module and N a locally compact B-module.*

1. *The $A \otimes B$-module structure on $b(M,N)$ extends uniquely to a topological $A^{\#} \widehat{\otimes} B$-module structure.*

(2) *The natural map $M^* \otimes N \to b(M,N)$, $f \otimes n \mapsto (m \mapsto f(m)n)$ extends uniquely to a topological isomorphism $(M^*)^\# \mathbin{\widehat{\otimes}} N \cong b(M,N)$ of $A^\# \mathbin{\widehat{\otimes}} B$-modules.*

Proof. Follows from Propositions 2.9.4 and 3.3.5. □

Lemma 3.6.6. *Let A, B be locally compact algebras and M a locally compact A-module. If M^* is projective of finite type as an A-module without topology then the natural map $M^* \otimes_A (A^\# \mathbin{\widehat{\otimes}} B) \to b(M,B)$ is an $A^\# \mathbin{\widehat{\otimes}} B$-module isomorphism.*

Proof. Follows from Proposition 3.6.5 and Lemma 3.2.4. □

Proposition 3.6.7. *Let A, B be lth algebras, M an lth A-module and N an lth B-module. Assume that either of the following conditions hold:*

(1) *A is discrete and M is compact.*
(2) *A is compact and M is discrete.*

Then the following is true:

(1) *The $A \otimes B$-module structure on $c(M,N)$ extends uniquely to a topological $A \mathbin{\widehat{\otimes}} B$-module structure.*
(2) *The natural map $M^* \otimes N \to c(M,N)$, $f \otimes n \mapsto (m \mapsto f(m)n)$ extends uniquely to a topological isomorphism $M^* \mathbin{\widehat{\otimes}} N \cong c(M,N)$ of $A \mathbin{\widehat{\otimes}} B$-modules.*

Proof. Follows from Pontrjagin duality (Theorem 2.4.4), Propositions 2.8.5 and 3.3.5. □

Lemma 3.6.8. *Let A, B be lth algebras and M an lth A-module. Assume the following:*

(1) *A is either discrete or compact.*
(2) *M^* is a topological direct summand of $A^{\oplus n}$ for some $n \geqslant 0$.*

Then the following holds:

(1) *If A is discrete then M is compact. If A is compact then M is discrete. In particular the $A \otimes B$-module structure on $c(M,B)$ extends uniquely to a topological $A \mathbin{\widehat{\otimes}} B$-module structure.*
(2) *The natural map $M^* \otimes_A (A \mathbin{\widehat{\otimes}} B) \to c(M,B)$ is an $A \mathbin{\widehat{\otimes}} B$-module isomorphism.*

Proof. Follows from Pontrjagin duality, Proposition 3.6.7 and Lemma 3.2.4. □

3.7. Germ spaces as modules

Definition 3.7.1. Let A, B be locally compact $\mathbb{F}_q$-algebras, M a locally compact A-module and N a locally compact B-module. We equip the germ space $g(M,N)$ with an $A \mathbin{\check{\otimes}} B$-module structure in the following way. Consider the short exact sequence of Proposition 2.11.2:

$$0 \to a(M,N) \to b(M,N) \to g(M,N) \to 0.$$

The first arrow in this sequence is an $A \mathbin{\check{\otimes}} B$-module homomorphism by construction. We equip $g(M,N)$ with the resulting $A \mathbin{\check{\otimes}} B$-module structure.

3.8. τ-ring and τ-module structures

Recall from Definition 1.1.1 that a τ-ring is a ring R equipped with an endomorphism $\tau\colon R \to R$. We would like to fix τ-ring structures on algebras of the form $A \mathbin{\check{\otimes}} B$, $A \mathbin{\widehat{\otimes}} B$.

Definition 3.8.1. Let A, B be locally compact $\mathbb{F}_q$-algebras. Let $\sigma\colon B \to B$ be the q-power map.

(1) We equip $A \check{\otimes} B$ with the τ-ring structure given by the endomorphism $1 \check{\otimes} \sigma$.
(2) Assuming $A \widehat{\otimes} B$ admits the natural topological $\mathbb{F}_q$-algebra structure we equip it with the τ-ring structure given by the endomorphism $1 \widehat{\otimes} \sigma$.

Lemma 3.8.2. *Let A, B be locally compact $\mathbb{F}_q$-algebras and M a locally compact A-module. Let $\sigma\colon B \to B$ be the q-power map. If $a(M,B)$, $b(M,B)$, $c(M,B)$ and $g(M,B)$ are equipped with endomorphisms τ given by composition with σ then the following is true:*

(1) *$a(M,B)$ is a left $A \check{\otimes} B\{\tau\}$-module.*
(2) *$b(M,B)$ is a left $A^{\#} \widehat{\otimes} B\{\tau\}$-module.*
(3) *$g(M,B)$ is a left $A \check{\otimes} B\{\tau\}$-module.*
(4) *$c(M,B)$ is a left $A \widehat{\otimes} B\{\tau\}$-module provided the assumptions of Proposition 3.6.7 on A and M hold.*

In all cases the τ-ring structures are as in Definition 3.8.1.

Definition 3.8.3. Under assumptions of Lemma 3.8.2 we equip the spaces $a(M,B)$, $b(M,B)$, $c(M,B)$ and $g(M,B)$ with the τ-module structures as described above. From now on we work with only these τ-module structures.

Proof of Lemma 3.8.2. (1) Let $f \in a(M,B)$, $x \in A \check{\otimes} B$. We need to prove that

$$\sigma \circ (x \cdot f) = \tau(x) \cdot (\sigma \circ f).$$

This is clear if $x \in A \otimes B$. As $A \otimes B \subset A \check{\otimes} B$ is dense and $\tau\colon A \check{\otimes} B \to A \check{\otimes} B$ is continuous the general statement follows. (2) and (4) follow in the same manner. (3) follows from (1), (2) and the short exact sequence of Proposition 2.11.2. $\square$

3.9. Residue and duality

In this section we show that in one special case the function spaces $a(M,N)$, $b(M,N)$ and $c(M,N)$ have particularly nice module structures. It is exactly the case which appears in our applications.

Let C be a smooth projective connected curve over $\mathbb{F}_q$. Fix a closed point $\infty \in C$. Let F be the local field of C at ∞, $\mathcal{O}_F \subset F$ the ring of integers and $A = \Gamma(C - \{\infty\}, \mathcal{O}_C)$ where $\mathcal{O}_C$ is the structure sheaf of C. The natural topology on F makes it into a locally compact $\mathbb{F}_q$-algebra with a compact open subalgebra $\mathcal{O}_F \subset F$ and a discrete cocompact subalgebra $A \subset F$.

Let Ω_C be the sheaf of Kähler differentials of C over $\mathbb{F}_q$. We use the following notation:

$$\omega_A = \Gamma(\operatorname{Spec} A,\, \Omega_C), \quad \omega_{\mathcal{O}_F} = \Gamma(\operatorname{Spec} \mathcal{O}_F,\, \Omega_C), \quad \omega_F = \Gamma(\operatorname{Spec} F,\, \Omega_C).$$

The F-module ω_F carries a natural locally compact topology with $\omega_{\mathcal{O}_F} \subset \omega_F$ a compact open $\mathcal{O}_F$-submodule and $\omega_A \subset \omega_F$ a discrete cocompact A-submodule. It comes equipped with a residue map $\omega_F \to k$ where k is the residue field at ∞. We denote $\operatorname{res}\colon \omega_F \to \mathbb{F}_q$ its composition with the trace map $\operatorname{tr}\colon k \to \mathbb{F}_q$. In our study we need the following duality theorem for res:

Theorem 3.9.1. *The pairing $\omega_F \times F \to \mathbb{F}_q$, $(\eta, x) \mapsto \operatorname{res}(x\eta)$ induces the following topological isomorphisms:*

$$\omega_A \cong (F/A)^*, \quad \omega_{\mathcal{O}_F} \cong (F/\mathcal{O}_F)^*, \quad \omega_F \cong F^*.$$

Proof. The result is well-known. Still we sketch a proof for the reader's convenience. Let $\mathcal{O}_C(1)$ be the Serre twist of $\mathcal{O}_C$ by the divisor ∞. Let $n \in \mathbb{Z}$. A Čech computation shows that

$$\mathrm{R}\Gamma(C,\, \mathcal{O}_C(n)) = \Big[A \oplus z^{-n}\mathcal{O}_F \to F\Big],$$
$$\mathrm{R}\Gamma(C,\, \Omega_C(-n)) = \Big[\omega_A \oplus z^{n}\omega_{\mathcal{O}_F} \to \omega_F\Big]$$

where $z \in \mathcal{O}_F$ is a uniformizer and the differentials send (x, y) to $x - y$. The residue pairing $\omega_F \times F \to \mathbb{F}_q$ induces the following perfect pairings:

$$\mathrm{H}^1(C,\, \Omega_C(-n)) \times \mathrm{H}^0(C,\, \mathcal{O}(n)) \to \mathbb{F}_q,$$
$$\mathrm{H}^0(C,\, \Omega_C(-n)) \times \mathrm{H}^1(C,\, \mathcal{O}(n)) \to \mathbb{F}_q.$$

Using the explicit descriptions of cohomology groups provided by the complexes above we rewrite these pairings as

$$\frac{\omega_F}{\omega_A + z^n\omega_{\mathcal{O}_F}} \times (A \cap z^{-n}\mathcal{O}_F) \to \mathbb{F}_q, \tag{3.1}$$

$$[\omega_A \cap z^n\omega_{\mathcal{O}_F}] \times \frac{F}{A + z^{-n}\mathcal{O}_F} \to \mathbb{F}_q. \tag{3.2}$$

The open subspaces $(A + z^{-n}\mathcal{O}_F)/A$ form a fundamental system which covers F/A. Taking the limit of (3.2) as $n \to -\infty$ we conclude that the residue pairing induces a topological isomorphism $\omega_A \cong (F/A)^*$. It remains to deduce the topological isomorphisms $\omega_F \cong F^*$ and $\omega_{\mathcal{O}_F} \cong (F/\mathcal{O}_F)^*$.

Let us denote $\rho\colon \omega_F \to \mathrm{Hom}_{\mathbb{F}_q}(F, \mathbb{F}_q)$ the map defined by the residue pairing. A priori we do not even know whether its image is contained in $F^* \subset \mathrm{Hom}_{\mathbb{F}_q}(F, \mathbb{F}_q)$. First we prove that ρ sends $z^n\omega_{\mathcal{O}_F}$ to $(F/z^{-n}\mathcal{O}_F)^* \subset F^*$. As ρ is F-linear it is enough to treat the case $n = 0$. In this case (3.1) implies that $\operatorname{res}(\eta) = 0$ for every $\eta \in \omega_{\mathcal{O}_F}$. Hence $\operatorname{res}(x\eta) = 0$ for all $x \in \mathcal{O}_F$ and $\eta \in \omega_{\mathcal{O}_F}$. We conclude that $\rho(\eta) \in (F/\mathcal{O}_F)^* \subset F^*$.

Our next step is to prove that for every $n \in \mathbb{Z}$ the induced map

$$\rho\colon \frac{z^n\omega_{\mathcal{O}_F}}{z^{n+1}\omega_{\mathcal{O}_F}} \to \Big(\frac{z^{-(n+1)}\mathcal{O}_F}{z^{-n}\mathcal{O}_F}\Big)^* \tag{3.3}$$

is injective. Since $\rho(\eta)(x) = \operatorname{res}(x\eta)$ it is enough to prove this for a single $n \in \mathbb{Z}$. As the divisor $\infty \in C$ is ample there exists an $n \gg 0$ such that $\mathrm{H}^1(C,\, \mathcal{O}_C(n)) = 0$. Now (3.2) implies that $\omega_A \cap z^n\omega_{\mathcal{O}_F} = 0$. If $\eta \in z^n\omega_{\mathcal{O}_F}$ is such that $\operatorname{res}(z^{-(n+1)}x\eta) = 0$ for any $x \in \mathcal{O}_F^\times$ then the pairing (3.1) implies that $\eta \in \omega_A + z^{n+1}\omega_{\mathcal{O}_F}$. Since $\omega_A \cap z^n\omega_{\mathcal{O}_F} = 0$ we conclude that $\eta \in z^{n+1}\omega_{\mathcal{O}_F}$. Whence (3.3) is injective.

At the same time (3.3) is a morphism of one-dimensional $\mathcal{O}_F/z$-vector spaces. It is therefore an isomorphism. We conclude that for every $n > 0$ the induced map

$$\rho\colon \frac{\omega_{\mathcal{O}_F}}{z^n\omega_{\mathcal{O}_F}} \to \Big(\frac{z^{-n}\mathcal{O}_F}{\mathcal{O}_F}\Big)^*$$

is an isomorphism. As $\omega_{\mathcal{O}_F}$ is complete it follows that $\rho\colon \omega_{\mathcal{O}_F} \to (F/\mathcal{O}_F)^*$ is a topological isomorphism. Since the open subspaces $z^n\omega_{\mathcal{O}_F}$ cover ω_F we deduce that $\rho\colon \omega_F \to F^*$ is a topological isomorphism. □

Corollary 3.9.2. *Let N be a locally compact module over a locally compact algebra B. The map $\omega \otimes n \mapsto (x \mapsto \operatorname{res}(x\omega)n)$ extends uniquely to the following topological isomorphisms:*

- *an isomorphism $\omega_A \otimes N \xrightarrow{\sim} a(F/A, N)$ of $A \otimes B$-modules.*
- *an isomorphism $\omega_A \mathbin{\widehat{\otimes}} N \xrightarrow{\sim} b(F/A, N)$ of $A \mathbin{\widehat{\otimes}} B$-modules.*
- *an isomorphism $\omega_F \mathbin{\breve{\otimes}} N \xrightarrow{\sim} a(F, N)$ of $F \mathbin{\breve{\otimes}} B$-modules.*

- *an isomorphism* $\omega_F^{\#} \mathbin{\widehat{\otimes}} N \xrightarrow{\sim} b(F, N)$ *of* $F^{\#} \mathbin{\widehat{\otimes}} B$*-modules.*
- *an isomorphism* $\omega_{\mathcal{O}_F} \mathbin{\check{\otimes}} N \xrightarrow{\sim} a(F/\mathcal{O}_F, N)$ *of* $\mathcal{O}_F \mathbin{\check{\otimes}} B$*-modules.*
- *an isomorphism* $\omega_{\mathcal{O}_F} \mathbin{\widehat{\otimes}} N \xrightarrow{\sim} c(F/\mathcal{O}_F, N)$ *of* $\mathcal{O}_F \mathbin{\widehat{\otimes}} B$*-modules.*

Note that $c(F/A, N) = b(F/A, N)$ and $b(F/\mathcal{O}_F, N) = a(F/\mathcal{O}_F, N)$.

Proof of Corollary 3.9.2. In view of Theorem 3.9.1 it follows from Propositions 3.6.3, 3.6.5 and 3.6.7. ☐

Let $R_0 \to R$ be a ring homomorphism and M an R-module. Recall that an R_0-submodule $M_0 \subset M$ is called a lattice if the natural map $R \otimes_{R_0} M_0 \to M$ is an isomorphism (Section 2 in the chapter "Notation and conventions").

Corollary 3.9.3. *Let B be a locally compact algebra.*

(1) ω_A *is an* A*-lattice in* $a(F/A, B)$ *and* $b(F/A, B)$.
(2) ω_F *is an* F*-lattice in* $a(F, B)$ *and* $b(F, B)$.
(3) $\omega_{\mathcal{O}_F}$ *is an* $\mathcal{O}_F$*-lattice in* $a(F/\mathcal{O}_F, B)$ *and* $c(F/\mathcal{O}_F, B)$.

Proof. Theorem 3.9.1 identifies ω_A with $(F/A)^* \subset a(F/A, B) \subset b(F/A, B)$. As ω_A is a locally free module over a discrete ring A the result (1) follows from Lemmas 3.6.4 and 3.6.6. Similarly (2) and (3) follow from Lemmas 3.6.4, 3.6.6 and 3.6.8. ☐

Corollary 3.9.4. *Let B be a locally compact algebra.*

(1) $a(F/A, B)$ *is a locally free* $A \otimes B$*-module of rank* 1.
(2) $b(F/A, B)$ *is a locally free* $A \mathbin{\widehat{\otimes}} B$*-module of rank* 1.
(3) $a(F, B)$ *is a free* $F \mathbin{\check{\otimes}} B$*-module of rank* 1.
(4) $b(F, B)$ *is a free* $F^{\#} \mathbin{\widehat{\otimes}} B$*-module of rank* 1.
(5) $a(F/\mathcal{O}_F, B)$ *is a free* $\mathcal{O}_F \mathbin{\check{\otimes}} B$*-module of rank* 1.
(6) $c(F/\mathcal{O}_F, B)$ *is a free* $\mathcal{O}_F \mathbin{\widehat{\otimes}} B$*-module of rank* 1.

Proof. Follows from Corollary 3.9.3 since Ω_C is a locally free sheaf of rank 1 on C. ☐

Corollary 3.9.5. *Let B be a locally compact algebra.*

(1) $a(F/A, B)$ *is an* $A \otimes B$*-lattice in* $b(F/A, B)$ *and* $a(F, B)$.
(2) $a(F/\mathcal{O}_F, B)$ *is an* $\mathcal{O}_F \mathbin{\check{\otimes}} B$*-lattice in* $a(F, B)$ *and* $c(F/\mathcal{O}_F, B)$.
(3) $a(F, B)$ *is an* $F \mathbin{\check{\otimes}} B$*-latticein* $b(F, B)$.
(4) $b(F/A, B)$ *is an* $A \mathbin{\widehat{\otimes}} B$*-lattice in* $b(F, B)$.

Proof. Follows from Corollary 3.9.3. ☐

CHAPTER 4

Cohomology of shtukas

Fix a locally compact noetherian $\mathbb{F}_q$-algebra Λ and a smooth projective curve X over $\mathbb{F}_q$. We call Λ the *coefficient ring* and X the *base curve*. Set $S = \operatorname{Spec}\Lambda$ and consider the product $S \times X$. We equip $S \times X$ with the τ-scheme structure given by the endomorphism which acts as the identity on S and as the q-power map on X.

In this chapter we study the cohomology of locally free shtukas on $S \times X$. The basis of our approach is a Čech method described in Section 4.3. We also introduce and study a few supplementary constructions. The reader should be warned that some of them will not reappear until the last chapter of the book. They are the compactly supported cohomology functor, the global germ map and the local-global compatibility theorem.

The Čech method presented here involves a choice of additional data, the points "at infinity" on X. So let us fix finitely many closed points $x_1, \ldots, x_n$ of X. The complement of $\{x_1, \ldots, x_n\}$ in X is an affine subscheme which we denote $Y = \operatorname{Spec} R$. The product of the local fields of X at $x_1, \ldots, x_n$ is denoted K with $\mathcal{O}_K \subset K$ the ring of integers and $\mathfrak{m}_K \subset \mathcal{O}_K$ the Jacobson radical. By construction $\operatorname{Spec}\mathcal{O}_K/\mathfrak{m}_K = \{x_1, \ldots, x_n\} \subset X$. The natural topology on $\mathcal{O}_K$ makes it into a compact open $\mathbb{F}_q$-subalgebra of a locally compact $\mathbb{F}_q$-algebra K.

We begin with a study of shtuka cohomology around the points "at infinity". Let $\mathcal{M}$ be a locally free shtuka on $\Lambda \mathbin{\breve{\otimes}} \mathcal{O}_K$. In Section 4.1 we introduce the germ cohomology complex $\mathrm{R}\Gamma_g(\Lambda \mathbin{\breve{\otimes}} K, \mathcal{M})$. As suggested by the notation it depends only on the restriction of $\mathcal{M}$ to $\Lambda \mathbin{\breve{\otimes}} K$. The germ cohomology is modelled on the germ spaces of Section 2.11. With some degree of caution it can be regarded as compactly supported cohomology for shtukas on $\Lambda \mathbin{\breve{\otimes}} K$ with respect to the compactification given by the ring $\Lambda \mathbin{\breve{\otimes}} \mathcal{O}_K$.

The germ cohomology is related to the usual cohomology via the local germ map

$$\mathrm{R}\Gamma(\Lambda \mathbin{\breve{\otimes}} \mathcal{O}_K,\, \mathcal{M}) \xrightarrow{\sim} \mathrm{R}\Gamma_g(\Lambda \mathbin{\breve{\otimes}} K,\, \mathcal{M})$$

which we construct in Section 4.2. This map is defined only if $\mathcal{M}(\Lambda \otimes \mathcal{O}_K/\mathfrak{m}_K)$ is nilpotent and is always a quasi-isomorphism. The local germ map will play a role in Chapters 8 – 12.

Starting from Section 4.3 we shift the focus to the global setting. Let $\mathcal{M}$ be a locally free shtuka on $S \times X$. We introduce a Čech method which computes $\mathrm{R}\Gamma(S \times X, \mathcal{M})$ as the mapping fiber

$$\Big[\mathrm{R}\Gamma(\Lambda \otimes R,\, \mathcal{M}) \oplus \mathrm{R}\Gamma(\Lambda^{\#} \mathbin{\widehat{\otimes}} \mathcal{O}_K,\, \mathcal{M}) \xrightarrow{\text{difference}} \mathrm{R}\Gamma(\Lambda^{\#} \mathbin{\widehat{\otimes}} K,\, \mathcal{M})\Big].$$

This is our main tool to handle the cohomology of shtukas on $S \times X$.

The compactly supported cohomology functor $\mathrm{R}\Gamma_{\mathrm{c}}(S \times Y, \mathcal{M})$ is introduced in Section 4.4. As suggested by the notation $\mathrm{R}\Gamma_{\mathrm{c}}(S \times Y, \mathcal{M})$ depends only on the restriction of $\mathcal{M}$ to $S \times Y$. It comes equipped with a natural map $\mathrm{R}\Gamma(S \times X, \mathcal{M}) \to \mathrm{R}\Gamma_{\mathrm{c}}(S \times Y, \mathcal{M})$. We prove that this map is a quasi-isomorphism if $\mathcal{M}(\Lambda \otimes \mathcal{O}_K/\mathfrak{m}_K)$ is nilpotent. One can interpet the nilpotence condition as saying that $\mathcal{M}$ is an extension by zero of a shtuka on $S \times Y$. We use $\mathrm{R}\Gamma_{\mathrm{c}}(S \times Y, \mathcal{M})$ to construct the global germ map

$$\mathrm{R}\Gamma(S \times X,\, \mathcal{M}) \to \mathrm{R}\Gamma_g(\Lambda \mathbin{\check{\otimes}} K,\, \mathcal{M}).$$

Similarly to its local counterpart the global germ map is defined under assumption that $\mathcal{M}(\Lambda \otimes \mathcal{O}_K/\mathfrak{m}_K)$ is nilpotent. However it is not a quasi-isomorphism in general.

Section 4.5 is devoted to the proof of Theorem 4.5.1, a compatibility statement for the local and global germ maps. This statement will be used in Chapter 12 in the proof of the class number formula.

In Section 4.6 we present an advanced version of the Čech method for shtukas on $\operatorname{Spec} \mathcal{O}_F \times X$ where $\mathcal{O}_F$ is the ring of integers of a local field F. This method enables us to prove the following: if $\mathcal{M}(\mathcal{O}_F/\mathfrak{m}_F \otimes R)$ is nilpotent then the natural map

$$\mathrm{R}\Gamma(\operatorname{Spec} \mathcal{O}_F \times X,\, \mathcal{M}) \to \mathrm{R}\Gamma(\mathcal{O}_F \mathbin{\widehat{\otimes}} \mathcal{O}_K,\, \mathcal{M})$$

is a quasi-isomorphism (Theorem 4.6.3). Informally speaking, the cohomology of $\mathcal{M}$ concentrates on $\mathcal{O}_F \mathbin{\widehat{\otimes}} \mathcal{O}_K$. This phenomenon is important to the theory of regulator developed in Chapters 5 and 6.

Finally in Sections 4.7 and 4.8 we study how the change of the coefficient ring S reflects on shtuka cohomology and ζ-isomorphisms. The results of these sections will be used in Chapters 6 and 12.

The cohomology functors in this chapter are typically given by mapping fibers

$$\Big[\mathrm{R}\Gamma(\mathcal{X},\, \mathcal{M}) \to \mathrm{R}\Gamma(\mathcal{X}',\, \mathcal{M})\Big]$$

where $\mathcal{X}$ and $\mathcal{X}'$ are affine schemes (for the mapping fiber see Definition 5.1 in Chapter "Notation and conventions"). A word of warning about them is necessary. The complexes $\mathrm{R}\Gamma(\mathcal{X},\, \mathcal{M})$ and $\mathrm{R}\Gamma(\mathcal{X}',\, \mathcal{M})$ are well-defined only as objects in the derived category. As a consequence one gets a problem with functoriality. If $\mathcal{M} \to \mathcal{N}$ is a morphism of shtukas then the induced maps

$$(*) \qquad \begin{aligned} \mathrm{R}\Gamma(\mathcal{X},\, \mathcal{M}) &\to \mathrm{R}\Gamma(\mathcal{X},\, \mathcal{N}), \\ \mathrm{R}\Gamma(\mathcal{X}',\, \mathcal{M}) &\to \mathrm{R}\Gamma(\mathcal{X}',\, \mathcal{N}) \end{aligned}$$

do not determine a unique morphism of the mapping fibers. The reason is that this morphism depends on the choice of non-derived representatives for the maps $(*)$.

We solve this problem in the following way. Since the schemes $\mathcal{X}$ and $\mathcal{X}'$ are affine Theorem 1.9.1 provides us with canonical non-derived representatives for the complexes $\mathrm{R}\Gamma(\mathcal{X},\, \mathcal{M})$ and $\mathrm{R}\Gamma(\mathcal{X}',\, \mathcal{M})$, namely the associated complexes $\Gamma_{\mathrm{a}}(\mathcal{X},\, \mathcal{M})$ and $\Gamma_{\mathrm{a}}(\mathcal{X}',\, \mathcal{M})$. The mapping fiber construction is functorial on the level of the non-derived category of complexes. Following the convention of Section 1.1.9 we identify $\mathrm{R}\Gamma(\mathcal{X},\, \mathcal{M})$ with $\Gamma_{\mathrm{a}}(\mathcal{X},\, \mathcal{M})$ and $\mathrm{R}\Gamma(\mathcal{X}',\, \mathcal{M})$ with $\Gamma_{\mathrm{a}}(\mathcal{X}',\, \mathcal{M})$. We thus regain the functoriality.

4.1. Germ cohomology

In this section we fix locally compact $\mathbb{F}_q$-algebras Λ and B. Following the conventions of Section 3.8 we equip $\Lambda \mathbin{\check{\otimes}} B$ and $\Lambda^{\#} \mathbin{\widehat{\otimes}} B$ with the τ-ring structures given by the endomorphisms which act as identity on Λ and as the q-power map on B.

Definition 4.1.1. Let $\mathcal{M}$ be a quasi-coherent shtuka on $\Lambda \mathbin{\check{\otimes}} B$. The *germ cohomology* complex of $\mathcal{M}$ is the Λ-module complex

$$\mathrm{R}\Gamma_g(\Lambda \mathbin{\check{\otimes}} B, \mathcal{M}) = \Big[\mathrm{R}\Gamma(\Lambda \mathbin{\check{\otimes}} B,\, \mathcal{M}) \to \mathrm{R}\Gamma(\Lambda^{\#} \mathbin{\widehat{\otimes}} B,\, \mathcal{M})\Big].$$

The differential in this complex is induced by the natural map $\Lambda \mathbin{\check{\otimes}} B \to \Lambda^{\#} \mathbin{\widehat{\otimes}} B$ which is the completion of the continuous bijection $\Lambda \otimes_{\mathrm{ic}} B \to \Lambda^{\#} \otimes_{\mathrm{c}} B$. The n-th cohomology group of $\mathrm{R}\Gamma_g(\Lambda \mathbin{\check{\otimes}} B, \mathcal{M})$ is denoted $\mathrm{H}^n_g(\Lambda \mathbin{\check{\otimes}} B,\, \mathcal{M})$.

Proposition 4.1.2. *If $\mathcal{M}$ is a locally free shtuka on $\Lambda \mathbin{\check{\otimes}} B$ then the natural map*

$$\mathrm{R}\Gamma_g(\Lambda \mathbin{\check{\otimes}} B,\, \mathcal{M}) \to \mathrm{R}\Gamma\Big(\Lambda \mathbin{\check{\otimes}} B,\, \tfrac{\mathcal{M}(\Lambda^\# \widehat{\otimes} B)}{\mathcal{M}(\Lambda \check{\otimes} B)}\Big)[-1]$$

is a quasi-isomorphism.

Proof. Tensoring the short exact sequence

$$0 \to \Lambda \mathbin{\check{\otimes}} B \to \Lambda^\# \mathbin{\widehat{\otimes}} B \to \frac{\Lambda^\# \mathbin{\widehat{\otimes}} B}{\Lambda \mathbin{\check{\otimes}} B} \to 0$$

with a locally free $\Lambda \mathbin{\check{\otimes}} B$-module of finite rank we get a short exact sequence. As $\mathcal{M}$ is locally free the claim follows. ☐

Proposition 4.1.3. *Let $f\colon \Lambda_1 \to \Lambda_2$ and $g\colon B_1 \to B_2$ be continuous homomorphisms of locally compact $\mathbb{F}_q$-algebras. Let $\mathcal{M}$ be a locally free shtuka on $\Lambda_1 \mathbin{\check{\otimes}} B_1$. If f^* and g are local isomorphisms of topological $\mathbb{F}_q$-vector spaces then the natural map*

$$\mathrm{R}\Gamma_g(\Lambda_1 \mathbin{\check{\otimes}} B_1,\, \mathcal{M}) \to \mathrm{R}\Gamma_g(\Lambda_2 \mathbin{\check{\otimes}} B_2,\, \mathcal{M})$$

induced by $f \mathbin{\check{\otimes}} g$ is a quasi-isomorphism.

Proof. By Proposition 2.11.6 the maps f and g induce a bijection

$$\frac{\Lambda_1^\# \mathbin{\widehat{\otimes}} B_1}{\Lambda_1 \mathbin{\check{\otimes}} B_1} \cong \frac{\Lambda_2^\# \mathbin{\widehat{\otimes}} B_2}{\Lambda_2 \mathbin{\check{\otimes}} B_2}.$$

As the shtuka $\mathcal{M}$ is locally free it follows that $f \mathbin{\check{\otimes}} g$ induces an isomorphism of shtukas

$$\frac{\mathcal{M}(\Lambda_1^\# \mathbin{\widehat{\otimes}} B_1)}{\mathcal{M}(\Lambda_1 \mathbin{\check{\otimes}} B_1)} \cong \frac{\mathcal{M}(\Lambda_2^\# \mathbin{\widehat{\otimes}} B_2)}{\mathcal{M}(\Lambda_2 \mathbin{\check{\otimes}} B_2)}.$$

The result now follows from Proposition 4.1.2. ☐

4.2. Local germ map

Fix a noetherian locally compact $\mathbb{F}_q$-algebra Λ. In applications this algebra will usually be a local field. Let K be a finite product of local fields, $\mathcal{O}_K \subset K$ the ring of integers and $\mathfrak{m}_K \subset \mathcal{O}_K$ the Jacobson radical.

The ideal $\mathfrak{m}_K \subset \mathcal{O}_K$ is open so that we have a natural map $\Lambda^\# \mathbin{\widehat{\otimes}} \mathcal{O}_K \to \Lambda \otimes \mathcal{O}_K/\mathfrak{m}_K$. Taking the completion of $\Lambda \otimes_{\mathrm{ic}} \mathcal{O}_K \to \Lambda \otimes_{\mathrm{ic}} \mathcal{O}_K/\mathfrak{m}_K$ we get a natural map $\Lambda \mathbin{\check{\otimes}} \mathcal{O}_K \to \Lambda \otimes \mathcal{O}_K/\mathfrak{m}_K$ since $\Lambda \otimes_{\mathrm{ic}} \mathcal{O}_K/\mathfrak{m}_K$ is discrete.

Proposition 4.2.1. *Let $\mathcal{M}$ be a locally free shtuka on $\Lambda^\# \widehat{\otimes} \mathcal{O}_K$. If $\mathcal{M}(\Lambda \otimes \mathcal{O}_K/\mathfrak{m}_K)$ is nilpotent then $\mathrm{R}\Gamma(\Lambda^\# \mathbin{\widehat{\otimes}} \mathcal{O}_K,\, \mathcal{M}) = 0$.*

Proof. According to Proposition 3.5.6 the ring $\Lambda^\# \widehat{\otimes} \mathcal{O}_K$ is noetherian and complete with respect to the ideal $\Lambda^\# \mathbin{\widehat{\otimes}} \mathfrak{m}_K$. By Proposition 3.5.4 the natural map $\Lambda^\# \mathbin{\widehat{\otimes}} \mathcal{O}_K \to \Lambda \otimes \mathcal{O}_K/\mathfrak{m}_K$ is surjective with kernel $\Lambda^\# \mathbin{\widehat{\otimes}} \mathfrak{m}_K$. So the result follows from Proposition 1.10.4. ☐

Lemma 4.2.2. *If $\mathcal{M}$ is a locally free shtuka on $\Lambda \mathbin{\check{\otimes}} \mathcal{O}_K$ then the natural map $\mathrm{R}\Gamma_g(\Lambda \mathbin{\check{\otimes}} \mathcal{O}_K,\, \mathcal{M}) \to \mathrm{R}\Gamma_g(\Lambda \mathbin{\check{\otimes}} K,\, \mathcal{M})$ is a quasi-isomorphism.*

Proof. The inclusion $\mathcal{O}_K \hookrightarrow K$ is a local isomorphism of topological $\mathbb{F}_q$-vector spaces. So the result is a consequence of Proposition 4.1.3. ☐

Let $\mathcal{M}$ be a locally free shtuka on $\Lambda \mathbin{\breve{\otimes}} \mathcal{O}_K$. According to Definition 4.1.1

$$\mathrm{R}\Gamma_g(\Lambda \mathbin{\breve{\otimes}} \mathcal{O}_K, \mathcal{M}) = \Big[\mathrm{R}\Gamma(\Lambda \mathbin{\breve{\otimes}} \mathcal{O}_K,\, \mathcal{M}) \to \mathrm{R}\Gamma(\Lambda^{\#} \mathbin{\widehat{\otimes}} \mathcal{O}_K,\, \mathcal{M})\Big].$$

The projection to the first argument of the mapping fiber defines a natural map

$$\mathrm{R}\Gamma_g(\Lambda \mathbin{\breve{\otimes}} \mathcal{O}_K,\, \mathcal{M}) \to \mathrm{R}\Gamma(\Lambda \mathbin{\breve{\otimes}} \mathcal{O}_K, \mathcal{M}).$$

Taking its composition with the quasi-isomorphism $\mathrm{R}\Gamma_g(\Lambda \mathbin{\breve{\otimes}} K,\, \mathcal{M}) \cong \mathrm{R}\Gamma_g(\Lambda \mathbin{\breve{\otimes}} \mathcal{O}_K,\, \mathcal{M})$ of Lemma 4.2.2 we obtain a map

$$\mathrm{R}\Gamma_g(\Lambda \mathbin{\breve{\otimes}} K,\, \mathcal{M}) \to \mathrm{R}\Gamma(\Lambda \mathbin{\breve{\otimes}} \mathcal{O}_K,\, \mathcal{M}). \tag{4.1}$$

Lemma 4.2.3. *Let $\mathcal{M}$ be a locally free shtuka on $\Lambda \mathbin{\breve{\otimes}} \mathcal{O}_K$. If $\mathcal{M}(\Lambda \otimes \mathcal{O}_K/\mathfrak{m}_K)$ is nilpotent then the natural map* (4.1) *is a quasi-isomorphism.*

Proof. By construction the natural map $\mathrm{R}\Gamma_g(\Lambda \mathbin{\breve{\otimes}} \mathcal{O}_K,\, \mathcal{M}) \to \mathrm{R}\Gamma(\Lambda \mathbin{\breve{\otimes}} \mathcal{O}_K,\, \mathcal{M})$ extends to a distinguished triangle

$$\mathrm{R}\Gamma_g(\Lambda \mathbin{\breve{\otimes}} \mathcal{O}_K,\, \mathcal{M}) \to \mathrm{R}\Gamma(\Lambda \mathbin{\breve{\otimes}} \mathcal{O}_K,\, \mathcal{M}) \to \mathrm{R}\Gamma(\Lambda^{\#} \mathbin{\widehat{\otimes}} \mathcal{O}_K,\, \mathcal{M}) \to [1].$$

Together with the quasi-isomorphism $\mathrm{R}\Gamma_g(\Lambda \mathbin{\breve{\otimes}} \mathcal{O}_K,\, \mathcal{M}) \cong \mathrm{R}\Gamma_g(\Lambda \mathbin{\breve{\otimes}} K,\, \mathcal{M})$ it gives us a distinguished triangle

$$\mathrm{R}\Gamma_g(\Lambda \mathbin{\breve{\otimes}} K,\, \mathcal{M}) \to \mathrm{R}\Gamma(\Lambda \mathbin{\breve{\otimes}} \mathcal{O}_K,\, \mathcal{M}) \to \mathrm{R}\Gamma(\Lambda^{\#} \mathbin{\widehat{\otimes}} \mathcal{O}_K,\, \mathcal{M}) \to [1].$$

Proposition 4.2.1 shows that $\mathrm{R}\Gamma(\Lambda^{\#} \mathbin{\widehat{\otimes}} \mathcal{O}_K,\, \mathcal{M}) = 0$, so the result follows. ☐

Definition 4.2.4. Let $\mathcal{M}$ be a locally free shtuka on $\Lambda \mathbin{\breve{\otimes}} \mathcal{O}_K$ such that $\mathcal{M}(\Lambda \otimes \mathcal{O}_K/\mathfrak{m}_K)$ is nilpotent. By Lemma 4.2.1 the natural map (4.1) is a quasi-isomorphism. We define *the local germ map*

$$\mathrm{R}\Gamma(\Lambda \mathbin{\breve{\otimes}} \mathcal{O}_K,\, \mathcal{M}) \xrightarrow{\sim} \mathrm{R}\Gamma_g(\Lambda \mathbin{\breve{\otimes}} K,\, \mathcal{M})$$

as its inverse. The adjective "local" signifies that it involves a shtuka defined over a semil-local ring $\mathcal{O}_K$. Observe that the local germ map is a quasi-isomorphism by construction.

Proposition 4.2.5. *Let $\mathcal{M}$ be a locally free shtuka on $\Lambda \mathbin{\breve{\otimes}} \mathcal{O}_K$. If $\mathcal{M}(\Lambda \otimes \mathcal{O}_K/\mathfrak{m}_K)$ is nilpotent then $\mathrm{R}\Gamma(\Lambda \mathbin{\breve{\otimes}} \mathcal{O}_K,\, \mathcal{M})$ and $\mathrm{R}\Gamma_g(\Lambda \mathbin{\breve{\otimes}} K,\, \mathcal{M})$ are concentrated in degree* 1.

Proof. The complex $\mathrm{R}\Gamma(\Lambda \mathbin{\breve{\otimes}} \mathcal{O}_K,\, \mathcal{M})$ is concentrated in degrees 0 and 1 since $\operatorname{Spec}(\Lambda \mathbin{\breve{\otimes}} \mathcal{O}_K)$ is affine. The complex $\mathrm{R}\Gamma_g(\Lambda \mathbin{\breve{\otimes}} K,\, \mathcal{M})$ is concentrated in degrees 1 and 2 by Proposition 4.1.2. As these complexes are quasi-isomorphic via the local germ map the conclusion follows. ☐

Proposition 4.2.6. *Let $\mathcal{M} = [\mathcal{M}_0 \overset{i}{\underset{j}{\rightrightarrows}} \mathcal{M}_1]$ be a locally free shtuka on $\Lambda \mathbin{\breve{\otimes}} \mathcal{O}_K$ and let $x \in \mathcal{M}_1$. Assume that $\mathcal{M}(\Lambda \otimes \mathcal{O}_K/\mathfrak{m}_K)$ is nilpotent.*

(1) *There exists a unique $y \in \mathcal{M}_0(\Lambda^{\#} \mathbin{\widehat{\otimes}} \mathcal{O}_K)$ such that $(i - j)(y) = x$.*
(2) *Consider the composition*

$$\mathrm{H}^1(\Lambda \mathbin{\breve{\otimes}} \mathcal{O}_K,\, \mathcal{M}) \xrightarrow{\text{local}} \mathrm{H}^1_g(\Lambda \mathbin{\breve{\otimes}} K,\, \mathcal{M}) \xrightarrow{\sim} \mathrm{H}^0\Big(\Lambda \mathbin{\breve{\otimes}} K,\, \tfrac{\mathcal{M}(\Lambda^{\#} \widehat{\otimes} K)}{\mathcal{M}(\Lambda \breve{\otimes} K)}\Big)$$

of the local germ map and the natural isomorphism of Proposition 4.1.2. This composition sends the class of x to the image of y in the quotient $\mathcal{M}_0(\Lambda^{\#} \mathbin{\widehat{\otimes}} K)/\mathcal{M}_0(\Lambda \mathbin{\breve{\otimes}} K)$.

Proof. (1) $\mathrm{R}\Gamma(\Lambda^{\#} \mathbin{\widehat{\otimes}} \mathcal{O}_K,\, \mathcal{M})$ is represented by the complex

$$\Big[\mathcal{M}_0(\Lambda^{\#} \mathbin{\widehat{\otimes}} \mathcal{O}_K) \xrightarrow{i-j} \mathcal{M}_1(\Lambda^{\#} \mathbin{\widehat{\otimes}} \mathcal{O}_K)\Big].$$

By Proposition 4.2.1 $\mathrm{R}\Gamma(\Lambda^{\#} \mathbin{\widehat{\otimes}} \mathcal{O}_K,\, \mathcal{M}) = 0$. So the map $i-j$ in the complex above is a bijection and (1) follows.

(2) Consider the maps

$$\mathrm{H}^1_g(\Lambda \mathbin{\check{\otimes}} \mathcal{O}_K,\, \mathcal{M}) \to \mathrm{H}^1(\Lambda \mathbin{\check{\otimes}} \mathcal{O}_K,\, \mathcal{M}),$$
$$\mathrm{H}^1_g(\Lambda \mathbin{\check{\otimes}} \mathcal{O}_K,\, \mathcal{M}) \to \mathrm{H}^0\Big(\Lambda \mathbin{\check{\otimes}} \mathcal{O}_K,\, \tfrac{\mathcal{M}(\Lambda^{\#} \widehat{\otimes} \mathcal{O}_K)}{\mathcal{M}(\Lambda \check{\otimes} \mathcal{O}_K)}\Big)$$

determined by the natural maps of complexes

$$\mathrm{R}\Gamma_g(\Lambda \mathbin{\check{\otimes}} \mathcal{O}_K,\, \mathcal{M}) \to \mathrm{R}\Gamma(\Lambda \mathbin{\check{\otimes}} \mathcal{O}_K,\, \mathcal{M}),$$
$$\mathrm{R}\Gamma_g(\Lambda \mathbin{\check{\otimes}} \mathcal{O}_K,\, \mathcal{M}) \to \mathrm{R}\Gamma\Big(\Lambda \mathbin{\check{\otimes}} \mathcal{O}_K,\, \tfrac{\mathcal{M}(\Lambda^{\#} \widehat{\otimes} \mathcal{O}_K)}{\mathcal{M}(\Lambda \check{\otimes} \mathcal{O}_K)}\Big)[-1].$$

In order to prove (2) it is enough to produce a cohomology class $h \in \mathrm{H}^1_g(\Lambda \mathbin{\check{\otimes}} \mathcal{O}_K,\, \mathcal{M})$ which maps to the class of x in $\mathrm{H}^1(\Lambda \mathbin{\check{\otimes}} \mathcal{O}_K,\, \mathcal{M})$ and to the class of y in $\mathcal{M}_0(\Lambda^{\#} \mathbin{\widehat{\otimes}} \mathcal{O}_K)/\mathcal{M}_0(\Lambda \mathbin{\check{\otimes}} \mathcal{O}_K)$.

By definition $\mathrm{R}\Gamma_g(\Lambda \mathbin{\check{\otimes}} \mathcal{O}_K,\, \mathcal{M})$ is represented by the total complex of the double complex

$$\begin{tikzcd}
\mathcal{M}_1(\Lambda \mathbin{\check{\otimes}} \mathcal{O}_K) \arrow[r] & \mathcal{M}_1(\Lambda^{\#} \mathbin{\widehat{\otimes}} \mathcal{O}_K) \\
\mathcal{M}_0(\Lambda \mathbin{\check{\otimes}} \mathcal{O}_K) \arrow[r] \arrow[u, "i-j"] & \mathcal{M}_0(\Lambda^{\#} \mathbin{\widehat{\otimes}} \mathcal{O}_K) \arrow[u, "j-i"']
\end{tikzcd}$$

The element $(x, y) \in \mathcal{M}_1(\Lambda \mathbin{\check{\otimes}} \mathcal{O}_K) \oplus \mathcal{M}_0(\Lambda^{\#} \mathbin{\widehat{\otimes}} \mathcal{O}_K)$ is a 1-cocyle in the total complex since $x + (j-i)(y) = 0$ by definition of y. By construction (x, y) maps to the class of x in $\mathrm{H}^1(\Lambda \mathbin{\check{\otimes}} \mathcal{O}_K,\, \mathcal{M})$ and to the class of y in the quotient $\mathcal{M}_0(\Lambda^{\#} \mathbin{\widehat{\otimes}} \mathcal{O}_K)/\mathcal{M}_0(\Lambda \mathbin{\check{\otimes}} \mathcal{O}_K)$. Thus (2) follows. ☐

4.3. Čech cohomology

In this section we work with a smooth projective curve X over $\mathbb{F}_q$ and a coefficient algebra Λ as described in the introduction. We set $S = \operatorname{Spec} \Lambda$. Our goal is to present a Čech method for computing the cohomology of shtukas on $S \times X$.

The coefficient algebra Λ is assumed to be *noetherian.* As usual the τ-structures on the tensor product rings $\Lambda \otimes R$, $\Lambda^{\#} \mathbin{\widehat{\otimes}} \mathcal{O}_K$, $\Lambda^{\#} \mathbin{\widehat{\otimes}} K$ and on the scheme $S \times X = \operatorname{Spec} \Lambda \times X$ are given by endomorphisms which act as the identity on Λ and as the q-power map on the other factor.

Definition 4.3.1. Let $\mathcal{M}$ be a quasi-coherent shtuka on $S \times X$. The *Čech chomology complex* of $\mathcal{M}$ is the Λ-module complex

$$\mathrm{R}\check{\Gamma}(S \times X,\, \mathcal{M}) = \Big[\mathrm{R}\Gamma(\Lambda \otimes R,\, \mathcal{M}) \oplus \mathrm{R}\Gamma(\Lambda^{\#} \mathbin{\widehat{\otimes}} \mathcal{O}_K,\, \mathcal{M}) \to \mathrm{R}\Gamma(\Lambda^{\#} \mathbin{\widehat{\otimes}} K,\, \mathcal{M})\Big]$$

where the unlabelled morphism is the difference of the natural maps. The n-th cohomology group of this complex is denoted $\check{\mathrm{H}}^n(S \times X,\, \mathcal{M})$.

Lemma 4.3.2. *The natural commutative square*

$$\begin{tikzcd}
\operatorname{Spec}(\Lambda^{\#} \mathbin{\widehat{\otimes}} K) \arrow[r, "\iota'"] \arrow[d, "f'"] & \operatorname{Spec}(\Lambda^{\#} \mathbin{\widehat{\otimes}} \mathcal{O}_K) \arrow[d, "f"] \\
\operatorname{Spec}(\Lambda \otimes R) \arrow[r, "\iota"] & S \times X
\end{tikzcd}$$

is cartesian. Furthermore $\operatorname{Spec}(\Lambda \otimes R)$ *and* $\operatorname{Spec}(\Lambda^{\#} \mathbin{\widehat{\otimes}} \mathcal{O}_K)$ *form a flat covering of* $S \times X$*.*

Proof. Proposition 3.5.1 implies that the square is cartesian. The complement of $\operatorname{Spec}(\Lambda \otimes R)$ in $S \times X$ is $\operatorname{Spec}(\Lambda \otimes \mathcal{O}_K/\mathfrak{m}_K)$ so the images of $\operatorname{Spec}(\Lambda \otimes R)$ and $\operatorname{Spec}(\Lambda^{\#} \mathbin{\widehat{\otimes}} \mathcal{O}_K)$ cover $S \times X$. It remains to prove that $\operatorname{Spec}(\Lambda^{\#} \mathbin{\widehat{\otimes}} \mathcal{O}_K)$ is flat over $S \times X$.

Pick an affine open subscheme $\operatorname{Spec} R' \subset X$ which contains $\operatorname{Spec} \mathcal{O}_K/\mathfrak{m}_K$. Shrinking $\operatorname{Spec} R'$ if necessary we can find an element $r' \in R'$ which is a uniformizer of $\mathcal{O}_K$. By Proposition 3.5.6 the ring $\Lambda^{\#} \mathbin{\widehat{\otimes}} \mathcal{O}_K$ is complete with respect to the ideal $\Lambda^{\#} \mathbin{\widehat{\otimes}} \mathfrak{m}_K$. This ideal is generated by $\mathfrak{m}_K$ according to Proposition 3.5.4. As r' is a generator of $\mathfrak{m}_K$ we deduce that $\Lambda^{\#} \mathbin{\widehat{\otimes}} \mathcal{O}_K$ is the completion of $\Lambda \otimes R'$ with respect to $\Lambda \otimes r'R'$. Now the fact that $\Lambda \otimes R'$ is noetherian implies that $\Lambda^{\#} \mathbin{\widehat{\otimes}} \mathcal{O}_K$ is flat over $\Lambda \otimes R'$ and therefore over $S \times X$. □

Let $\mathcal{F}$ be a quasi-coherent sheaf on $S \times X$. We define a complex of sheaves on $S \times X$:

$$\mathcal{C}(\mathcal{F}) = \Big[\iota_* \iota^* \mathcal{F} \oplus f_* f^* \mathcal{F} \to g_* g^* \mathcal{F}\Big]$$

where $g\colon \operatorname{Spec}(\Lambda^{\#} \mathbin{\widehat{\otimes}} K) \to S \times X$ is the natural map and the differential is the difference of the natural maps as in the definition of $\mathrm{R}\breve{\Gamma}$. The sum of adjunction units provides us with a natural morphism $\mathcal{F}[0] \to \mathcal{C}(\mathcal{F})$.

Lemma 4.3.3. *If* $\mathcal{F}$ *is a quasi-coherent sheaf on* $S \times X$ *then the natural map* $\mathcal{F}[0] \to \mathcal{C}(\mathcal{F})$ *is a quasi-isomorphism.*

Proof. We first show that natural sequence

$$0 \to \mathcal{O}_{S\times X} \to \iota_* \iota^* \mathcal{O}_{S\times X} \oplus f_* f^* \mathcal{O}_{S\times X} \to g_* g^* \mathcal{O}_{S\times X} \to 0 \tag{4.2}$$

is exact. As the commutative diagram of Lemma 4.3.2 is cartesian and the morphism $f\colon \operatorname{Spec}(\Lambda^{\#} \mathbin{\widehat{\otimes}} \mathcal{O}_K) \to S \times X$ is affine the pullback of (4.2) to $\operatorname{Spec}(\Lambda^{\#} \mathbin{\widehat{\otimes}} \mathcal{O}_K)$ is

$$0 \to \Lambda^{\#} \mathbin{\widehat{\otimes}} \mathcal{O}_K \xrightarrow{(1,\iota')} (\Lambda^{\#} \mathbin{\widehat{\otimes}} \mathcal{O}_K) \oplus (\Lambda^{\#} \mathbin{\widehat{\otimes}} K) \xrightarrow{(\iota',-1)} \Lambda^{\#} \mathbin{\widehat{\otimes}} K \to 0.$$

This sequence is clearly exact. The same argument shows that the pullback of (4.2) to $\operatorname{Spec}(\Lambda \otimes R)$ is exact. As $\operatorname{Spec}(\Lambda \otimes R)$ and $\operatorname{Spec}(\Lambda^{\#} \mathbin{\widehat{\otimes}} \mathcal{O}_K)$ form a flat covering of $S \times X$ it follows that (4.2) is exact.

Now let $\mathcal{F}$ be a quasi-coherent sheaf on the scheme $S \times X$. Consider the morphism $g\colon \operatorname{Spec}(\Lambda^{\#} \mathbin{\widehat{\otimes}} K) \to S \times X$. As $S \times X$ is separated over $\mathbb{F}_q$ the morphism g is affine. Thus the natural map

$$\big(g_* \mathcal{O}_{\operatorname{Spec} \Lambda^{\#} \widehat{\otimes} K}\big) \otimes_{\mathcal{O}_{S\times X}} \mathcal{F} \to g_* g^* \mathcal{F}$$

is an isomorphism. The same argument applies to the maps ι and f. We conclude that

$$\mathcal{C}(\mathcal{O}_{S\times X}) \otimes_{\mathcal{O}_{S\times X}} \mathcal{F} = \mathcal{C}(\mathcal{F}).$$

Consider the distinguished triangle

$$\mathcal{O}_{S\times X}[0] \to \mathcal{C}(\mathcal{O}_{S\times X}) \to C \to [1]$$

extending the natural quasi-isomorphism $\mathcal{O}_{S\times X}[0] \to \mathcal{C}(\mathcal{O}_{S\times X})$. Applying the functor $- \otimes_{\mathcal{O}_{\Lambda\times X}} \mathcal{F}$ we obtain a distinguished triangle

$$\mathcal{F}[0] \to \mathcal{C}(\mathcal{F}) \to C \otimes_{\mathcal{O}_{S\times X}} \mathcal{F} \to [1]$$

where the first arrow is the natural map $\mathcal{F}[0] \to \mathcal{C}(\mathcal{F})$. By construction C is a bounded acyclic complex of flat $\mathcal{O}_{S\times X}$-modules. Hence the complex $C \otimes_{\mathcal{O}_{S\times X}} \mathcal{F}$ is acyclic and the first arrow in the triangle above is a quasi-isomorphism. □

Definition 4.3.4. Let $\mathcal{F}$ be a quasi-coherent sheaf on $S \times X$.

(1) The *Čech cohomology complex* of $\mathcal{F}$ is the Λ-module complex

$$\mathrm{R}\check{\Gamma}(S \times X,\, \mathcal{F}) = \Gamma(S \times X,\, \mathcal{C}(\mathcal{F})).$$

(2) We define a natural map $\mathrm{R}\check{\Gamma}(S \times X,\, \mathcal{F}) \to \mathrm{R}\Gamma(S \times X,\, \mathcal{F})$ as the composition

$$\Gamma(S \times X,\, \mathcal{C}(\mathcal{F})) \to \mathrm{R}\Gamma(S \times X,\, \mathcal{C}(\mathcal{F})) \xleftarrow{\simeq} \mathrm{R}\Gamma(S \times X,\, \mathcal{F})$$

of the natural map $\Gamma \to \mathrm{R}\Gamma$ and the quasi-isomorphism provided by Lemma 4.3.3.

More explicitly

$$\mathrm{R}\check{\Gamma}(S \times X,\, \mathcal{F}) = \Big[\Gamma(\Lambda \otimes R,\, \mathcal{F}) \oplus \Gamma(\Lambda^{\#} \mathbin{\widehat{\otimes}} \mathcal{O}_K,\, \mathcal{F}) \to \Gamma(\Lambda^{\#} \mathbin{\widehat{\otimes}} K,\, \mathcal{F})\Big].$$

The differential is as in the definition of $\mathrm{R}\check{\Gamma}$ for shtukas.

To make the expressions in the rest of the section more legible we will generally omit the argument $S \times X$ of the functors Γ, $\mathrm{R}\check{\Gamma}$ and $\mathrm{R}\Gamma$ for quasi-coherent sheaves and shtukas. The same applies to the associated complex functor Γ_{a}.

Theorem 4.3.5. *The natural map* $\mathrm{R}\check{\Gamma}(\mathcal{F}) \to \mathrm{R}\Gamma(\mathcal{F})$ *is a quasi-isomorphism for every quasi-coherent sheaf* $\mathcal{F}$.

Proof. By construction $\mathcal{C}(\mathcal{F})$ sits in a distinguished triangle

$$\mathcal{C}(\mathcal{F}) \to (\iota_*\iota^*\mathcal{F} \oplus f_*f^*\mathcal{F})[0] \to g_*g^*\mathcal{F}[0] \to [1].$$

Applying Γ and $\mathrm{R}\Gamma$ we obtain a morphism of distinguished triangles

$$\begin{tikzcd}
\Gamma(\mathcal{C}(\mathcal{F})) \arrow[r] \arrow[d] & \Gamma(\iota_*\iota^*\mathcal{F} \oplus f_*f^*\mathcal{F})[0] \arrow[r] \arrow[d] & \Gamma(g_*g^*\mathcal{F})[0] \arrow[r] \arrow[d] & {[1]} \\
\mathrm{R}\Gamma(\mathcal{C}(\mathcal{F})) \arrow[r] & \mathrm{R}\Gamma(\iota_*\iota^*\mathcal{F} \oplus f_*f^*\mathcal{F}) \arrow[r] & \mathrm{R}\Gamma(g_*g^*\mathcal{F}) \arrow[r] & {[1]}
\end{tikzcd}$$

We will prove that the second and third vertical arrows in this diagram are quasi-isomorphisms. It follows that the first arrow is a quasi-isomorphism and so the lemma is proven.

Consider the third vertical arrow. The map g is affine so that $g_*g^*\mathcal{F}[0] = \mathrm{R}g_*g^*\mathcal{F}$. Hence $\mathrm{R}\Gamma(g_*g^*\mathcal{F}) = \mathrm{R}\Gamma(\mathrm{R}g_*g^*\mathcal{F}) = \mathrm{R}\Gamma(\Lambda^{\#} \mathbin{\widehat{\otimes}} K, \mathcal{F})$. As $\mathrm{Spec}(\Lambda^{\#} \mathbin{\widehat{\otimes}} K)$ is affine the natural map $\Gamma(\Lambda^{\#} \mathbin{\widehat{\otimes}} K, \mathcal{F})[0] \to \mathrm{R}\Gamma(\Lambda^{\#} \mathbin{\widehat{\otimes}} K, \mathcal{F})$ is a quasi-isomorphism. Hence the third vertical map in the diagram above is a quasi-isomorphism. The maps ι and f are also affine whence the same argument shows that the second vertical map is a quasi-isomorphism. □

Let $\mathcal{M}$ be a quasi-coherent shtuka on $S \times X$. Define a complex of shtukas on $S \times X$:

$$\mathcal{C}(\mathcal{M}) = \Big[\iota_*\iota^*\mathcal{M} \oplus f_*f^*\mathcal{M} \to g_*g^*\mathcal{M}\Big].$$

Here the differential is the difference of the natural maps as in the definition of $\mathrm{R}\check{\Gamma}(S \times X,\, \mathcal{M})$. The sum of the adjunction units gives a natural morphism $\mathcal{M}[0] \to \mathcal{C}(\mathcal{M})$.

Lemma 4.3.6. *If* $\mathcal{M}$ *is a quasi-coherent shtuka on* $S \times X$ *then* $\mathrm{R}\check{\Gamma}(\mathcal{M}) = \Gamma_{\mathrm{a}}(\mathcal{C}(\mathcal{M}))$.

Proof. Let $f\colon \mathcal{N} \to \mathcal{N}'$ be a morphism of shtukas and let $C = [\mathcal{N} \to \mathcal{N}']$ be its mapping fiber. The associated complex functor Γ_{a} is defined in such a way that $\Gamma_{\mathrm{a}}(C)$ is the mapping fiber of $\Gamma_{\mathrm{a}}(f)$. Applying this observation to $C = \mathcal{C}(\mathcal{M})$ we get the result. □

Lemma 4.3.7. *If $\mathcal{M}$ is a quasi-coherent shtuka on $S \times X$ then the natural map $\mathcal{M}[0] \to \mathcal{C}(\mathcal{M})$ is a quasi-isomorphism.*

Proof. Follows instantly from Lemma 4.3.3. $\square$

Definition 4.3.8. We define a natural map $\mathrm{R}\breve{\Gamma}(\mathcal{M}) \to \mathrm{R}\Gamma(\mathcal{M})$ as the composition

$$\Gamma_{\mathrm{a}}(\mathcal{C}(\mathcal{M})) \to \mathrm{R}\Gamma(\mathcal{C}(\mathcal{M})) \xleftarrow{\sim} \mathrm{R}\Gamma(\mathcal{M})$$

of the natural map $\Gamma_{\mathrm{a}} \to \mathrm{R}\Gamma$ and the quasi-isomorphism provided by Lemma 4.3.7.

Theorem 4.3.9. *For every quasi-coherent shtuka $\mathcal{M}$ on $S \times X$ the natural map $\mathrm{R}\breve{\Gamma}(\mathcal{M}) \to \mathrm{R}\Gamma(\mathcal{M})$ is a quasi-isomorphism.*

Proof. According to Proposition 1.6.4 the natural diagram

$$\begin{tikzcd}
\Gamma_{\mathrm{a}}(\mathcal{C}(\mathcal{M})) \arrow[r] \arrow[d] & \Gamma(\mathcal{C}(\mathcal{M}_0)) \arrow[r, "i-j"] \arrow[d] & \Gamma(\mathcal{C}(\mathcal{M}_1)) \arrow[r] \arrow[d] & {[1]} \\
\mathrm{R}\Gamma(\mathcal{C}(\mathcal{M})) \arrow[r] & \mathrm{R}\Gamma(\mathcal{C}(\mathcal{M}_0)) \arrow[r, "i-j"] & \mathrm{R}\Gamma(\mathcal{C}(\mathcal{M}_1)) \arrow[r] & {[1]}
\end{tikzcd}$$

is a morphism of distinguished triangles. Furthermore the canonical triangles are natural. Hence the quasi-isomorphism $\mathcal{M}[0] \to \mathcal{C}(\mathcal{M})$ induces an isomorphism of distinguished triangles

$$\begin{tikzcd}
\mathrm{R}\Gamma(\mathcal{C}(\mathcal{M})) \arrow[r] & \mathrm{R}\Gamma(\mathcal{C}(\mathcal{M}_0)) \arrow[r, "i-j"] & \mathrm{R}\Gamma(\mathcal{C}(\mathcal{M}_1)) \arrow[r] & {[1]} \\
\mathrm{R}\Gamma(\mathcal{M}) \arrow[r] \arrow[u, "\wr"] & \mathrm{R}\Gamma(\mathcal{M}_0) \arrow[r, "i-j"] \arrow[u, "\wr"] & \mathrm{R}\Gamma(\mathcal{M}_1) \arrow[r] \arrow[u, "\wr"'] & {[1]}
\end{tikzcd}$$

Taking the composition of the two diagrams above we get a morphism of distinguished triangles

$$\begin{tikzcd}
\mathrm{R}\breve{\Gamma}(\mathcal{M}) \arrow[r] \arrow[d] & \mathrm{R}\breve{\Gamma}(\mathcal{M}_0) \arrow[r, "i-j"] \arrow[d] & \mathrm{R}\breve{\Gamma}(\mathcal{M}_1) \arrow[r] \arrow[d] & {[1]} \\
\mathrm{R}\Gamma(\mathcal{M}) \arrow[r] & \mathrm{R}\Gamma(\mathcal{M}_0) \arrow[r, "i-j"] & \mathrm{R}\Gamma(\mathcal{M}_1) \arrow[r] & {[1]}
\end{tikzcd}$$

The second and third vertical arrows are quasi-isomorphisms by Theorem 4.3.5. Whence the result. $\square$

4.4. Compactly supported cohomology

We continue using the notation and the conventions of Section 4.3. In this section we assume that the coefficient algebra Λ carries a structure of a locally compact $\mathbb{F}_q$-algebra. A typical example of Λ relevant to our applications is the discrete algebra $\mathbb{F}_q[t]$ and the locally compact algebra $\mathbb{F}_q(\!(t^{-1})\!)$. We denote $Y = \operatorname{Spec} R$ as in the introduction.

Definition 4.4.1. Let $\mathcal{M}$ be a quasi-coherent shtuka on $S \times Y$. The *compactly supported cohomology complex* of $\mathcal{M}$ is the Λ-module complex

$$\mathrm{R}\Gamma_{\mathrm{c}}(S \times Y, \mathcal{M}) = \Big[\mathrm{R}\Gamma(\Lambda \otimes R, \mathcal{M}) \to \mathrm{R}\Gamma(\Lambda^{\#} \mathbin{\widehat{\otimes}} K, \mathcal{M})\Big].$$

Here the differential is induced by the natural inclusion $\Lambda \otimes R \to \Lambda^{\#} \mathbin{\widehat{\otimes}} K$. The n-th cohomology group of $\mathrm{R}\Gamma_{\mathrm{c}}(S \times Y, \mathcal{M})$ is denoted $\mathrm{H}^n_c(S \times Y, \mathcal{M})$.

Proposition 4.4.2. *If $\mathcal{M}$ is a locally free shtuka on $S \times Y$ then the natural map*

$$\mathrm{R}\Gamma_{\mathrm{c}}(S \times Y,\, \mathcal{M}) \to \mathrm{R}\Gamma\Big(S \times Y,\, \tfrac{\mathcal{M}(\Lambda^{\#} \widehat{\otimes} K)}{\mathcal{M}(\Lambda \otimes R)}\Big)[-1]$$

is a quasi-isomorphism. □

Definition 4.4.3. We define a map $\mathrm{R}\Gamma_{\mathrm{c}}(S \times Y,\, \mathcal{M}) \to \mathrm{R}\Gamma_g(\Lambda \mathbin{\breve{\otimes}} K,\, \mathcal{M})$ by the diagram

$$\begin{tikzcd} \big[\mathrm{R}\Gamma(\Lambda \otimes R,\, \mathcal{M}) \arrow[r] \arrow[d] & \mathrm{R}\Gamma(\Lambda^{\#} \widehat{\otimes} K,\, \mathcal{M})\big] \arrow[d, equal] \\ \big[\mathrm{R}\Gamma(\Lambda \mathbin{\breve{\otimes}} K,\, \mathcal{M}) \arrow[r] & \mathrm{R}\Gamma(\Lambda^{\#} \widehat{\otimes} K,\, \mathcal{M})\big] \end{tikzcd}$$

Definition 4.4.4. Let $\mathcal{M}$ be a quasi-coherent shtuka on $S \times X$. We define a map $\mathrm{R}\Gamma_{\mathrm{c}}(S \times Y,\, \mathcal{M}) \to \mathrm{R}\Gamma(S \times X,\, \mathcal{M})$ as the composition

$$\mathrm{R}\Gamma_{\mathrm{c}}(S \times Y,\, \mathcal{M}) \xrightarrow{\text{embedding}} \mathrm{R}\breve{\Gamma}(S \times X,\, \mathcal{M}) \xrightarrow{\text{Thm. 4.3.9}} \mathrm{R}\Gamma(S \times X,\, \mathcal{M}).$$

Proposition 4.4.5. *Let $\mathcal{M}$ be a locally free shtuka on $S \times X$. If $\mathcal{M}(\Lambda \otimes \mathcal{O}_K/\mathfrak{m}_K)$ is nilpotent then $\mathrm{R}\Gamma_{\mathrm{c}}(S \times Y,\, \mathcal{M}) \to \mathrm{R}\Gamma(S \times X,\, \mathcal{M})$ is a quasi-isomorphism.*

The condition that $\mathcal{M}(\Lambda \otimes \mathcal{O}_K/\mathfrak{m}_K)$ is nilpotent may be interpreted as saying that $\mathcal{M}$ is an extension by zero of a shtuka on the open τ-subscheme $S \times Y \subset S \times X$.

Proof of Proposition 4.4.5. The map in question extends to a distinguished triangle

$$\mathrm{R}\Gamma_{\mathrm{c}}(S \times Y,\, \mathcal{M}) \to \mathrm{R}\breve{\Gamma}(S \times X,\, \mathcal{M}) \to \mathrm{R}\Gamma(\Lambda^{\#} \widehat{\otimes} \mathcal{O}_K,\, \mathcal{M}) \to [1].$$

The result follows since $\mathrm{R}\Gamma(\Lambda^{\#} \widehat{\otimes} \mathcal{O}_K,\, \mathcal{M}) = 0$ by Proposition 4.2.1. □

Definition 4.4.6. Let $\mathcal{M}$ be a locally free shtuka on $S \times X$ such that $\mathcal{M}(\Lambda \otimes \mathcal{O}_K/\mathfrak{m}_K)$ is nilpotent. The *global germ map* is defined as the composition

$$\mathrm{R}\Gamma(S \times X,\, \mathcal{M}) \xleftarrow{\sim} \mathrm{R}\Gamma_{\mathrm{c}}(S \times Y,\, \mathcal{M}) \xrightarrow{\text{Def. 4.4.3}} \mathrm{R}\Gamma_g(\Lambda \mathbin{\breve{\otimes}} K,\, \mathcal{M})$$

in $\mathrm{D}(\Lambda)$ where the first arrow is the quasi-isomorphism of Definition 4.4.4. The adjective "global" indicates that this map involves a shtuka on the whole $S \times X$ as opposed to $\Lambda \mathbin{\breve{\otimes}} \mathcal{O}_K$.

4.5. Local-global compatibility

We keep the conventions and the notation of Section 4.3. Our coefficient algebra will be either a local field F or its ring of integers $\mathcal{O}_F$. We denote $D = \operatorname{Spec} \mathcal{O}_F$ and $D^\circ = \operatorname{Spec} F$. The letter "$D$" stands for a disk.

Let $\mathcal{M}$ be a locally free shtuka on $D^\circ \times X$ such that $\mathcal{M}(F \otimes \mathcal{O}_K/\mathfrak{m}_K)$ is nilpotent. We have two maps from the cohomology of $\mathcal{M}$ to the germ cohomology $\mathrm{R}\Gamma_g(F \mathbin{\breve{\otimes}} K,\, \mathcal{M})$:

- the local germ map of Definition 4.2.4,
- the global germ map of Definition 4.4.6.

They form a square in the derived category of F-vector spaces:

$$\begin{tikzcd} \mathrm{R}\Gamma(D^\circ \times X,\, \mathcal{M}) \arrow[r, "\text{global}"] \arrow[d] & \mathrm{R}\Gamma_g(F \mathbin{\breve{\otimes}} K,\, \mathcal{M}) \arrow[d, equal] \\ \mathrm{R}\Gamma(F \mathbin{\breve{\otimes}} \mathcal{O}_K,\, \mathcal{M}) \arrow[r, "\text{local}", "\sim"'] & \mathrm{R}\Gamma_g(F \mathbin{\breve{\otimes}} K,\, \mathcal{M}) \end{tikzcd} \tag{4.3}$$

The left arrow in this square is the pullback map.

The definitions of the local and the global germ map have nothing in common so there is no a priori reason for (4.3) to be commutative. Nevertheless we will prove that (4.3) commutes under the assumption that $\mathcal{M}$ extends to a locally free shtuka on $D \times X$.

Theorem 4.5.1. *If $\mathcal{M}$ is a locally free shtuka on $D\times X$ such that $\mathcal{M}(F\otimes\mathcal{O}_K/\mathfrak{m}_K)$ is nilpotent then the square* (4.3) *is commutative.*

Later in this chapter we will show that the left arrow in (4.3) is a quasi-isomorphism provided $\mathcal{M}(\mathcal{O}_F/\mathfrak{m}_F \otimes R)$ is nilpotent (Theorem 4.6.3). The local germ map is a quasi-isomorphism by construction. The commutativity of (4.3) then implies that the global germ map is a quasi-isomorphism, a property which is not evident from its definition.

Proof of Theorem 4.5.1. Take H^1 of (4.3) and extend it to the left as follows:

$$\begin{tikzcd} \mathrm{H}^1(D\times X,\,\mathcal{M}) \arrow[r, dashed] \arrow[d, dashed] & \mathrm{H}^1(D^\circ\times X,\,\mathcal{M}) \arrow[r] \arrow[d] & \mathrm{H}^1_g(F\,\check{\otimes}\,K,\,\mathcal{M}) \arrow[d, equal] \\ \mathrm{H}^1(\mathcal{O}_F\,\check{\otimes}\,\mathcal{O}_K,\,\mathcal{M}) \arrow[r, dashed] & \mathrm{H}^1(F\,\check{\otimes}\,\mathcal{O}_K,\,\mathcal{M}) \arrow[r] & \mathrm{H}^1_g(F\,\check{\otimes}\,K,\,\mathcal{M}) \end{tikzcd} \tag{4.4}$$

The three additional maps are the pullback morphisms. We proceed to prove that the outer rectangle of (4.4) commutes.

Theorem 4.3.9 equips us with natural isomorphisms

$$\mathrm{H}^1(D\times X,\,\mathcal{M}) \cong \check{\mathrm{H}}^1(D\times X,\,\mathcal{M}),$$
$$\mathrm{H}^1(D^\circ\times X,\,\mathcal{M}) \cong \check{\mathrm{H}}^1(D^\circ\times X,\,\mathcal{M}),$$

while Proposition 4.1.2 provides a natural isomorphism

$$\mathrm{H}^1_g(F\,\check{\otimes}\,K,\,\mathcal{M}) \cong \mathrm{H}^0(F\,\check{\otimes}\,K,\,\mathcal{Q})$$

where

$$\mathcal{Q} = \frac{\mathcal{M}(F^{\#}\,\widehat{\otimes}\,K)}{\mathcal{M}(F\,\check{\otimes}\,K)}.$$

Using them we rewrite (4.4) as

$$\begin{tikzcd} \check{\mathrm{H}}^1(D\times X,\,\mathcal{M}) \arrow[r] \arrow[d] & \check{\mathrm{H}}^1(D^\circ\times X,\,\mathcal{M}) \arrow[r] & \mathrm{H}^0(F\,\check{\otimes}\,K,\,\mathcal{Q}) \arrow[d, equal] \\ \mathrm{H}^1(\mathcal{O}_F\,\check{\otimes}\,\mathcal{O}_K,\,\mathcal{M}) \arrow[r] & \mathrm{H}^1(F\,\check{\otimes}\,\mathcal{O}_K,\,\mathcal{M}) \arrow[r] & \mathrm{H}^0(F\,\check{\otimes}\,K,\,\mathcal{Q}) \end{tikzcd} \tag{4.5}$$

The middle arrow is omitted since it is not easy to describe in terms of Čech cohomology.

Let the shtuka $\mathcal{M}$ be given by a diagram

$$\mathcal{M}_0 \underset{j}{\overset{i}{\rightrightarrows}} \mathcal{M}_1.$$

Let Λ be either $\mathcal{O}_F$ or F. By definition $\mathrm{R}\check{\Gamma}(\operatorname{Spec}\Lambda\times X,\,\mathcal{M})$ is the total complex of the double complex

$$\begin{tikzcd} \mathcal{M}_1(\Lambda\otimes R)\oplus\mathcal{M}_1(\Lambda^{\#}\,\widehat{\otimes}\,\mathcal{O}_K) \arrow[r, "\text{difference}"] & \mathcal{M}_1(\Lambda^{\#}\,\widehat{\otimes}\,K) \\ \mathcal{M}_0(\Lambda\otimes R)\oplus\mathcal{M}_0(\Lambda^{\#}\,\widehat{\otimes}\,\mathcal{O}_K) \arrow[r, "\text{difference}"] \arrow[u, "i-j"] & \mathcal{M}_0(\Lambda^{\#}\,\widehat{\otimes}\,K) \arrow[u, "j-i"'] \end{tikzcd}$$

So a cohomology class in $\check{\mathrm{H}}^1(\operatorname{Spec}\Lambda\times X,\,\mathcal{M})$ is represented by a triple

$$(a,x,b)\in\mathcal{M}_1(\Lambda\otimes R)\oplus\mathcal{M}_1(\Lambda^{\#}\,\widehat{\otimes}\,\mathcal{O}_K)\oplus\mathcal{M}_0(\Lambda^{\#}\,\widehat{\otimes}\,K)$$

satisfying $a - x + (j-i)(b) = 0$.

Fix a cohomology class $h \in \check{\mathrm{H}}^1(D \times X,\, \mathcal{M})$. We want to compute its image under the composition

$$\check{\mathrm{H}}^1(D \times X,\, \mathcal{M}) \to \check{\mathrm{H}}^1(D^\circ \times X,\, \mathcal{M}) \to \mathrm{H}^0(F \breve{\otimes} K,\, \mathcal{Q})$$

of the two top arrows in (4.5). Let (a, x, b) be a triple representing h. The image of h in $\check{\mathrm{H}}^1(D^\circ \times X,\, \mathcal{M})$ is represented by the same triple (a, x, b). From Definition 4.4.6 it follows that the map $\check{\mathrm{H}}^1(D^\circ \times X,\, \mathcal{M}) \to \mathrm{H}^1_g(F \breve{\otimes} K,\, \mathcal{M})$ of (4.5) is a composition

$$\check{\mathrm{H}}^1(D^\circ \times X,\, \mathcal{M}) \xleftarrow{\sim} \mathrm{H}^1_c(F \otimes R,\, \mathcal{M}) \to \mathrm{H}^0(F \breve{\otimes} K,\, \mathcal{Q}) \tag{4.6}$$

By construction $\mathrm{R}\Gamma_c(F \otimes R,\, \mathcal{M})$ is the total complex of the double complex

$$\begin{tikzcd} \mathcal{M}_1(F \otimes R) \arrow[r] & \mathcal{M}_1(F^\# \widehat{\otimes} K) \\ \mathcal{M}_0(F \otimes R) \arrow[r] \arrow[u, "i-j"] & \mathcal{M}_0(F^\# \widehat{\otimes} K) \arrow[u, "j-i"'] \end{tikzcd}$$

So a cohomology class in $\mathrm{H}^1_c(F \otimes R,\, \mathcal{M})$ is represented by a pair

$$(a', b') \in \mathcal{M}_1(F \otimes R) \oplus \mathcal{M}_0(F^\# \widehat{\otimes} K)$$

such that $a' + (j-i)(b') = 0$.

The isomorphism $\mathrm{H}^1_c(F \otimes R,\, \mathcal{M}) \cong \check{\mathrm{H}}^1(D^\circ \times X,\, \mathcal{M})$ of (4.6) sends (a', b') to $(a', 0, b')$. Thus in order to compute the image of h in $\mathrm{H}^1_c(F \otimes R,\, \mathcal{M})$ we need to replace (a, x, b) with a cohomologous triple of the form $(a', 0, b')$. A triple is a coboundary if and only if it has the form

$$\big((i-j)(a'), (i-j)(y), a' - y\big)$$

where $a' \in \mathcal{M}_0(F \otimes R)$ and $y \in \mathcal{M}_0(F^\# \widehat{\otimes} \mathcal{O}_K)$. By Proposition 4.2.6 (1) there is a unique $y \in \mathcal{M}_0(F^\# \widehat{\otimes} \mathcal{O}_K)$ such that $(i-j)(y) = x$. The triple $(0, x, -y)$ is then a coboundary so that (a, x, b) is cohomologous to $(a, 0, b + y)$ and the image of h in $\mathrm{H}^1_c(F \otimes R,\, \mathcal{M})$ is represented by $(a, b + y)$.

We are finally ready to compute the image of $h \in \check{\mathrm{H}}^1(D \times X,\, \mathcal{M})$ under the composition (4.6):

$$\check{\mathrm{H}}^1(D^\circ \times X,\, \mathcal{M}) \xleftarrow{\sim} \mathrm{H}^1_c(F \otimes R,\, \mathcal{M}) \to \mathrm{H}^0(F \breve{\otimes} K,\, \mathcal{Q})$$

By definition of $\mathcal{Q}$ we have

$$\mathrm{R}\Gamma(F \breve{\otimes} K,\, \mathcal{Q}) = \Big[\tfrac{\mathcal{M}_0(F^\# \widehat{\otimes} K)}{\mathcal{M}_0(F \breve{\otimes} K)} \xrightarrow{i-j} \tfrac{\mathcal{M}_1(F^\# \widehat{\otimes} K)}{\mathcal{M}_1(F \breve{\otimes} K)}\Big].$$

The second arrow in (4.6) sends a pair (a', b') representing a class in $\mathrm{H}^1_c(F \otimes R,\, \mathcal{M})$ to the equivalence class $[b']$ of b' in the quotient $\mathcal{M}_0(F^\# \widehat{\otimes} K)/\mathcal{M}_0(F \breve{\otimes} K)$. Above we demonstrated that the image of h in $\mathrm{H}^1_c(F \otimes R,\, \mathcal{M})$ is represented by the pair $(a, b + y)$. Hence the image of h in $\mathrm{H}^0(F \breve{\otimes} K,\, \mathcal{Q})$ is given by the equivalence class $[b + y]$.

The key observation in this proof is that $[b + y] = [y]$. Indeed the left arrow in the natural commutative square

$$\begin{tikzcd} \mathcal{O}_F \breve{\otimes} K \arrow[r] \arrow[d] & F \breve{\otimes} K \arrow[d] \\ \mathcal{O}_F^\# \widehat{\otimes} K \arrow[r] & F^\# \widehat{\otimes} K. \end{tikzcd}$$

is an isomorphism by Proposition 3.3.10. Hence the homomorphism $\mathcal{O}_F^\# \widehat{\otimes} K \to F^\# \widehat{\otimes} K$ factors through $F \breve{\otimes} K$ and the natural map

$$\mathcal{M}_0(\mathcal{O}_F^\# \widehat{\otimes} K) \to \frac{\mathcal{M}_0(F^\# \widehat{\otimes} K)}{\mathcal{M}_0(F \breve{\otimes} K)}$$

is zero. As $b \in \mathcal{M}_0(\mathcal{O}_F^\# \widehat{\otimes} K)$ by construction we conclude that $[b + y] = [y]$.

So far we have demonstrated the following. Let $h \in \check{\mathrm{H}}^1(D \times X, \mathcal{M})$ be a cohomology class. If h is represented by a triple

$$(a, x, b) \in \mathcal{M}_1(\mathcal{O}_F \otimes R) \oplus \mathcal{M}_1(\mathcal{O}_F^\# \widehat{\otimes} \mathcal{O}_K) \oplus \mathcal{M}_0(\mathcal{O}_F^\# \widehat{\otimes} K)$$

then the image of h under the composition

$$\check{\mathrm{H}}^1(D \times X, \mathcal{M}) \to \check{\mathrm{H}}^1(D^\circ \times X, \mathcal{M}) \to \mathrm{H}^0(F \breve{\otimes} K, \mathcal{Q})$$

of the two top arrows in the square (4.5) is given by the equivalence class

$$[y] \in \frac{\mathcal{M}_0(F^\# \widehat{\otimes} K)}{\mathcal{M}_0(F \breve{\otimes} K)}$$

where $y \in \mathcal{M}_0(F^\# \widehat{\otimes} \mathcal{O}_K)$ is the unique element satisfying $(i - j)(y) = x$. We are now in position to prove that the square (4.5) is commutative.

The cohomology classes in $\mathrm{H}^1(\mathcal{O}_F \breve{\otimes} \mathcal{O}_K, \mathcal{M})$ are represented by elements of $\mathcal{M}_1(\mathcal{O}_F \breve{\otimes} \mathcal{O}_K)$. By Proposition 3.3.10 the natural map $\mathcal{O}_F \breve{\otimes} \mathcal{O}_K \to \mathcal{O}_F^\# \widehat{\otimes} \mathcal{O}_K$ is an isomorphism. Hence we can identify $\mathcal{M}_1(\mathcal{O}_F \breve{\otimes} \mathcal{O}_K)$ with $\mathcal{M}_1(\mathcal{O}_F^\# \widehat{\otimes} \mathcal{O}_K)$. The left arrow $\check{\mathrm{H}}^1(D \times X, \mathcal{M}) \to \mathrm{H}^1(\mathcal{O}_F \breve{\otimes} \mathcal{O}_K, \mathcal{M})$ of the square (4.5) sends the cocycle (a, x, b) to $x \in \mathcal{M}_1(\mathcal{O}_F^\# \widehat{\otimes} \mathcal{O}_K) = \mathcal{M}_1(\mathcal{O}_F \breve{\otimes} \mathcal{O}_K)$. Now Proposition 4.2.6 (2) implies that the image of x under the composition of the two bottom arrows in (4.5) is $[y]$. Therefore the square (4.5) is commutative.

We deduce that the outer rectangle of (4.4) is commutative. Since the F-linear extension

$$F \otimes_{\mathcal{O}_F} \mathrm{H}^1(D \times X, \mathcal{M}) \to \mathrm{H}^1(D^\circ \times X, \mathcal{M})$$

of the top horizontal map in this square is an isomorphism the right square of (4.4) is commutative too. By Proposition 4.2.5 the complex $\mathrm{R}\Gamma_g(F \breve{\otimes} K, \mathcal{M})$ is concentrated in degree 1. Therefore commutativity of the right square of (4.4) implies commutativity of the main diagram (4.3) in the derived category of F-vector spaces. $\square$

4.6. Completed Čech cohomology

We keep the notation and the conventions of Sections 4.3 and 4.5. In this section we present a refined version of the Čech method for computing the cohomology of *coherent* shtukas on $D \times X$. In essense it is the Čech method of Section 4.3 developed in the setting of formal schemes over $D = \operatorname{Spec} \mathcal{O}_F$.

Definition 4.6.1. Let $\mathcal{M}$ be a coherent shtuka on $D \times X$. The *completed Čech chomology complex* of $\mathcal{M}$ is

$$\mathrm{R}\widehat{\Gamma}(D \times X, \mathcal{M}) = \Big[\mathrm{R}\Gamma(\mathcal{O}_F \widehat{\otimes} R, \mathcal{M}) \oplus \mathrm{R}\Gamma(\mathcal{O}_F \widehat{\otimes} \mathcal{O}_K, \mathcal{M}) \to \mathrm{R}\Gamma(\mathcal{O}_F \widehat{\otimes} K, \mathcal{M}) \Big].$$

Here the differential is the difference of the natural maps.

Recall that the Čech complex of $\mathcal{M}$ is

$$\mathrm{R}\check{\Gamma}(D \times X, \mathcal{M}) = \Big[\mathrm{R}\Gamma(\mathcal{O}_F \otimes R, \mathcal{M}) \oplus \mathrm{R}\Gamma(\mathcal{O}_F^\# \widehat{\otimes} \mathcal{O}_K, \mathcal{M}) \to \mathrm{R}\Gamma(\mathcal{O}_F^\# \widehat{\otimes} K, \mathcal{M}) \Big].$$

We thus have a natural map $\mathrm{R}\check{\Gamma}(D \times X, \mathcal{M}) \to \mathrm{R}\widehat{\Gamma}(D \times X, \mathcal{M})$.

Theorem 4.6.2. *Let $\mathcal{M}$ be a shtuka on $D \times X$. If $\mathcal{M}$ is coherent then the natural map* $\mathrm{R}\check{\Gamma}(D \times X, \mathcal{M}) \to \mathrm{R}\widehat{\Gamma}(D \times X, \mathcal{M})$ *is a quasi-isomorphism.*

Rather than using completed Čech cohomology directly we will rely on Theorem 4.6.3 which captures a "cohomology concentration" phenomenon for locally free shtukas on $D \times X$. It will play a role in Chapter 6.

Theorem 4.6.3. *Let $\mathcal{M}$ be a locally free shtuka on $D \times X$. If $\mathcal{M}(\mathcal{O}_F/\mathfrak{m}_F \otimes R)$ is nilpotent then the natural map* $\mathrm{R}\Gamma(D \times X, \mathcal{M}) \to \mathrm{R}\Gamma(\mathcal{O}_F \mathbin{\widehat{\otimes}} \mathcal{O}_K, \mathcal{M})$ *is a quasi-isomorphism.*

Proof. Due to Theorem 4.6.2 it is enough to prove that the natural map

$$\mathrm{R}\widehat{\Gamma}(D \times X, \mathcal{M}) \to \mathrm{R}\Gamma(\mathcal{O}_F \mathbin{\widehat{\otimes}} \mathcal{O}_K, \mathcal{M})$$

is a quasi-isomorphism. By definition

$$\mathrm{R}\widehat{\Gamma}(D \times X, \mathcal{M}) = \Big[\mathrm{R}\Gamma(\mathcal{O}_F \mathbin{\widehat{\otimes}} R, \mathcal{M}) \oplus \mathrm{R}\Gamma(\mathcal{O}_F \mathbin{\widehat{\otimes}} \mathcal{O}_K, \mathcal{M}) \to \mathrm{R}\Gamma(\mathcal{O}_F \mathbin{\widehat{\otimes}} K, \mathcal{M})\Big].$$

Hence it is enough to show that the complexes $\mathrm{R}\Gamma(\mathcal{O}_F \mathbin{\widehat{\otimes}} R, \mathcal{M})$ and $\mathrm{R}\Gamma(\mathcal{O}_F \mathbin{\widehat{\otimes}} K, \mathcal{M})$ are acyclic.

According to Proposition 3.5.6 the ring $\mathcal{O}_F \mathbin{\widehat{\otimes}} R$ is noetherian and complete with respect to the ideal $\mathfrak{m}_F \mathbin{\widehat{\otimes}} R$. By Proposition 3.5.4 the natural map $\mathcal{O}_F \mathbin{\widehat{\otimes}} R \to \mathcal{O}_F/\mathfrak{m}_F \otimes R$ is surjective with kernel $\mathfrak{m} \mathbin{\widehat{\otimes}} R$. Thus $\mathrm{R}\Gamma(\mathcal{O}_F \mathbin{\widehat{\otimes}} R, \mathcal{M}) = 0$ by Proposition 1.10.4. Applying the same argument to $\mathcal{O}_F \mathbin{\widehat{\otimes}} K^{\#}$ we deduce that $\mathrm{R}\Gamma(\mathcal{O}_F \mathbin{\widehat{\otimes}} K^{\#}, \mathcal{M}) = 0$. The natural map $\mathcal{O}_F \mathbin{\widehat{\otimes}} K^{\#} \to \mathcal{O}_F \mathbin{\widehat{\otimes}} K$ is an isomorphism by Proposition 3.3.9 whence $\mathrm{R}\Gamma(\mathcal{O}_F \mathbin{\widehat{\otimes}} K, \mathcal{M}) = 0$. ☐

We now turn to the proof of Theorem 4.6.2. We will derive it from a similar statement for coherent sheaves.

Let $\mathcal{F}$ be a quasi-coherent sheaf on $D \times X$. Recall that

$$\mathrm{R}\check{\Gamma}(D \times X, \mathcal{F}) = \Big[\Gamma(\mathcal{O}_F \otimes R, \mathcal{F}) \oplus \Gamma(\mathcal{O}_F^{\#} \mathbin{\widehat{\otimes}} \mathcal{O}_K, \mathcal{F}) \to \Gamma(\mathcal{O}_F^{\#} \mathbin{\widehat{\otimes}} K, \mathcal{F})\Big].$$

We set

$$\mathrm{R}\widehat{\Gamma}(D \times X, \mathcal{F}) = \Big[\Gamma(\mathcal{O}_F \mathbin{\widehat{\otimes}} R, \mathcal{F}) \oplus \Gamma(\mathcal{O}_F \mathbin{\widehat{\otimes}} \mathcal{O}_K, \mathcal{F}) \to \Gamma(\mathcal{O}_F \mathbin{\widehat{\otimes}} K, \mathcal{F})\Big]$$

with the same differentials as in the definition of $\mathrm{R}\widehat{\Gamma}$ for shtukas. To improve the legibility we will generally omit the argument $D \times X$ of the functors $\mathrm{R}\check{\Gamma}$ and $\mathrm{R}\widehat{\Gamma}$. By construction we have a natural map $\mathrm{R}\check{\Gamma}(\mathcal{F}) \to \mathrm{R}\widehat{\Gamma}(\mathcal{F})$.

For technical reasons it will be more convenient for us to work with different presentations of the complexes $\mathrm{R}\check{\Gamma}(\mathcal{F})$ and $\mathrm{R}\widehat{\Gamma}(\mathcal{F})$. We define the complexes

$$B(\mathcal{F}) = \Big[\Gamma(\mathcal{O}_F \otimes R, \mathcal{F}) \oplus \Gamma(\mathcal{O}_F \mathbin{\check{\otimes}} \mathcal{O}_K, \mathcal{F}) \to \Gamma(\mathcal{O}_F \mathbin{\check{\otimes}} K, \mathcal{F})\Big],$$
$$\widehat{B}(\mathcal{F}) = \Big[\Gamma(\mathcal{O}_F \mathbin{\widehat{\otimes}} R, \mathcal{F}) \oplus \Gamma(\mathcal{O}_F \mathbin{\widehat{\otimes}} \mathcal{O}_K^{\#}, \mathcal{F}) \to \Gamma(\mathcal{O}_F \mathbin{\widehat{\otimes}} K^{\#}, \mathcal{F})\Big]$$

with the same differentials as $\mathrm{R}\check{\Gamma}(\mathcal{F})$ and $\mathrm{R}\widehat{\Gamma}(\mathcal{F})$.

Lemma 4.6.4. *The natural map $B(\mathcal{F}) \to \mathrm{R}\check{\Gamma}(\mathcal{F})$ is an isomorphism.*

Proof. Follows from Proposition 3.3.10 since $\mathcal{O}_F$ is compact. ☐

Lemma 4.6.5. *The natural map $\widehat{B}(\mathcal{F}) \to \mathrm{R}\widehat{\Gamma}(\mathcal{F})$ is an isomorphism.*

Proof. Follows from Proposition 3.3.9 since $\mathcal{O}_F$ is compact. ☐

Lemma 4.6.6. *For every quasi-coherent sheaf $\mathcal{F}$ on $D \times X$ there exists a natural quasi-isomorphism* $B(\mathcal{F}) \xrightarrow{\sim} \mathrm{R}\Gamma(D \times X, \mathcal{F})$.

Proof. The natural map $B(\mathcal{F}) \to \mathrm{R}\check{\Gamma}(\mathcal{F})$ is an isomorphism by Lemma 4.6.4. So the result is a consequence of Theorem 4.3.5. $\square$

Lemma 4.6.7. *Let $\mathcal{F}$ be a quasi-coherent sheaf on $D \times X$. If $\mathfrak{m}_F^n \mathcal{F} = 0$ for some $n \gg 0$ then the natural map $B(\mathcal{F}) \to \widehat{B}(\mathcal{F})$ is an isomorphism.*

Proof. Consider the natural diagram

$$\Gamma(\mathcal{O}_F \mathbin{\check{\otimes}} K, \mathcal{F}) \to \Gamma(\mathcal{O}_F \mathbin{\widehat{\otimes}} K^{\#}, \mathcal{F}) \to \Gamma(\mathcal{O}_F/\mathfrak{m}_F^n \otimes K, \mathcal{F}).$$

By Proposition 3.5.4 the second arrow in this diagram is the reduction modulo $\mathfrak{m}_F^n$. The composite arrow is the reduction modulo $\mathfrak{m}_F^n$ by Propostion 3.5.5. Both arrows are isomorphisms since $\mathfrak{m}_F^n \mathcal{F} = 0$. Hence so is the first arrow. The same argument shows that the natural maps $\Gamma(\mathcal{O}_F \mathbin{\check{\otimes}} \mathcal{O}_K, \mathcal{F}) \to \Gamma(\mathcal{O}_F \mathbin{\widehat{\otimes}} \mathcal{O}_K^{\#}, \mathcal{F})$ and $\Gamma(\mathcal{O}_F \otimes R, \mathcal{F}) \to \Gamma(\mathcal{O}_F \mathbin{\widehat{\otimes}} R, \mathcal{F})$ are isomorphisms. $\square$

Lemma 4.6.8. *If $\mathcal{F}$ is a coherent sheaf on $D \times X$ then the natural map $\widehat{B}(\mathcal{F}) \to \lim_n \widehat{B}(\mathcal{F}/\mathfrak{m}_F^n)$ is an isomorphism.*

Proof. Proposition 3.5.6 shows that the ring $\mathcal{O}_F \mathbin{\widehat{\otimes}} K^{\#}$ is noetherian and complete with respect to the ideal $\mathfrak{m}_F \mathbin{\widehat{\otimes}} K^{\#}$. According to Proposition 3.5.4 this ideal is generated by $\mathfrak{m}_F$. The $\mathcal{O}_F \mathbin{\widehat{\otimes}} K^{\#}$-module $\Gamma(\mathcal{O}_F \mathbin{\widehat{\otimes}} K^{\#}, \mathcal{F})$ is finitely generated. As a consequence it is complete with respect to $\mathfrak{m}_F(\mathcal{O}_F \mathbin{\widehat{\otimes}} K^{\#})$. Hence the natural map

$$\Gamma(\mathcal{O}_F \mathbin{\widehat{\otimes}} K^{\#}, \mathcal{F}) \to \lim_n \Gamma(\mathcal{O}_F \mathbin{\widehat{\otimes}} K^{\#}, \mathcal{F}/\mathfrak{m}_F^n) = \lim_n \Gamma(\mathcal{O}_F \mathbin{\widehat{\otimes}} K, \mathcal{F})/\mathfrak{m}_F^n$$

is an isomorphism. The same argument applies to $\mathcal{O}_F \mathbin{\widehat{\otimes}} R$ and $\mathcal{O}_F \mathbin{\widehat{\otimes}} \mathcal{O}_K^{\#}$. $\square$

The next two lemmas use the derived limit functor Rlim for abelian groups. We use the Stacks Project [07KV] as a reference for Rlim.

Lemma 4.6.9. *If $\mathcal{F}$ is a quasi-coherent sheaf on $D \times X$ then the natural map*

$$\lim_n \widehat{B}(\mathcal{F}/\mathfrak{m}_F^n) \to \mathrm{Rlim}_n \widehat{B}(\mathcal{F}/\mathfrak{m}^n)$$

is a quasi-isomorphism.

Proof. Let us denote

$$\begin{aligned} A_n &= \Gamma(\mathcal{O}_F \mathbin{\widehat{\otimes}} R, \mathcal{F}/\mathfrak{m}_F^n), \\ B_n &= \Gamma(\mathcal{O}_F \mathbin{\widehat{\otimes}} \mathcal{O}_K^{\#}, \mathcal{F}/\mathfrak{m}_F^n), \\ C_n &= \Gamma(\mathcal{O}_F \mathbin{\widehat{\otimes}} K^{\#}, \mathcal{F}/\mathfrak{m}_F^n). \end{aligned}$$

The natural map in question extends to a morphism of distinguished triangles

$$\begin{tikzcd} \lim_n \widehat{B}(\mathcal{F}/\mathfrak{m}_F^n) \arrow[r] \arrow[d] & \lim_n A_n \oplus B_n \arrow[r] \arrow[d] & \lim_n C_n \arrow[r] \arrow[d] & {[1]} \\ \mathrm{Rlim}_n \widehat{B}(\mathcal{F}/\mathfrak{m}_F^n) \arrow[r] & \mathrm{Rlim}_n A_n \oplus B_n \arrow[r] & \mathrm{Rlim}_n C_n \arrow[r] & {[1]} \end{tikzcd}$$

So in order to show that the first vertical arrow is a quasi-isomorphism it is enough to prove that so are the second and the third vertical arrows.

The transition maps in the projective system $\{B_n\}_{n \geqslant 1}$ are surjective by construction. Hence this system satisfies the Mittag-Leffler condition [02N0]. As a consequence $\mathrm{R}^1\lim_n B_n = 0$ [07KW]. The natural map $\lim_n B_n \to \mathrm{Rlim}_n B_n$ is thus a quasi-isomorphism. The same argument applies to $\{A_n\}$ and $\{C_n\}$. $\square$

Lemma 4.6.10. *If $\mathcal{F}$ is a coherent sheaf on $D \times X$ then the natural map*

$$\mathrm{H}^i(\widehat{B}(\mathcal{F})) \to \lim_n \mathrm{H}^i(\widehat{B}(\mathcal{F}/\mathfrak{m}_F^n))$$

is an isomorphism for every i.

Proof. Lemma 4.6.8 implies that the map

$$\mathrm{H}^i(\widehat{B}(\mathcal{F})) \to \mathrm{H}^i(\lim_n \widehat{B}(\mathcal{F}/\mathfrak{m}_F^n))$$

is an isomorphism for every i. At the same time the map

$$\mathrm{H}^i(\lim_n \widehat{B}(\mathcal{F}/\mathfrak{m}_F^n)) \to \mathrm{H}^i(\mathrm{Rlim}_n \widehat{B}(\mathcal{F}/\mathfrak{m}_F^n))$$

is an isomorphism by Lemma 4.6.9. The cohomology group $\mathrm{H}^i(\mathrm{Rlim}_n \widehat{B}(\mathcal{F}/\mathfrak{m}_F^n))$ sits in a natural short exact sequence [07KY]

$$0 \to \mathrm{R}^1\lim_n \mathrm{H}^{i-1}(\widehat{B}(\mathcal{F}/\mathfrak{m}_F^n)) \to \mathrm{H}^i(\mathrm{Rlim}_n \widehat{B}(\mathcal{F}/\mathfrak{m}_F^n)) \to \lim_n \mathrm{H}^i(\widehat{B}(\mathcal{F}/\mathfrak{m}_F^n)) \to 0.$$

We thus need to prove that the first term in this sequence vanishes.

Lemma 4.6.6 provides us with natural isomorphisms

$$\mathrm{H}^{i-1}(D \times X,\, \mathcal{F}/\mathfrak{m}_F^n) \xrightarrow{\sim} \mathrm{H}^{i-1}(\widehat{B}(\mathcal{F}/\mathfrak{m}_F^n)).$$

As $D \times X$ is proper over $\mathcal{O}_F$ it follows that $\mathrm{H}^{i-1}(\widehat{B}(\mathcal{F}/\mathfrak{m}_F^n))$ is a finitely generated $\mathcal{O}_F/\mathfrak{m}_F^n$-module for every n. Thus the image of $\mathrm{H}^{i-1}(\widehat{B}(\mathcal{F}/\mathfrak{m}_F^m))$ in $\mathrm{H}^{i-1}(\widehat{B}(\mathcal{F}/\mathfrak{m}_F^n))$ is independent of m for $m \gg n$. In other words the projective system

$$\{\mathrm{H}^{i-1}(\widehat{B}(\mathcal{F}/\mathfrak{m}_F^n))\}_{n \geqslant 1}$$

satisfies the Mittag-Leffler condition [02N0]. Hence its first derived limit $\mathrm{R}^1\lim$ is zero [07KW]. □

Lemma 4.6.11. *If $\mathcal{F}$ is a coherent sheaf on $D \times X$ then the natural map $\mathrm{R}\check{\Gamma}(\mathcal{F}) \to \mathrm{R}\widehat{\Gamma}(\mathcal{F})$ is a quasi-isomorphism.*

Proof. In view of Lemmas 4.6.4 and 4.6.5 it is enough to prove that the natural map $B(\mathcal{F}) \to \widehat{B}(\mathcal{F})$ is a quasi-isomorphism. Let $i \in \mathbb{Z}$. We have a natural commutative diagram

$$\begin{tikzcd}
\mathrm{H}^i(B(\mathcal{F})) \arrow[r] \arrow[d] & \mathrm{H}^i(\widehat{B}(\mathcal{F})) \arrow[d] \\
\lim_n \mathrm{H}^i(B(\mathcal{F}/\mathfrak{m}_F^n)) \arrow[r] & \lim_n \mathrm{H}^i(\widehat{B}(\mathcal{F}/\mathfrak{m}_F^n)).
\end{tikzcd}$$

The right arrow is an isomorphism by Lemma 4.6.10 while the bottom arrow is an isomorphism by Lemma 4.6.7. Thus in order to prove that the top arrow is an isomorphism it is enough to show that the left arrow is so. This arrow fits into a natural commutative square

$$\begin{tikzcd}
\mathrm{H}^i(D \times X, \mathcal{F}) \arrow[r] \arrow[d] & \mathrm{H}^i(B(\mathcal{F})) \arrow[d] \\
\lim_n \mathrm{H}^i(D \times X, \mathcal{F}/\mathfrak{m}_F^n) \arrow[r] & \lim_n \mathrm{H}^i(B(\mathcal{F}/\mathfrak{m}_F^n))
\end{tikzcd}$$

where the horizontal arrows are the natural isomorphisms of Lemma 4.6.6. According to the Theorem on formal functions [02OC] the left arrow in this square is an isomorphism. Whence the result follows. □

Proof of Theorem 4.6.2. Let the shtuka $\mathcal{M}$ be given by a diagram

$$\mathcal{M}_0 \overset{i}{\underset{j}{\rightrightarrows}} \mathcal{M}_1.$$

We have a natural morphism of distinguished triangles

$$\begin{tikzcd}
\mathrm{R}\check{\Gamma}(\mathcal{M}) \arrow[r] \arrow[d] & \mathrm{R}\check{\Gamma}(\mathcal{M}_0) \arrow[r, "i-j"] \arrow[d] & \mathrm{R}\check{\Gamma}(\mathcal{M}_1) \arrow[r] \arrow[d] & {[1]} \\
\mathrm{R}\widehat{\Gamma}(\mathcal{M}) \arrow[r] & \mathrm{R}\widehat{\Gamma}(\mathcal{M}_0) \arrow[r, "i-j"] & \mathrm{R}\widehat{\Gamma}(\mathcal{M}_1) \arrow[r] & {[1]}
\end{tikzcd}$$

The second and third vertical arrows are quasi-isomorphisms by Lemma 4.6.11 so we are done. □

4.7. Change of coefficients

Fix a noetherian $\mathbb{F}_q$-algebra Λ. In this section we study how the cohomology of shtukas on $\operatorname{Spec}\Lambda \times X$ changes under the pullback to $\operatorname{Spec}\Lambda' \times X$ where Λ' is an Λ-algebra. We denote $S = \operatorname{Spec}\Lambda$ and $S' = \operatorname{Spec}\Lambda'$.

Definition 4.7.1. Let $\mathcal{M}$ be an $\mathcal{O}_{S\times X}$-module shtuka. We define a natural morphism

$$\mathrm{R}\Gamma(S \times X,\, \mathcal{M}) \otimes^{\mathbf{L}}_{\Lambda} \Lambda' \to \mathrm{R}\Gamma(S' \times X,\, \mathcal{M})$$

by extension of scalars of the pullback morphism $\mathrm{R}\Gamma(S\times X,\, \mathcal{M}) \to \mathrm{R}\Gamma(S'\times X,\, \mathcal{M})$. Given an $\mathcal{O}_{S\times X}$-module $\mathcal{E}$ we define a natural morphism $\mathrm{R}\Gamma(S \times X, \mathcal{E}) \otimes^{\mathbf{L}}_{\Lambda} \Lambda' \to \mathrm{R}\Gamma(S' \times X, \mathcal{E})$ in the same way.

Lemma 4.7.2. *Let Λ' be an Λ-algebra. If $\mathcal{M}$ is an $\mathcal{O}_{S\times X}$-module shtuka given by a diagram*

$$\mathcal{M}_0 \underset{j}{\overset{i}{\rightrightarrows}} \mathcal{M}_1$$

then the natural diagram

$$\begin{tikzcd}
\mathrm{R}\Gamma(S \times X, \mathcal{M}) \otimes^{\mathbf{L}}_{\Lambda} \Lambda' \arrow[r] \arrow[d] & \mathrm{R}\Gamma(S' \times X, \mathcal{M}) \arrow[d] \\
\mathrm{R}\Gamma(S \times X, \mathcal{M}_0) \otimes^{\mathbf{L}}_{\Lambda} \Lambda' \arrow[r] \arrow[d, "i-j"'] & \mathrm{R}\Gamma(S' \times X, \mathcal{M}_0) \arrow[d, "i-j"] \\
\mathrm{R}\Gamma(S \times X, \mathcal{M}_1) \otimes^{\mathbf{L}}_{\Lambda} \Lambda' \arrow[r] \arrow[d] & \mathrm{R}\Gamma(S' \times X, \mathcal{M}_1) \arrow[d] \\
{[1]} & {[1]}
\end{tikzcd} \tag{4.7}$$

is a morphism of distinguished triangles. Here the left column is the image under $-\otimes^{\mathbf{L}}_{\Lambda} \Lambda'$ of the canonical triangle for $\mathcal{M}$ and the right column is the canonical triangle for the pullback of $\mathcal{M}$ to $S' \times X$.

Proof. Follows from Proposition 1.8.4. □

Proposition 4.7.3. *Let $\mathcal{M}$ be an $\mathcal{O}_{S\times X}$-module shtuka given by a diagram*

$$\mathcal{M}_0 \underset{j}{\overset{i}{\rightrightarrows}} \mathcal{M}_1.$$

If $\mathcal{M}_0$, $\mathcal{M}_1$ are coherent and flat over Λ then the following holds:

(1) $\mathrm{R}\Gamma(S \times X, \mathcal{M})$ *is a perfect Λ-module complex.*

(2) *For every Λ-algebra Λ' the natural map*

$$\mathrm{R}\Gamma(S \times X, \mathcal{M}) \otimes^{\mathbf{L}}_{\Lambda} \Lambda' \to \mathrm{R}\Gamma(S' \times X, \mathcal{M})$$

is a quasi-isomorphism. Moreover the diagram (4.7) *is an isomorphism of distinguished triangles.*

Proof. Since $\mathcal{M}_0$ is coherent and flat over Λ the base change theorem for coherent cohomology [07VK] shows that $\mathrm{R}\Gamma(S\times X,\, \mathcal{M}_0)$ is a perfect Λ-module complex and the natural map $\mathrm{R}\Gamma(S\times X,\, \mathcal{M}_0)\otimes^{\mathbf{L}}_{\Lambda} \Lambda' \to \mathrm{R}\Gamma(S'\times X,\, \mathcal{M}_0)$ is a quasi-isomorphism. The same applies to $\mathcal{M}_1$. As $\mathrm{R}\Gamma(S \times X, \mathcal{M})$ fits to a distinguished triangle

$$\mathrm{R}\Gamma(S \times X, \mathcal{M}) \to \mathrm{R}\Gamma(S \times X, \mathcal{M}_0) \xrightarrow{i-j} \mathrm{R}\Gamma(S \times X, \mathcal{M}_1) \to [1]$$

we conclude that it is a perfect Λ-module complex. Finally Lemma 4.7.2 implies that (4.7) is an isomorphism of distinguished triangles. $\square$

4.8. ζ-isomorphisms

Let Λ be a noetherian $\mathbb{F}_q$-algebra and let $S = \operatorname{Spec}\Lambda$. In this section we study ζ-isomorphisms for shtukas over $S\times X$. We will prove that under suitable conditions they are stable under change of Λ.

Let $\mathcal{M}$ be an $\mathcal{O}_{S\times X}$-module shtuka given by a diagram

$$\mathcal{M}_0 \overset{i}{\underset{j}{\rightrightarrows}} \mathcal{M}_1.$$

Assume that the Λ-module complexes $\mathrm{R}\Gamma(\mathcal{M})$, $\mathrm{R}\Gamma(\nabla\mathcal{M})$, $\mathrm{R}\Gamma(\mathcal{M}_0)$ and $\mathrm{R}\Gamma(\mathcal{M}_1)$ are bounded with perfect cohomology modules. In this situation we have a ζ-isomorphism

$$\zeta_{\mathcal{M}}\colon\ \det_\Lambda \mathrm{R}\Gamma(\mathcal{M}) \xrightarrow{\sim} \det_\Lambda \mathrm{R}\Gamma(\nabla\mathcal{M}).$$

It is the composition

$$\det_\Lambda \mathrm{R}\Gamma(\mathcal{M}) \xrightarrow{\sim} \det_\Lambda \mathrm{R}\Gamma(\mathcal{M}_0) \otimes_\Lambda \det_\Lambda^{-1} \mathrm{R}\Gamma(\mathcal{M}_1) \xrightarrow{\sim} \det_\Lambda \mathrm{R}\Gamma(\nabla\mathcal{M})$$

of isomorphisms induced by the canonical triangles

$$\mathrm{R}\Gamma(\mathcal{M}) \to \mathrm{R}\Gamma(\mathcal{M}_0) \xrightarrow{i-j} \mathrm{R}\Gamma(\mathcal{M}_1) \to [1],$$
$$\mathrm{R}\Gamma(\nabla\mathcal{M}) \to \mathrm{R}\Gamma(\mathcal{M}_0) \xrightarrow{i} \mathrm{R}\Gamma(\mathcal{M}_1) \to [1].$$

Proposition 4.8.1. *If Λ is regular then the ζ-isomorphism is defined for every coherent shtuka on $S \times X$.*

We will also need ζ-isomorphisms for coefficient rings Λ which are not regular. The example of such an Λ relevant to our study is a local artinian ring which is not a field.

Proof of Proposition 4.8.1. Suppose that a coherent shtuka $\mathcal{M}$ is given by a diagram

$$\mathcal{M}_0 \overset{i}{\underset{j}{\rightrightarrows}} \mathcal{M}_1.$$

Grothendieck vanishing theorem [02UZ] shows that the Λ-modules $\mathrm{H}^n(\mathcal{M}_0)$ and $\mathrm{H}^n(\mathcal{M}_1)$ are zero for $n > 1$. By [02O5] they are finitely generated for $n = 0, 1$. Thus $\mathrm{H}^n(\mathcal{M})$ and $\mathrm{H}^n(\nabla\mathcal{M})$ are zero for $n \notin \{0, 1, 2\}$ and finitely generated for $n = 0, 1, 2$. The ring Λ has finite global dimension since it is regular [00O7]. Whence $\mathrm{H}^n(\mathcal{M})$, $\mathrm{H}^n(\nabla\mathcal{M})$, $\mathrm{H}^n(\mathcal{M}_0)$ and $\mathrm{H}^n(\mathcal{M}_1)$ are perfect Λ-modules. $\square$

Proposition 4.8.2. *Let Λ' be an Λ-algebra and let $S' = \operatorname{Spec}\Lambda'$. Let $\mathcal{M}$ be a shtuka on $S \times X$ given by a diagram*

$$\mathcal{M}_0 \overset{i}{\underset{j}{\rightrightarrows}} \mathcal{M}_1.$$

Let $\mathcal{M}_{S'}$ be the pullback of $\mathcal{M}$ to $S' \times X$. Assume that

1. *the ζ-isomorphisms are defined for $\mathcal{M}$ and $\mathcal{M}_{S'}$,*
2. *$\mathcal{M}_0$ and $\mathcal{M}_1$ are coherent and flat over Λ.*

Then the following holds:

1. *The natural maps*
$$\mathrm{R}\Gamma(\mathcal{M}) \otimes^{\mathbf{L}}_{\Lambda} \Lambda' \to \mathrm{R}\Gamma(\mathcal{M}_{S'}),$$
$$\mathrm{R}\Gamma(\nabla\mathcal{M}) \otimes^{\mathbf{L}}_{\Lambda} \Lambda' \to \mathrm{R}\Gamma(\nabla\mathcal{M}_{S'})$$
of Definition 4.7.1 are quasi-isomorphisms.
2. *The natural square*
$$\begin{tikzcd} \det_\Lambda \mathrm{R}\Gamma(\mathcal{M}) \otimes_\Lambda \Lambda' \arrow[r, "\zeta_{\mathcal{M}} \otimes 1"] \arrow[d, "\wr"'] & \det_\Lambda \mathrm{R}\Gamma(\nabla\mathcal{M}) \otimes_\Lambda \Lambda' \arrow[d, "\wr"] \\ \det_{\Lambda'} \mathrm{R}\Gamma(\mathcal{M}_{S'}) \arrow[r, "\zeta_{\mathcal{M}_{S'}}"] & \det_{\Lambda'} \mathrm{R}\Gamma(\nabla\mathcal{M}_{S'}) \end{tikzcd}$$
is commutative. Here the vertical arrows are induced by the quasi-isomorphisms of (1).

Proof. The natural isomorphisms of determinants

$$\det_\Lambda \mathrm{R}\Gamma(\mathcal{M}) \to \det_\Lambda \mathrm{R}\Gamma(\mathcal{M}_0) \otimes_\Lambda \det_\Lambda^{-1} \mathrm{R}\Gamma(\mathcal{M}_1),$$
$$\det_\Lambda \mathrm{R}\Gamma(\nabla\mathcal{M}) \to \det_\Lambda \mathrm{R}\Gamma(\mathcal{M}_0) \otimes_\Lambda \det_\Lambda^{-1} \mathrm{R}\Gamma(\mathcal{M}_1)$$

induced by the triangles of $\mathcal{M}$ and $\nabla\mathcal{M}$ are stable under the pullback to Λ' by construction (see the proof of Corollary 2 after Theorem 2 in [17]). So the result follows from Proposition 4.7.3. ∎

CHAPTER 5

Regulator theory

Let F be a local field and let K be a finite product of local fields. As usual we assume F and K to contain $\mathbb{F}_q$. We denote $\mathcal{O}_F \subset F$ and $\mathcal{O}_K \subset K$ the rings of integers, $\mathfrak{m}$ the maximal ideal of $\mathcal{O}_F$ and $\mathfrak{m}_K$ the Jacobson radical of $\mathcal{O}_K$. We omit the subscript F for the ideal $\mathfrak{m} \subset \mathcal{O}_F$ to improve the legibility.

We mainly work with $\mathcal{O}_F \mathbin{\widehat{\otimes}} \mathcal{O}_K$-module shtukas. In agreement with the conventions of Section 3.8 the τ-structure on $\mathcal{O}_F \mathbin{\widehat{\otimes}} \mathcal{O}_K$ is given by the endomorphism which acts as the identity on $\mathcal{O}_F$ and as the q-Frobenius on $\mathcal{O}_K$.

The aim of this chapter is to construct for a certain class of shtukas $\mathcal{M}$ on $\mathcal{O}_F \mathbin{\widehat{\otimes}} \mathcal{O}_K$ a natural quasi-isomorphism

$$\mathrm{R}\Gamma(\mathcal{M}) \xrightarrow{\ \rho\ } \mathrm{R}\Gamma(\nabla\mathcal{M})$$

called the *regulator*. We do it for *elliptic shtukas*, a class of shtukas generalizing the models of Drinfeld modules in the sense of Chapter 9. The key definitions and results of this chapter are as follows:

- Theorem 5.4.3 describes the cohomology of a certain class of shtukas on $\mathcal{O}_F \mathbin{\widehat{\otimes}} \mathcal{O}_K$. It applies in particular to elliptic shtukas.
- Definition 5.6.1 introduces elliptic shtukas.
- Definition 5.14.1 introduces the regulator for elliptic shtukas.
- The existence and unicity of the regulator is affirmed by Theorem 5.14.3.

It is easy to characterize the regulator as a natural transformation of functors on the category of elliptic shtukas (see Definition 5.14.1). However its construction is a bit involved.

The content of this chapter is new save for the preliminary Sections 5.1 and 5.2. In the case of Section 5.1 we are not aware of a reference for Lemma 5.1.1 in the literature but it should have certainly appeared before. Our search for a shtuka-theoretic regulator was motivated by the article [18] of V. Lafforgue.

Remark. After the work on this text was finished the author found out that V. Lafforgue has constructed a map [18, Lemme 4.8] which resembles in some respects the regulator isomorphism of this chapter. It is an interesting question whether the two maps are actually the same.

5.1. Topological preliminaries

In this section we give a topological criterion for an $\mathcal{O}_F$-module to be finitely generated. We will use it to prove that cohomology modules of certain shtukas are finitely generated.

Lemma 5.1.1. *Let M be a compact Hausdorff $\mathcal{O}_F$-module. The following are equivalent:*

(1) *$M/\mathfrak{m}$ is finite as a set.*
(2) *M is finitely generated as an $\mathcal{O}_F$-module without topology.*

Proof. (1) $\Rightarrow$ (2). Let z be a uniformizer of $\mathcal{O}_F$. The submodule $\mathfrak{m}M \subset M$ is the image of M under multiplication by z so it is closed, compact and Hausdorff. Furthermore multiplication by z defines a surjective map $M/\mathfrak{m} \to (\mathfrak{m}M)/\mathfrak{m}$ so that $(\mathfrak{m}M)/\mathfrak{m}$ is finite. By induction we conclude that the submodules $\mathfrak{m}^n M \subset M$ are closed and of finite index, hence open.

Let us show that the open submodules $\mathfrak{m}^n M$ form a fundamental system of neighbourhoods of zero. Proposition 2.4.1 and Lemma 3.2.7 imply that M admits a fundamental system of open $\mathcal{O}_F$-submodules. If $U \subset M$ is an open submodule then M/U is finite so there exists an $n > 0$ such that z^n acts by zero on M/U. Hence $\mathfrak{m}^n M \subset U$.

Now M is a compact $\mathcal{O}_F$-module so it is complete as a topological $\mathbb{F}_q$-vector space. As the submodules $\mathfrak{m}^n M$ form a fundamental system we conclude that $M = \lim_{n>0} M/\mathfrak{m}^n$.

Let r be the dimension of $M/\mathfrak{m}$ as an $\mathcal{O}_F/\mathfrak{m}$-vector space. For every $n > 0$ let H_n be the set of surjective $\mathcal{O}_F$-linear maps from $\mathcal{O}_F^{\oplus r}$ to $M/\mathfrak{m}^n$. The sets H_n form a projective system H_* in a natural way. Every point of the limit of H_* defines a continuous morphism from $\mathcal{O}_F^{\oplus r}$ to M. Such a morphism is surjective since it has dense image by construction and its domain $\mathcal{O}_F^{\oplus r}$ is compact.

By definition the set H_1 is nonempty. Nakayama's lemma implies that the transition maps $H_{n+1} \to H_n$ are surjective for all $n > 0$. Therefore the projective system H_* has a nonempty limit. $\square$

It is worth mentioning that Lemma 5.1.1 works for any nonarchimedean local field F and more generally for any local noetherian ring $\mathcal{O}_F$ with finite residue field. Indeed one can show that a (locally) compact Hausdorff $\mathcal{O}_F$-module M admits a fundamental system of open submodules and the rest of the argument applies essentially as is.

5.2. Algebraic preliminaries

Let us review some elementary algebraic properties of the ring $\mathcal{O}_F \mathbin{\widehat{\otimes}} \mathcal{O}_K$.

Lemma 5.2.1. *An ideal $\mathfrak{a} \subset \mathcal{O}_K$ is open if and only if it is a free $\mathcal{O}_K$-module of rank 1.*

Proof. By definition $\mathcal{O}_K$ is a finite product of complete discrete valuation rings. The ideal $\mathfrak{a}$ is open if and only if it projects to a nonzero ideal in every factor. Whence the result. $\square$

Let $\mathfrak{a} \subset \mathcal{O}_K$ be an open ideal. We will often use natural homomorphisms

$$\begin{aligned} f_{\mathfrak{a}} &\colon \mathcal{O}_F \mathbin{\widehat{\otimes}} \mathcal{O}_K \to \mathcal{O}_F \otimes \mathcal{O}_K/\mathfrak{a}, \\ g_{\mathfrak{a}} &\colon \mathcal{O}_F \mathbin{\widehat{\otimes}} \mathcal{O}_K \to F \otimes \mathcal{O}_K/\mathfrak{a}. \end{aligned}$$

The homomorphism $f_{\mathfrak{a}}$ is the completion of the natural map $\mathcal{O}_F \otimes_{\mathrm{c}} \mathcal{O}_K \to \mathcal{O}_F \otimes_{\mathrm{c}} \mathcal{O}_K/\mathfrak{a}$. We use the fact that $\mathcal{O}_K/\mathfrak{a}$ is finite to identify $\mathcal{O}_F \widehat{\otimes} (\mathcal{O}_K/\mathfrak{a})$ with $\mathcal{O}_F \otimes \mathcal{O}_K/\mathfrak{a}$. The homomorphism $g_{\mathfrak{a}}$ is the composition of $f_{\mathfrak{a}}$ with the natural map $\mathcal{O}_F \otimes \mathcal{O}_K/\mathfrak{a} \to F \otimes \mathcal{O}_K/\mathfrak{a}$. By construction $g_{\mathfrak{a}}$ factors over $F \otimes_{\mathcal{O}_F} (\mathcal{O}_F \mathbin{\widehat{\otimes}} \mathcal{O}_K)$. The notation $f_{\mathfrak{a}}$ and $g_{\mathfrak{a}}$ will not be used. The same constructions apply to a nonzero ideal $\mathfrak{b} \subset \mathcal{O}_F$.

By definition $\mathcal{O}_F = k[[z]]$ for a finite field extension k of $\mathbb{F}_q$ and a uniformizer $z \in \mathcal{O}_F$. In a similar way

$$\mathcal{O}_K = \prod_{i=1}^{d} k_i[[\zeta_i]]$$

where k_i are finite field extensions of $\mathbb{F}_q$ and ζ_i are uniformizers of the factors of $\mathcal{O}_K$. As a consequence

$$\mathcal{O}_F \mathbin{\widehat{\otimes}} \mathcal{O}_K = \prod_{i=1}^{d} (k \otimes k_i)[[z, \zeta_i]]$$

is a finite product of power series rings in two variables. With this observation in mind the following lemmas become obvious.

Lemma 5.2.2. *Let $\mathfrak{a} \subset \mathcal{O}_K$ be an open ideal.*

(1) *The ideal $\mathfrak{a} \cdot (\mathcal{O}_F \mathbin{\widehat{\otimes}} \mathcal{O}_K) = \mathcal{O}_F \mathbin{\widehat{\otimes}} \mathfrak{a}$ is a free $\mathcal{O}_F \mathbin{\widehat{\otimes}} \mathcal{O}_K$-module of rank 1.*
(2) *The sequence $0 \to \mathfrak{a} \cdot (\mathcal{O}_F \mathbin{\widehat{\otimes}} \mathcal{O}_K) \to \mathcal{O}_F \mathbin{\widehat{\otimes}} \mathcal{O}_K \to \mathcal{O}_F \otimes \mathcal{O}_K/\mathfrak{a} \to 0$ is exact.* □

Lemma 5.2.3. *Let $\mathfrak{b} \subset \mathcal{O}_F$ be an open ideal.*

(1) *The ideal $\mathfrak{b} \cdot (\mathcal{O}_F \mathbin{\widehat{\otimes}} \mathcal{O}_K) = \mathfrak{b} \mathbin{\widehat{\otimes}} \mathcal{O}_K$ is a free module of rank 1.*
(2) *The sequence $0 \to \mathfrak{b} \cdot (\mathcal{O}_F \mathbin{\widehat{\otimes}} \mathcal{O}_K) \to \mathcal{O}_F \mathbin{\widehat{\otimes}} \mathcal{O}_K \to \mathcal{O}_F/\mathfrak{b} \otimes \mathcal{O}_K \to 0$ is exact.* □

The next lemma will only be used in the proof of Proposition 5.10.3.

Lemma 5.2.4. *If $\mathfrak{a} \subset \mathcal{O}_K$ and $\mathfrak{b} \subset \mathcal{O}_F$ are open ideals then the natural sequence*

$$0 \to \frac{\mathcal{O}_F \mathbin{\widehat{\otimes}} \mathfrak{a}}{\mathfrak{b} \mathbin{\widehat{\otimes}} \mathfrak{a}} \oplus \frac{\mathfrak{b} \mathbin{\widehat{\otimes}} \mathcal{O}_K}{\mathfrak{b} \mathbin{\widehat{\otimes}} \mathfrak{a}} \to \frac{\mathcal{O}_F \mathbin{\widehat{\otimes}} \mathcal{O}_K}{\mathfrak{b} \mathbin{\widehat{\otimes}} \mathfrak{a}} \to \frac{\mathcal{O}_F \mathbin{\widehat{\otimes}} \mathcal{O}_K}{\mathfrak{b} \mathbin{\widehat{\otimes}} \mathcal{O}_K + \mathcal{O}_F \mathbin{\widehat{\otimes}} \mathfrak{a}} \to 0$$

is exact.

Proof. Follows since $\mathfrak{b} \mathbin{\widehat{\otimes}} \mathfrak{a} = \mathfrak{b} \mathbin{\widehat{\otimes}} \mathcal{O}_K \cap \mathcal{O}_F \mathbin{\widehat{\otimes}} \mathfrak{a}$. □

5.3. Cohomology with artinian coefficients

Fix a finite $\mathbb{F}_q$-algebra Λ which is a local artinian ring. Let $\mathfrak{m} \subset \Lambda$ be the maximal ideal. In this section we work with the ring $\Lambda \otimes \mathcal{O}_K$. We equip it with the τ-ring structure given by the endomorphism which acts as the identity on Λ and as the q-Frobenius on $\mathcal{O}_K$.

Lemma 5.3.1. *Let $\mathcal{M}$ be a locally free shtuka on $\Lambda \otimes \mathcal{O}_K$. If $\mathcal{M}(\Lambda/\mathfrak{m} \otimes K)$ is nilpotent then $\mathcal{M}(\Lambda \otimes K)$ is nilpotent.*

Proof. The ring $\Lambda \otimes K$ is noetherian and complete with respect to the ideal $\mathfrak{m} \otimes K$. As the ideal $\mathfrak{m}$ is nilpotent the result follows from Proposition 1.10.4. □

Theorem 5.3.2. *Let $\mathcal{M}$ be a locally free shtuka on $\Lambda \otimes \mathcal{O}_K$. If $\mathcal{M}(\Lambda/\mathfrak{m} \otimes K)$ is nilpotent then:*

(1) $\mathrm{H}^0(\mathcal{M}) = 0$,
(2) $\mathrm{H}^1(\mathcal{M})$ *is a free Λ-module of finite rank.*

Proof. (1) Since $\mathcal{M}$ is locally free the natural map $\mathrm{H}^0(\Lambda \otimes \mathcal{O}_K, \mathcal{M}) \to \mathrm{H}^0(\Lambda \otimes K, \mathcal{M})$ injective. However $\mathcal{M}(\Lambda \otimes K)$ is nilpotent by Lemma 5.3.1 so $\mathrm{H}^0(\Lambda \otimes K, \mathcal{M}) = 0$ by Propostion 1.10.3.

(2) First let us prove that $\mathrm{H}^1(\mathcal{M})$ is a flat Λ-module. Suppose that $\mathcal{M}$ is given by a diagram

$$M_0 \underset{j}{\overset{i}{\rightrightarrows}} M_1.$$

The cohomology complex $\mathrm{R}\Gamma(\mathcal{M})$ is represented by the associated complex

$$\Gamma_{\mathrm{a}}(\mathcal{M}) = [M_0 \xrightarrow{i-j} M_1].$$

It is a comlex of flat Λ-modules since $\mathcal{M}$ is locally free by assumption. As a consequence

$$\Gamma_{\mathrm{a}}(\mathcal{M}) \otimes^{\mathbf{L}}_{\Lambda} \Lambda/\mathfrak{m} = \Gamma_{\mathrm{a}}(\mathcal{M}) \otimes_{\Lambda} \Lambda/\mathfrak{m}.$$

However $\Gamma_{\mathrm{a}}(\mathcal{M}) \otimes_{\Lambda} \Lambda/\mathfrak{m}$ is the complex representing $\mathrm{R}\Gamma(\Lambda/\mathfrak{m} \otimes \mathcal{O}_K, \mathcal{M})$. Applying the argument (1) above to the shtuka $\mathcal{M}(\Lambda/\mathfrak{m} \otimes \mathcal{O}_K)$ we deduce that $\mathrm{R}\Gamma(\Lambda/\mathfrak{m} \otimes \mathcal{O}_K, \mathcal{M})$ is concentrated in degree 1. Hence $\mathrm{H}^1(\mathcal{M}) \otimes^{\mathbf{L}}_{\Lambda} \Lambda/\mathfrak{m}$ is concentrated in degree 0 or in other words $\mathrm{Tor}_n(\mathrm{H}^1(\mathcal{M}), \Lambda/\mathfrak{m}) = 0$ for $n > 0$. Therefore $\mathrm{H}^1(\mathcal{M})$ is a flat Λ-module [051K].

Next we prove that $\mathrm{H}^1(\mathcal{M})$ is finitely generated. The $\mathcal{O}_K$-modules M_0 and M_1 are finitely generated by assumption. They carry a natural compact Hausdorff topology given by the powers of the ideal $\mathfrak{m}_K$. We would like to prove that the map $(i-j)\colon M_0 \to M_1$ is open. Since M_1 is a compact $\mathcal{O}_K$-module it then follows that $M_1/(i-j)M_0 = \mathrm{H}^1(\mathcal{M})$ is a finite set.

Consider the locally compact K-vector spaces $V_0 = M_0 \otimes_{\mathcal{O}_K} K$ and $V_1 = M_1 \otimes_{\mathcal{O}_K} K$. By Lemma 5.3.1 the shtuka $\mathcal{M}(\Lambda \otimes K)$ is nilpotent whence $i\colon V_0 \to V_1$ is an isomorphism and the endomorphism $i^{-1}j$ of V_0 is nilpotent. The isomorphism $i^{-1}\colon V_1 \to V_0$ is continuous by K-linearity. The map $j\colon V_0 \to V_1$ is continuous since it is a Frobenius-linear morphism of finite-dimensional K-vector spaces. As a consequence $i^{-1}j\colon V_0 \to V_0$ is continuous. Since it is nilpotent we conclude that the endomorphism $(1-i^{-1}j)^{-1}$ is continuous. Therefore $1 - i^{-1}j$ is open. However

$$i - j = i \circ (1 - i^{-1}j).$$

So $(i-j)\colon V_0 \to V_1$ is a composition of open maps. We conclude that $(i-j)\colon M_0 \to M_1$ is open. □

5.4. Finiteness of cohomology

In this section we prove that under certain natural conditions the cohomology groups of shtukas over $\mathcal{O}_F \mathbin{\widehat{\otimes}} \mathcal{O}_K$ are finitely generated free $\mathcal{O}_F$-modules.

Lemma 5.4.1. *For every locally free shtuka $\mathcal{M}$ on $\mathcal{O}_F \mathbin{\widehat{\otimes}} \mathcal{O}_K$ the natural map $\mathrm{R}\Gamma(\mathcal{M}) \otimes^{\mathbf{L}}_{\mathcal{O}_F} \mathcal{O}_F/\mathfrak{m} \to \mathrm{R}\Gamma(\mathcal{O}_F/\mathfrak{m} \otimes \mathcal{O}_K, \mathcal{M})$ is a quasi-isomorphism.*

Proof. Let $z \in \mathcal{O}_F$ be a uniformizer. Observe that $\mathrm{R}\Gamma(\mathcal{M}) \otimes^{\mathbf{L}}_{\mathcal{O}_F} \mathcal{O}_F/\mathfrak{m}$ is the cone of the multiplication map $z\colon \mathrm{R}\Gamma(\mathcal{M}) \to \mathrm{R}\Gamma(\mathcal{M})$. Applying $\mathrm{R}\Gamma(-)$ to the short exact sequence $0 \to \mathcal{M} \xrightarrow{z} \mathcal{M} \to \mathcal{M}/z \to 0$ we conclude that the natural map $\mathrm{R}\Gamma(\mathcal{M}) \otimes^{\mathbf{L}}_{\mathcal{O}_F} \mathcal{O}_F/\mathfrak{m} \to \mathrm{R}\Gamma(\mathcal{M}/z)$ is a quasi-isomorphism. By Lemma 5.2.3 the quotient $\mathcal{M}/z$ is the restriction of $\mathcal{M}$ to $\mathcal{O}_F/\mathfrak{m} \otimes \mathcal{O}_K$. Whence the result. □

Proposition 5.4.2. *Let $\mathcal{M}$ be a locally free shtuka on $\mathcal{O}_F \mathbin{\widehat{\otimes}} \mathcal{O}_K$. The following are equivalent:*

1. $\mathrm{R}\Gamma(\mathcal{O}_F/\mathfrak{m} \otimes \mathcal{O}_K, \mathcal{M})$ *is a prerfect complex of $\mathcal{O}_F/\mathfrak{m}$-vector spaces.*
2. $\mathrm{R}\Gamma(\mathcal{M})$ *is a perfect complex of $\mathcal{O}_F$-modules.*

Proof. (2) $\Rightarrow$ (1) follows from Lemma 5.4.1. (1) $\Rightarrow$ (2). Suppose that $\mathcal{M}$ is given by a diagram

$$M_0 \underset{j}{\overset{i}{\rightrightarrows}} M_1.$$

The $\mathcal{O}_F \mathbin{\widehat{\otimes}} \mathcal{O}_K$-modules M_0, M_1 come equipped with a canonical topology given by powers of the ideal

$$\mathfrak{a} = \mathfrak{m} \mathbin{\widehat{\otimes}} \mathcal{O}_K + \mathcal{O}_F \mathbin{\widehat{\otimes}} \mathfrak{m}_K.$$

The map i is continuous in this topology since it is $\mathcal{O}_F \mathbin{\widehat{\otimes}} \mathcal{O}_K$-linear. The partial Frobenius $\tau\colon \mathcal{O}_F \mathbin{\widehat{\otimes}} \mathcal{O}_K \to \mathcal{O}_F \mathbin{\widehat{\otimes}} \mathcal{O}_K$ maps $\mathfrak{a}$ to itself. As j is τ-linear it follows that j is continuous. Hence $i - j$ is continuous.

Consider the complex

$$\big[M_0 \xrightarrow{i-j} M_1\big].$$

The ring $\mathcal{O}_F \mathbin{\widehat{\otimes}} \mathcal{O}_K$ is compact with respect to the $\mathfrak{a}$-adic topology. Therefore the finitely generated $\mathcal{O}_F \widehat{\otimes} \mathcal{O}_K$-modules M_0, M_1 are compact Hausdorff. It follows that the image of M_0 in M_1 is closed and the quotient topology on $\mathrm{H}^1(\mathcal{M})$ is compact Hausdorff. So is the subspace topology on $\mathrm{H}^0(\mathcal{M})$.

The natural map $\mathcal{O}_F \to \mathcal{O}_F \widehat{\otimes} \mathcal{O}_K$ is continuous and the differential $i-j$ in the complex above is $\mathcal{O}_F$-linear. As a consequence $\mathrm{H}^0(\mathcal{M})$ and $\mathrm{H}^1(\mathcal{M})$ are topological $\mathcal{O}_F$-modules. Now $\mathrm{H}^0(\mathcal{M})$ and $\mathrm{H}^1(\mathcal{M})$ are compact Hausdorff so Lemma 5.1.1 shows that they are finitely generated. □

The first main result of this chapter is the following:

Theorem 5.4.3. *Let $\mathcal{M}$ be a locally free shtuka on $\mathcal{O}_F \mathbin{\widehat{\otimes}} \mathcal{O}_K$. If $\mathcal{M}(\mathcal{O}_F/\mathfrak{m} \otimes K)$ is nilpotent then the following holds:*

(1) $\mathrm{H}^0(\mathcal{M}) = 0$,
(2) $\mathrm{H}^1(\mathcal{M})$ *is a finitely generated free $\mathcal{O}_F$-module.*

Proof. The shtuka $\mathcal{M}(\mathcal{O}_F/\mathfrak{m} \otimes K)$ is nilpotent so Theorem 5.3.2 shows that

(i) $\mathrm{H}^0(\mathcal{O}_F/\mathfrak{m} \otimes \mathcal{O}_K,\, \mathcal{M}) = 0$,
(ii) $\mathrm{H}^1(\mathcal{O}_F/\mathfrak{m} \otimes \mathcal{O}_K,\, \mathcal{M})$ is finite as a set.

Using Proposition 5.4.2 we deduce that $\mathrm{H}^0(\mathcal{M})$ and $\mathrm{H}^1(\mathcal{M})$ are finitely generated $\mathcal{O}_F$-modules. At the same time (i) in combination with Lemma 5.4.1 implies that $\mathrm{H}^0(\mathcal{M})$ is divisible and $\mathrm{H}^1(\mathcal{M})$ is torsion-free. So the result follows. □

5.5. Artinian regulators

Let us fix a finite $\mathbb{F}_q$-algebra Λ which is a local artinian ring. We denote $\mathfrak{m} \subset \Lambda$ the maximal ideal. In this section we work over the ring $\Lambda \otimes \mathcal{O}_K$. We equip it with the τ-ring structure given by the endomorphism which acts as the identity on Λ and as the q-Frobenius on $\mathcal{O}_K$.

We study locally free shtukas on $\Lambda \otimes \mathcal{O}_K$ which restrict to nilpotent shtukas on $\Lambda/\mathfrak{m} \otimes K$. Under certain conditions we will define a regulator map for such shtukas, the *artinian regulator*.

Lemma 5.5.1. *Let $\mathcal{M}$ be a locally free shtuka on $\Lambda \otimes \mathcal{O}_K$ given by a diagram*

$$M_0 \underset{j}{\overset{i}{\rightrightarrows}} M_1.$$

If $\mathcal{M}(\Lambda/\mathfrak{m} \otimes K)$ is nilpotent then the map $i\colon M_0 \otimes_{\mathcal{O}_K} K \to M_1 \otimes_{\mathcal{O}_K} K$ is an isomorphism.

Proof. By Lemma 5.3.1 the shtuka $\mathcal{M}(\Lambda \otimes K)$ is nilpotent. The i-map of such a shtuka is an isomorphism by definition. □

Definition 5.5.2. Let a locally free shtuka $\mathcal{M}$ on $\Lambda \otimes \mathcal{O}_K$ be given by a diagram

$$M_0 \underset{j}{\overset{i}{\rightrightarrows}} M_1.$$

Suppose that $\mathcal{M}(\Lambda/\mathfrak{m} \otimes K)$ is nilpotent. We say that the artinian regulator is defined for $\mathcal{M}$ if the endomorphism $i^{-1}j$ of the Λ-module $M_0 \otimes_{\mathcal{O}_K} K$ preserves the submodule M_0. In this case we define the *artinian regulator* $\rho_{\mathcal{M}} \colon \Gamma_{\mathrm{a}}(\mathcal{M}) \to \Gamma_{\mathrm{a}}(\nabla\mathcal{M})$ by the diagram

$$\begin{tikzcd}
{[M_0} \arrow[r, "i-j"] \arrow[d, "1-i^{-1}j"'] & {M_1]} \arrow[d, "1"] \\
{[M_0} \arrow[r, "i"] & {M_1].}
\end{tikzcd}$$

In a moment we will give a sufficient condition for the regulator to be defined. Before that let us study its properties.

Lemma 5.5.3. *The regulator of Definition 5.5.2 has the following properties.*

(1) $\rho_{\mathcal{M}}$ *is natural in* $\mathcal{M}$.
(2) $\rho_{\mathcal{M}}$ *is an isomorphism.*

Proof. (1) Clear. (2) Suppose that $\mathcal{M}$ is given by a diagram

$$M_0 \overset{i}{\underset{j}{\rightrightarrows}} M_1.$$

The shtuka $\mathcal{M}(\Lambda \otimes K)$ is nilpotent by Lemma 5.3.1 whence the endomorphism $i^{-1}j$ of $M_0 \otimes_{\mathcal{O}_K} K$ is nilpotent. As a consequence the endomorphism $1 - i^{-1}j$ is in fact an automorphism. □

Definition 5.5.4. Let $\mathfrak{a} \subset \mathcal{O}_K$ be an ideal. Let a quasi-coherent shtuka $\mathcal{M}$ on $\Lambda \otimes \mathcal{O}_K$ be given by a diagram

$$M_0 \overset{i}{\underset{j}{\rightrightarrows}} M_1.$$

We define

$$\mathfrak{a}\mathcal{M} = \Big[\mathfrak{a}M_0 \overset{i}{\underset{j}{\rightrightarrows}} \mathfrak{a}M_1\Big].$$

According to Lemma 5.2.1 an ideal $\mathfrak{a} \subset \mathcal{O}_K$ is open if and only if it is free of rank 1.

Lemma 5.5.5. *Let $\mathcal{M}$ be a locally free shtuka on $\Lambda \otimes \mathcal{O}_K$ such that $\mathcal{M}(\Lambda/\mathfrak{m} \otimes K)$ is nilpotent. Let $\mathfrak{a} \subset \mathcal{O}_K$ be an open ideal.*

(1) *$\mathfrak{a}\mathcal{M}$ is a locally free shtuka which restricts to a nilpotent shtuka on $\Lambda/\mathfrak{m} \otimes K$.*
(2) *If the regulator is defined for $\mathcal{M}$ then it is defined for $\mathfrak{a}\mathcal{M}$.*

Proof. (1) Clear. (2) Suppose that $\mathcal{M}$ is given by a diagram

$$M_0 \overset{i}{\underset{j}{\rightrightarrows}} M_1.$$

We know that $j(M_0) \subset i(M_0)$. As a consequence $j(\mathfrak{a}M_0) \subset i(\mathfrak{a}^q M_0)$ which implies that $\mathfrak{a}M_0$ is invariant under $i^{-1}j$. □

Lemma 5.5.6. *Let $\mathfrak{a} \subset \mathcal{O}_K$ be an open ideal and let $\mathcal{M}$ be a locally free shtuka on $\Lambda \otimes \mathcal{O}_K$. If $\mathcal{M}(\Lambda/\mathfrak{m} \otimes K)$ is nilpotent then the short exact sequence of shtukas*

$$0 \to \mathfrak{a}\mathcal{M} \to \mathcal{M} \to \mathcal{M}/\mathfrak{a} \to 0$$

induces a long exact sequence of cohomology

$$0 \to \mathrm{H}^0(\mathcal{M}/\mathfrak{a}) \xrightarrow{\delta} \mathrm{H}^1(\mathfrak{a}\mathcal{M}) \longrightarrow \mathrm{H}^1(\mathcal{M}) \xrightarrow{\text{red.}} \mathrm{H}^1(\mathcal{M}/\mathfrak{a}) \to 0. \tag{5.1}$$

The exact sequence (5.1) will play an important role in our theory. It will mainly appear as the sequence (5.3) for elliptic shtukas on $\mathcal{O}_F \mathbin{\widehat{\otimes}} \mathcal{O}_K$.

Proof of Lemma 5.5.6. By Lemma 5.5.5 $\mathfrak{a}\mathcal{M}$ is a locally free shtuka on $\Lambda \otimes \mathcal{O}_K$ whose restriction to $\Lambda/\mathfrak{m} \otimes K$ is nilpotent. Hence $\mathrm{H}^0(\mathfrak{a}\mathcal{M})$ and $\mathrm{H}^0(\mathcal{M})$ vanish by Theorem 5.3.2. □

If a shtuka $\mathcal{M}$ on $\Lambda \otimes \mathcal{O}_K$ is linear then $\mathrm{R}\Gamma(\mathcal{M})$ is represented by a complex with an $\mathcal{O}_K$-linear differential. As a consequence $\mathrm{R}\Gamma(\mathcal{M})$ carries a natural action of $\mathcal{O}_K$.

Lemma 5.5.7. *Let $\mathcal{M}$ be a locally free shtuka on $\Lambda \otimes \mathcal{O}_K$ such that $\mathcal{M}(\Lambda/\mathfrak{m} \otimes K)$ is nilpotent. Let $\mathfrak{a} \subset \mathcal{O}_K$ be an open ideal. If $\mathcal{M}$ is linear then the following are equivalent:*

(1) $\mathfrak{a} \cdot \mathrm{H}^1(\mathcal{M}) = 0$.
(2) *The map* $\mathrm{H}^0(\mathcal{M}/\mathfrak{a}) \xrightarrow{\delta} \mathrm{H}^1(\mathfrak{a}\mathcal{M})$ *in* (5.1) *is an isomorphism.*
(3) *The map* $\mathrm{H}^1(\mathcal{M}) \xrightarrow{\text{red.}} \mathrm{H}^1(\mathcal{M}/\mathfrak{a})$ *in* (5.1) *is an isomorphism.*

Proof. Suppose that $\mathcal{M}$ is given by a diagram

$$M_0 \underset{0}{\overset{i}{\rightrightarrows}} M_1.$$

By definition $\mathrm{H}^1(\mathcal{M}) = M_1/i(M_0)$. Hence the following are equivalent:

(1) $\mathfrak{a} \cdot \mathrm{H}^1(\mathcal{M}) = 0$.
(1′) $\mathfrak{a}M_1 \subset i(M_0)$.
(1″) The natural map $\mathrm{H}^1(\mathfrak{a}\mathcal{M}) \to \mathrm{H}^1(\mathcal{M})$ is zero.

By Lemma 5.5.6 either of the conditions (2) or (3) is equivalent to (1″). □

Lemma 5.5.8. *Let $\mathcal{M}$ be a locally free shtuka on $\Lambda \otimes \mathcal{O}_K$ such that $\mathcal{M}(\Lambda/\mathfrak{m} \otimes K)$ is nilpotent. Suppose that the regulator is defined for $\mathcal{M}$. Let $\mathfrak{a} \subset \mathcal{O}_K$ be an open ideal. The following are equivalent:*

(1) $\mathfrak{a} \cdot \mathrm{H}^1(\nabla\mathcal{M}) = 0$.
(2) *The map* $\mathrm{H}^0(\mathcal{M}/\mathfrak{a}) \xrightarrow{\delta} \mathrm{H}^1(\mathfrak{a}\mathcal{M})$ *in* (5.1) *is an isomorphism.*
(3) *The map* $\mathrm{H}^1(\mathcal{M}) \xrightarrow{\text{red.}} \mathrm{H}^1(\mathcal{M}/\mathfrak{a})$ *in* (5.1) *is an isomorphism.*

Proof. Lemma 5.5.5 tells that the regulator is defined for the shtuka $\mathfrak{a}\mathcal{M}$ as well. By Lemma 5.5.3 regulators are natural. We thus get a commutative diagram of complexes

$$\begin{tikzcd}
0 \arrow[r] & \Gamma_{\mathrm{a}}(\mathfrak{a}\mathcal{M}) \arrow[r] \arrow[d, "\rho_{\mathfrak{a}\mathcal{M}}"'] & \Gamma_{\mathrm{a}}(\mathcal{M}) \arrow[r] \arrow[d, "\rho_{\mathcal{M}}"'] & \Gamma_{\mathrm{a}}(\mathcal{M}/\mathfrak{a}) \arrow[r] \arrow[d, "\rho_{\mathcal{M}/\mathfrak{a}}"] & 0 \\
0 \arrow[r] & \Gamma_{\mathrm{a}}(\nabla\mathfrak{a}\mathcal{M}) \arrow[r] & \Gamma_{\mathrm{a}}(\nabla\mathcal{M}) \arrow[r] & \Gamma_{\mathrm{a}}(\nabla\mathcal{M}/\mathfrak{a}) \arrow[r] & 0
\end{tikzcd}$$

where $\rho_{\mathcal{M}/\mathfrak{a}}$ is induced by $\rho_{\mathcal{M}}$. The regulators $\rho_{\mathcal{M}}$ and $\rho_{\mathfrak{a}\mathcal{M}}$ are isomorphisms by Lemma 5.5.3. As a consequence $\rho_{\mathcal{M}/\mathfrak{a}}$ is an isomorphism. Taking H^1 of the diagram above we conclude that the following are equivalent:

(3) The reduction map $\mathrm{H}^1(\mathcal{M}) \to \mathrm{H}^1(\mathcal{M}/\mathfrak{a})$ is an isomorphism.
(3′) The reduction map $\mathrm{H}^1(\nabla\mathcal{M}) \to \mathrm{H}^1(\nabla\mathcal{M}/\mathfrak{a})$ is an isomorphism.

According to Lemma 5.5.7 the condition (1) is equivalent to (3′). Hence the condition (1) is equivalent to (3). The conditions (2) and (3) are equivalent by Lemma 5.5.6. □

Proposition 5.5.9. *Let $\mathcal{M}$ be a locally free shtuka on $\Lambda \otimes \mathcal{O}_K$ such that $\mathcal{M}(\Lambda/\mathfrak{m} \otimes K)$ is nilpotent. Let $\mathfrak{a} \subset \mathcal{O}_K$ be an open ideal. Assume that*

(1) $\mathfrak{a} \cdot \mathrm{H}^1(\nabla\mathcal{M}) = 0$,
(2) $\mathcal{M}/\mathfrak{a}$ *is linear.*

Then the following holds:

(1) *The regulator is defined for* $\mathcal{M}$.
(2) *The map* $\mathrm{H}^0(\mathcal{M}/\mathfrak{a}) \xrightarrow{\delta} \mathrm{H}^1(\mathfrak{a}\mathcal{M})$ *in* (5.1) *is an isomorphism.*
(3) *The map* $\mathrm{H}^1(\mathcal{M}) \xrightarrow{\text{red.}} \mathrm{H}^1(\mathcal{M}/\mathfrak{a})$ *in* (5.1) *is an isomorphism.*

Proof. Suppse that $\mathcal{M}$ is given by a diagram

$$M_0 \underset{j}{\overset{i}{\rightrightarrows}} M_1.$$

Now $j(M_0) \subset \mathfrak{a}M_1$ by assumption (2) and $\mathfrak{a}M_1 \subset i(M_0)$ by assumption (1). Hence M_0 is preserved by $i^{-1}j$. We conclude that the regulator is defined for $\mathcal{M}$. In view of this fact the results (2) and (3) follow from the assumption (1) by Lemma 5.5.8. $\square$

Proposition 5.5.10. *Let* $\mathcal{M}$ *be a locally free shtuka on* $\Lambda\otimes\mathcal{O}_K$ *such that* $\mathcal{M}(\Lambda/\mathfrak{m}\otimes K)$ *is nilpotent. Let* $\mathfrak{a} \subset \mathcal{O}_K$ *be an open ideal. Assume that* $\mathcal{M}/\mathfrak{a}$ *is linear. If the regulator is defined for* $\mathcal{M}$ *then the diagram*

$$\begin{tikzcd}
\mathrm{H}^1(\mathcal{M}) \arrow[r, "\text{red.}"] \arrow[d, "\rho_{\mathcal{M}}"'] & \mathrm{H}^1(\mathcal{M}/\mathfrak{a}) \arrow[d, "1"] \\
\mathrm{H}^1(\nabla\mathcal{M}) \arrow[r, "\text{red.}"] & \mathrm{H}^1(\nabla\mathcal{M}/\mathfrak{a}).
\end{tikzcd}$$

is commutative.

Proof. The regulator is defined for $\mathfrak{a}\mathcal{M}$ by Lemma 5.5.5. Since the regulators are natural by Lemma 5.5.3 we obtain a commutative diagram of complexes

$$\begin{tikzcd}
0 \arrow[r] & \Gamma_{\mathrm{a}}(\mathfrak{a}\mathcal{M}) \arrow[r] \arrow[d, "\rho_{\mathfrak{a}\mathcal{M}}"'] & \Gamma_{\mathrm{a}}(\mathcal{M}) \arrow[r] \arrow[d, "\rho_{\mathcal{M}}"'] & \Gamma_{\mathrm{a}}(\mathcal{M}/\mathfrak{a}) \arrow[r] \arrow[d, "\rho_{\mathcal{M}/\mathfrak{a}}"] & 0 \\
0 \arrow[r] & \Gamma_{\mathrm{a}}(\nabla\mathfrak{a}\mathcal{M}) \arrow[r] & \Gamma_{\mathrm{a}}(\nabla\mathcal{M}) \arrow[r] & \Gamma_{\mathrm{a}}(\nabla\mathcal{M}/\mathfrak{a}) \arrow[r] & 0
\end{tikzcd}$$

where $\rho_{\mathcal{M}/\mathfrak{a}}$ is the morphism induced by $\rho_{\mathcal{M}}$. The regulator $\rho_{\mathcal{M}}$ is given by the identity map in degree 1. So the same is true for $\rho_{\mathcal{M}/\mathfrak{a}}$. Taking H^1 of the diagram above we get the result. $\square$

Proposition 5.5.11. *Let* $\mathcal{M}$ *be a locally free shtuka on* $\Lambda\otimes\mathcal{O}_K$ *such that* $\mathcal{M}(\Lambda/\mathfrak{m}\otimes K)$ *is nilpotent. Let* $\mathfrak{a} \subset \mathcal{O}_K$ *be an open ideal. Assume that*

(1) $\mathfrak{a}\cdot \mathrm{H}^1(\nabla\mathcal{M}) = 0$,
(2) $\mathcal{M}/\mathfrak{a}^2$ *is linear.*

Then the regulator is defined for $\mathfrak{a}\mathcal{M}$ *and the square*

$$\begin{tikzcd}
\mathrm{H}^0(\mathcal{M}/\mathfrak{a}) \arrow[r, "\delta"] \arrow[d, "1"'] & \mathrm{H}^1(\mathfrak{a}\mathcal{M}) \arrow[d, "\rho_{\mathfrak{a}\mathcal{M}}"] \\
\mathrm{H}^0(\nabla\mathcal{M}/\mathfrak{a}) \arrow[r, "\delta"] & \mathrm{H}^1(\nabla\mathfrak{a}\mathcal{M})
\end{tikzcd}$$

is commutative. Here the maps δ *are the boundary homomorphisms of the long exact sequence* (5.1).

Proof. The regulator is defined for $\mathcal{M}$ by Proposition 5.5.9. Hence it is defined for $\mathfrak{a}\mathcal{M}$ by Lemma 5.5.5. We then have a diagram of complexes

$$\begin{tikzcd}
0 \arrow[r] & \Gamma_{\mathrm{a}}(\mathfrak{a}\mathcal{M}) \arrow[r] \arrow[d, "\rho_{\mathfrak{a}\mathcal{M}}"'] & \Gamma_{\mathrm{a}}(\mathcal{M}) \arrow[r] \arrow[d, "\rho_{\mathcal{M}}"'] & \Gamma_{\mathrm{a}}(\mathcal{M}/\mathfrak{a}) \arrow[r] \arrow[d, "1"] & 0 \\
0 \arrow[r] & \Gamma_{\mathrm{a}}(\nabla\mathfrak{a}\mathcal{M}) \arrow[r] & \Gamma_{\mathrm{a}}(\nabla\mathcal{M}) \arrow[r] & \Gamma_{\mathrm{a}}(\nabla\mathcal{M}/\mathfrak{a}) \arrow[r] & 0
\end{tikzcd} \tag{5.2}$$

with short exact rows. We will show that the right square commutes. The result then follows since (5.2) induces a morphism of long exact cohomology sequences.

The right square of (5.2) commutes in degree 1 since $\rho_{\mathcal{M}}$ is given by the identity map in degree 1. We thus need to show commutativity in degree 0. Suppose that $\mathcal{M}$ is given by a diagram

$$M_0 \underset{j}{\overset{i}{\rightrightarrows}} M_1.$$

To show commutativity of (5.2) is is enough to prove that $i^{-1}j(M_0) \subset \mathfrak{a}M_0$. By assumption (2) we have $j(M_0) \subset \mathfrak{a}^2 M_1$. Assumption (1) implies that $\mathfrak{a}^2 M_1 \subset i(\mathfrak{a}M_0)$. Hence $i^{-1}j(M_0) \subset \mathfrak{a}M_0$. □

5.6. Elliptic shtukas

Starting from this section we work over the ring $\mathcal{O}_F \widehat{\otimes} \mathcal{O}_K$. Recall that an ideal of $\mathfrak{a} \subset \mathcal{O}_K$ is open if and only if it is free of rank 1 (Lemma 5.2.1).

Definition 5.6.1. Let $\mathcal{M}$ be a shtuka on $\mathcal{O}_F \widehat{\otimes} \mathcal{O}_K$ and let $\mathfrak{e} \subset \mathcal{O}_K$ be an open ideal. We say that $\mathcal{M}$ is an *elliptic shtuka of ramification ideal* $\mathfrak{e}$ if the following holds:

(E0) $\mathcal{M}$ is locally free.
(E1) $\mathcal{M}(\mathcal{O}_F/\mathfrak{m} \otimes K)$ is nilpotent.
(E2) $\mathcal{M}(F \otimes \mathcal{O}_K/\mathfrak{m}_K)$ is nilpotent.
(E3) $\mathfrak{m} \cdot \mathrm{H}^1(\nabla\mathcal{M}) = \mathfrak{e} \cdot \mathrm{H}^1(\nabla\mathcal{M})$.
(E4) $\mathcal{M}(\mathcal{O}_F \otimes \mathcal{O}_K/\mathfrak{e})$ is linear.

The ramification ideal $\mathfrak{e} \subset \mathcal{O}_K$ is fixed throughout the rest of the chapter. In the following we speak simply of elliptic shtukas instead of elliptic shtukas of ramification ideal $\mathfrak{e}$.

Example. Let $\mathcal{O}_F = \mathbb{F}_q[[z]]$, $\mathcal{O}_K = \mathbb{F}_q[[\zeta]]$. We have $\mathcal{O}_F \widehat{\otimes} \mathcal{O}_K = \mathbb{F}_q[[z,\zeta]]$. The endomorphism τ of this ring preserves $\mathbb{F}_q[[z]]$ and sends ζ to ζ^q. Our ramification ideal $\mathfrak{e} \subset \mathbb{F}_q[[\zeta]]$ will be the ideal generated by ζ.

Consider the shtuka

$$\mathcal{M} = \Big[\ \mathbb{F}_q[[z,\zeta]] \underset{z\zeta\cdot\tau}{\overset{\zeta - z}{\rightrightarrows}} \mathbb{F}_q[[z,\zeta]]\ \Big].$$

In fact $\mathcal{M}$ is (a part of) a model of the Carlitz module. We claim that $\mathcal{M}$ is an elliptic shtuka of ramification ideal $\mathfrak{e}$. Indeed $\mathcal{M}$ is locally free by construction. Furthermore

$$\mathcal{M}(\mathcal{O}_F/\mathfrak{m} \otimes K) = \Big[\ \mathbb{F}_q(\!(\zeta)\!) \underset{0}{\overset{\zeta}{\rightrightarrows}} \mathbb{F}_q(\!(\zeta)\!)\ \Big]$$
$$\mathcal{M}(F \otimes \mathcal{O}_K/\mathfrak{m}_K) = \Big[\ \mathbb{F}_q(\!(z)\!) \underset{0}{\overset{-z}{\rightrightarrows}} \mathbb{F}_q(\!(z)\!)\ \Big].$$

Hence the restrictions of $\mathcal{M}$ to $\mathcal{O}_F/\mathfrak{m} \otimes K$ and $F \otimes \mathcal{O}_K/\mathfrak{m}_K$ are nilpotent. The cohomology of $\nabla\mathcal{M}$ is easy to compute:

$$\mathrm{H}^1(\nabla\mathcal{M}) = \mathbb{F}_q[[z,\zeta]]/(\zeta - z) = \mathbb{F}_q[[\zeta]].$$

The element $z \in \mathbb{F}_q[[z]]$ acts on $\mathrm{H}^1(\nabla\mathcal{M})$ by multiplication by ζ. So $\mathfrak{m} \cdot \mathrm{H}^1(\nabla\mathcal{M}) = \mathfrak{e} \cdot \mathrm{H}^1(\nabla\mathcal{M})$. Finally the linearity condition holds since

$$\mathcal{M}(\mathcal{O}_F \otimes \mathcal{O}_K/\mathfrak{e}) = \Big[\ \mathbb{F}_q[[z]] \underset{0}{\overset{-z}{\rightrightarrows}} \mathbb{F}_q[[z]]\ \Big].$$

Proposition 5.6.2. *If a shtuka $\mathcal{M}$ is elliptic then so is $\nabla\mathcal{M}$.*

Proof. Indeed the functor ∇ commutes with arbitrary restrictions and preserves nilpotence so that $\nabla\mathcal{M}$ satisfies the conditions (E1) and (E2) of Definition 5.6.1. The condition (E3) is tautologically satisfied and (E4) follows since the shtuka $\nabla\mathcal{M}$ is already linear. □

Theorem 5.6.3. *If $\mathcal{M}$ is an elliptic shtuka then the following holds:*

(1) $\mathrm{H}^0(\mathcal{M}) = 0$
(2) $\mathrm{H}^1(\mathcal{M})$ *is a finitely generated free $\mathcal{O}_F$-module.*

Proof. Indeed $\mathcal{M}$ is locally free by (E0) and $\mathcal{M}(\mathcal{O}_F/\mathfrak{m} \otimes K)$ is nilpotent by (E1) so the result follows from Theorem 5.4.3. □

5.7. Twists and quotients

Definition 5.7.1. Let $I \subset \mathcal{O}_F \mathbin{\widehat{\otimes}} \mathcal{O}_K$ be a τ-invariant ideal. Let $\mathcal{M}$ be a shtuka on $\mathcal{O}_F \mathbin{\widehat{\otimes}} \mathcal{O}_K$ given by a diagram

$$M_0 \overset{i}{\underset{j}{\rightrightarrows}} M_1.$$

We define

$$I\mathcal{M} = \Big[IM_0 \overset{i}{\underset{j}{\rightrightarrows}} IM_1 \Big].$$

We call $I\mathcal{M}$ the *twist* of $\mathcal{M}$ by I. The shtuka $I\mathcal{M}$ comes equipped with a natural embedding $I\mathcal{M} \hookrightarrow \mathcal{M}$. We denote the quotient $\mathcal{M}/I$.

Warning. Given invariant ideals $I, J \subset \mathcal{O}_F \mathbin{\widehat{\otimes}} \mathcal{O}_K$ we denote $I\mathcal{M}/J$ the quotient $(I\mathcal{M})/(IJ\mathcal{M})$. In other words we first do the twist by I and then take the quotient by J.

We will use the following invariant ideals:

- $\mathfrak{m}^n \mathbin{\widehat{\otimes}} \mathcal{O}_K$ for $n \geqslant 0$.
- $\mathcal{O}_F \mathbin{\widehat{\otimes}} \mathfrak{a}$ for $\mathfrak{a} \subset \mathcal{O}_K$ an open ideal.
- $\mathfrak{m}^n \mathbin{\widehat{\otimes}} \mathcal{O}_K + \mathcal{O}_F \mathbin{\widehat{\otimes}} \mathfrak{a}$ for $\mathfrak{m}^n$ and $\mathfrak{a}$ as above.
- $\mathfrak{m}^n \mathbin{\widehat{\otimes}} \mathfrak{a}$ for $\mathfrak{m}^n$ and $\mathfrak{a}$ as above.

To simplify the notation we will write $\mathfrak{m}^n\mathcal{M}$ instead of $(\mathfrak{m}^n \mathbin{\widehat{\otimes}} \mathcal{O}_K)\mathcal{M}$. The same applies to $\mathfrak{a}$, $\mathfrak{m}^n + \mathfrak{a}$ and the quotients by the ideals of these three types. The twist of $\mathcal{M}$ by the last ideal will be denoted $\mathfrak{m}^n\mathfrak{a}\mathcal{M}$ and the quotient will be denoted $\mathcal{M}/\mathfrak{m}^n\mathfrak{a}$.

Lemma 5.7.2. *Let $n \geqslant 0$ and let $\mathfrak{a} \subset \mathcal{O}_K$ be an open ideal. If $\mathcal{M}$ is a shtuka on $\mathcal{O}_F \mathbin{\widehat{\otimes}} \mathcal{O}_K$ then*

$$\mathcal{M}/\mathfrak{m}^n = \mathcal{M}(\mathcal{O}_F/\mathfrak{m}^n \otimes \mathcal{O}_K),$$
$$\mathcal{M}/\mathfrak{a} = \mathcal{M}(\mathcal{O}_F \otimes \mathcal{O}_K/\mathfrak{a}),$$
$$\mathcal{M}/(\mathfrak{m}^n + \mathfrak{a}) = \mathcal{M}(\mathcal{O}_F/\mathfrak{m}^n \otimes \mathcal{O}_K/\mathfrak{a}).$$

Proof. By Lemma 5.2.3 we have $\mathcal{O}_F/\mathfrak{m}^n \otimes_{\mathcal{O}_F} (\mathcal{O}_F \mathbin{\widehat{\otimes}} \mathcal{O}_K) = \mathcal{O}_F/\mathfrak{m}^n \otimes \mathcal{O}_K$ so the first formula holds. In a similar way Lemma 5.2.2 implies the second formula. The last formula follows from the first two. □

Proposition 5.7.3. *If $\mathcal{M}$ is an elliptic shtuka then so is $\mathfrak{m}\mathcal{M}$.*

Proof. Indeed the shtukas $\mathcal{M}$ and $\mathfrak{m}\mathcal{M}$ are isomorphic. □

Lemma 5.7.4. *Let $\mathcal{M}$ be a locally free shtuka on $\mathcal{O}_F \mathbin{\widehat{\otimes}} \mathcal{O}_K$ such that $\mathcal{M}(F \otimes \mathcal{O}_K/\mathfrak{m}_K)$ is nilpotent. For every open ideal $\mathfrak{a} \subset \mathcal{O}_K$ the following holds:*

(1) $\mathcal{M}(F \otimes \mathcal{O}_K/\mathfrak{a})$ *is nilpotent,*
(2) $\mathrm{H}^0(\mathcal{M}/\mathfrak{a}) = 0$.

Proof. (1) It is enough to assume that K is a single local field. In this case $\mathfrak{a} = \mathfrak{m}_K^n$ for some $n \geqslant 0$. The ring $F \otimes \mathcal{O}_K/\mathfrak{m}_K^n$ is noetherian and complete with respect to the τ-invariant ideal $F \otimes \mathfrak{m}_K/\mathfrak{m}_K^n$. Since $\mathcal{M}(F \otimes \mathcal{O}_K/\mathfrak{m}_K)$ is nilpotent Proposition 1.10.4 implies that $\mathcal{M}(F \otimes \mathcal{O}_K/\mathfrak{m}_K^n)$ is nilpotent.

(2) Lemma 5.7.2 shows that $\mathcal{M}/\mathfrak{a} = \mathcal{M}(\mathcal{O}_F \otimes \mathcal{O}_K/\mathfrak{a})$. The natural map

$$\mathrm{H}^0(\mathcal{O}_F \otimes \mathcal{O}_K/\mathfrak{a},\, \mathcal{M}) \to \mathrm{H}^0(F \otimes \mathcal{O}_K/\mathfrak{a},\, \mathcal{M})$$

is injective since $\mathcal{M}$ is locally free. However the shtuka $\mathcal{M}(F \otimes \mathcal{O}_K/\mathfrak{a})$ is nilpotent by (1). So Proposition 1.10.3 shows that $\mathrm{H}^0(F \otimes \mathcal{O}_K/\mathfrak{a},\, \mathcal{M}) = 0$. $\square$

Lemma 5.7.5. *Let $\mathcal{M}$ be a shtuka on $\mathcal{O}_F \widehat{\otimes} \mathcal{O}_K$. If $\mathcal{M}(F \otimes \mathcal{O}_K/\mathfrak{m}_K)$ is nilpotent then $(\mathfrak{e}\mathcal{M})(F \otimes \mathcal{O}_K/\mathfrak{m}_K)$ is nilpotent.*

Proof. We have a short exact sequence of shtukas

$$0 \to \mathfrak{e}\mathcal{M}/\mathfrak{m}_K\mathfrak{e}\mathcal{M} \to \mathcal{M}/\mathfrak{m}_K\mathfrak{e}\mathcal{M} \to \mathcal{M}/\mathfrak{m}_K\mathcal{M} \to 0.$$

Using Lemma 5.7.2 we rewrite it as follows:

$$0 \to (\mathfrak{e}\mathcal{M})(\mathcal{O}_F \otimes \mathcal{O}_K/\mathfrak{m}_K) \to \mathcal{M}(\mathcal{O}_F \otimes \mathcal{O}_K/\mathfrak{m}_K\mathfrak{e}) \to \mathcal{M}(\mathcal{O}_F \otimes \mathcal{O}_K/\mathfrak{m}_K) \to 0.$$

Localizing at a uniformizer of $\mathcal{O}_F$ we get a short exact sequence

$$0 \to (\mathfrak{e}\mathcal{M})(F \otimes \mathcal{O}_K/\mathfrak{m}_K) \to \mathcal{M}(F \otimes \mathcal{O}_K/\mathfrak{m}_K\mathfrak{e}) \to \mathcal{M}(F \otimes \mathcal{O}_K/\mathfrak{m}_K) \to 0.$$

The third shtuka is nilpotent by assumption while the second shtuka is nilpotent by Lemma 5.7.4. Hence the first shtuka is nilpotent. $\square$

Proposition 5.7.6. *Let $\mathcal{M}$ be a shtuka on $\mathcal{O}_F \widehat{\otimes} \mathcal{O}_K$. For every $n \geqslant 0$ the shtuka $(\mathfrak{e}^n\mathcal{M})/\mathfrak{e}^n$ is linear.*

Proof. Suppose that $\mathcal{M}$ is given by a diagram

$$M_0 \underset{j}{\overset{i}{\rightrightarrows}} M_1.$$

We need to prove that $j(\mathfrak{e}^n M_0) \subset \mathfrak{e}^{2n} M_1$. The endomorphism τ of $\mathcal{O}_F \widehat{\otimes} \mathcal{O}_K$ sends $\mathfrak{e}$ to $\mathfrak{e}^q$. Since j is τ-linear it follows that $j(\mathfrak{e}^n M_0) \subset \mathfrak{e}^{nq} M_1$. The result follows since $q > 1$. $\square$

Proposition 5.7.7. *If $\mathcal{M}$ is an elliptic shtuka then so is $\mathfrak{e}\mathcal{M}$.*

Proof. Let us verify the conditions of Definition 5.6.1 for $\mathfrak{e}\mathcal{M}$.

(E0) Lemma 5.2.2 shows that $\mathcal{O}_F \widehat{\otimes}\, \mathfrak{e}$ is a free $\mathcal{O}_F \widehat{\otimes} \mathcal{O}_K$-module of rank 1. Hence $\mathfrak{e}\mathcal{M}$ is a locally free shtuka.

(E1) The shtukas $\mathcal{M}$ and $\mathfrak{e}\mathcal{M}$ coincide on $(\mathcal{O}_F \widehat{\otimes} \mathcal{O}_K) \otimes_{\mathcal{O}_K} K$ so the restriction of $\mathfrak{e}\mathcal{M}$ to $\mathcal{O}_F/\mathfrak{m}_F \otimes K$ is nilpotent.

(E2) Follows by Lemma 5.7.5.

(E3) Consider the short exact sequence of shtukas

$$0 \to \nabla\mathfrak{e}\mathcal{M} \to \nabla\mathcal{M} \to \nabla\mathcal{M}/\mathfrak{e} \to 0.$$

The module $\mathrm{H}^0(\nabla\mathcal{M}/\mathfrak{e})$ vanishes by Lemma 5.7.4. Taking the cohomology sequence we conclude that the natural map $\mathrm{H}^1(\nabla\mathfrak{e}\mathcal{M}) \to \mathrm{H}^1(\nabla\mathcal{M})$ is injective. This map is $\mathcal{O}_F \otimes \mathcal{O}_K$-linear by construction. Now the image of $\mathrm{H}^1(\nabla\mathfrak{e}\mathcal{M})$ in $\mathrm{H}^1(\nabla\mathcal{M})$ is $\mathfrak{e} \cdot \mathrm{H}^1(\nabla\mathcal{M})$ by definition. Therefore

$$\mathfrak{m} \cdot \mathrm{H}^1(\nabla\mathfrak{e}\mathcal{M}) = \mathfrak{m}\mathfrak{e} \cdot \mathrm{H}^1(\nabla\mathcal{M}) = \mathfrak{e}\mathfrak{m} \cdot \mathrm{H}^1(\nabla\mathcal{M}) = \mathfrak{e}\mathfrak{e} \cdot \mathrm{H}^1(\nabla\mathcal{M}) = \mathfrak{e} \cdot \mathrm{H}^1(\nabla\mathfrak{e}\mathcal{M}).$$

(E4) Indeed the shtuka $(\mathfrak{e}\mathcal{M})/\mathfrak{e}$ is linear according to Proposition 5.7.6. $\square$

5.8. Filtration on cohomology

An elliptic shtuka $\mathcal{M}$ carries a natural filtration by elliptic subshtukas $\mathfrak{e}^n\mathcal{M}$. In this section we would like to describe the induced filtration on $\mathrm{H}^1(\mathcal{M})$. If the elliptic shtuka $\mathcal{M}$ is linear then

$$\mathfrak{m}\cdot\mathrm{H}^1(\mathcal{M}) = \mathfrak{e}\cdot\mathrm{H}^1(\mathcal{M}) = \mathrm{H}^1(\mathfrak{e}\mathcal{M})$$

by the condition (E3). As a consequence the filtration on $\mathrm{H}^1(\mathcal{M})$ induced by $\mathfrak{e}^n\mathcal{M}$ is the filtration by powers of $\mathfrak{m}$. Our goal is to prove that the same is true without the linearity assumption.

Lemma 5.8.1. *Let $\mathcal{M}$ be an elliptic shtuka and let $n \geqslant 0$. The shtuka $\mathcal{N} = \mathcal{M}/\mathfrak{m}^n$ has the following properties:*

(1) *$\mathcal{N}$ is a locally free $\mathcal{O}_F/\mathfrak{m}^n \otimes \mathcal{O}_K$-shtuka.*
(2) *$\mathcal{N}(\mathcal{O}_F/\mathfrak{m}\otimes K)$ is nilpotent.*
(3) *$\mathfrak{e}^n\cdot\mathrm{H}^1(\nabla\mathcal{N}) = 0$.*

Proof. By Lemma 5.7.2 we have $\mathcal{N} = \mathcal{M}(\mathcal{O}_F/\mathfrak{m}^n\otimes\mathcal{O}_K)$. So (1) and (2) follow since $\mathcal{M}$ is an elliptic shtuka.

The natural map $\mathrm{H}^1(\nabla\mathcal{M})\to\mathrm{H}^1(\nabla\mathcal{N})$ is surjective and $\mathcal{O}_F\otimes\mathcal{O}_K$-linear. According to the condition (E3) we have $\mathfrak{e}\cdot\mathrm{H}^1(\nabla\mathcal{M}) = \mathfrak{m}\cdot\mathrm{H}^1(\nabla\mathcal{M})$. As a consequence $\mathfrak{e}^n\cdot\mathrm{H}^1(\nabla\mathcal{N}) = \mathfrak{m}^n\cdot\mathrm{H}^1(\nabla\mathcal{N})$. However $\mathfrak{m}^n$ acts on this module by zero since $\nabla\mathcal{N}$ is a shtuka on $\mathcal{O}_F/\mathfrak{m}^n\otimes\mathcal{O}_K$. We thus get (3). □

Proposition 5.8.2. *Let $\mathcal{M}$ be an elliptic shtuka and let $n\geqslant 0$.*

(1) *The sequence $0\to\mathrm{H}^1(\mathfrak{m}^n\mathcal{M})\to\mathrm{H}^1(\mathcal{M})\to\mathrm{H}^1(\mathcal{M}/\mathfrak{m}^n)\to 0$ is exact.*
(2) *The image of $\mathrm{H}^1(\mathfrak{m}^n\mathcal{M})$ in $\mathrm{H}^1(\mathcal{M})$ is $\mathfrak{m}^n\mathrm{H}^1(\mathcal{M})$.*

Proof. (1) By Lemma 5.8.1 the quotient $\mathcal{M}/\mathfrak{m}^n$ is a locally free $\mathcal{O}_F/\mathfrak{m}^n\otimes\mathcal{O}_K$-shtuka whose restriction to $\mathcal{O}_F/\mathfrak{m}\otimes K$ is nilpotent. Hence $\mathrm{H}^0(\mathcal{M}/\mathfrak{m}^n) = 0$ by Theorem 5.3.2. Taking the cohomology sequence of the short exact sequence $0\to\mathfrak{m}^n\mathcal{M}\to\mathcal{M}\to\mathcal{M}/\mathfrak{m}^n\to 0$ we get the result. (2) is clear. □

Lemma 5.8.3. *Let $\mathcal{M}$ be an elliptic shtuka and let $n\geqslant 0$. If $\mathcal{M}/\mathfrak{e}^n$ is linear then the shtuka $\mathcal{N} = \mathcal{M}/\mathfrak{m}^n$ has the following properties:*

(1) *The artinian regulator is defined for $\mathcal{N}$ (Definition 5.5.2).*
(2) *The reduction map $\mathrm{H}^1(\mathcal{N})\to\mathrm{H}^1(\mathcal{N}/\mathfrak{e}^n)$ is an isomorphism.*

Proof. We claim that the shtuka $\mathcal{N}$ has the following properties:

(i) $\mathcal{N}$ is a locally free $\mathcal{O}_F/\mathfrak{m}^n\otimes\mathcal{O}_K$-module shtuka whose restriction to $\mathcal{O}_F/\mathfrak{m}\otimes K$ is niltpotent.
(ii) $\mathfrak{e}^n\cdot\mathrm{H}^1(\nabla\mathcal{N}) = 0$.
(iii) $\mathcal{N}/\mathfrak{e}^n$ is linear.

Indeed Lemma 5.8.1 implies (i) and (ii) while (iii) follows since the quotient $\mathcal{M}/\mathfrak{e}^n$ is linear by assumption. We then apply Proposition 5.5.9 to $\mathcal{N}$ with $\Lambda = \mathcal{O}_F/\mathfrak{m}^n$ and $\mathfrak{a} = \mathfrak{e}^n$ and conclude that $\mathcal{N}$ has the properties (1) and (2). □

Lemma 5.8.4. *If $\mathcal{M}$ is an elliptic shtuka then the reduction map $\mathrm{H}^1(\mathcal{M}/\mathfrak{e})\to\mathrm{H}^1(\mathcal{M}/(\mathfrak{m}+\mathfrak{e}))$ is an isomorphism.*

Proof. The short exact sequence of shtukas $0\to(\mathfrak{m}\mathcal{M})/\mathfrak{e}\to\mathcal{M}/\mathfrak{e}\to\mathcal{M}/(\mathfrak{m}+\mathfrak{e})\to 0$ induces an exact sequence of cohomology

$$\mathrm{H}^1(\mathfrak{m}\mathcal{M}/\mathfrak{e})\to\mathrm{H}^1(\mathcal{M}/\mathfrak{e})\to\mathrm{H}^1(\mathcal{M}/(\mathfrak{m}+\mathfrak{e}))\to 0.$$

So to prove the lemma it is enough to demonstrate that the first map in this sequence is zero.

The shtukas $\mathcal{M}/\mathfrak{e}$ and $(\mathfrak{m}\mathcal{M})/\mathfrak{e}$ are linear since $\mathcal{M}$ is elliptic. So we can assume without loss of generality that $\mathcal{M}$ is itself linear. The natural map $\mathrm{H}^1(\mathcal{M})\to$

$\mathrm{H}^1(\mathcal{M}/\mathfrak{e})$ is $\mathcal{O}_F \otimes \mathcal{O}_K$-linear and surjective. Furthermore $\mathfrak{m} \cdot \mathrm{H}^1(\mathcal{M}) = \mathfrak{e} \cdot \mathrm{H}^1(\mathcal{M})$ by definition of $\mathfrak{e}$. As a consequence

$$\mathfrak{m} \cdot \mathrm{H}^1(\mathcal{M}/\mathfrak{e}) = \mathfrak{e} \cdot \mathrm{H}^1(\mathcal{M}/\mathfrak{e}).$$

However $\mathcal{M}/\mathfrak{e}$ is a linear shtuka on $\mathcal{O}_F \otimes \mathcal{O}_K/\mathfrak{e}$ so $\mathfrak{e}$ acts on $\mathrm{H}^1(\mathcal{M}/\mathfrak{e})$ by zero. Thus $\mathfrak{m} \cdot \mathrm{H}^1(\mathcal{M}/\mathfrak{e}) = 0$ which implies that the natural map $\mathrm{H}^1(\mathfrak{m}\mathcal{M}/\mathfrak{e}) \to \mathrm{H}^1(\mathcal{M}/\mathfrak{e})$ is zero. □

Theorem 5.8.5. *Let $\mathcal{M}$ be an elliptic shtuka and let $n \geqslant 0$.*

(1) *The sequence* $0 \to \mathrm{H}^1(\mathfrak{e}^n\mathcal{M}) \to \mathrm{H}^1(\mathcal{M}) \to \mathrm{H}^1(\mathcal{M}/\mathfrak{e}^n) \to 0$ *is exact.*
(2) *The image of* $\mathrm{H}^1(\mathfrak{e}^n\mathcal{M})$ *in* $\mathrm{H}^1(\mathcal{M})$ *is* $\mathfrak{m}^n\mathrm{H}^1(\mathcal{M})$.

So as we claimed at the beginning of this section the filtration on $\mathrm{H}^1(\mathcal{M})$ induced by the subshtukas $\mathfrak{e}^n\mathcal{M}$ is the filtration by powers of $\mathfrak{m}$.

Proof of Theorem 5.8.5. (1) Lemma 5.7.4 claims that $\mathrm{H}^0(\mathcal{M}/\mathfrak{e}^n) = 0$. So the natural sequence above is exact. (2) By Proposition 5.7.7 the shtuka $\mathfrak{e}^n\mathcal{M}$ is elliptic. It is thus enough to treat the case $n = 1$. Consider the natural commutative square

$$\begin{tikzcd}
\mathrm{H}^1(\mathcal{M}) \arrow[r] \arrow[d] & \mathrm{H}^1(\mathcal{M}/\mathfrak{e}) \arrow[d] \\
\mathrm{H}^1(\mathcal{M}/\mathfrak{m}) \arrow[r] & \mathrm{H}^1(\mathcal{M}/\mathfrak{m}+\mathfrak{e})
\end{tikzcd}$$

We just demonstrated that the top arrow in this square is surjective with kernel $\mathrm{H}^1(\mathfrak{e}\mathcal{M})$. According to Proposition 5.8.2 the left arrow is surjective with kernel $\mathfrak{m}\mathrm{H}^1(\mathcal{M})$. The right arrow is an isomorphism by Lemma 5.8.4. Since $\mathcal{M}/\mathfrak{e}$ is linear by (E4) Lemma 5.8.3 shows that the bottom arrow is an isomorphism. So the result follows. □

5.9. Overview of regulators

Now as our discussion of elliptic shtukas has gained some substance we can give an overview of the regulator theory which will follow. Let $\mathcal{M}$ be an elliptic shtuka. If $\mathcal{M}$ is linear then one tautologically has a natural isomorphism $\rho\colon \mathrm{H}^1(\mathcal{M}) \to \mathrm{H}^1(\nabla\mathcal{M})$ induced by the identity of the shtukas $\mathcal{M}$ and $\nabla\mathcal{M}$, the *regulator* of $\mathcal{M}$. We would like to extend it to all elliptic shtukas $\mathcal{M}$. A rough idea is to approximate $\mathcal{M}$ with linear pieces.

Definition. A natural transformation

$$\mathrm{H}^1(\mathcal{M}) \xrightarrow{\rho} \mathrm{H}^1(\nabla\mathcal{M})$$

of functors on the category of elliptic shtukas is called a *regulator* if for every $\mathcal{M}$ such that $\mathcal{M}/\mathfrak{e}^{2n}$ is linear the diagram

$$\begin{tikzcd}
\mathrm{H}^1(\mathcal{M}) \arrow[r, "\rho"] \arrow[d, "\text{red.}"'] & \mathrm{H}^1(\nabla\mathcal{M}) \arrow[d, "\text{red.}"] \\
\mathrm{H}^1(\mathcal{M}/\mathfrak{e}^n) \arrow[r, "1"] & \mathrm{H}^1(\nabla\mathcal{M}/\mathfrak{e}^n)
\end{tikzcd}$$

is commutative.

Note that we demand the quotient $\mathcal{M}/\mathfrak{e}^{2n}$ to be linear, not just $\mathcal{M}/\mathfrak{e}^n$. Even though the square above makes sense if merely $\mathcal{M}/\mathfrak{e}^n$ is linear the regulator will fail to exist if one demands all such squares to commute.

A regulator ρ will send the submodule $\mathrm{H}^1(\mathfrak{e}^n\mathcal{M}) \subset \mathrm{H}^1(\mathcal{M})$ to $\mathrm{H}^1(\nabla\mathfrak{e}^n\mathcal{M}) \subset \mathrm{H}^1(\nabla\mathcal{M})$ since it is natural. By Proposition 5.7.6 the subquotients $(\mathfrak{e}^{2n}\mathcal{M})/\mathfrak{e}^{2n} = \mathfrak{e}^{2n}\mathcal{M}/\mathfrak{e}^{4n}\mathcal{M}$ are linear for all $n \geqslant 0$. Hence the regulator is an isomorphism and is unique. However the existence of the regulator is a different question altogether. Its construction occupies the rest of the chapter.

In Sections 5.10 and 5.11 we present some auxillary results on cohomology of elliptic shtukas and their subquotients. These sections are of a technical nature. The level of technicality reaches its high point in Section 5.12 where we study the tautological regulators. They are the basic building blocks for the regulator on $\mathrm{H}^1(\mathcal{M})$. Already in Section 5.13 the statements and proofs become much more natural. The construction of the regulator on $\mathrm{H}^1(\mathcal{M})$ in Section 5.14 is actually quite simple. Besides this construction, the main results of Section 5.14 are Theorem 5.14.4 which gives a criterion for an $\mathcal{O}_F$-linear map $f\colon \mathrm{H}^1(\mathcal{M}) \to \mathrm{H}^1(\nabla\mathcal{M})$ to coincide with the regulator, and Theorem 5.14.5 which serves as a link with the trace formula of Chapter 6.

5.10. Cohomology of subquotients

In this section we present several technical propositions which will be used in the construction of the regulator map. First we use Theorem 5.8.5 to derive two exact sequences of cohomology modules.

Proposition 5.10.1. *Let $n \geqslant 0$. If $\mathcal{M}$ is an elliptic shtuka then the short exact sequence of shtukas*

$$0 \to \mathfrak{e}^n\mathcal{M}/\mathfrak{m}^n \to \mathcal{M}/\mathfrak{m}^n \to \mathcal{M}/(\mathfrak{m}^n + \mathfrak{e}^n) \to 0$$

induces an exact sequence of cohomology modules

$$\begin{aligned}(5.3)\quad 0 \to \mathrm{H}^0(\mathcal{M}/(\mathfrak{m}^n + \mathfrak{e}^n)) \xrightarrow{\delta} \mathrm{H}^1(\mathfrak{e}^n\mathcal{M}/\mathfrak{m}^n) \xrightarrow{0}& \\ \longrightarrow \mathrm{H}^1(\mathcal{M}/\mathfrak{m}^n) \xrightarrow{\text{red.}} \mathrm{H}^1(\mathcal{M}/(\mathfrak{m}^n + \mathfrak{e}^n)) \to 0.&\end{aligned}$$

The middle map in this sequence is zero and the adjacent maps are isomorphisms.

Proof. According to Lemma 5.8.1 the shtuka $\mathcal{M}/\mathfrak{m}^n$ is a locally free $\mathcal{O}_F/\mathfrak{m}^n \otimes \mathcal{O}_K$-module shtuka whose restriction to $\mathcal{O}_F/\mathfrak{m} \otimes K$ is nilpotent. Theorem 5.3.2 implies that $\mathrm{H}^0(\mathcal{M}/\mathfrak{m}^n) = 0$. Therefore the sequence (5.3) is exact. To prove the result it is enough to show that the middle map in this sequence is zero. This map fits into a natural commutative square

$$\begin{tikzcd} \mathrm{H}^1(\mathfrak{e}^n\mathcal{M}) \arrow[r, dashed] \arrow[d, dashed] & \mathrm{H}^1(\mathcal{M}) \arrow[d, dashed] \\ \mathrm{H}^1(\mathfrak{e}^n\mathcal{M}/\mathfrak{m}^n) \arrow[r] & \mathrm{H}^1(\mathcal{M}/\mathfrak{m}^n) \end{tikzcd}$$

By Proposition 5.8.2 the vertical maps in this square are surjections and the kernel of the right map is $\mathfrak{m}^n\mathrm{H}^1(\mathcal{M})$. However Theorem 5.8.5 shows that the image of the top map is $\mathfrak{m}^n\mathrm{H}^1(\mathcal{M})$ whence the composition of the top and the right maps is zero. Since the left map is surjective we conclude that the bottom map is zero. $\square$

Proposition 5.10.2. *Let $n \geqslant 0$. If $\mathcal{M}$ is an elliptic shtuka then the short exact sequence of shtukas*

$$0 \to \mathfrak{m}^n\mathcal{M}/\mathfrak{e}^n \to \mathcal{M}/\mathfrak{e}^n \to \mathcal{M}/(\mathfrak{m}^n + \mathfrak{e}^n) \to 0$$

induces an exact sequence of cohomology modules

$$\begin{aligned}(5.4)\quad 0 \to \mathrm{H}^0(\mathcal{M}/(\mathfrak{m}^n + \mathfrak{e}^n)) \xrightarrow{\delta} \mathrm{H}^1(\mathfrak{m}^n\mathcal{M}/\mathfrak{e}^n) \xrightarrow{0}& \\ \longrightarrow \mathrm{H}^1(\mathcal{M}/\mathfrak{e}^n) \xrightarrow{\text{red.}} \mathrm{H}^1(\mathcal{M}/(\mathfrak{m}^n + \mathfrak{e}^n)) \to 0.&\end{aligned}$$

The middle map in this sequence is zero and the adjacent maps are isomorphisms.

Proof. The cohomology module $\mathrm{H}^0(\mathcal{M}/\mathfrak{e}^n)$ vanishes according to Lemma 5.7.4. Thus the sequence (5.4) is exact. To prove the proposition it is enough to show that the middle map in (5.4) is zero. This map fits into a natural commutative square

$$\begin{tikzcd} \mathrm{H}^1(\mathfrak{m}^n\mathcal{M}) \arrow[r, dashed] \arrow[d, dashed] & \mathrm{H}^1(\mathcal{M}) \arrow[d, dashed] \\ \mathrm{H}^1(\mathfrak{m}^n\mathcal{M}/\mathfrak{e}^n) \arrow[r] & \mathrm{H}^1(\mathcal{M}/\mathfrak{e}^n). \end{tikzcd}$$

Theorem 5.8.5 shows that the vertical maps are surjective and that the kernel of the right map is $\mathfrak{m}^n\mathrm{H}^1(\mathcal{M})$. So the composition of the top and the right maps is zero. As the left map is surjective we conclude that the bottom map is zero. $\square$

Next we study the boundary homomorphisms δ of the sequences (5.3) and (5.4).

Proposition 5.10.3. *Let $n \geqslant 0$. If $\mathcal{M}$ is an elliptic shtuka then the natural diagram*

$$\begin{tikzcd} \mathrm{H}^0(\mathcal{M}/(\mathfrak{m}^n+\mathfrak{e}^n)) \arrow[r, "\delta"] \arrow[d, "\delta"'] & \mathrm{H}^1(\mathfrak{m}^n\mathcal{M}/\mathfrak{e}^n) \arrow[d] \\ \mathrm{H}^1(\mathfrak{e}^n\mathcal{M}/\mathfrak{m}^n) \arrow[r] & \mathrm{H}^1(\mathcal{M}/\mathfrak{m}^n\mathfrak{e}^n) \end{tikzcd}$$

is anticommutative. *Here the vertical δ is the boundary homomorphism of* (5.3) *while the horizontal δ is the boundary homomorphism of* (5.4)*.*

Proof. To improve legibility we will write $\mathfrak{a}$ for $\mathfrak{e}^n$ and $\mathfrak{b}$ for $\mathfrak{m}^n$. Since the shtuka $\mathcal{M}$ is locally free Lemma 5.2.4 implies that the natural sequence

$$0 \to \mathfrak{a}\mathcal{M}/\mathfrak{b} \oplus \mathfrak{b}\mathcal{M}/\mathfrak{a} \to \mathcal{M}/\mathfrak{a}\mathfrak{b} \to \mathcal{M}/(\mathfrak{a}+\mathfrak{b}) \to 0$$

is exact. Now consider the natural diagram

$$\begin{tikzcd} 0 \arrow[r] & \mathfrak{a}\mathcal{M}/\mathfrak{b} \oplus \mathfrak{b}\mathcal{M}/\mathfrak{a} \arrow[r] \arrow[d, "1"] & \mathcal{M}/\mathfrak{a}\mathfrak{b} \arrow[r] \arrow[d, "(\mathrm{red.},\mathrm{red.})"] & \mathcal{M}/(\mathfrak{a}+\mathfrak{b}) \arrow[r] \arrow[d, "\Delta"] & 0 \\ 0 \arrow[r] & \mathfrak{a}\mathcal{M}/\mathfrak{b} \oplus \mathfrak{b}\mathcal{M}/\mathfrak{a} \arrow[r] & \mathcal{M}/\mathfrak{a} \oplus \mathcal{M}/\mathfrak{b} \arrow[r] & (\mathcal{M}/\mathfrak{a}+\mathfrak{b})^{\oplus 2} \arrow[r] & 0 \end{tikzcd}$$

Here Δ means the diagonal map. Observe that the lower row is the direct sum of the short exact sequences

$$0 \to \mathfrak{b}\mathcal{M}/\mathfrak{a} \to \mathcal{M}/\mathfrak{b} \to \mathcal{M}/(\mathfrak{a}+\mathfrak{b}) \to 0,$$
$$0 \to \mathfrak{a}\mathcal{M}/\mathfrak{b} \to \mathcal{M}/\mathfrak{a} \to \mathcal{M}/(\mathfrak{a}+\mathfrak{b}) \to 0$$

which give rise to the cohomology sequences (5.3) and (5.4) respectively. The diagram above is clearly commutative. So it induces a morphism of long exact sequences. A part of it looks like this:

$$\begin{tikzcd} \mathrm{H}^0(\mathcal{M}/(\mathfrak{a}+\mathfrak{b})) \arrow[r] \arrow[d, "\Delta"] & \mathrm{H}^1(\mathfrak{a}\mathcal{M}/\mathfrak{b})\oplus\mathrm{H}^1(\mathfrak{b}\mathcal{M}/\mathfrak{a}) \arrow[r] \arrow[d, "1"] & \mathrm{H}^1(\mathcal{M}/\mathfrak{a}\mathfrak{b}) \arrow[d, "(\mathrm{red.},\mathrm{red.})"] \\ \mathrm{H}^0(\mathcal{M}/(\mathfrak{a}+\mathfrak{b}))^{\oplus 2} \arrow[r, "\delta\oplus\delta"] & \mathrm{H}^1(\mathfrak{a}\mathcal{M}/\mathfrak{b})\oplus\mathrm{H}^1(\mathfrak{b}\mathcal{M}/\mathfrak{a}) \arrow[r] & \mathrm{H}^1(\mathcal{M}/\mathfrak{b})\oplus\mathrm{H}^1(\mathcal{M}/\mathfrak{a}) \end{tikzcd}$$

As a consequence the boundary homomorphism $\mathrm{H}^0(\mathcal{M}/(\mathfrak{a}+\mathfrak{b})) \to \mathrm{H}^1(\mathfrak{a}\mathcal{M}/\mathfrak{b}) \oplus \mathrm{H}^1(\mathfrak{b}\mathcal{M}/\mathfrak{a})$ in the top row coincides with (δ,δ). Since the composition of the adjacent homomorphisms

$$\mathrm{H}^0(\mathcal{M}/(\mathfrak{a}+\mathfrak{b})) \xrightarrow{(\delta,\delta)} \mathrm{H}^1(\mathfrak{a}\mathcal{M}/\mathfrak{b}) \oplus \mathrm{H}^1(\mathfrak{b}\mathcal{M}/\mathfrak{a}) \longrightarrow \mathrm{H}^1(\mathcal{M}/\mathfrak{a}\mathfrak{b})$$

is zero we get the result. $\square$

Proposition 5.10.4. *Let $n \geqslant 0$. If $\mathcal{M}$ is an elliptic shtuka then the natural maps*

$$\mathrm{H}^1(\mathfrak{e}^n\mathcal{M}/\mathfrak{m}^n) \to \mathrm{H}^1(\mathcal{M}/\mathfrak{m}^n\mathfrak{e}^n),$$
$$\mathrm{H}^1(\mathfrak{m}^n\mathcal{M}/\mathfrak{e}^n) \to \mathrm{H}^1(\mathcal{M}/\mathfrak{m}^n\mathfrak{e}^n)$$

are injective.

Proof. Consider the short exact sequence $0 \to \mathfrak{e}^n\mathcal{M}/\mathfrak{m}^n \to \mathcal{M}/\mathfrak{m}^n\mathfrak{e}^n \to \mathcal{M}/\mathfrak{e}^n \to 0$. By Lemma 5.7.4 the module $\mathrm{H}^0(\mathcal{M}/\mathfrak{e}^n)$ vanishes. Hence the natural map $\mathrm{H}^1(\mathfrak{e}^n\mathcal{M}/\mathfrak{m}^n) \to \mathrm{H}^1(\mathcal{M}/\mathfrak{m}^n\mathfrak{e}^n)$ is injective. In a similar way Theorem 5.3.2 implies that the map $\mathrm{H}^1(\mathfrak{m}^n\mathcal{M}/\mathfrak{e}^n) \to \mathrm{H}^1(\mathcal{M}/\mathfrak{m}^n\mathfrak{e}^n)$ is injective. □

5.11. Natural isomorphisms on cohomology

Let $\mathcal{M}$ be an elliptic shtuka. According to Theorem 5.8.5 the images of $\mathrm{H}^1(\mathfrak{m}\mathcal{M})$ and $\mathrm{H}^1(\mathfrak{e}\mathcal{M})$ inside $\mathrm{H}^1(\mathcal{M})$ are equal to $\mathfrak{m}\mathrm{H}^1(\mathcal{M})$.

Definition 5.11.1. We define the natural isomorphism

$$\gamma_{\mathcal{M}} \colon \mathrm{H}^1(\mathfrak{m}\mathcal{M}) \to \mathrm{H}^1(\mathfrak{e}\mathcal{M})$$

as the unique map which makes the triangle

$$\begin{tikzcd} \mathrm{H}^1(\mathfrak{m}\mathcal{M}) \arrow[rr, dashed, "\gamma_{\mathcal{M}}", "\sim"'] \arrow[dr] & & \mathrm{H}^1(\mathfrak{e}\mathcal{M}) \arrow[dl] \\ & \mathrm{H}^1(\mathcal{M}) & \end{tikzcd}$$

commutative.

To simplify the expressions we denote $\gamma^n_{\mathcal{M}}$ the composition

$$\mathrm{H}^1(\mathfrak{m}^n\mathcal{M}) \xrightarrow{\gamma_{\mathfrak{m}^{n-1}\mathcal{M}}} \mathrm{H}^1(\mathfrak{e}\mathfrak{m}^{n-1}\mathcal{M}) \xrightarrow{\gamma_{\mathfrak{e}\mathfrak{m}^{n-2}\mathcal{M}}} \dots \xrightarrow{\gamma_{\mathfrak{e}^{n-1}\mathcal{M}}} \mathrm{H}^1(\mathfrak{e}^n\mathcal{M}).$$

We write γ^n instead of $\gamma^n_{\mathcal{M}}$ if the corresponding shtuka $\mathcal{M}$ is clear from the context. The map $\gamma^n_{\mathcal{M}}$ will only be used in Sections 5.12 and 5.13.

Lemma 5.11.2. *If $n, k \geqslant 0$ are integers then the composition*

$$\mathrm{H}^1(\mathfrak{m}^{n+k}\mathcal{M}) \xrightarrow{\gamma^n_{\mathfrak{m}^k\mathcal{M}}} \mathrm{H}^1(\mathfrak{e}^n\mathfrak{m}^k\mathcal{M}) \xrightarrow{\gamma^k_{\mathfrak{e}^n\mathcal{M}}} \mathrm{H}^1(\mathfrak{e}^{n+k}\mathcal{M})$$

is equal to $\gamma^{n+k}_{\mathcal{M}}$. □

Our goal is to relate γ^n to the maps which appear in the natural sequences (5.3) and (5.4).

Lemma 5.11.3. *Let $\mathcal{M}$ be an elliptic shtuka. For every $n \geqslant 0$ there exists a unique natural isomorphism $\overline{\gamma}_n \colon \mathrm{H}^1(\mathfrak{m}^n\mathcal{M}/\mathfrak{e}^n) \xrightarrow{\sim} \mathrm{H}^1(\mathfrak{e}^n\mathcal{M}/\mathfrak{m}^n)$ such that the diagram*

$$\begin{tikzcd} \mathrm{H}^1(\mathfrak{m}^n\mathcal{M}) \arrow[r, "\gamma^n", "\sim"'] \arrow[d, "\text{red.}"'] & \mathrm{H}^1(\mathfrak{e}^n\mathcal{M}) \arrow[d, "\text{red.}"] \\ \mathrm{H}^1(\mathfrak{m}^n\mathcal{M}/\mathfrak{e}^n) \arrow[r, "\overline{\gamma}_n", "\sim"'] & \mathrm{H}^1(\mathfrak{e}^n\mathcal{M}/\mathfrak{m}^n) \end{tikzcd}$$

is commutative.

The map $\overline{\gamma}_n$ will only be used in Section 5.12. The same remark applies to the related map $\overline{\varepsilon}_n$ which we introduce below.

Proof of Lemma 5.11.3. By Theorem 5.8.5 the left reduction map is surjective with kernel $\mathfrak{m}^n\mathrm{H}^1(\mathfrak{m}^n\mathcal{M})$. Similarly Proposition 5.8.2 shows that the right reduction map is surjective with kernel $\mathfrak{m}^n\mathrm{H}^1(\mathfrak{e}^n\mathcal{M})$. Since γ^n is an isomorphism the result follows. □

Proposition 5.11.4. *Let $\mathcal{M}$ be an elliptic shtuka. For every $n \geqslant 0$ the maps δ of* (5.3), (5.4) *are isomorphisms and the triangle*

$$\begin{tikzcd} & \mathrm{H}^0(\mathcal{M}/\mathfrak{m}^n + \mathfrak{e}^n) \arrow[dl, "\delta"'] \arrow[dr, "\delta"] & \\ \mathrm{H}^1(\mathfrak{m}^n\mathcal{M}/\mathfrak{e}^n) \arrow[rr, "\overline{\gamma}_n"] & & \mathrm{H}^1(\mathfrak{e}^n\mathcal{M}/\mathfrak{m}^n) \end{tikzcd}$$

is anticommutative.

Proof. The maps δ are isomorphisms by Propositions 5.10.1 and 5.10.2. Next, consider the diagram

$$\begin{tikzcd} & \mathrm{H}^0(\mathcal{M}/\mathfrak{m}^n + \mathfrak{e}^n) \arrow[dl, "\delta"'] \arrow[dr, "\delta"] & \\ \mathrm{H}^1(\mathfrak{m}^n\mathcal{M}/\mathfrak{e}^n) \arrow[rr, "\overline{\gamma}_n"] \arrow[dr] & & \mathrm{H}^1(\mathfrak{e}^n\mathcal{M}/\mathfrak{m}^n) \arrow[dl] \\ & \mathrm{H}^1(\mathcal{M}/\mathfrak{m}^n\mathfrak{e}^n) & \end{tikzcd}$$

The square in this diagram commutes by Proposition 5.10.3. The bottom diagonal maps are injective by Proposition 5.10.4. Hence it is enough to prove that the bottom triangle commutes. By definition of $\overline{\gamma}_n$ we need to show the commutativity of the triangle

$$\begin{tikzcd} \mathrm{H}^1(\mathfrak{m}^n\mathcal{M}) \arrow[rr, "\gamma^n"] \arrow[dr] & & \mathrm{H}^1(\mathfrak{e}^n\mathcal{M}) \arrow[dl] \\ & \mathrm{H}^1(\mathcal{M}/\mathfrak{m}^n\mathfrak{e}^n) & \end{tikzcd}$$

However the two diagonal maps factor over the natural maps to $\mathrm{H}^1(\mathcal{M})$. Hence the commutativity follows from the definition of γ^n. □

Lemma 5.11.5. *Let $\mathcal{M}$ be an elliptic shtuka. For every $n \geqslant 0$ there exists a unique natural isomorphism $\overline{\varepsilon}_n\colon \mathrm{H}^1(\mathfrak{e}^n\mathcal{M}/\mathfrak{m}^n) \xrightarrow{\sim} \mathrm{H}^1(\mathfrak{e}^n\mathcal{M}/\mathfrak{e}^n)$ such that the square*

$$\begin{tikzcd} \mathrm{H}^1(\mathfrak{e}^n\mathcal{M}) \arrow[r, equal] \arrow[d, "\text{red.}"'] & \mathrm{H}^1(\mathfrak{e}^n\mathcal{M}) \arrow[d, "\text{red.}"] \\ \mathrm{H}^1(\mathfrak{e}^n\mathcal{M}/\mathfrak{m}^n) \arrow[r, "\overline{\varepsilon}_n", "\sim"'] & \mathrm{H}^1(\mathfrak{e}^n\mathcal{M}/\mathfrak{e}^n) \end{tikzcd}$$

is commutative.

Proof. Indeed the left reduction map is surjective with kernel $\mathfrak{m}^n\mathrm{H}^1(\mathfrak{e}^n\mathcal{M})$ by Proposition 5.8.2 while the right reduction map is surjective with the same kernel by Theorem 5.8.5. □

Lemma 5.11.6. *Let $\mathcal{M}$ be an elliptic shtuka. For every $n \geqslant 0$ the square*

$$\begin{tikzcd} \mathrm{H}^1(\mathfrak{m}^n\mathcal{M}) \arrow[r, "\gamma^n"] \arrow[d, "\text{red.}"'] & \mathrm{H}^1(\mathfrak{e}^n\mathcal{M}) \arrow[d, "\text{red.}"] \\ \mathrm{H}^1(\mathfrak{m}^n\mathcal{M}/\mathfrak{e}^n) \arrow[r, "\overline{\varepsilon}_n\circ\overline{\gamma}_n"] & \mathrm{H}^1(\mathfrak{e}^n\mathcal{M}/\mathfrak{e}^n) \end{tikzcd}$$

is commutative.

Proof. Follows instantly from the definitions of $\overline{\gamma}_n$ and $\overline{\varepsilon}_n$. $\square$

Proposition 5.11.7. *Let $\mathcal{M}$ be an elliptic shtuka and let $n \geqslant 0$.*

(1) *The reduction maps*

$$\mathrm{H}^1(\mathfrak{e}^n\mathcal{M}/\mathfrak{m}^n) \xrightarrow{\text{red.}} \mathrm{H}^1(\mathfrak{e}^n\mathcal{M}/\mathfrak{m}^n + \mathfrak{e}^n),$$
$$\mathrm{H}^1(\mathfrak{e}^n\mathcal{M}/\mathfrak{e}^n) \xrightarrow{\text{red.}} \mathrm{H}^1(\mathfrak{e}^n\mathcal{M}/\mathfrak{m}^n + \mathfrak{e}^n)$$

are isomorphisms.

(2) *The diagram*

$$\begin{tikzcd}
\mathrm{H}^1(\mathfrak{e}^n\mathcal{M}/\mathfrak{m}^n) \arrow[rr, "\overline{\varepsilon}_n"] \arrow[dr, "\text{red.}"'] & & \mathrm{H}^1(\mathfrak{e}^n\mathcal{M}/\mathfrak{e}^n) \arrow[dl, "\text{red.}"] \\
& \mathrm{H}^1(\mathfrak{e}^n\mathcal{M}/\mathfrak{m}^n + \mathfrak{e}^n) &
\end{tikzcd}$$

is commutative.

Proof. (1) follows from Propositions 5.10.1 and 5.10.2. (2) Indeed the outer square in the diagram

$$\begin{tikzcd}
& \mathrm{H}^1(\mathfrak{e}^n\mathcal{M}) \arrow[dl, "\text{red.}"'] \arrow[dr, "\text{red.}"] & \\
\mathrm{H}^1(\mathfrak{e}^n\mathcal{M}/\mathfrak{m}^n) \arrow[rr, "\overline{\varepsilon}_n"] \arrow[dr, "\text{red.}"'] & & \mathrm{H}^1(\mathfrak{e}^n\mathcal{M}/\mathfrak{e}^n) \arrow[dl, "\text{red.}"] \\
& \mathrm{H}^1(\mathfrak{e}^n\mathcal{M}/\mathfrak{m}^n + \mathfrak{e}^n) &
\end{tikzcd}$$

commutes by definition of $\overline{\varepsilon}_n$. The top left reduction map is surjective by Proposition 5.8.2 So the result follows. $\square$

5.12. Tautological regulators

Lemma 5.12.1. *Let $d \geqslant 1$. If $\mathcal{M}$ is an elliptic shtuka then the reduction map $\mathrm{H}^1(\mathcal{M}) \to \mathrm{H}^1(\mathcal{M}/\mathfrak{e}^d)$ induces an isomorphism $\mathrm{H}^1(\mathcal{M})/\mathfrak{m}^d \to \mathrm{H}^1(\mathcal{M}/\mathfrak{e}^d)$.*

Proof. Indeed Theorem 5.8.5 states that the reduction map is surjective with kernel $\mathfrak{m}^d\mathrm{H}^1(\mathcal{M})$. $\square$

Let $\mathcal{M}$ be an elliptic shtuka and let $d \geqslant 1$. If $\mathcal{M}/\mathfrak{e}^d$ is linear then the shtukas $\mathcal{M}/\mathfrak{e}^d$ and $\nabla\mathcal{M}/\mathfrak{e}^d$ coincide tautologically. We thus get a natural isomorphism

$$\mathrm{H}^1(\mathcal{M}/\mathfrak{e}^d) \xrightarrow{1} \mathrm{H}^1(\nabla\mathcal{M}/\mathfrak{e}^d).$$

Using it we will now define a natural isomorphism $\mathrm{H}^1(\mathcal{M})/\mathfrak{m}^d \xrightarrow{\sim} \mathrm{H}^1(\nabla\mathcal{M})/\mathfrak{m}^d$.

Definition 5.12.2. Let $d \geqslant 1$ and let $\mathcal{M}$ be an elliptic shtuka such that $\mathcal{M}/\mathfrak{e}^d$ is linear. We define the map $\overline{\rho}_d$ by the commutative diagram

$$\begin{tikzcd}
\mathrm{H}^1(\mathcal{M})/\mathfrak{m}^d \arrow[r, "\overline{\rho}_d"] \arrow[d, "\text{red.}"', "\wr"] & \mathrm{H}^1(\nabla\mathcal{M})/\mathfrak{m}^d \arrow[d, "\text{red.}", "\wr"'] \\
\mathrm{H}^1(\mathcal{M}/\mathfrak{e}^d) \arrow[r, "1"] & \mathrm{H}^1(\nabla\mathcal{M}/\mathfrak{e}^d).
\end{tikzcd}$$

Here the vertical maps are the isomorphisms of Lemma 5.12.1. We call $\overline{\rho}_d$ the *tautological regulator of order d*. By construction $\overline{\rho}_d$ is a natural isomorphism.

For the duration of this section and Section 5.13 we fix a uniformizer $z \in \mathcal{O}_F$. As it will be shown in Section 5.13 our results do not depend on the choice of z. However this choice simplifies the exposition. With the uniformizer z fixed we have for every elliptic shtuka $\mathcal{M}$ a natural isomorphism $\varpi\colon \mathcal{M} \to \mathfrak{m}\mathcal{M}$.

Definition 5.12.3. Let $\mathcal{M}$ be an elliptic shtuka. We define a natural isomorphism

$$\mathrm{H}^1(\mathcal{M}) \xrightarrow{\ s\ } \mathrm{H}^1(\mathfrak{e}\mathcal{M})$$

as the composition $\mathrm{H}^1(\mathcal{M}) \xrightarrow{\varpi} \mathrm{H}^1(\mathfrak{m}\mathcal{M}) \xrightarrow{\gamma} \mathrm{H}^1(\mathfrak{e}\mathcal{M})$. Here γ is the natural isomorphism of Definition 5.11.1. We call s the *sliding isomorphism.*

Apart from this section the sliding isomorphism s will only be used in Section 5.13.

Lemma 5.12.4. *Let $\mathcal{M}$ be an elliptic shtuka. For every $n \geqslant 0$ the natural diagram*

$$\begin{tikzcd}
\mathrm{H}^1(\mathcal{M}) \arrow[d, "\varpi^n"'] \arrow[dr, "s^n"] & \\
\mathrm{H}^1(\mathfrak{m}^n\mathcal{M}) \arrow[r, "\gamma^n"] & \mathrm{H}^1(\mathfrak{e}^n\mathcal{M})
\end{tikzcd}$$

is commutative.

Proof. This diagram commutes for $n = 0$. Assuming that it commutes for some n we prove that it does so for $n+1$. Consider the natural diagram

$$\begin{tikzcd}
\mathrm{H}^1(\mathcal{M}) \arrow[r, "\varpi^n"] \arrow[dr, "\varpi^{n+1}"'] & \mathrm{H}^1(\mathfrak{m}^n\mathcal{M}) \arrow[r, "\gamma^n_{\mathcal{M}}"] \arrow[d, "\varpi"] & \mathrm{H}^1(\mathfrak{e}^n\mathcal{M}) \arrow[d, "\varpi"] \arrow[dr, "s"] & \\
 & \mathrm{H}^1(\mathfrak{m}^{n+1}\mathcal{M}) \arrow[r, "\gamma^n_{\mathfrak{m}\mathcal{M}}"] & \mathrm{H}^1(\mathfrak{m}\mathfrak{e}^n\mathcal{M}) \arrow[r, "\gamma_{\mathfrak{e}^n\mathcal{M}}"] & \mathrm{H}^1(\mathfrak{e}^{n+1}\mathcal{M})
\end{tikzcd}$$

The left triangle in this diagram commutes by definition of ϖ, the square commutes by naturality of γ and the right triangle commutes by definition of s. By assumption $s^n = \gamma^n(\mathcal{M}) \circ \varpi^n$. However

$$\gamma_{\mathfrak{e}^n\mathcal{M}} \circ \gamma^n_{\mathfrak{m}\mathcal{M}} = \gamma^{n+1}_{\mathcal{M}}$$

by definition of γ so the result follows. $\square$

Recall that the natural isomorphism $\overline{\rho}_d$ is defined only under assumption that $\mathcal{M}/\mathfrak{e}^d$ is linear. We would like to extend it to all elliptic shtukas. To that end we will prove that the diagram

$$\begin{tikzcd}
\mathrm{H}^1(\mathcal{M})/\mathfrak{m}^d \arrow[r, "\overline{\rho}_d"] \arrow[d, "s^d"', "\wr"] & \mathrm{H}^1(\nabla\mathcal{M})/\mathfrak{m}^d \arrow[d, "\wr"', "s^d"] \\
\mathrm{H}^1(\mathfrak{e}^d\mathcal{M})/\mathfrak{m}^d \arrow[r, "\overline{\rho}_d"] & \mathrm{H}^1(\nabla\mathfrak{e}^d\mathcal{M})/\mathfrak{m}^d
\end{tikzcd}$$

commutes provided the shtuka $\mathcal{M}/\mathfrak{e}^{2d}$ is linear. The proof is a bit technical. We split it into a chain of auxillary lemmas. In the following the integer d and the elliptic shtuka $\mathcal{M}$ will be fixed. To improve the legibility we will write $\mathfrak{a}$ in place of $\mathfrak{e}^d$.

Lemma 5.12.5. *The shtuka $(\mathfrak{a}\mathcal{M})/\mathfrak{a}$ is linear.*

Proof. Follows instantly from Proposition 5.7.6. $\square$

Consider the shtuka $\mathfrak{a}\mathcal{M}/\mathfrak{m}^d$. According to Lemma 5.8.1 it is a locally free shtuka on $\mathcal{O}_F/\mathfrak{m}^d \otimes \mathcal{O}_K$ whose restriction to $\mathcal{O}_F/\mathfrak{m}^d \otimes K$ is nilpotent. In Section 5.5 we equipped the shtukas of this kind with a natural isomorphism

$$\rho\colon \mathrm{H}^1(\mathfrak{a}\mathcal{M}/\mathfrak{m}^d) \to \mathrm{H}^1(\nabla\mathfrak{a}\mathcal{M}/\mathfrak{m}^d)$$

called the artinian regulator. It is defined only under certain conditions.

Lemma 5.12.6. *If $\mathcal{M}/\mathfrak{a}$ is linear then the artinian regulator is defined for $\mathfrak{a}\mathcal{M}/\mathfrak{m}^d$.*

Proof. Indeed $(\mathfrak{a}\mathcal{M})/\mathfrak{a}$ is linear by Lemma 5.12.5 whence the result follows from Lemma 5.8.3 applied to the shtuka $\mathfrak{a}\mathcal{M}$. □

In Section 5.11 we introduced natural isomorphisms

$$\mathrm{H}^1(\mathfrak{m}^d\mathcal{M}/\mathfrak{a}) \xrightarrow{\overline{\gamma}_d} \mathrm{H}^1(\mathfrak{a}\mathcal{M}/\mathfrak{m}^d) \xrightarrow{\overline{\varepsilon}_d} \mathrm{H}^1(\mathfrak{a}\mathcal{M}/\mathfrak{a}).$$

In the following we drop the indices d for legibility. Our next step is to study how the artinian regulator ρ of the shtuka $\mathfrak{a}\mathcal{M}/\mathfrak{m}^d$ interacts with $\overline{\gamma}$ and $\overline{\varepsilon}$.

Lemma 5.12.7. *If $\mathcal{M}/\mathfrak{a}$ is linear then the diagram*

$$\begin{tikzcd}
\mathrm{H}^1(\mathfrak{a}\mathcal{M}/\mathfrak{m}^d) \arrow[r, "\rho"] \arrow[d, "\overline{\varepsilon}"', "\wr"] & \mathrm{H}^1(\nabla\mathfrak{a}\mathcal{M}/\mathfrak{m}^d) \arrow[d, "\wr"', "\overline{\varepsilon}"] \\
\mathrm{H}^1(\mathfrak{a}\mathcal{M}/\mathfrak{a}) \arrow[r, "1"] & \mathrm{H}^1(\nabla\mathfrak{a}\mathcal{M}/\mathfrak{a})
\end{tikzcd}$$

is commutative.

Proof. Consider the diagram

$$\begin{tikzcd}
\mathrm{H}^1(\mathfrak{a}\mathcal{M}/\mathfrak{m}^d) \arrow[r, "\rho"] \arrow[d, "\text{red.}"', "\wr"] & \mathrm{H}^1(\nabla\mathfrak{a}\mathcal{M}/\mathfrak{m}^d) \arrow[d, "\wr"', "\text{red.}"] \\
\mathrm{H}^1(\mathfrak{a}\mathcal{M}/(\mathfrak{m}^d+\mathfrak{a})) \arrow[r, "1"] & \mathrm{H}^1(\nabla\mathfrak{a}\mathcal{M}/(\mathfrak{m}^d+\mathfrak{a})) \\
\mathrm{H}^1(\mathfrak{a}\mathcal{M}/\mathfrak{a}) \arrow[r, "1"] \arrow[u, "\text{red.}", "\wr"'] & \mathrm{H}^1(\nabla\mathfrak{a}\mathcal{M}/\mathfrak{a}) \arrow[u, "\wr", "\text{red.}"']
\end{tikzcd}$$

The bottom square is clearly commutative. Applying Proposition 5.5.10 to $\mathfrak{a}\mathcal{M}/\mathfrak{m}^d$ we conclude that the top square is commutative. Now according to Proposition 5.11.7 the isomorphism $\overline{\varepsilon}$ is the composition of reduction isomorphisms

$$\mathrm{H}^1(\mathfrak{a}\mathcal{M}/\mathfrak{m}^d) \xrightarrow{\sim} \mathrm{H}^1(\mathfrak{a}\mathcal{M}/(\mathfrak{m}^d+\mathfrak{a})) \xleftarrow{\sim} \mathrm{H}^1(\mathfrak{a}\mathcal{M}/\mathfrak{a}).$$

Applying the same Proposition to $\nabla\mathfrak{a}\mathcal{M}/\mathfrak{m}^d$ we get the result. □

Lemma 5.12.8. *If $\mathcal{M}/\mathfrak{a}^2$ is linear then the diagram*

$$\begin{tikzcd}
\mathrm{H}^1(\mathfrak{m}^d\mathcal{M}/\mathfrak{a}) \arrow[r, "1"] \arrow[d, "\overline{\gamma}"', "\wr"] & \mathrm{H}^1(\nabla\mathfrak{m}^d\mathcal{M}/\mathfrak{a}) \arrow[d, "\wr"', "\overline{\gamma}"] \\
\mathrm{H}^1(\mathfrak{a}\mathcal{M}/\mathfrak{m}^d) \arrow[r, "\rho"] & \mathrm{H}^1(\nabla\mathfrak{a}\mathcal{M}/\mathfrak{m}^d)
\end{tikzcd}$$

is commutative.

Proof. Consider the boundary homomorphisms

$$\mathrm{H}^0(\mathcal{M}/(\mathfrak{m}^d+\mathfrak{a})) \xrightarrow{\delta} \mathrm{H}^1(\mathfrak{a}\mathcal{M}/\mathfrak{m}^d),$$
$$\mathrm{H}^0(\mathcal{M}/(\mathfrak{m}^d+\mathfrak{a})) \xrightarrow{\delta} \mathrm{H}^1(\mathfrak{m}^d\mathcal{M}/\mathfrak{a})$$

of the exact sequences (5.3) and (5.4). Using them we construct a diagram

$$\begin{tikzcd}
\mathrm{H}^1(\mathfrak{m}^d\mathcal{M}/\mathfrak{a}) \arrow[r, "1"] & \mathrm{H}^1(\nabla\mathfrak{m}^d\mathcal{M}/\mathfrak{a}) \\
\mathrm{H}^0(\mathcal{M}/(\mathfrak{m}^d+\mathfrak{a})) \arrow[r, "1"] \arrow[u, "\delta", "\wr"'] \arrow[d, "\delta"', "\wr"] & \mathrm{H}^0(\nabla\mathcal{M}/(\mathfrak{m}^d+\mathfrak{a})) \arrow[u, "\delta", "\wr"'] \arrow[d, "\delta"', "\wr"] \\
\mathrm{H}^1(\mathfrak{a}\mathcal{M}/\mathfrak{m}^d) \arrow[r, "\rho"] & \mathrm{H}^1(\nabla\mathfrak{a}\mathcal{M}/\mathfrak{m}^d)
\end{tikzcd} \tag{5.5}$$

The top square is clearly commutative. We apply Proposition 5.5.11 to deduce the commutativity of the bottom square. To use it we need to verify that the following conditions hold for the shtuka $\mathcal{N} = \mathcal{M}/\mathfrak{m}^d$:

(1) $\mathfrak{a}\cdot\mathrm{H}^1(\nabla\mathcal{N}) = 0$.
(2) $\mathcal{N}/\mathfrak{a}^2$ is linear.

Lemma 5.8.1 implies (1) while (2) follows since $\mathcal{M}/\mathfrak{a}^2$ is linear. We conclude that (5.5) is commutative. Now Proposition 5.11.4 shows that the maps δ are isomorphisms and that the composition

$$\mathrm{H}^1(\mathfrak{m}^d\mathcal{M}/\mathfrak{a}) \xrightarrow{\delta^{-1}} \mathrm{H}^0(\mathcal{M}/(\mathfrak{m}^d+\mathfrak{a})) \xrightarrow{\delta} \mathrm{H}^1(\mathfrak{a}\mathcal{M}/\mathfrak{m}^d)$$

is equal to $-\overline{\gamma}$. The same Proposition applies to $\nabla\mathcal{M}$ as well. So the commutativity of (5.5) implies our result. □

We are finally ready to obtain the key result of this section.

Proposition 5.12.9. *Let $d \geqslant 1$ and let $\mathcal{M}$ be an elliptic shtuka. If $\mathcal{M}/\mathfrak{e}^{2d}$ is linear then the square*

$$\begin{tikzcd}
\mathrm{H}^1(\mathcal{M})/\mathfrak{m}^d \arrow[r, "\overline{\rho}_d"] \arrow[d, "s^d"', "\wr"] & \mathrm{H}^1(\nabla\mathcal{M})/\mathfrak{m}^d \arrow[d, "s^d", "\wr"'] \\
\mathrm{H}^1(\mathfrak{e}^d\mathcal{M})/\mathfrak{m}^d \arrow[r, "\overline{\rho}_d"] & \mathrm{H}^1(\nabla\mathfrak{e}^d\mathcal{M})/\mathfrak{m}^d.
\end{tikzcd}$$

is commutative.

Proof. We continue to use the notation $\mathfrak{a}$ for $\mathfrak{e}^d$. By Lemma 5.12.5 the shtuka $(\mathfrak{a}\mathcal{M})/\mathfrak{a}$ is linear so that the square makes sense. We proceed by repeated splitting of this square till the problem is reduced to its core.

Using Lemma 5.12.4 we split the square as follows:

$$\begin{tikzcd}
\mathrm{H}^1(\mathcal{M})/\mathfrak{m}^d \arrow[r, "\overline{\rho}_d"] \arrow[d, "\varpi^d"', "\wr"] & \mathrm{H}^1(\nabla\mathcal{M})/\mathfrak{m}^d \arrow[d, "\varpi^d", "\wr"'] \\
\mathrm{H}^1(\mathfrak{m}^d\mathcal{M})/\mathfrak{m}^d \arrow[r, "\overline{\rho}_d"] \arrow[d, "\gamma^d"', "\wr"] & \mathrm{H}^1(\nabla\mathfrak{m}^d\mathcal{M})/\mathfrak{m}^d \arrow[d, "\gamma^d", "\wr"'] \\
\mathrm{H}^1(\mathfrak{a}\mathcal{M})/\mathfrak{m}^d \arrow[r, "\overline{\rho}_d"] & \mathrm{H}^1(\nabla\mathfrak{a}\mathcal{M})/\mathfrak{m}^d.
\end{tikzcd} \tag{5.6}$$

The top square commutes by functoriality of $\overline{\rho}_d$ so we concentrate on the bottom square. It is necessary to split this square even further.

Recall that $\overline{\rho}_d$ is defined as the composition

$$\mathrm{H}^1(\mathcal{M})/\mathfrak{m}^d \xrightarrow{\text{red.}} \mathrm{H}^1(\mathcal{M}/\mathfrak{a}) \xrightarrow{1} \mathrm{H}^1(\nabla\mathcal{M}/\mathfrak{a}) \xleftarrow{\text{red.}} \mathrm{H}^1(\nabla\mathcal{M})/\mathfrak{m}^d$$

So we can rewrite the bottom square as follows:

$$\begin{tikzcd}
\mathrm{H}^1(\mathfrak{m}^d\mathcal{M})/\mathfrak{m}^d \arrow[r, "\text{red.}"] \arrow[d, "\gamma^d"', "\wr"] & \mathrm{H}^1(\mathfrak{m}^d\mathcal{M}/\mathfrak{a}) \arrow[r, "1"] \arrow[d, "(\overline{\varepsilon}\circ\overline{\gamma})_{\mathcal{M}}"', "\wr"] & \mathrm{H}^1(\nabla\mathfrak{m}^d\mathcal{M}/\mathfrak{a}) \arrow[d, "\wr"', "(\overline{\varepsilon}\circ\overline{\gamma})_{\nabla\mathcal{M}}"] & \mathrm{H}^1(\nabla\mathfrak{m}^d\mathcal{M})/\mathfrak{m}^d \arrow[l, "\text{red.}"'] \arrow[d, "\wr"', "\gamma^d"] \\
\mathrm{H}^1(\mathfrak{a}\mathcal{M})/\mathfrak{m}^d \arrow[r, "\text{red.}"] & \mathrm{H}^1(\mathfrak{a}\mathcal{M}/\mathfrak{a}) \arrow[r, "1"] & \mathrm{H}^1(\nabla\mathfrak{a}\mathcal{M}/\mathfrak{a}) & \mathrm{H}^1(\nabla\mathfrak{a}\mathcal{M})/\mathfrak{m}^d \arrow[l, "\text{red.}"']
\end{tikzcd}$$

Here the notation gets a bit confusing, so let us elaborate on it. The compositions $(\overline{\varepsilon}\circ\overline{\gamma})_{\mathcal{M}}$ and $(\overline{\varepsilon}\circ\overline{\gamma})_{\nabla\mathcal{M}}$ have the same source and target. So one would expect that the middle square comutes tautologically. However the actual situation is more complicated. The maps $(\overline{\varepsilon}\circ\overline{\gamma})_{\mathcal{M}}$ and $(\overline{\varepsilon}\circ\overline{\gamma})_{\nabla\mathcal{M}}$ are defined in terms of data which comes from completely different shtukas $\mathcal{M}$ and $\nabla\mathcal{M}$. We add the subscripts $\mathcal{M}$ and $\nabla\mathcal{M}$ to emphasise this fact.

The left and right squares in the diagram above commute by Lemma 5.11.6. Let us consider the middle square. We split it in two:

$$\begin{tikzcd}
\mathrm{H}^1(\mathfrak{m}^d\mathcal{M}/\mathfrak{a}) \arrow[r, "1"] \arrow[d, "\overline{\gamma}"', "\wr"] & \mathrm{H}^1(\nabla\mathfrak{m}^d\mathcal{M}/\mathfrak{a}) \arrow[d, "\wr"', "\overline{\gamma}"] \\
\mathrm{H}^1(\mathfrak{a}\mathcal{M}/\mathfrak{m}^d) \arrow[r, "\rho"] \arrow[d, "\overline{\varepsilon}"', "\wr"] & \mathrm{H}^1(\nabla\mathfrak{a}\mathcal{M}/\mathfrak{m}^d) \arrow[d, "\wr"', "\overline{\varepsilon}"] \\
\mathrm{H}^1(\mathfrak{a}\mathcal{M}/\mathfrak{a}) \arrow[r, "1"] & \mathrm{H}^1(\nabla\mathfrak{a}\mathcal{M}/\mathfrak{a}).
\end{tikzcd}$$

Here ρ is the artinian regulator of the shtuka $\mathfrak{a}\mathcal{M}/\mathfrak{m}^d$ which is defined by Lemma 5.12.6. The bottom square commutes by Lemma 5.12.7. Since $\mathcal{M}/\mathfrak{a}^2$ is linear the top square commutes by Lemma 5.12.8. So we are done. $\square$

5.13. Regulators of finite order

Definition 5.13.1. Let $d \geqslant 1$. An $\mathcal{O}_F/\mathfrak{m}^d$-linear natural transformation

$$\rho_d \colon \mathrm{H}^1(\mathcal{M})/\mathfrak{m}^d \longrightarrow \mathrm{H}^1(\nabla\mathcal{M})/\mathfrak{m}^d$$

of functors on the category of elliptic shtukas is called a *regulator of order d* if the following holds:

(1) If $\mathcal{M}$ is an elliptic shtuka such that $\mathcal{M}/\mathfrak{e}^{2d}$ is linear then ρ_d coincides with the tautological regulator $\overline{\rho}_d$ of Definition 5.12.2.
(2) For every elliptic shtuka $\mathcal{M}$ the natural diagram

$$\begin{tikzcd}
\mathrm{H}^1(\mathcal{M})/\mathfrak{m}^d \arrow[r, "\rho_d"] \arrow[d, "s^d"', "\wr"] & \mathrm{H}^1(\nabla\mathcal{M})/\mathfrak{m}^d \arrow[d, "\wr"', "s^d"] \\
\mathrm{H}^1(\mathfrak{e}^d\mathcal{M})/\mathfrak{m}^d \arrow[r, "\rho_d"] & \mathrm{H}^1(\nabla\mathfrak{e}^d\mathcal{M})/\mathfrak{m}^d
\end{tikzcd}$$

is commutative.

The exponent $2d$ in the condition (1) does not look natural. Indeed the tautological regulator is defined even if $\mathcal{M}/\mathfrak{e}^d$ is linear. However with the exponent d the regulators will fail to exist.

It is worth noting that the definition of the regulator for the quotient $\mathrm{H}^1(\mathcal{M})/\mathfrak{m}^d$ is actually more complicated than the definition for the module $\mathrm{H}^1(\mathcal{M})$ itself (Definition 5.14.1). In the latter case one does not need the condition (2).

Proposition 5.13.2. *Let $d \geqslant 1$. A regulator of order d exists, is unique and is an isomorphism.*

Proof. Let $\mathcal{M}$ be an elliptic shtuka. According to Proposition 5.7.6 the shtuka $(\mathfrak{e}^{2d}\mathcal{M})/\mathfrak{e}^{2d}$ is linear. In particular we have the tautological regulator $\overline{\rho}_d$ for $\mathfrak{e}^{2d}\mathcal{M}$. We define the regulator ρ_d for $\mathcal{M}$ by the commutative diagram

$$\begin{tikzcd}
\mathrm{H}^1(\mathcal{M})/\mathfrak{m}^d \arrow[r, "\rho_d"] \arrow[d, "s^{2d}"', "\wr"] & \mathrm{H}^1(\nabla\mathcal{M})/\mathfrak{m}^d \arrow[d, "s^{2d}", "\wr"'] \\
\mathrm{H}^1(\mathfrak{e}^{2d}\mathcal{M})/\mathfrak{m}^d \arrow[r, "\overline{\rho}_d"] & \mathrm{H}^1(\nabla\mathfrak{e}^{2d}\mathcal{M})/\mathfrak{m}^d
\end{tikzcd}$$

Due to condition (1) of Definition 5.13.1 this diagram should commute for any regulator of order d. We thus get the unicity of ρ_d.

Let us prove that the map ρ_d we just defined is a regulator. It is a natural $\mathcal{O}_F/\mathfrak{m}^d$-linear isomorphism since the maps s^{2d} and $\overline{\rho}_d$ are so. If $\mathcal{M}/\mathfrak{e}^{2d}$ is itself linear then the diagram

$$\begin{tikzcd}
\mathrm{H}^1(\mathcal{M})/\mathfrak{m}^d \arrow[r, "\overline{\rho}_d"] \arrow[d, "s^{2d}"', "\wr"] & \mathrm{H}^1(\nabla\mathcal{M})/\mathfrak{m}^d \arrow[d, "s^{2d}", "\wr"'] \\
\mathrm{H}^1(\mathfrak{e}^{2d}\mathcal{M})/\mathfrak{m}^d \arrow[r, "\overline{\rho}_d"] & \mathrm{H}^1(\nabla\mathfrak{e}^{2d}\mathcal{M})/\mathfrak{m}^d
\end{tikzcd}$$

commutes by Proposition 5.12.9. Hence the condition (1) of Definition 5.13.1 is satisfied. To check the condition (2) consider the diagram

$$\begin{tikzcd}
\mathrm{H}^1(\mathcal{M})/\mathfrak{m}^d \arrow[r, "\rho_d"] \arrow[d, "s^{d}"', "\wr"] & \mathrm{H}^1(\nabla\mathcal{M})/\mathfrak{m}^d \arrow[d, "s^{d}", "\wr"'] \\
\mathrm{H}^1(\mathfrak{e}^{d}\mathcal{M})/\mathfrak{m}^d \arrow[r, "\rho_d"] \arrow[d, "s^{d}"', "\wr"] & \mathrm{H}^1(\nabla\mathfrak{e}^{d}\mathcal{M})/\mathfrak{m}^d \arrow[d, "s^{d}", "\wr"'] \\
\mathrm{H}^1(\mathfrak{e}^{2d}\mathcal{M})/\mathfrak{m}^d \arrow[r, "\overline{\rho}_d"] \arrow[d, "s^{d}"', "\wr"] & \mathrm{H}^1(\nabla\mathfrak{e}^{2d}\mathcal{M})/\mathfrak{m}^d \arrow[d, "s^{d}", "\wr"'] \\
\mathrm{H}^1(\mathfrak{e}^{3d}\mathcal{M})/\mathfrak{m}^d \arrow[r, "\overline{\rho}_d"] & \mathrm{H}^1(\nabla\mathfrak{e}^{3d}\mathcal{M})/\mathfrak{m}^d
\end{tikzcd}$$

We need to prove that the top square commutes. The rectangles of height 2 commute by definition of ρ_d. The bottom square commutes by Proposition 5.12.9. As a consequence the middle square commutes which implies the commutativity of the top square. $\square$

Proposition 5.13.3. *The regulator ρ_d does not depend on the choice of the uniformizer $z \in \mathcal{O}_F$ in the definition of the sliding isomorphism s (Definition 5.12.3).*

Proof. We will show that ρ_d satisfies the condition (2) of Definition 5.13.1 with any choice of z. According to Lemma 5.12.4 the sliding isomorphism s^d is the composition

$$\mathrm{H}^1(\mathcal{M}) \xrightarrow{\varpi^d} \mathrm{H}^1(\mathfrak{m}^d\mathcal{M}) \xrightarrow{\gamma^d} \mathrm{H}^1(\mathfrak{e}^d\mathcal{M})$$

Here γ^d does not depend on the choice of z. The natural isomorpism $\varpi\colon \mathcal{M} \to \mathfrak{m}\mathcal{M}$ is the unique map whose composition with the natural embedding $\mathfrak{m}\mathcal{M} \hookrightarrow \mathcal{M}$ is the multiplication by z.

Now let $u \in \mathcal{O}_F^\times$. The regulator ρ_d commutes with multiplication by u^d since it is $\mathcal{O}_F/\mathfrak{m}^d$-linear. However $u\varpi$ is the natural isomorphism $\mathcal{M} \to \mathfrak{m}\mathcal{M}$ with the choice of uniformizer uz. We conclude that ρ_d satisfies the condition (2) of Definition 5.13.1 with the uniformizer uz as well. $\square$

To construct the regulator isomorphism on the entire module $\mathrm{H}^1(\mathcal{M})$ we would like to take the limit of regulators ρ_d for $d \to \infty$. To do it we first need to show that these regulators agree.

Proposition 5.13.4. *If $k \geqslant d \geqslant 1$ are integers then the regulators ρ_d and ρ_k coincide modulo $\mathfrak{m}^d$.*

Proof. Set $n = 2dk$. Let $\mathcal{M}$ be an elliptic shtuka. According to Proposition 5.7.6 the shtuka $(\mathfrak{e}^n\mathcal{M})/\mathfrak{e}^n$ is linear. The tautological regulators $\overline{\rho}_d$ and $\overline{\rho}_{dk}$ are defined for $\mathfrak{e}^n\mathcal{M}$. Since d and dk divide n the diagrams

$$\begin{tikzcd}
\mathrm{H}^1(\mathcal{M})/\mathfrak{m}^d \arrow[r, "\rho_d"] \arrow[d, "s^n", "\wr"'] & \mathrm{H}^1(\nabla\mathcal{M})/\mathfrak{m}^d \arrow[d, "s^n", "\wr"'] \\
\mathrm{H}^1(\mathfrak{e}^n\mathcal{M})/\mathfrak{m}^d \arrow[r, "\overline{\rho}_d"] & \mathrm{H}^1(\nabla\mathfrak{e}^n\mathcal{M})/\mathfrak{m}^d
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathrm{H}^1(\mathcal{M})/\mathfrak{m}^{dk} \arrow[r, "\rho_{dk}"] \arrow[d, "s^n", "\wr"'] & \mathrm{H}^1(\nabla\mathcal{M})/\mathfrak{m}^{dk} \arrow[d, "s^n", "\wr"'] \\
\mathrm{H}^1(\mathfrak{e}^n\mathcal{M})/\mathfrak{m}^{dk} \arrow[r, "\overline{\rho}_{dk}"] & \mathrm{H}^1(\nabla\mathfrak{e}^n\mathcal{M})/\mathfrak{m}^{dk}
\end{tikzcd}$$

commute by definition of ρ_d and ρ_{dk}. However $\overline{\rho}_d \equiv \overline{\rho}_{dk} \pmod{\mathfrak{m}^d}$ by construction. We conclude that $\rho_d \equiv \rho_{dk} \pmod{\mathfrak{m}^d}$. Applying the same argument with d and k interchanged we deduce that $\rho_k \equiv \rho_{dk} \equiv \rho_d \pmod{\mathfrak{m}^d}$. $\square$

5.14. Regulators

Definition 5.14.1. An $\mathcal{O}_F$-linear natural transformation

$$\mathrm{H}^1(\mathcal{M}) \xrightarrow{\ \rho\ } \mathrm{H}^1(\nabla\mathcal{M})$$

of functors on the category of elliptic shtukas is called a *regulator* if for every $\mathcal{M}$ such that $\mathcal{M}/\mathfrak{e}^{2n}$ is linear the diagram

$$\begin{tikzcd}
\mathrm{H}^1(\mathcal{M}) \arrow[r, "\rho"] \arrow[d, "\text{red.}"'] & \mathrm{H}^1(\nabla\mathcal{M}) \arrow[d, "\text{red.}"] \\
\mathrm{H}^1(\mathcal{M}/\mathfrak{e}^n) \arrow[r, "1"] & \mathrm{H}^1(\nabla\mathcal{M}/\mathfrak{e}^n)
\end{tikzcd}$$

is commutative.

Lemma 5.14.2. *Let $f\colon M \to N$ be a morphism of $\mathcal{O}_F$-modules. If M and N are finitely generated free then the following are equivalent:*

(1) $f = 0$.
(2) *For every $d > 0$ there exists an $n \geqslant 0$ such that $f(\mathfrak{m}^n M) \subset \mathfrak{m}^{n+d}N$.* $\square$

Theorem 5.14.3. *The regulator exists, is unique and is an isomorphism.*

Proof. Let $\mathcal{M}$ be an elliptic shtuka. By Proposition 5.7.6 the shtuka $(\mathfrak{e}^{2d}\mathcal{M})/\mathfrak{e}^{2d}$ is linear for every $d \geqslant 1$. Theorem 5.8.5 shows that:

- $\mathrm{H}^1(\mathfrak{e}^{2d}\mathcal{M}) = \mathfrak{m}^{2d}\mathrm{H}^1(\mathcal{M})$ as submodules of $\mathrm{H}^1(\mathcal{M})$,
- the kernel of the reduction map $\mathrm{H}^1(\nabla\mathfrak{e}^{2d}\mathcal{M}) \to \mathrm{H}^1(\nabla(\mathfrak{e}^{2d}\mathcal{M})/\mathfrak{e}^{2d})$ is the submodule $\mathfrak{m}^{4d}\mathrm{H}^1(\nabla\mathcal{M})$.

So the unicity follows from Lemma 5.14.2.

Now let us construct the regulator and prove that it is an isomorphism. According to Proposition 5.13.2 for every $d \geqslant 1$ there exists a unique regulator of order d

$$\rho_d\colon \mathrm{H}^1(\mathcal{M})/\mathfrak{m}^d \to \mathrm{H}^1(\nabla\mathcal{M})/\mathfrak{m}^d.$$

It is a natural $\mathcal{O}_F/\mathfrak{m}^d$-linear isomorphism. The regulators of different orders are compatible by Proposition 5.13.4 and do not depend on the auxillary choice of a uniformizer $z \in \mathcal{O}_F$ by Proposition 5.13.3. Now take their limit for $d \to \infty$. Since $\mathrm{H}^1(\mathcal{M})$ and $\mathrm{H}^1(\nabla\mathcal{M})$ are finitely generated $\mathcal{O}_F$-modules we get a natural $\mathcal{O}_F$-linear isomorphism $\rho\colon \mathrm{H}^1(\mathcal{M}) \to \mathrm{H}^1(\nabla\mathcal{M})$. It satisfies the condition of Definition 5.14.1 since every ρ_d satisfies it up to order d. □

Theorem 5.14.4. *Let $\mathcal{M}$ be an elliptic shtuka and let $f\colon \mathrm{H}^1(\mathcal{M}) \to \mathrm{H}^1(\nabla\mathcal{M})$ be an $\mathcal{O}_F$-linear map.*

(1) *For every $n \geqslant 0$ the map f sends the submodule $\mathrm{H}^1(\mathfrak{e}^n\mathcal{M})$ to $\mathrm{H}^1(\nabla\mathfrak{e}^n\mathcal{M})$.*
(2) *If for every $d > 0$ there exists an $n \geqslant 0$ such that the shtuka $\mathfrak{e}^n\mathcal{M}/\mathfrak{e}^{2d}$ is linear and the diagram*

$$\begin{tikzcd} \mathrm{H}^1(\mathfrak{e}^n\mathcal{M}) \arrow[r, "f"] \arrow[d, "\text{red.}"'] & \mathrm{H}^1(\nabla\mathfrak{e}^n\mathcal{M}) \arrow[d, "\text{red.}"] \\ \mathrm{H}^1(\mathfrak{e}^n\mathcal{M}/\mathfrak{e}^d) \arrow[r, "1"] & \mathrm{H}^1(\nabla\mathfrak{e}^n\mathcal{M}/\mathfrak{e}^d) \end{tikzcd}$$

is commutative then f coincides with the regulator ρ of $\mathcal{M}$.

Proof. Follows from Lemma 5.14.2 in view of Theorem 5.8.5. □

The next theorem relates the regulator ρ to the artinian regulator of Section 5.5. It is essential to the proof of the trace formula in Chapter 6. It can also be used as an alternative definition of the regulator.

Theorem 5.14.5. *Let $\mathcal{M}$ be an elliptic shtuka of ramification ideal $\mathfrak{e}$ and let $n > 0$. If $\mathcal{M}/\mathfrak{e}^{2n}$ is linear then the following holds:*

(1) *The artinian regulator $\rho_{\mathcal{M}/\mathfrak{m}^n}$ is defined for the shtuka $\mathcal{M}/\mathfrak{m}^n$.*
(2) *The diagram*

$$\begin{tikzcd} \mathrm{H}^1(\mathcal{M}) \arrow[r, "\rho"] \arrow[d, "\text{red.}"'] & \mathrm{H}^1(\nabla\mathcal{M}) \arrow[d, "\text{red.}"] \\ \mathrm{H}^1(\mathcal{M}/\mathfrak{m}^n) \arrow[r, "\rho_{\mathcal{M}/\mathfrak{m}^n}"] & \mathrm{H}^1(\nabla\mathcal{M}/\mathfrak{m}^n) \end{tikzcd}$$

is commutative.

Proof. (1) follows from Lemma 5.8.3 since $\mathcal{M}/\mathfrak{e}^n$ is linear. Let us concentrate on (2). Consider the diagram

$$\begin{tikzcd} \mathrm{H}^1(\mathcal{M}) \arrow[r, "\rho"] \arrow[d, "\text{red.}"'] & \mathrm{H}^1(\nabla\mathcal{M}) \arrow[d, "\text{red.}"] \\ \mathrm{H}^1(\mathcal{M}/\mathfrak{m}^n) \arrow[r, "\rho_{\mathcal{M}/\mathfrak{m}^d}"] \arrow[d, "\text{red.}"', "\wr"] & \mathrm{H}^1(\nabla\mathcal{M}/\mathfrak{m}^n) \arrow[d, "\text{red.}", "\wr"'] \\ \mathrm{H}^1(\mathcal{M}/(\mathfrak{m}^n+\mathfrak{e}^n)) \arrow[r, "1"] & \mathrm{H}^1(\nabla\mathcal{M}/(\mathfrak{m}^n+\mathfrak{e}^n)) \end{tikzcd}$$

The bottom square in this diagram commutes by Proposition 5.5.10. The vertical arrows in the bottom square are isomorphisms by Proposition 5.10.1. Hence the top square of this diagram commutes if and only if the outer rectangle commutes. Now we can split the outer rectangle in two other squares:

$$\begin{tikzcd}
\mathrm{H}^1(\mathcal{M}) \arrow[r, "\rho"] \arrow[d, "\text{red.}"'] & \mathrm{H}^1(\nabla\mathcal{M}) \arrow[d, "\text{red.}"] \\
\mathrm{H}^1(\mathcal{M}/\mathfrak{e}^n) \arrow[r, "1"] \arrow[d, "\text{red.}"', "\wr"] & \mathrm{H}^1(\nabla\mathcal{M}/\mathfrak{e}^n) \arrow[d, "\text{red.}", "\wr"'] \\
\mathrm{H}^1(\mathcal{M}/(\mathfrak{m}^n + \mathfrak{e}^n)) \arrow[r, "1"] & \mathrm{H}^1(\nabla\mathcal{M}/(\mathfrak{m}^n + \mathfrak{e}^n))
\end{tikzcd}$$

The bottom square commutes tautologically. As the shtuka $\mathcal{M}/\mathfrak{e}^{2n}$ is linear the top square commutes by definition of ρ. Hence we are done. $\square$

CHAPTER 6

Trace formula

Let X be a smooth projective curve over $\mathbb{F}_q$. As in Chapter 4 we fix an open dense affine subscheme $\operatorname{Spec} R \subset X$. Its complement consists finitely many closed points. We denote K the product of the local fields of X at these points, $\mathcal{O}_K \subset K$ the ring of integers and $\mathfrak{m}_K \subset \mathcal{O}_K$ the Jacobson radical.

Fix a local field F containing $\mathbb{F}_q$. Let $\mathcal{O}_F \subset F$ be the ring of integers and $\mathfrak{m}_F \subset \mathcal{O}_F$ the maximal ideal. We set $D = \operatorname{Spec} \mathcal{O}_F$. In this chapter we mainly work with the τ-scheme $D \times X$. As usual the endomorphism $\tau \colon D \times X \to D \times X$ acts as the identity on the left hand side of the product $\times$ and as the q-Frobenius on the right hand side. The same applies to the schemes of the form $\operatorname{Spec} \Lambda \times X$ for an $\mathbb{F}_q$-algebra Λ and to tensor product rings.

Let $\mathfrak{e} \subset \mathfrak{m}_K$ be an open ideal. We say that a shtuka $\mathcal{M}$ on $D \times X$ is *elliptic of ramification ideal* $\mathfrak{e}$ if it has the following properties:

(1) $\mathcal{M}$ is locally free,
(2) $\mathcal{M}(\mathcal{O}_F/\mathfrak{m}_F \otimes R)$ is nilpotent,
(3) $\mathcal{M}(\mathcal{O}_F \mathbin{\widehat{\otimes}} \mathcal{O}_K)$ is an elliptic shtuka of ramification ideal $\mathfrak{e}$ in the sense of Definition 5.6.1.

Using the theory of Chapter 5 we will construct for every elliptic shtuka $\mathcal{M}$ a natural quasi-isomorphism

$$\rho_{\mathcal{M}} \colon \mathrm{R}\Gamma(\mathcal{M}) \xrightarrow{\sim} \mathrm{R}\Gamma(\nabla\mathcal{M}),$$

the *regulator* of $\mathcal{M}$. We will also define a numerical invariant $L(\mathcal{M}) \in \mathcal{O}_F^\times$. This invariant is given by an infinite product of local factors

$$\prod_{\mathfrak{m}} L(\mathcal{M}(\mathcal{O}_F \otimes R/\mathfrak{m}))^{-1}$$

where $\mathfrak{m}$ runs over the maximal ideals of R.

The main result of this chapter is Theorem 6.10.3 which states that

$$\zeta_{\mathcal{M}} = L(\mathcal{M}) \cdot \det\nolimits_{\mathcal{O}_F}(\rho_{\mathcal{M}})$$

under a certain technical condition on $\mathcal{M}$. Here $\zeta_{\mathcal{M}}$ is the ζ-isomorphism of $\mathcal{M}$ in the sense of Definition 1.11.2. We call this expression the trace formula for regulators of elliptic shtukas. Theorem 6.10.3 is based on Anderson's trace formula [2] in the form given to it by Böckle-Pink [3] and V. Lafforgue [18]. The statement of Theorem 6.10.3 has its roots in the article [18] of V. Lafforgue as well.

6.1. Preliminaries

In this section we prove an auxillary statement on cohomology of coherent sheaves over $S \times X$ where S is the spectrum of a local noetherian ring Λ. We denote $\mathfrak{m}_\Lambda \subset \Lambda$ the maximal ideal.

Lemma 6.1.1. *Let $\mathcal{F}$ be a coherent sheaf on $S \times X$. If $\mathcal{F}$ is Λ-flat then the following are equivalent:*

(1) $\mathrm{H}^0(\operatorname{Spec}(\Lambda/\mathfrak{m}_\Lambda) \times X,\, \mathcal{F}) = 0$.
(2) $\mathrm{H}^0(\mathcal{F}) = 0$ *and* $\mathrm{H}^1(\mathcal{F})$ *is a free Λ-module of finite rank.*

Proof. The base change theorem for coherent cohomology [07VK] states that $\mathrm{R}\Gamma(\mathcal{F})$ is a perfect complex of Λ-modules and

$$\mathrm{R}\Gamma(\mathcal{F}) \otimes^{\mathbf{L}}_{\Lambda} \Lambda/\mathfrak{m}_\Lambda = \mathrm{R}\Gamma(\mathrm{Spec}(\Lambda/\mathfrak{m}_\Lambda) \times X, \mathcal{F}).$$

Hence (2) $\Rightarrow$ (1). Let us prove the other direction. By base change we know that $\mathrm{R}\Gamma(\mathcal{F}) \otimes^{\mathbf{L}}_{\Lambda} \Lambda/\mathfrak{m}_\Lambda$ is concentrated in degree 1. As $\mathrm{R}\Gamma(\mathcal{F})$ is a perfect complex it follows [068V] that $\mathrm{R}\Gamma(\mathcal{F})$ has Tor amplitude $[1, 1]$. Now [0658] shows that $\mathrm{R}\Gamma(\mathcal{F})$ is quasi-isomorphic to a finitely generated free Λ-module placed in degree 1. ☐

6.2. Euler products in the artinian case

We work with a finite $\mathbb{F}_q$-algebra Λ which is a local artinian ring. As before we denote $\mathfrak{m}_\Lambda \subset \Lambda$ the maximal ideal.

Lemma 6.2.1. *Let k be a finite field extension of $\mathbb{F}_q$. Let $\mathcal{M}$ be a locally free shtuka on $\Lambda \otimes k$ given by a diagram*

$$M_0 \underset{j}{\overset{i}{\rightrightarrows}} M_1.$$

If $\mathcal{M}(\Lambda/\mathfrak{m}_\Lambda \otimes k)$ is nilpotent then the following holds:

(1) *M_0 is a free Λ-module of finite rank,*
(2) *$i\colon M_0 \to M_1$ is an isomorphism,*
(3) *$i^{-1}j\colon M_0 \to M_0$ is a Λ-linear nilpotent endomorphism.*

Proof. (1) is clear. The ring $\Lambda \otimes k$ is noetherian and complete with respect to the nilpotent ideal $\mathfrak{m}_\Lambda \otimes R$ so Proposition 1.10.4 implies that $\mathcal{M}$ is nilpotent. (2) and (3) follow by definition of a nilpotent shtuka. ☐

Definition 6.2.2. Let k be a finite field extension of $\mathbb{F}_q$. Let $\mathcal{M}$ be a locally free shtuka on $\Lambda \otimes k$ given by a diagram

$$M_0 \underset{j}{\overset{i}{\rightrightarrows}} M_1.$$

Assuming that $\mathcal{M}(\Lambda/\mathfrak{m}_\Lambda \otimes k)$ is nilpotent we define

$$L(\mathcal{M}) = \det_\Lambda(1 - i^{-1}j \mid M_0) \in \Lambda^\times.$$

Lemma 6.2.3. *Let $\mathcal{M}$ be a locally free shtuka on $\Lambda \otimes R$. If $\mathcal{M}(\Lambda/\mathfrak{m}_\Lambda \otimes R)$ is nilpotent then $\mathcal{M}$ itself is nilpotent.*

Proof. The ring $\Lambda \otimes R$ is noetherian and complete with respect to the nilpotent ideal $\mathfrak{m}_\Lambda \otimes R$. Whence Proposition 1.10.4 implies that $\mathcal{M}$ is nilpotent. ☐

Lemma 6.2.4. *Let $\mathcal{M}$ be a locally free shtuka on $\Lambda \otimes R$. If $\mathcal{M}(\Lambda/\mathfrak{m}_\Lambda \otimes R)$ is nilpotent then for almost all maximal ideals $\mathfrak{m} \subset R$ we have $L(\mathcal{M}(\Lambda \otimes R/\mathfrak{m})) = 1$.*

Proof. Suppose that $\mathcal{M}$ is given by a diagram

$$M_0 \underset{j}{\overset{i}{\rightrightarrows}} M_1.$$

Let $\mathfrak{m} \subset R$ be a maximal ideal. According to [3, Lemma 8.1.3] we have

$$\det_\Lambda(1 - i^{-1}j \mid M_0/\mathfrak{m}) = \det_{\Lambda \otimes R/\mathfrak{m}}\big(1 - (i^{-1}j)^d \mid M_0/\mathfrak{m}\big)$$

where d is the degree of the finite field $R/\mathfrak{m}$ over $\mathbb{F}_q$. However $i^{-1}j\colon M_0 \to M_0$ is a nilpotent endomorphism by Lemma 6.2.3. Hence $(i^{-1}j)^d = 0$ for $d \gg 0$ and we get the result. ☐

Definition 6.2.5. Let $\mathcal{M}$ be a locally free shtuka on $\Lambda \otimes R$. Assuming that $\mathcal{M}(\Lambda/\mathfrak{m}_\Lambda \otimes R)$ is nilpotent we define

$$L(\mathcal{M}) = \prod_{\mathfrak{m}} L(\mathcal{M}(\Lambda \otimes R/\mathfrak{m}))^{-1} \in \Lambda^\times$$

where $\mathfrak{m} \subset R$ ranges over the maximal ideals. This product is well-defined by Lemma 6.2.4.

Lemma 6.2.6. *Let $\mathcal{M}$ be a locally free shtuka on $\Lambda \otimes R$ such that $\mathcal{M}(\Lambda/\mathfrak{m}_\Lambda \otimes R)$ is nilpotent. If Λ is a field then $L(\mathcal{M}) = 1$.*

Proof. If V is a vector space over a field and N a nilpotent endomorphism of V then $\det(1 - N \,|\, V) = 1$. Hence for every maximal ideal $\mathfrak{m} \subset R$ the invariant $L(\mathcal{M}(\Lambda \otimes R/\mathfrak{m}))$ is equal to 1. ☐

Let $S = \operatorname{Spec} \Lambda$ and let $\mathcal{M}$ be a locally free shtuka on $S \times X$ such that $\mathcal{M}(\Lambda/\mathfrak{m}_\Lambda \otimes R)$ is nilpotent. We write $L(\mathcal{M})$ in place of $L(\mathcal{M}(\Lambda \otimes R))$ to make the formulas more legible. Note that the closed points of X in the complement of $\operatorname{Spec} R$ are not taken into account.

6.3. Anderson's trace formula

We continue working with a finite $\mathbb{F}_q$-algebra Λ which is a local artinian ring. As usual $\mathfrak{m}_\Lambda \subset \Lambda$ stands for the maximal ideal and S denotes the spectrum of Λ. Our trace formula for the shtuka-theoretic regulator is based on the following narrow variant of Anderson's trace formula:

Theorem 6.3.1. *Let $\mathcal{M}$ be a locally free shtuka on $S \times X$ given by a diagram*

$$\mathcal{E} \underset{j}{\overset{1}{\rightrightarrows}} \mathcal{E}.$$

Suppose that

(1) $\mathrm{H}^0(\operatorname{Spec}(\Lambda/\mathfrak{m}_\Lambda) \times X, \mathcal{E}) = 0$,
(2) *$\mathcal{M}$ is nilpotent,*
(3) *$\mathcal{M}(\Lambda \otimes \mathcal{O}_K/\mathfrak{m}_K)$ is linear.*

Then $\mathrm{H}^1(\mathcal{E})$ is a free Λ-module of finite rank and $L(\mathcal{M}) = \det_\Lambda(1 - j \mid \mathrm{H}^1(\mathcal{E}))$.

We will deduce it from Proposition 3.2 in the article [18] of V. Lafforgue. However our setting differs slightly from Lafforgue's. In [18, Section 3] the coefficient ring (denoted A) is a power series ring while we work with a local artinian ring Λ. The next lemma helps to bridge this gap. In the following $\mathcal{I} \subset \mathcal{O}_{S\times X}$ stands for the unique invertible ideal sheaf such that

$$\mathcal{I}(\Lambda \otimes R) = \Lambda \otimes R, \quad \mathcal{I}(\Lambda \otimes \mathcal{O}_K) = \Lambda \otimes \mathfrak{m}_K.$$

Lemma 6.3.2. *Let $\mathcal{E}$ be a locally free sheaf on $S \times X$. If $\mathrm{H}^0(\operatorname{Spec}(\Lambda/\mathfrak{m}_\Lambda) \times X, \mathcal{E}) = 0$ then for every $n \geqslant 0$ the following holds:*

(1) *$\mathrm{H}^0(\mathcal{I}^n\mathcal{E}) = 0$ and $\mathrm{H}^1(\mathcal{I}^n\mathcal{E})$ is a free Λ-module of finite rank.*
(2) *The natural map $\mathrm{H}^1(\mathcal{I}^n\mathcal{E}) \to \mathrm{H}^1(\mathcal{E})$ is a split surjection.*

Proof. Consider the short exact sequence of coherent sheaves $0 \to \mathcal{I}^n\mathcal{E} \to \mathcal{E} \to \mathcal{E}/\mathcal{I}^n\mathcal{E} \to 0$. The assumption implies that $\mathrm{H}^0(\operatorname{Spec}(\Lambda/\mathfrak{m}_\Lambda) \times X, \mathcal{I}^n\mathcal{E}) = 0$ so (1) follows from Lemma 6.1.1. Since $\mathcal{E}/\mathcal{I}^n\mathcal{E}$ is supported at a closed affine subscheme of $S \times X$ the long exact cohomology sequence implies that the maps $\mathrm{H}^1(\mathcal{I}^n\mathcal{E}) \to \mathrm{H}^1(\mathcal{E})$ are onto. They split since $\mathrm{H}^1(\mathcal{E})$ is free. ☐

We also need a lemma on shtukas:

Lemma 6.3.3. *Let $\mathcal{M}$ be a locally free shtuka on $S \times X$ given by a diagram*

$$\mathcal{E} \underset{j}{\overset{1}{\rightrightarrows}} \mathcal{E}.$$

Suppose that

(1) $\mathrm{H}^0(\mathrm{Spec}(\Lambda/\mathfrak{m}_\Lambda) \times X, \mathcal{E}) = 0$,
(2) $\mathcal{M}(\Lambda \otimes \mathcal{O}_K/\mathfrak{m}_K)$ *is linear.*

Then for every $n \geqslant 0$ the following holds:

(1) $\mathrm{H}^1(\mathcal{I}^n\mathcal{E})$ *is a free Λ-module of finite rank,*
(2) $\det_\Lambda(1 - j \mid \mathrm{H}^1(\mathcal{I}^n\mathcal{E})) = \det_\Lambda(1 - j \mid \mathrm{H}^1(\mathcal{E}))$.

Proof. Part (1) is immediate from Lemma 6.3.2. The same lemma implies that the short exact sequence of coherent sheaves $0 \to \mathcal{I}^{n+1}\mathcal{E} \to \mathcal{I}^n\mathcal{E} \to (\mathcal{I}^n\mathcal{E})/\mathcal{I} \to 0$ induces a short exact sequence of cohomology modules

$$0 \to \mathrm{H}^0(\mathcal{I}^n\mathcal{E}/\mathcal{I}) \to \mathrm{H}^1(\mathcal{I}^{n+1}\mathcal{E}) \to \mathrm{H}^1(\mathcal{I}^n\mathcal{E}) \to 0$$

with $\mathrm{H}^0(\mathcal{I}^n\mathcal{E}/\mathcal{I})$ a free Λ-module of finite rank. As a consequence

$$\det_\Lambda\big(1 - j \mid \mathrm{H}^1(\mathcal{I}^{n+1}\mathcal{E}\big) = \det_\Lambda\big(1 - j \mid \mathrm{H}^0(\mathcal{I}^n\mathcal{E}/\mathcal{I})\big) \cdot \det_\Lambda\big(1 - j \mid \mathrm{H}^1(\mathcal{I}^n\mathcal{E})\big).$$

By assumption $j(\mathcal{E}) \subset \mathcal{I}\mathcal{E}$. Since $\tau(\mathcal{I}) \subset \mathcal{I}^q$ and j is τ-linear we conclude that $j(\mathcal{I}^n\mathcal{E}) \subset \mathcal{I}^{qn+1}\mathcal{E}$. Hence j is zero on the quotient $\mathcal{I}^n\mathcal{E}/\mathcal{I}$ and we get the result. □

Proof of Theorem 6.3.1. Let $\Omega_{S\times X}$ be the canonical sheaf of $S \times X$ over S and let

$$\mathcal{V} = \mathrm{H}^0(\mathrm{Spec}\,\Lambda \otimes R, \mathcal{H}\mathrm{om}(\mathcal{E}, \Omega_{S\times X})),$$
$$\mathcal{V}_t = \mathrm{H}^0(S \times X, \mathcal{H}\mathrm{om}(\mathcal{I}^t\mathcal{E}, \Omega_{S\times X})).$$

Grothendieck-Serre duality identifies $\mathcal{V}_t$ with the (-1)-st cohomology module of the complex

$$\mathrm{RHom}_\Lambda(\mathrm{R}\Gamma(\mathcal{I}^t\mathcal{E}), \Lambda).$$

However Lemma 6.3.2 shows that $\mathrm{H}^0(\mathcal{I}^t\mathcal{E}) = 0$ and $\mathrm{H}^1(\mathcal{I}^t\mathcal{E})$ is a free Λ-module of finite rank. Hence Grothendieck-Serre duality gives a natural isomorphism

$$\mathrm{Hom}_\Lambda(\mathrm{H}^1(\mathcal{I}^t\mathcal{E}), \Lambda) \xrightarrow{\sim} \mathcal{V}_t.$$

In particular every $\mathcal{V}_t$ is a free Λ-module of finite rank. By Lemma 6.3.2 the natural maps $\mathrm{H}^1(\mathcal{I}^{t+1}\mathcal{E}) \to \mathrm{H}^1(\mathcal{I}^t\mathcal{E})$ are split surjections whence the inclusions $\mathcal{V}_t \subset \mathcal{V}_{t+1}$ are split.

For every $t \geqslant 0$ the endomorphism j of $\mathcal{E}$ induces an endomorphism of $\mathrm{H}^1(\mathcal{I}^t\mathcal{E})$ and Grothendieck-Serre duality identifies it with the Cartier-linear endomorphisms $\kappa_\mathcal{V} \colon \mathcal{V}_t \to \mathcal{V}_t$ which are used in Section 3 of [18].

As we explained above the Λ-modules $\mathcal{V}_t$ are free of finite rank and the inclusions $\mathcal{V}_t \subset \mathcal{V}_{t+1}$ are split. So the argument of [18, Proposition 3.2] applies with only one minor change. Namely, one needs to ensure that the auxillary locally free sheaves $\mathcal{F}$ on $S \times X$ constructed in the course of the proof have the property that $\mathrm{H}^0(\mathcal{F}) = 0$ and $\mathrm{H}^1(\mathcal{F})$ is a free Λ-module of finite rank. This can be done as follows. The dual of the ideal sheaf $\mathcal{I}$ is ample relative to S. Hence $\mathrm{H}^0(\mathrm{Spec}(\Lambda/\mathfrak{m}_\Lambda) \times X, \mathcal{I}^n\mathcal{F})$ is zero for $n \gg 0$. Lemma 6.1.1 implies that $\mathcal{I}^n\mathcal{F}$ has the desired property.

The rest of the argument works without change. It shows that for $t \gg 0$ we have an equality of power series in $\Lambda[[T]]$:

$$\det_\Lambda(1 - T\kappa_\mathcal{V} \mid \mathcal{V}_t) = \prod_{\mathfrak{m}} \det_\Lambda\big(1 - Tj \mid \mathcal{E}(\Lambda \otimes R/\mathfrak{m})\big)^{-1}$$

where $\mathfrak{m} \subset R$ runs over the maximal ideals.

Now recall the endomorphism $j \colon \mathcal{E} \to \tau_*\mathcal{E}$ is assumed to be nilpotent. Furthermore the maximal ideal of Λ is nilpotent too. As a consequence [3, Lemma 8.1.4] implies that only finitely many factors in the product on the right hand side are different from 1. Thus we can evaluate the equality above at $T = 1$ and conclude that

$$\det_\Lambda(1 - \kappa_{\mathcal{V}} \mid \mathcal{V}_t) = \prod_{\mathfrak{m}} \det_\Lambda\big(1 - j \mid \mathcal{E}(\Lambda \otimes R/\mathfrak{m})\big)^{-1} = L(\mathcal{M}).$$

Therefore $\det_\Lambda(1 - j \mid \mathrm{H}^1(\mathcal{I}^t\mathcal{E})) = L(\mathcal{M})$. Lemma 6.3.3 shows that this equality holds already for $t = 0$. □

6.4. Artinian regulators

We keep the assumption that Λ is a finite $\mathbb{F}_q$-algebra which is local artinian.

Lemma 6.4.1. *Let $\mathcal{M}$ be a locally free shtuka on $S \times X$ given by a diagram*

$$\mathcal{M}_0 \underset{j}{\overset{i}{\rightrightarrows}} \mathcal{M}_1.$$

If $\mathcal{M}(\Lambda/\mathfrak{m}_\Lambda \otimes R)$ is nilpotent then the restriction of i to $\Lambda \otimes R$ is an isomorphism.

Proof. Indeed $\mathcal{M}(\Lambda \otimes R)$ is nilpotent by Lemma 6.2.3, and the i-arrow of such a shtuka is an isomorphism by definition. □

Lemma 6.4.2. *Let $\mathcal{M}$ be a locally free shtuka on $S \times X$ given by a diagram*

$$\mathcal{M}_0 \underset{j}{\overset{i}{\rightrightarrows}} \mathcal{M}_1.$$

Suppose that the shtuka $\mathcal{M}(\Lambda/\mathfrak{m}_\Lambda \otimes R)$ is nilpotent. If the endomorphism $i^{-1}j$ of $\mathcal{M}_0(\operatorname{Spec} \Lambda \otimes K)$ preserves the submodule $\mathcal{M}_0(\operatorname{Spec} \Lambda \otimes \mathcal{O}_K)$ then the following holds:

1. *The τ-linear endomorphism $i^{-1}j$ of $\mathcal{M}_0|_{\operatorname{Spec} \Lambda \otimes R}$ extends to a unique τ-linear endomorphism of $\mathcal{M}_0$.*
2. *The endomorphism $i^{-1}j$ of $\mathcal{M}_0$ is nilpotent.*

Proof. (1) According to Lemma 4.3.2 the fibre product of the schemes $\operatorname{Spec}(\Lambda \otimes R)$ and $\operatorname{Spec}(\Lambda \otimes \mathcal{O}_K)$ over $S \times X$ is $\operatorname{Spec}(\Lambda \otimes K)$. Furthermore the closed subscheme $\operatorname{Spec}(\Lambda \otimes \mathcal{O}_K/\mathfrak{m}_K)$ of $S \times X$ is defined locally by a principal ideal sheaf. Hence Beauville-Laszlo glueing theorem [0BP2] implies that there exists a unique morphism $(i^{-1}j)^a \colon \tau^*\mathcal{M}_0 \to \mathcal{M}_0$ restricting to the adjoints

$$(i^{-1}j)^a \colon \tau^*\mathcal{M}_0|_{\operatorname{Spec} \Lambda \otimes R} \to \mathcal{M}_0|_{\operatorname{Spec} \Lambda \otimes R},$$
$$(i^{-1}j)^a \colon \tau^*\mathcal{M}_0|_{\operatorname{Spec} \Lambda \otimes \mathcal{O}_K} \to \mathcal{M}_0|_{\operatorname{Spec} \Lambda \otimes \mathcal{O}_K}$$

of the endomorphisms $i^{-1}j$ on $\Lambda \otimes R$ respectively $\Lambda \otimes \mathcal{O}_K$. (2) follows since $i^{-1}j$ is nilpotent on $\Lambda \otimes R$. □

Observe that τ acts as the identity on the underlying topological space of $S \times X$. We are thus in position to apply the constructions of Section 1.15: given a shtuka

$$\mathcal{M} = \Big[\mathcal{M}_0 \underset{j}{\overset{i}{\rightrightarrows}} \mathcal{M}_1\Big]$$

on the scheme $S \times X$ we have the associated complex of Λ-module sheaves

$$\mathcal{G}_a(\mathcal{M}) = \big[\mathcal{M}_0 \xrightarrow{i-j} \mathcal{M}_1\big]$$

on the underlying topological space of $S \times X$.

Definition 6.4.3. Let $\mathcal{M}$ be a locally free shtuka on $S \times X$ given by a diagram

$$\mathcal{M}_0 \overset{i}{\underset{j}{\rightrightarrows}} \mathcal{M}_1.$$

Suppose that $\mathcal{M}(\Lambda/\mathfrak{m}_\Lambda \otimes R)$ is nilpotent. We say that the artinian regulator is defined for $\mathcal{M}$ if the τ-linear endomorphism $i^{-1}j$ of $\mathcal{M}_0(\operatorname{Spec}\Lambda \otimes K)$ preserves the submodule $\mathcal{M}_0(\operatorname{Spec}\Lambda \otimes \mathcal{O}_K)$. In this case we define the *artinian regulator* $\rho_{\mathcal{M}} \colon \mathcal{G}_a(\mathcal{M}) \to \mathcal{G}_a(\nabla\mathcal{M})$ by the diagram

$$\begin{tikzcd} \big[\mathcal{M}_0 \arrow[r, "i-j"] \arrow[d, "1-i^{-1}j"'] & \mathcal{M}_1\big] \arrow[d, "1"] \\ \big[\mathcal{M}_0 \arrow[r, "i"] & \mathcal{M}_1\big] \end{tikzcd}$$

Here $i^{-1}j \colon \mathcal{M}_0 \to \mathcal{M}_0$ is the τ-linear endomorphism constructed in Lemma 6.4.2.

Theorem 1.15.4 identifies $\mathrm{R}\Gamma(S \times X, \mathcal{M})$ and $\mathrm{R}\Gamma(S \times X, \mathcal{G}_a(\mathcal{M}))$. So the artinian regulator $\rho_{\mathcal{M}}$ induces a morphism $\mathrm{R}\Gamma(\mathcal{M}) \to \mathrm{R}\Gamma(\nabla\mathcal{M})$ on shtuka cohomology. We also denote it $\rho_{\mathcal{M}}$.

Lemma 6.4.4. *Let $\mathcal{M}$ be a locally free shtuka on $S \times X$ such that $\mathcal{M}(\Lambda/\mathfrak{m}_\Lambda \otimes R)$ is nilpotent. If the artinian regulator is defined for $\mathcal{M}$ then it is defined for $\mathcal{M}(\Lambda \otimes \mathcal{O}_K)$ and the diagram*

$$\begin{tikzcd} \mathrm{R}\Gamma(\mathcal{M}) \arrow[r, "\rho_{\mathcal{M}}"] \arrow[d] & \mathrm{R}\Gamma(\nabla\mathcal{M}) \arrow[d] \\ \mathrm{R}\Gamma(\Lambda \otimes \mathcal{O}_K,\, \mathcal{M}) \arrow[r, "\rho_{\mathcal{M}(\Lambda\otimes\mathcal{O}_K)}"] & \mathrm{R}\Gamma(\Lambda \otimes \mathcal{O}_K,\, \nabla\mathcal{M}) \end{tikzcd}$$

is commutative.

Proof. Suppose that $\mathcal{M}$ is given by a diagram

$$\mathcal{M}_0 \overset{i}{\underset{j}{\rightrightarrows}} \mathcal{M}_1.$$

By assumption the endomorphism $i^{-1}j$ of $\mathcal{M}_0(\operatorname{Spec}\Lambda \otimes K)$ preserves the submodule $\mathcal{M}_0(\operatorname{Spec}\Lambda \otimes \mathcal{O}_K)$. Hence the artinian regulator in the sense of Definition 5.5.2 is defined for $\mathcal{M}(\Lambda \otimes \mathcal{O}_K)$.

Let U be an affine open neighbourhood of the closed subscheme $\operatorname{Spec}\mathcal{O}_K/\mathfrak{m}_K$ inside X. Applying the same construction as for the shtukas on $S \times X$ we get an artinian regulator ρ_U for the restriction of $\mathcal{M}$ to $S \times U$. We now claim that the squares in the diagram

$$\begin{tikzcd} \mathrm{R}\Gamma(S \times X,\, \mathcal{M}) \arrow[r, "\rho_{\mathcal{M}}"] \arrow[d] & \mathrm{R}\Gamma(S \times X,\, \nabla\mathcal{M}) \arrow[d] \\ \mathrm{R}\Gamma(S \times U,\, \mathcal{M}) \arrow[r, "\rho_U"] \arrow[d] & \mathrm{R}\Gamma(S \times U,\, \nabla\mathcal{M}) \arrow[d] \\ \mathrm{R}\Gamma(\Lambda \otimes \mathcal{O}_K,\, \mathcal{M}) \arrow[r, "\rho_{\mathcal{M}(\Lambda\otimes\mathcal{O}_K)}"] & \mathrm{R}\Gamma(\Lambda \otimes \mathcal{O}_K,\, \nabla\mathcal{M}) \end{tikzcd} \tag{6.1}$$

are commutative. For the top square it follows from Theorem 1.15.4 and the definition of ρ, ρ_U. For the bottom square we argue as follows.

The higher cohomology of quasi-coherent sheaves on the affine scheme $S \times U$ is zero. As a consequence

$$\begin{aligned} \mathrm{R}\Gamma(S \times U,\, \mathcal{G}_a(\mathcal{M})) &= \Gamma(S \times U,\, \mathcal{G}_a(\mathcal{M})), \\ \mathrm{R}\Gamma(S \times U,\, \mathcal{G}_a(\nabla\mathcal{M})) &= \Gamma(S \times U,\, \mathcal{G}_a(\nabla\mathcal{M})). \end{aligned}$$

The complexes on the right hand side are the associated complexes of $\mathcal{M}$ respectively $\nabla\mathcal{M}$. Denoting $M_0 = \mathcal{M}_0(S \times U)$ and $M_1 = \mathcal{M}_1(S \times U)$ we conclude that ρ_U comes from a morphism of the associated complexes given by the diagram

$$\begin{tikzcd} {[M_0} \arrow[r, "i-j"] \arrow[d, "1-i^{-1}j"'] & {M_1]} \arrow[d, "1"] \\ {[M_0} \arrow[r, "i"] & {M_1].} \end{tikzcd}$$

Therefore the bottom square of (6.1) commutes by definition of artinian regulators for shtukas on $\Lambda \otimes \mathcal{O}_K$ (Definition 5.5.2). $\square$

Lemma 6.4.5. *Let $\mathcal{M}$ be a locally free shtuka on $S \times X$ given by the diagram*

$$\mathcal{M}_0 \underset{j}{\overset{i}{\rightrightarrows}} \mathcal{M}_1.$$

Suppose that $\mathcal{M}(\Lambda/\mathfrak{m}_\Lambda \otimes R)$ is nilpotent. If the artinian regulator is defined for $\mathcal{M}$ then the diagram

$$\begin{tikzcd} \mathrm{R}\Gamma(\mathcal{M}) \arrow[r] \arrow[d, "\rho_{\mathcal{M}}"'] & \mathrm{R}\Gamma(\mathcal{M}_0) \arrow[r, "i-j"] \arrow[d, "1-i^{-1}j"'] & \mathrm{R}\Gamma(\mathcal{M}_1) \arrow[r] \arrow[d, "1"] & {[1]} \\ \mathrm{R}\Gamma(\nabla\mathcal{M}) \arrow[r] & \mathrm{R}\Gamma(\mathcal{M}_0) \arrow[r, "i"] & \mathrm{R}\Gamma(\mathcal{M}_1) \arrow[r] & {[1]} \end{tikzcd}$$

is a morphism of canonical triangles.

Proof. Consider a diagram of sheaf complexes

$$\begin{tikzcd} \mathcal{G}_a(\mathcal{M}) \arrow[r] \arrow[d, "\rho_{\mathcal{M}}"'] & \mathcal{M}_0[0] \arrow[r, "i-j"] \arrow[d, "1-i^{-1}j"'] & \mathcal{M}_1[0] \arrow[r] \arrow[d, "1"] & {[1]} \\ \mathcal{G}_a(\nabla\mathcal{M}) \arrow[r] & \mathcal{M}_0[0] \arrow[r, "i"] & \mathcal{M}_1[0] \arrow[r] & {[1]} \end{tikzcd} \tag{6.2}$$

The rows are the distinguished triangles of the mapping fiber complexes $\mathcal{G}_a(\mathcal{M})$ respectively $\mathcal{G}_a(\nabla\mathcal{M})$. By construction (6.2) is a morphism of distinguished triangles. Now Theorem 1.15.4 shows that applying the sheaf cohomology functor $\mathrm{R}\Gamma(S\times X, -)$ to (6.2) we get exactly the diagram in the statement of this lemma. $\square$

6.5. Trace formula for artinian regulators

We continue working with a finite $\mathbb{F}_q$-algebra Λ which is a local artinian ring. As before $\mathfrak{m}_\Lambda \subset \Lambda$ denotes the maximal ideal and S stands for the spectrum of Λ.

Proposition 6.5.1. *Let $\mathcal{M}$ be a locally free shtuka on $S \times X$. If $\mathcal{M}(\Lambda/\mathfrak{m}_\Lambda \otimes R)$ is nilpotent then the following holds:*

(1) *The natural map $\mathrm{R}\Gamma(\mathcal{M}) \to \mathrm{R}\Gamma(\Lambda \otimes \mathcal{O}_K,\, \mathcal{M})$ is a quasi-isomorphism.*
(2) *The complex $\mathrm{R}\Gamma(\mathcal{M})$ is concentrated in degree 1.*
(3) *$\mathrm{H}^1(\mathcal{M})$ is a free Λ-module of finite rank.*

Proof. (1) Consider the Čech complex

$$\mathrm{R}\check{\Gamma}(\mathcal{M}) = \Big[\mathrm{R}\Gamma(\Lambda \otimes R,\, \mathcal{M}) \oplus \mathrm{R}\Gamma(\Lambda \otimes \mathcal{O}_K,\, \mathcal{M}) \to \mathrm{R}\Gamma(\Lambda \otimes K,\, \mathcal{M})\Big].$$

According to Theorem 4.3.9 there is a natural quasi-isomorphism $\mathrm{R}\Gamma(\mathcal{M}) \xrightarrow{\sim} \mathrm{R}\check{\Gamma}(\mathcal{M})$. Its composition with the natural map $\mathrm{R}\check{\Gamma}(\mathcal{M}) \to \mathrm{R}\Gamma(\Lambda \otimes \mathcal{O}_K,\, \mathcal{M})$ is the pullback map $\mathrm{R}\Gamma(\mathcal{M}) \to \mathrm{R}\Gamma(\Lambda \otimes \mathcal{O}_K,\, \mathcal{M})$. So to prove that this pullback map is a quasi-isomorphism it is enough to demonstrate that

$$\mathrm{R}\Gamma(\Lambda \otimes R,\, \mathcal{M}) = 0, \quad \mathrm{R}\Gamma(\Lambda \otimes K,\, \mathcal{M}) = 0.$$

Now the ring $\Lambda \otimes R$ is noetherian and complete with respect to the ideal $\mathfrak{m}_\Lambda \otimes R$. As the shtuka $\mathcal{M}(\Lambda/\mathfrak{m}_\Lambda \otimes R)$ is nilpotent Proposition 1.10.4 implies that $\mathrm{R}\Gamma(\Lambda \otimes R,\, \mathcal{M}) = 0$. The shtuka $\mathcal{M}(\Lambda/\mathfrak{m}_\Lambda \otimes K)$ is nilpotent since nilpotence is preserved under pullbacks. So the same argument shows that $\mathrm{R}\Gamma(\Lambda \otimes K,\, \mathcal{M}) = 0$. Whence the result. In view of (1) the results (2) and (3) follow from Theorem 5.3.2. □

Let $\mathcal{M}$ be a locally free shtuka on $S \times X$ such that $\mathcal{M}(\Lambda/\mathfrak{m}_\Lambda \otimes R)$ is nilpotent. Our goal is to compare the artinian regulator $\rho_{\mathcal{M}}$ with the ζ-isomorphism of $\mathcal{M}$. This isomorphism is constructed in the following way (see Definition 1.11.2). Suppose that $\mathcal{M}$ is given by a diagram

$$\mathcal{M}_0 \underset{j}{\overset{i}{\rightrightarrows}} \mathcal{M}_1.$$

We make the additional assumption that the Λ-modules $\mathrm{H}^n(\mathcal{M}_0)$, $\mathrm{H}^n(\mathcal{M}_1)$, $\mathrm{H}^n(\mathcal{M})$, $\mathrm{H}^n(\nabla\mathcal{M})$, $n \geqslant 0$, are finitely generated free. The ζ-isomorphism

$$\zeta_{\mathcal{M}} \colon \det_\Lambda \mathrm{R}\Gamma(\mathcal{M}) \xrightarrow{\sim} \det_\Lambda \mathrm{R}\Gamma(\nabla\mathcal{M})$$

is then defined as the composition of the isomorphisms

$$\det_\Lambda \mathrm{R}\Gamma(\mathcal{M}) \xrightarrow{\sim} \det_\Lambda \mathrm{R}\Gamma(\mathcal{M}_0) \otimes_\Lambda \det_\Lambda^{-1} \mathrm{R}\Gamma(\mathcal{M}_1) \xleftarrow{\sim} \det_\Lambda \mathrm{R}\Gamma(\nabla\mathcal{M})$$

induced by the canonical triangles

$$\mathrm{R}\Gamma(\mathcal{M}) \to \mathrm{R}\Gamma(\mathcal{M}_0) \xrightarrow{i-j} \mathrm{R}\Gamma(\mathcal{M}_1) \to [1],$$
$$\mathrm{R}\Gamma(\nabla\mathcal{M}) \to \mathrm{R}\Gamma(\mathcal{M}_0) \xrightarrow{i} \mathrm{R}\Gamma(\mathcal{M}_1) \to [1].$$

The freeness hypothesis on the cohomology is necessary to make the construction of $\zeta_{\mathcal{M}}$ work. The reason is that in the theory of Knudsen-Mumford [17, Corollary 2 after Theorem 2] a distinguished triangle $A \to B \to C \to [1]$ of perfect Λ-module complexes determines a canonical isomorphisms

$$\det_\Lambda A \xrightarrow{\sim} \det_\Lambda B \otimes_\Lambda \det_\Lambda^{-1} C$$

only if the cohomology modules of the complexes A, B and C are themselves perfect. A module over the local artinian ring Λ is perfect if and only if it is free of finite rank while the cohomology modules of a perfect Λ-module complex can be arbitrary finitely generated Λ-modules.

Theorem 6.5.2. *Let $\mathcal{M}$ be a locally free shtuka on $S \times X$ given by a diagram*

$$\mathcal{M}_0 \underset{j}{\overset{i}{\rightrightarrows}} \mathcal{M}_1.$$

Assume that

1. $\mathrm{H}^0(\mathrm{Spec}(\Lambda/\mathfrak{m}_\Lambda) \times X,\, \mathcal{M}_*) = 0$ *for* $* \in \{0,1\}$,
2. $\mathcal{M}(\Lambda/\mathfrak{m}_\Lambda \otimes R)$ *is nilpotent,*
3. *There exists an open ideal* $\mathfrak{a} \subset \mathfrak{m}_K$ *such that* $\mathfrak{a} \cdot \mathrm{H}^1(\Lambda \otimes \mathcal{O}_K, \nabla\mathcal{M}) = 0$ *and* $\mathcal{M}(\Lambda \otimes \mathcal{O}_K/\mathfrak{a}^2)$ *is linear.*

Then the following holds:

(1) *The artinian regulator* $\rho_{\mathcal{M}}$ *is defined for* $\mathcal{M}$.
(2) *The* ζ*-isomorphism* $\zeta_{\mathcal{M}}$ *is defined for* $\mathcal{M}$.
(3) $\zeta_{\mathcal{M}} = L(\mathcal{M}) \cdot \det_\Lambda(\rho_{\mathcal{M}})$.

Proof. (1) According to Proposition 5.5.9 the assumptions (2) and (3) imply that the artinian regulator is defined for $\mathcal{M}(\Lambda \otimes \mathcal{O}_K)$ or equivalently that the endomorphism $i^{-1}j$ of $\mathcal{M}_0(\operatorname{Spec}\Lambda \otimes K)$ preserves the submodule $\mathcal{M}_0(\operatorname{Spec}\Lambda \otimes \mathcal{O}_K)$. Hence the artinian regulator is defined for $\mathcal{M}$.

(2) By assumption (1) the module $\mathrm{H}^0(\operatorname{Spec}(\Lambda/\mathfrak{m}_\Lambda) \times X, \mathcal{M}_0)$ is zero. Hence Lemma 6.1.1 shows that $\mathrm{H}^0(\mathcal{M}_0) = 0$ and $\mathrm{H}^1(\mathcal{M}_0)$ is a free Λ-module of finite rank. The same applies to $\mathcal{M}_1$. Since $\mathcal{M}(\Lambda/\mathfrak{m}_\Lambda \otimes R)$ is nilpotent by assumption (2) Proposition 6.5.1 demonstrates that the cohomology modules of $\mathrm{R}\Gamma(\mathcal{M})$ and $\mathrm{R}\Gamma(\nabla\mathcal{M})$ are also perfect.

(3) Lemma 6.4.5 implies that $\det_\Lambda(\rho_{\mathcal{M}}) = \det_\Lambda\big(1 - i^{-1}j \bigm| \mathrm{R}\Gamma(\mathcal{M}_0)\big) \cdot \zeta_{\mathcal{M}}$. However $\mathrm{H}^0(\mathcal{M}_0) = 0$ and $\mathrm{H}^1(\mathcal{M}_0)$ is a free Λ-module. Therefore

$$\zeta_{\mathcal{M}} = \det_\Lambda\big(1 - i^{-1}j \bigm| \mathrm{H}^1(\mathcal{M}_0)\big) \cdot \det_\Lambda(\rho_{\mathcal{M}}).$$

We thus need to compute $\det_\Lambda\big(1 - i^{-1}j \bigm| \mathrm{H}^1(\mathcal{M}_0)\big)$. Consider a locally free shtuka

$$\mathcal{N} = \Big[\mathcal{M}_0 \overset{1}{\underset{i^{-1}j}{\rightrightarrows}} \mathcal{M}_0\Big].$$

This shtuka has the following properties:

(i) $\mathrm{H}^0(\operatorname{Spec}(\Lambda/\mathfrak{m}_\Lambda) \times X, \mathcal{M}_0) = 0$,
(ii) $\mathcal{N}$ is nilpotent,
(iii) $\mathcal{N}(\Lambda \otimes \mathcal{O}_K/\mathfrak{m}_K)$ is linear.

Indeed (i) is just the assumption (1) and (ii) holds by Lemma 6.4.2. Let us prove that (iii) also holds. We temporarily denote

$$M_0 = \mathcal{M}_0(\operatorname{Spec}\Lambda \otimes \mathcal{O}_K), \quad M_1 = \mathcal{M}_1(\operatorname{Spec}\Lambda \otimes \mathcal{O}_K).$$

According to the assumption (3) we have $\mathfrak{a}M_1 \subset i(M_0)$. Hence $\mathfrak{a}^2 M_1 \subset i(\mathfrak{a}M_0)$. At the same time the assumption (3) implies that $j(M_0) \subset \mathfrak{a}^2 M_1$. As a consequence $j(M_0) \subset i(\mathfrak{a}M_0)$. We conclude that the shtuka $\mathcal{N}(\Lambda \otimes \mathcal{O}_K/\mathfrak{a})$ is linear. However $\mathfrak{a} \subset \mathfrak{m}_K$ by assumption so we get the property (iii).

We now apply Theorem 6.3.1 to $\mathcal{N}$ and conclude that

$$\det_\Lambda\big(1 - i^{-1}j \bigm| \mathrm{H}^1(\mathcal{M}_0)\big) = L(\mathcal{N}).$$

To get the result it remains to observe that $L(\mathcal{N}) = L(\mathcal{M})$ by construction. □

6.6. A lemma on determinants

Our aim for the moment is to prove a technical statement on determinants of finite-dimensional F-vector spaces. In this section we omit the subscript F of det and $\otimes$ in order to simlify the notation.

In the theory of Knudsen-Mumford [17] a distinguished triangle $A \to B \to C \to [1]$ of bounded complexes with perfect cohomology modules determines a natural isomorphism $\det A \xrightarrow{\sim} \det B \otimes \det^{-1} C$ [17, Corollary 2 after Theorem 2]. In particular a short exact sequence of finite-dimensional F-vector spaces gives rise to such a natural isomorphism.

Lemma 6.6.1. *Consider a diagram of finite-dimensional F-vector spaces:*

$$\begin{tikzcd}
 & & B \arrow[r, equal] \arrow[d] & B \arrow[d] & \\
0 \arrow[r] & A \arrow[r, "p"] \arrow[d, equal] & B \oplus A_0 \arrow[r, "q"] \arrow[d] & B \oplus A_1 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & A \arrow[r, "f"] & A_0 \arrow[r, "g"] & A_1 \arrow[r] & 0
\end{tikzcd}$$

Assume the following:

(1) *The rows are short exact sequences.*
(2) *The vertical arrows in the second and third column are natural inclusions respectively projections.*
(3) *The diagram is commutative.*

Then the natural square of determinants

$$\begin{tikzcd}
\det A \arrow[r, "\sim"] \arrow[d, equal] & \det(B \oplus A_0) \otimes \det^{-1}(B \oplus A_1) \arrow[d, "\wr"] \\
\det A \arrow[r, "\sim"] & \det A_0 \otimes \det^{-1} A_1
\end{tikzcd}$$

commutes up to the sign $(-1)^{nm}$ where $n = \dim A$, $m = \dim B$. The right vertical arrow in this square is induced by the natural isomorphism $\det B \otimes \det^{-1} B \xrightarrow{\sim} F$.

Morally this lemma is a special case of [17, Proposition 1 (ii)]. There is, however, a gap between morality and reality which one has to fill with a sound argument.

Proof of Lemma 6.6.1. Let $s\colon A_1 \to A_0$ be a section of the surjection $g\colon A_0 \to A_1$. Pick an element $a_1 \in \det A_1$. It is easy to show that the natural isomorphism $\det A \xrightarrow{\sim} \det A_0 \otimes \det^{-1} A_1$ sends $a \in \det A$ to the element

$$\big(f(a) \wedge s(a_1)\big) \otimes a_1^* \tag{6.3}$$

where $a_1^*\colon \det A_1 \to F$ is the unique linear map such that $a_1^*(a_1) = 1$. This element depends neither on the choice of s nor on the choice of a_1. We would like to obtain a similar explicit formula for the map determined by the short exact sequence

$$0 \to A \xrightarrow{p} B \oplus A_0 \xrightarrow{q} B \oplus A_1 \to 0.$$

The commutativity of the diagram implies that the map q is given by a matrix

$$\begin{pmatrix} 1 & h \\ 0 & g \end{pmatrix}$$

where $h\colon A_0 \to B$ is a certain map. Similarly

$$p = \begin{pmatrix} \delta \\ f \end{pmatrix}$$

where $\delta\colon A \to B$ is a certain map.

Let $s\colon A_1 \to A_0$ be a section of g. A quick computation shows that the map

$$t = \begin{pmatrix} 1 & -hs \\ 0 & s \end{pmatrix} : B \oplus A_1 \to B \oplus A_0$$

is a section of q. Next, fix elements $b \in \det B$ and $a_1 \in \det A_1$. Let $m = \dim B$ and $d_1 = \dim A_1$. By a slight abuse of notation we write $b \wedge a_1 \in \Lambda^{m+d_1}(B \oplus A_1)$ for the element of $\det(B \oplus A_1)$ defined by b, a_1. For every $a \in \det A$ we need to compute

$$p(a) \wedge t(b \wedge a_1).$$

Suppose that

$$a = a^1 \wedge \ldots \wedge a^n, \quad b = b^1 \wedge \ldots \wedge b^m, \quad a_1 = a_1^1 \wedge \ldots \wedge a_1^{d_1}.$$

In this case

$$t(b \wedge a_1) = b_1 \wedge \ldots \wedge b^m \wedge \big(s(a_1^1) - hs(a_1^1)\big) \wedge \ldots \wedge \big(s(a_1^{d_1}) - hs(a_1^{d_1})\big) = b \wedge s(a_1).$$

Furthermore

$$p(a) = \big(\delta(a^1) + f(a^1)\big) \wedge \ldots \wedge \big(\delta(a^n) + f(a^n)\big)$$

so that

$$p(a) \wedge t(b \wedge a_1) = f(a) \wedge b \wedge s(a_1) = (-1)^{nm} b \wedge f(a) \wedge s(a_1).$$

We conclude that the natural isomorphism $\det A \xrightarrow{\sim} \det(B \oplus A_0) \otimes \det^{-1}(B \oplus A_1)$ sends the element $a \in \det A$ to

$$(-1)^{nm} b \otimes \big(f(a) \wedge s(a_1)\big) \otimes b^* \otimes a_1^*$$

where $b^* \colon \det B \to F$ and $a_1^* \colon \det A_1 \to F$ are the unique elements such that $b^*(b) = 1$, $a_1^*(a_1) = 1$. Comparing this formula with (6.3) we get the result. □

6.7. A functoriality statement for ζ-isomorphisms

We work with a coefficient ring F which is a local field. As before we denote $D^\circ = \operatorname{Spec} F$.

Lemma 6.7.1. *Let $\mathcal{M}$ be a shtuka on $D^\circ \times X$. If $\mathcal{M}$ is coherent then the ζ-isomorphism is defined for $\mathcal{M}$.*

Proof. The ring F is regular so the result is a consequence of Proposition 4.8.1. □

Lemma 6.7.2. *Let $0 \to \mathcal{N} \to \mathcal{M} \to \mathcal{Q} \to 0$ be a short exact sequence of coherent shtukas on $D^\circ \times X$. Suppose that $\mathcal{N}$, $\mathcal{M}$ and $\mathcal{Q}$ are given by diagrams*

$$\mathcal{N} = \Big[\mathcal{N}_0 \underset{j}{\overset{i}{\rightrightarrows}} \mathcal{N}_1\Big], \quad \mathcal{M} = \Big[\mathcal{M}_0 \underset{j}{\overset{i}{\rightrightarrows}} \mathcal{M}_1\Big], \quad \mathcal{Q} = \Big[\mathcal{Q}_0 \underset{j}{\overset{i}{\rightrightarrows}} \mathcal{Q}_1\Big].$$

Assume the following:

1. $\mathrm{R\Gamma}(\mathcal{M})$ *and* $\mathrm{R\Gamma}(\nabla\mathcal{M})$ *are concentrated in degree* 1.
2. $\mathrm{H}^0(\mathcal{M}_*) = 0$ *for* $* \in \{0,1\}$.
3. $\mathrm{R\Gamma}(\mathcal{Q}) = 0$ *and* $\mathrm{R\Gamma}(\nabla\mathcal{Q}) = 0$.
4. $\mathrm{H}^1(\mathcal{Q}_*) = 0$ *for* $* \in \{0,1\}$.
5. $\mathcal{Q}$ *is linear.*

Then the following holds:

1. *The natural map* $\mathrm{R\Gamma}(\mathcal{N}) \to \mathrm{R\Gamma}(\mathcal{M})$ *is a quasi-isomorphism.*
2. *The natural map* $\mathrm{R\Gamma}(\nabla\mathcal{N}) \to \mathrm{R\Gamma}(\nabla\mathcal{M})$ *is a quasi-isomorphism.*
3. *The natural square*

$$\begin{tikzcd} \det_F \mathrm{R\Gamma}(\mathcal{N}) \arrow[r, "\sim"] \arrow[d, "\zeta_{\mathcal{N}}"', "\wr"] & \det_F \mathrm{R\Gamma}(\mathcal{M}) \arrow[d, "\wr"', "\zeta_{\mathcal{M}}"] \\ \det_F \mathrm{R\Gamma}(\nabla\mathcal{N}) \arrow[r, "\sim"] & \det_F \mathrm{R\Gamma}(\nabla\mathcal{M}) \end{tikzcd}$$

commutes up to the sign $(-1)^{nm}$ *where*

$$n = \dim_F \mathrm{H}^1(\nabla\mathcal{M}) + \dim_F \mathrm{H}^1(\mathcal{M}),$$
$$m = \dim_F \mathrm{H}^0(\mathcal{Q}_0).$$

Lemma 6.7.2 should certainly hold without the conditions (1), (2) and (4). However to establish it in this generality one should prove a stronger version of Lemma 6.6.1. Unfortunately such a proof appears to be quite messy. We opt not to do it since the present version of Lemma 6.7.2 is all we need to prove the main result of this text, the class number formula.

Proof of Lemma 6.7.2. In the following we drop the subscript F of det and $\otimes$ to improve the legibility. The claims (1) and (2) are immediate consequences of the assumption (3). Let us now prove (3). Assumptions (1) and (2) imply that we have a short exact sequence

$$0 \to \mathrm{H}^1(\mathcal{M}) \to \mathrm{H}^1(\mathcal{M}_0) \xrightarrow{i-j} \mathrm{H}^1(\mathcal{M}_1) \to 0.$$

Assumption (2) also implies that $\mathrm{H}^0(\mathcal{N}_*) = 0$ for $* \in \{0,1\}$. The natural map $\mathrm{R}\Gamma(\mathcal{N}) \to \mathrm{R}\Gamma(\mathcal{M})$ is a quasi-isomorphism due to assumption (3). We thus have a short exact sequence

$$0 \to \mathrm{H}^1(\mathcal{N}) \to \mathrm{H}^1(\mathcal{N}_0) \xrightarrow{i-j} \mathrm{H}^1(\mathcal{N}_1) \to 0.$$

As $\mathrm{H}^0(\mathcal{N}_*) = 0$ the assumption (4) implies that the cohomology sequence of the short exact sequence $0 \to \mathcal{N}_* \to \mathcal{M}_* \to \mathcal{Q}_* \to 0$ has the form

$$0 \to \mathrm{H}^0(\mathcal{Q}_*) \xrightarrow{\delta} \mathrm{H}^1(\mathcal{N}_*) \to \mathrm{H}^1(\mathcal{M}_*) \to 0.$$

Altogether we have a commutative diagram

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
 & 0 \arrow[r] \arrow[d] & \mathrm{H}^0(\mathcal{Q}_0) \arrow[r, "i", "\sim"'] \arrow[d, "\delta"'] & \mathrm{H}^0(\mathcal{Q}_1) \arrow[r] \arrow[d, "\delta"] & 0 \\
0 \arrow[r] & \mathrm{H}^1(\mathcal{N}) \arrow[r] \arrow[d, "\wr"] & \mathrm{H}^1(\mathcal{N}_0) \arrow[r, "i-j"] \arrow[d] & \mathrm{H}^1(\mathcal{N}_1) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathrm{H}^1(\mathcal{M}) \arrow[r] \arrow[d] & \mathrm{H}^1(\mathcal{M}_0) \arrow[r, "i-j"] \arrow[d] & \mathrm{H}^1(\mathcal{M}_1) \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

The arrow $\mathrm{H}^0(\mathcal{Q}_0) \to \mathrm{H}^0(\mathcal{Q}_1)$ is labelled i since the shtuka $\mathcal{Q}$ is linear by assumption (5). Let us denote $B = \mathrm{H}^0(\mathcal{Q}_0)$. Taking splittings of the maps δ we replace the diagram above with an isomorphic diagram

$$\begin{tikzcd}
 & & B \arrow[r, equal] \arrow[d] & B \arrow[d] & \\
0 \arrow[r] & \mathrm{H}^1(\mathcal{N}) \arrow[r] \arrow[d, "\wr"] & B \oplus \mathrm{H}^1(\mathcal{M}_0) \arrow[r, "i-j"] \arrow[d] & B \oplus \mathrm{H}^1(\mathcal{M}_1) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathrm{H}^1(\mathcal{M}) \arrow[r] & \mathrm{H}^1(\mathcal{M}_0) \arrow[r, "i-j"] & \mathrm{H}^1(\mathcal{M}_1) \arrow[r] & 0
\end{tikzcd} \tag{6.4}$$

The same argument applied to the shtukas $\nabla\mathcal{N}$, $\nabla\mathcal{M}$, $\nabla\mathcal{Q}$ shows that we have a commutative diagram

$$\begin{tikzcd}
 & & B \arrow[r, equal] \arrow[d] & B \arrow[d] & \\
0 \arrow[r] & \mathrm{H}^1(\nabla\mathcal{N}) \arrow[r] \arrow[d, "\wr"] & B \oplus \mathrm{H}^1(\mathcal{M}_0) \arrow[r, "i"] \arrow[d] & B \oplus \mathrm{H}^1(\mathcal{M}_1) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathrm{H}^1(\nabla\mathcal{M}) \arrow[r] & \mathrm{H}^1(\mathcal{M}_0) \arrow[r, "i"] & \mathrm{H}^1(\mathcal{M}_1) \arrow[r] & 0
\end{tikzcd} \tag{6.5}$$

Observe that the vertical arrows in the second and third column of (6.5) can be chosen to be the same as the corresponding arrows in (6.4). Indeed they only depend on the chosen splittings of the injections $\mathrm{H}^0(\mathcal{Q}_0) \to \mathrm{H}^1(\mathcal{N}_0)$ and $\mathrm{H}^0(\mathcal{Q}_1) \to \mathrm{H}^1(\mathcal{N}_1)$ and on the map $i\colon \mathrm{H}^0(\mathcal{Q}_0) \to \mathrm{H}^0(\mathcal{Q}_1)$. Here we again use the assumption (5) that $\mathcal{Q}$ is linear.

Applying Lemma 6.6.1 to (6.4) we conclude that the natural square of determinants

$$\begin{tikzcd}
\det \mathrm{H}^1(\mathcal{N}) \arrow[r, "\sim"] \arrow[d, "\wr"'] & \det\big(B \oplus \mathrm{H}^1(\mathcal{M}_0)\big) \otimes \det^{-1}\big(B \oplus \mathrm{H}^1(\mathcal{M}_1)\big) \arrow[d, "\wr"] \\
\det \mathrm{H}^1(\mathcal{M}) \arrow[r, "\sim"] & \det \mathrm{H}^1(\mathcal{M}_0) \otimes \det^{-1} \mathrm{H}^1(\mathcal{M}_1)
\end{tikzcd} \tag{6.6}$$

commutes up to the sign $(-1)^{n_1 m}$ where $n_1 = \dim \mathrm{H}^1(\mathcal{M})$, $m = \dim B$. Lemma 6.6.1 applied to (6.5) shows that the natural square of determinants

$$\begin{tikzcd}
\det \mathrm{H}^1(\nabla\mathcal{N}) \arrow[r, "\sim"] \arrow[d, "\wr"'] & \det\big(B \oplus \mathrm{H}^1(\mathcal{M}_0)\big) \otimes \det^{-1}\big(B \oplus \mathrm{H}^1(\mathcal{M}_1)\big) \arrow[d, "\wr"] \\
\det \mathrm{H}^1(\nabla\mathcal{M}) \arrow[r, "\sim"] & \det \mathrm{H}^1(\mathcal{M}_0) \otimes \det^{-1} \mathrm{H}^1(\mathcal{M}_1)
\end{tikzcd} \tag{6.7}$$

commutes up to $(-1)^{n_2 m}$ with $n_2 = \dim \mathrm{H}^1(\nabla\mathcal{M})$.

Now the composition of the bottom horizontal isomorphisms in (6.6) and (6.7) is the ζ-isomorphism of $\mathcal{M}$ by definition while the composition of the top horizontal isomorphisms is the ζ-isomorphism of $\mathcal{N}$ by construction of the diagrams (6.4) and (6.5). As the right vertical isomorphisms in (6.6) and (6.7) are the same we get the result. $\square$

6.8. Elliptic shtukas

Starting from this section we focus on the coefficient ring $\mathcal{O}_F$ which is the ring of integers of a local field F. As before we denote $D = \operatorname{Spec} \mathcal{O}_F$ and $D^\circ = \operatorname{Spec} F$.

Definition 6.8.1. Let $\mathcal{M}$ be a shtuka on $D \times X$ and let $\mathfrak{e} \subset \mathcal{O}_K$ be an open ideal. We say that $\mathcal{M}$ is an *elliptic shtuka of ramification ideal* $\mathfrak{e}$ if the following holds:

(1) $\mathcal{M}$ is locally free.
(2) $\mathcal{M}(\mathcal{O}_F/\mathfrak{m}_F \otimes R)$ is nilpotent.
(3) $\mathcal{M}(\mathcal{O}_F \mathbin{\widehat{\otimes}} \mathcal{O}_K)$ is an elliptic shtuka of ramification ideal $\mathfrak{e}$ in the sense of Definition 5.6.1.

We fix the ramification ideal $\mathfrak{e}$ throughout the rest of the chapter. In the following we speak simply of elliptic shtukas rather than elliptic shtukas of ramification ideal $\mathfrak{e}$.

Lemma 6.8.2. *If a shtuka $\mathcal{M}$ is elliptic then $\nabla\mathcal{M}$ is elliptic.*

Proof. Follows immediately from Proposition 5.6.2. □

Proposition 6.8.3. *For every elliptic shtuka $\mathcal{M}$ on $D \times X$ the following holds:*

(1) *The natural map* $\mathrm{R\Gamma}(\mathcal{M}) \to \mathrm{R\Gamma}(\mathcal{O}_F \widehat{\otimes} \mathcal{O}_K,\, \mathcal{M})$ *is a quasi-isomorphism.*
(2) $\mathrm{R\Gamma}(\mathcal{M})$ *is concentrated in degree* 1.
(3) $\mathrm{H}^1(\mathcal{M})$ *is a free* $\mathcal{O}_F$*-module of finite rank.*

Proof. Indeed $\mathcal{M}(\mathcal{O}_F/\mathfrak{m}_F \otimes R)$ is nilpotent so Theorem 4.6.3 shows that the natural map $\mathrm{R\Gamma}(\mathcal{M}) \to \mathrm{R\Gamma}(\mathcal{O}_F \widehat{\otimes} \mathcal{O}_K,\, \mathcal{M})$ is a quasi-isomorphism. Now $\mathcal{M}(\mathcal{O}_F \widehat{\otimes} \mathcal{O}_K)$ is an elliptic shtuka of ramification ideal $\mathfrak{e}$ by definition. So Theorem 5.6.3 shows that the complex $\mathrm{R\Gamma}(\mathcal{O}_F \widehat{\otimes} \mathcal{O}_K,\, \mathcal{M})$ is concentrated in degree 1 and $\mathrm{H}^1(\mathcal{O}_F \widehat{\otimes} \mathcal{O}_K,\, \mathcal{M})$ is a free $\mathcal{O}_F$-module of finite rank. □

As in the case of elliptic shtukas on $\mathcal{O}_F\widehat{\otimes}\mathcal{O}_K$ we will need a twisting construction for elliptic shtukas on $D \times X$.

Definition 6.8.4. Let $\mathcal{M}$ be a quasi-coherent shtuka on $D \times X$ given by a diagram

$$\mathcal{M}_0 \overset{i}{\underset{j}{\rightrightarrows}} \mathcal{M}_1.$$

We define the shtuka $\mathfrak{e}\mathcal{M}$ by the diagram

$$\mathcal{I}\mathcal{M}_0 \overset{i}{\underset{j}{\rightrightarrows}} \mathcal{I}\mathcal{M}_1$$

where $\mathcal{I} \subset \mathcal{O}_{S \times X}$ is the unique ideal sheaf such that

$$\mathcal{I}(\operatorname{Spec} \mathcal{O}_F \otimes R) = \mathcal{O}_F \otimes R, \quad \mathcal{I}(\operatorname{Spec} \mathcal{O}_F \otimes \mathcal{O}_K) = \mathcal{O}_F \otimes \mathfrak{e}.$$

The shtuka $\mathfrak{e}\mathcal{M}$ will be called the *twist* of $\mathcal{M}$.

Let $n \geqslant 1$. By construction we have a natural embedding $\mathfrak{e}^n\mathcal{M} \hookrightarrow \mathcal{M}$. We denote $\mathcal{M}/\mathfrak{e}^n$ the quotient $\mathcal{M}/\mathfrak{e}^n\mathcal{M}$. Observe that $\mathcal{M}/\mathfrak{e}^n$ coincides with the restriction of $\mathcal{M}$ to the closed affine subscheme $\operatorname{Spec}(\mathcal{O}_F \otimes \mathcal{O}_K/\mathfrak{e}^n)$ of $D \times X$.

Lemma 6.8.5. *Let $\mathcal{M}$ be a shtuka over $D \times X$. If $\mathcal{M}$ is elliptic then $\mathfrak{e}\mathcal{M}$ is elliptic.*

Proof. The shtuka $\mathfrak{e}\mathcal{M}$ is locally free since the ideal sheaf $\mathcal{I}$ above is invertible. Furthermore $\mathcal{M}(\mathcal{O}_F \otimes R) = (\mathfrak{e}\mathcal{M})(\mathcal{O}_F \otimes R)$ so that $(\mathfrak{e}\mathcal{M})(\mathcal{O}_F/\mathfrak{m}_F \otimes R)$ is nilpotent. Finally $(\mathfrak{e}\mathcal{M})(\mathcal{O}_F \widehat{\otimes} \mathcal{O}_K)$ is the twist of $\mathcal{M}(\mathcal{O}_F \widehat{\otimes} \mathcal{O}_K)$ by $\mathfrak{e}$ in the sense of Definition 5.7.1. Hence Proposition 5.7.3 shows that $(\mathfrak{e}\mathcal{M})(\mathcal{O}_F \widehat{\otimes} \mathcal{O}_K)$ is an elliptic shtuka in the sense of Definition 5.6.1. □

Definition 6.8.6. An $\mathcal{O}_F$-linear natural transformation

$$\mathrm{R\Gamma}(\mathcal{M}) \xrightarrow{\ \rho\ } \mathrm{R\Gamma}(\nabla\mathcal{M})$$

of functors on the category of elliptic shtukas is called a *regulator* if for every $\mathcal{M}$ such that $\mathcal{M}/\mathfrak{e}^{2n}$ is linear the diagram

$$\begin{tikzcd}
\mathrm{R\Gamma}(\mathcal{M}) \arrow[r, "\rho"] \arrow[d] & \mathrm{R\Gamma}(\nabla\mathcal{M}) \arrow[d] \\
\mathrm{R\Gamma}(\mathcal{M}/\mathfrak{e}^n) \arrow[r, "1"] & \mathrm{R\Gamma}(\nabla\mathcal{M}/\mathfrak{e}^n)
\end{tikzcd}$$

is commutative.

Theorem 6.8.7. *There exists a unique regulator. Moreover:*

(1) *The regulator is a quasi-isomorphism.*

(2) *For every $\mathcal{M}$ the square*

$$\begin{tikzcd}
\mathrm{R\Gamma}(\mathcal{M}) \arrow[r, "\rho"] \arrow[d] & \mathrm{R\Gamma}(\nabla\mathcal{M}) \arrow[d] \\
\mathrm{R\Gamma}(\mathcal{O}_F \mathbin{\widehat{\otimes}} \mathcal{O}_K,\, \mathcal{M}) \arrow[r, "\rho|_{\mathcal{O}_F \widehat{\otimes} \mathcal{O}_K}"] & \mathrm{R\Gamma}(\mathcal{O}_F \mathbin{\widehat{\otimes}} \mathcal{O}_K,\, \nabla\mathcal{M})
\end{tikzcd} \tag{6.8}$$

is commutative. Here the bottom arrow is the regulator of the elliptic shtuka $\mathcal{M}(\mathcal{O}_F \mathbin{\widehat{\otimes}} \mathcal{O}_K)$.

Proof. Proposition 6.8.3 shows that the vertical arrows in the diagram (6.8) are quasi-isomorphisms. This fact has two consequences. First we can define a natural transformation $\rho\colon \mathrm{R\Gamma}(\mathcal{M}) \to \mathrm{R\Gamma}(\nabla\mathcal{M})$ using (6.8). The definition of a regulator for shtukas on $\mathcal{O}_F \mathbin{\widehat{\otimes}} \mathcal{O}_K$ then implies that ρ is indeed a regulator in the sense of Definition 6.8.6.

Second, any regulator $\rho\colon \mathrm{R\Gamma}(\mathcal{M}) \to \mathrm{R\Gamma}(\nabla\mathcal{M})$ induces an $\mathcal{O}_F$-linear morphism $\mathrm{R\Gamma}(\mathcal{O}_F \mathbin{\widehat{\otimes}} \mathcal{O}_K,\, \mathcal{M}) \to \mathrm{R\Gamma}(\mathcal{O}_F \mathbin{\widehat{\otimes}} \mathcal{O}_K,\, \nabla\mathcal{M})$. Let us denote it ρ'. Observe that $\mathfrak{e}\mathcal{M}(\mathcal{O}_F \mathbin{\widehat{\otimes}} \mathcal{O}_K)$ is the twist of $\mathcal{M}(\mathcal{O}_F \mathbin{\widehat{\otimes}} \mathcal{O}_K)$ by $\mathfrak{e}$ in the sense of Definition 5.7.1. So Proposition 5.7.6 implies that $(\mathfrak{e}^{2n}\mathcal{M})/\mathfrak{e}^{2n}$ is linear for all $n \geqslant 0$. As a consequence the diagram

$$\begin{tikzcd}
\mathrm{R\Gamma}(\mathcal{O}_F \mathbin{\widehat{\otimes}} \mathcal{O}_K,\, \mathfrak{e}^{2n}\mathcal{M}) \arrow[r, "\rho'"] \arrow[d] & \mathrm{R\Gamma}(\mathcal{O}_F \mathbin{\widehat{\otimes}} \mathcal{O}_K,\, \mathfrak{e}^{2n}\,\nabla\mathcal{M}) \arrow[d] \\
\mathrm{R\Gamma}(\mathcal{O}_F \mathbin{\widehat{\otimes}} \mathcal{O}_K,\, (\mathfrak{e}^{2n}\mathcal{M})/\mathfrak{e}^{n}) \arrow[r, "1"] & \mathrm{R\Gamma}(\mathcal{O}_F \mathbin{\widehat{\otimes}} \mathcal{O}_K,\, (\mathfrak{e}^{2n}\,\nabla\mathcal{M})/\mathfrak{e}^{n})
\end{tikzcd}$$

is commutative for every $n \geqslant 0$. Theorem 5.14.4 now implies that ρ' coincides with the regulator of the elliptic shtuka $\mathcal{M}(\mathcal{O}_F \mathbin{\widehat{\otimes}} \mathcal{O}_K)$ so that we get the property (2). Now the unicity and (1) follow by Theorem 5.14.3. □

Definition 6.8.8. Let $\mathfrak{b} \subset \mathcal{O}_F$ be a nonzero ideal.

(1) For a sheaf of modules $\mathcal{E}$ on $D \times X$ we denote $\mathcal{E}/\mathfrak{b}$ the restriction of $\mathcal{E}$ to the closed subscheme $\mathrm{Spec}(\mathcal{O}_F/\mathfrak{b}) \times X$.
(2) For a shtuka $\mathcal{M}$ on $D \times X$ we denote $\mathcal{M}/\mathfrak{b}$ its restriction to the closed subscheme $\mathrm{Spec}(\mathcal{O}_F/\mathfrak{b}) \times X$.

Lemma 6.8.9. *Let $\mathfrak{b} \subset \mathcal{O}_F$ be a nonzero ideal. If $\mathcal{M}$ is a locally free shtuka on $D \times X$ then the pullback map $\mathrm{R\Gamma}(\mathcal{M}) \otimes^{\mathbf{L}}_{\mathcal{O}_F} \mathcal{O}_F/\mathfrak{b} \to \mathrm{R\Gamma}(\mathcal{M}/\mathfrak{b})$ is a quasi-isomorphism.*

Proof. It is an immediate consequence of Proposition 4.7.3. □

Lemma 6.8.10. *Let $\mathfrak{b} = \mathfrak{m}_F^d \subset \mathcal{O}_F$ be a nonzero ideal. Let $\mathcal{M}$ be an elliptic shtuka. If $\mathcal{M}(\mathcal{O}_F \otimes \mathcal{O}_K/\mathfrak{e}^{2d})$ is linear then the following holds:*

(1) *The artinian regulator $\rho_{\mathcal{M}/\mathfrak{b}}$ is defined for $\mathcal{M}/\mathfrak{b}$.*
(2) *The natural diagram*

$$\begin{tikzcd}
\mathrm{R\Gamma}(\mathcal{M}) \otimes^{\mathbf{L}}_{\mathcal{O}_F} \mathcal{O}_F/\mathfrak{b} \arrow[r, "\rho_{\mathcal{M}}"] \arrow[d, "\wr"'] & \mathrm{R\Gamma}(\nabla\mathcal{M}) \otimes^{\mathbf{L}}_{\mathcal{O}_F} \mathcal{O}_F/\mathfrak{b} \arrow[d, "\wr"] \\
\mathrm{R\Gamma}(\mathcal{M}/\mathfrak{b}) \arrow[r, "\rho_{\mathcal{M}/\mathfrak{b}}"] & \mathrm{R\Gamma}(\nabla\mathcal{M}/\mathfrak{b})
\end{tikzcd}$$

is commutative.

Proof. (1) Suppose that $\mathcal{M}$ is given by a diagram

$$\mathcal{M}_0 \underset{j}{\overset{i}{\rightrightarrows}} \mathcal{M}_1.$$

As $\mathcal{M}(\mathcal{O}_F \otimes \mathcal{O}_K/\mathfrak{e}^{2d})$ is linear Theorem 5.14.5 shows that the artinian regulator is defined for $\mathcal{M}(\mathcal{O}_F/\mathfrak{b} \otimes \mathcal{O}_K)$. In other words the endomorphism $i^{-1}j$ of $\mathcal{M}_0(\operatorname{Spec} \mathcal{O}_F/\mathfrak{b} \otimes K)$ preserves the submodule $\mathcal{M}_0(\operatorname{Spec} \mathcal{O}_F/\mathfrak{b} \otimes \mathcal{O}_K)$. So the artinian regulator is defined for $\mathcal{M}/\mathfrak{b}$.

(2) We need to prove that the square

$$\begin{tikzcd}
\mathrm{R}\Gamma(\mathcal{M}) \arrow[r, "\rho_{\mathcal{M}}"] \arrow[d] & \mathrm{R}\Gamma(\nabla\mathcal{M}) \arrow[d] \\
\mathrm{R}\Gamma(\mathcal{M}/\mathfrak{b}) \arrow[r, "\rho_{\mathcal{M}/\mathfrak{b}}"] & \mathrm{R}\Gamma(\nabla\mathcal{M}/\mathfrak{b})
\end{tikzcd}$$

is commutative. Theorem 6.8.7 in combination with Proposition 6.8.3 identifies the top arrow with the regulator of $\mathcal{M}(\mathcal{O}_F \widehat{\otimes} \mathcal{O}_K)$. Similarly Lemma 6.4.4 and Proposition 6.5.1 identify the bottom arrow with the artinian regulator of $\mathcal{M}(\mathcal{O}_F/\mathfrak{b} \otimes \mathcal{O}_K)$. So the result is a consequence of Theorem 5.14.5. □

Our goal is to compare the determinant of the regulator with the ζ-isomorphism.

Lemma 6.8.11. *If $\mathcal{M}$ is a locally free shtuka on $D \times X$ then the ζ-isomorphism is defined for $\mathcal{M}$.*

Proof. Follows from Proposition 4.8.1 since $\mathcal{O}_F$ is regular and $\mathcal{M}$ is coherent. □

Lemma 6.8.12. *Let $\mathfrak{b} \subset \mathcal{O}_F$ be a nonzero ideal. Let $\mathcal{M}$ be a locally free shtuka on $D \times X$ given by a diagram $\mathcal{M}_0 \rightrightarrows \mathcal{M}_1$. Assume that*

(1) *$\mathcal{M}(\mathcal{O}_F/\mathfrak{m}_F \otimes R)$ is nilpotent,*
(2) *$\mathrm{H}^0(\mathcal{M}_0/\mathfrak{m}_F) = 0$ and $\mathrm{H}^0(\mathcal{M}_1/\mathfrak{m}_F) = 0$.*

Then the following holds:

(1) *The ζ-isomorphism is defined for $\mathcal{M}/\mathfrak{b}$.*
(2) *The natural diagram*

$$\begin{tikzcd}
\big(\det_{\mathcal{O}_F} \mathrm{R}\Gamma(\mathcal{M})\big)/\mathfrak{b} \arrow[r, "\zeta_{\mathcal{M}}"] \arrow[d] & \big(\det_{\mathcal{O}_F} \mathrm{R}\Gamma(\nabla\mathcal{M})\big)/\mathfrak{b} \arrow[d] \\
\det_{\mathcal{O}_F/\mathfrak{b}} \mathrm{R}\Gamma(\mathcal{M}/\mathfrak{b}) \arrow[r, "\zeta_{\mathcal{M}/\mathfrak{b}}"] & \det_{\mathcal{O}_F/\mathfrak{b}} \mathrm{R}\Gamma(\nabla\mathcal{M}/\mathfrak{b})
\end{tikzcd}$$

is commutative.

Proof. (1) Let $* \in \{0,1\}$. Lemma 6.1.1 shows that $\mathrm{H}^0(\mathcal{M}_*) = 0$ and $\mathrm{H}^1(\mathcal{M}_*)$ is a free $\mathcal{O}_F$-module of finite rank. The base change theorem [07VK] then implies that $\mathrm{H}^0(\mathcal{M}_*/\mathfrak{b}) = 0$ and $\mathrm{H}^1(\mathcal{M}_*/\mathfrak{b})$ is a free $\mathcal{O}_F/\mathfrak{b}$-module of finite rank. As $\mathcal{M}(\mathcal{O}_F/\mathfrak{m}_F \otimes R)$ is nilpotent Proposition 6.5.1 implies that $\mathrm{H}^0(\mathcal{M}/\mathfrak{b}) = 0$ and $\mathrm{H}^1(\mathcal{M}/\mathfrak{b})$ is a free $\mathcal{O}_F/\mathfrak{b}$-module of finite rank. Hence the ζ-isomorphism is defined for $\mathcal{M}/\mathfrak{b}$. (2) Follows immediately from Proposition 4.8.2. □

Lemma 6.8.13. *Let $\mathcal{M}$ be an elliptic shtuka.*

(1) *$F \otimes_{\mathcal{O}_F} \mathrm{R}\Gamma(\mathcal{M}/\mathfrak{e}) = 0$.*
(2) *The natural map $F \otimes_{\mathcal{O}_F} \mathrm{R}\Gamma(\mathfrak{e}\mathcal{M}) \to F \otimes_{\mathcal{O}_F} \mathrm{R}\Gamma(\mathcal{M})$ is a quasi-isomorphism.*

Proof. In view of Proposition 6.8.3 the result follows from Theorem 5.8.5. □

Lemma 6.8.14. *Let $\mathcal{M}$ be an elliptic shtuka given by a diagram $\mathcal{M}_0 \rightrightarrows \mathcal{M}_1$. If $\mathrm{H}^0(\mathcal{M}_0/\mathfrak{m}_F) = 0$ and $\mathrm{H}^0(\mathcal{M}_1/\mathfrak{m}_F) = 0$ then then the natural square*

$$\begin{tikzcd} F \otimes_{\mathcal{O}_F} \det_{\mathcal{O}_F} \mathrm{R}\Gamma(\mathfrak{e}\mathcal{M}) \arrow[r, "\sim"] \arrow[d, "\zeta_{\mathfrak{e}\mathcal{M}}"', "\wr"] & F \otimes_{\mathcal{O}_F} \det_{\mathcal{O}_F} \mathrm{R}\Gamma(\mathcal{M}) \arrow[d, "\wr"', "\zeta_{\mathcal{M}}"] \\ F \otimes_{\mathcal{O}_F} \det_{\mathcal{O}_F} \mathrm{R}\Gamma(\nabla\mathfrak{e}\mathcal{M}) \arrow[r, "\sim"] & F \otimes_{\mathcal{O}_F} \det_{\mathcal{O}_F} \mathrm{R}\Gamma(\nabla\mathcal{M}) \end{tikzcd}$$

is commutative.

Proof. Let $\mathcal{M}^\circ$, $\mathfrak{e}\mathcal{M}^\circ$, $\mathcal{M}^\circ/\mathfrak{e}$ denote the pullbacks of the respective shtukas to $D^\circ \times X$. By construction we have a short exact sequence $0 \to \mathfrak{e}\mathcal{M}^\circ \to \mathcal{M}^\circ \to \mathcal{M}^\circ/\mathfrak{e} \to 0$. We would like to apply Lemma 6.7.2 to this short exact sequence. To do it we should verify that the following conditions are met:

(1) $\mathrm{R}\Gamma(\mathcal{M}^\circ)$ and $\mathrm{R}\Gamma(\nabla\mathcal{M}^\circ)$ are concentrated in degree 1.
(2) The sheaves of $\mathcal{M}^\circ$ have cohomology concentrated in degree 1.
(3) $\mathrm{R}\Gamma(\mathcal{M}/\mathfrak{e}) = 0$ and $\mathrm{R}\Gamma(\nabla\mathcal{M}/\mathfrak{e}) = 0$.
(4) The sheaves of $\mathcal{M}^\circ/\mathfrak{e}$ have cohomology concentrated in degree 0.
(5) $\mathcal{M}^\circ/\mathfrak{e}$ is linear.

(1) follows by Proposition 6.8.3, Lemma 6.1.1 implies (2) and (3) holds by Lemma 6.8.13. By construction $\mathcal{M}^\circ/\mathfrak{e}$ is supported at a closed affine subscheme $\operatorname{Spec} F \otimes \mathcal{O}_K/\mathfrak{e}$ of $D^\circ \times X$. Thus the condition (4) is satisfied. Finally the condition (5) follows since $\mathcal{M}$ is elliptic. Hence Lemma 6.7.2 demonstrates that the natural square

$$\begin{tikzcd} \det_F \mathrm{R}\Gamma(\mathfrak{e}\mathcal{M}^\circ) \arrow[r, "\sim"] \arrow[d, "\zeta_{\mathfrak{e}\mathcal{M}^\circ}"', "\wr"] & \det_F \mathrm{R}\Gamma(\mathcal{M}^\circ) \arrow[d, "\wr"', "\zeta_{\mathcal{M}^\circ}"] \\ \det_F \mathrm{R}\Gamma(\nabla\mathfrak{e}\mathcal{M}^\circ) \arrow[r, "\sim"] & \det_F \mathrm{R}\Gamma(\nabla\mathcal{M}^\circ) \end{tikzcd}$$

commutes up to the sign $(-1)^{nm}$ where

$$n = \dim_F \mathrm{H}^1(\mathcal{M}^\circ) + \dim_F \mathrm{H}^1(\nabla\mathcal{M}^\circ).$$

However $\dim_F \mathrm{H}^1(\mathcal{M}^\circ) = \dim_F \mathrm{H}^1(\nabla\mathcal{M}^\circ)$ since the shtuka $\mathcal{M}$ is elliptic and so admits a regulator isomorphism $\rho\colon \mathrm{H}^1(\mathcal{M}) \xrightarrow{\sim} \mathrm{H}^1(\nabla\mathcal{M})$. Thus the square above is in fact commutative. Proposition 4.8.2 now implies the result. $\square$

6.9. Euler products for $\mathcal{O}_F$

In this section we define the Euler products for shtukas over $\mathcal{O}_F \otimes R$.

Lemma 6.9.1. *Let k be a finite field extension of $\mathbb{F}_q$. Let $\mathcal{M}$ be a locally free shtuka on $\mathcal{O}_F \otimes k$ given by a diagram*

$$M_0 \underset{j}{\overset{i}{\rightrightarrows}} M_1.$$

If $\mathcal{M}(\mathcal{O}_F/\mathfrak{m}_F \otimes k)$ is nilpotent then

(1) *$i\colon M_0 \to M_1$ is an isomorphism,*
(2) *$(1 - i^{-1}j)\colon M_0 \to M_0$ is an $\mathcal{O}_F$-linear isomorphism.*

Proof. Indeed the ring $\mathcal{O}_F \otimes k$ is noetherian and complete with respect to a τ-invariant ideal $\mathfrak{m}_F \otimes k$. So Proposition 1.10.4 implies that $\mathrm{R}\Gamma(\nabla\mathcal{M}) = 0$ and $\mathrm{R}\Gamma(\mathcal{M}) = 0$.

Definition 6.9.2. Let k be a finite field extension of $\mathbb{F}_q$. Let a locally free shtuka $\mathcal{M}$ on $\mathcal{O}_F \otimes k$ be given by a diagram

$$M_0 \overset{i}{\underset{j}{\rightrightarrows}} M_1.$$

Assuming that $\mathcal{M}(\mathcal{O}_F/\mathfrak{m}_F \otimes k)$ is nilpotent we define

$$L(\mathcal{M}) = \det_{\mathcal{O}_F}(1 - i^{-1}j \mid M_0) \in \mathcal{O}_F^\times.$$

Lemma 6.9.3. *Let $n \geqslant 1$ be an integer. If $\mathcal{M}$ is a locally free shtuka on $\mathcal{O}_F \otimes R$ such that $\mathcal{M}(\mathcal{O}_F/\mathfrak{m}_F \otimes R)$ is nilpotent then for almost all maximal ideals $\mathfrak{m} \subset R$ we have $L\big(\mathcal{M}(\mathcal{O}_F \otimes R/\mathfrak{m})\big) \equiv 1 \pmod{\mathfrak{m}_F^n}$*

Proof. Let $\mathfrak{m} \subset R$ be a maximal ideal. By construction

$$L\big(\mathcal{M}(\mathcal{O}_F \otimes R/\mathfrak{m})\big) \equiv L\big(\mathcal{M}(\mathcal{O}_F/\mathfrak{m}_F^n \otimes R/\mathfrak{m})\big) \pmod{\mathfrak{m}_F^n}$$

Applying Lemma 6.2.4 to $\mathcal{M}(\mathcal{O}_F/\mathfrak{m}_F^n \otimes R)$ we get the result. □

Definition 6.9.4. Let $\mathcal{M}$ be a locally free shtuka on $\mathcal{O}_F \otimes R$. Assuming that $\mathcal{M}(\mathcal{O}_F/\mathfrak{m}_F \otimes R)$ is nilpotent we define

$$L(\mathcal{M}) = \prod_{\mathfrak{m}} L(\mathcal{M}(\mathcal{O}_F \otimes R/\mathfrak{m}))^{-1} \in \mathcal{O}_F^\times$$

where $\mathfrak{m} \subset R$ ranges over the maximal ideals. This product converges by Lemma 6.9.3.

Given a locally free shtuka $\mathcal{M}$ on $D \times X$ such that $\mathcal{M}(\mathcal{O}_F/\mathfrak{m}_F \otimes R)$ is nilpotent we write $L(\mathcal{M})$ instead of $L(\mathcal{M}(\mathcal{O}_F \otimes R))$ to simplify the notation. It is important to note that the closed points of X in the complement of $\operatorname{Spec} R$ are not taken into account.

Lemma 6.9.5. *If $\mathcal{M}$ is a locally free shtuka on $\mathcal{O}_F \otimes R$ such that $\mathcal{M}(\mathcal{O}_F/\mathfrak{m}_F \otimes R)$ is nilpotent then for every $n \geqslant 1$ we have $L(\mathcal{M}) \equiv L\big(\mathcal{M}(\mathcal{O}_F/\mathfrak{m}_F^n \otimes R)\big) \pmod{\mathfrak{m}_F^n}$.* □

Proposition 6.9.6. *If $\mathcal{M}$ is a locally free shtuka on $\mathcal{O}_F \otimes R$ such that $\mathcal{M}(\mathcal{O}_F/\mathfrak{m}_F \otimes R)$ is nilpotent then $L(\mathcal{M}) \equiv 1 \pmod{\mathfrak{m}_F}$.*

Proof. Indeed $L(\mathcal{M}) \equiv L(\mathcal{M}(\mathcal{O}_F/\mathfrak{m}_F \otimes R)) \pmod{\mathfrak{m}_F}$. Since $\mathcal{O}_F/\mathfrak{m}_F$ is a field Lemma 6.2.6 implies that $L(\mathcal{M}(\mathcal{O}_F/\mathfrak{m}_F \otimes R)) = 1$. □

6.10. Trace formula

Lemma 6.10.1. *Let $d \geqslant 1$. Let $\mathcal{M}$ be an elliptic shtuka given by a diagram*

$$\mathcal{M}_0 \rightrightarrows \mathcal{M}_1.$$

Assume that the following holds:

(1) $\mathrm{H}^0(\mathcal{M}_0/\mathfrak{m}_F) = 0$ *and* $\mathrm{H}^0(\mathcal{M}_1/\mathfrak{m}_F) = 0$,
(2) $\mathcal{M}/\mathfrak{e}^{2d}$ *is linear.*

If the ramification ideal $\mathfrak{e}$ is contained in $\mathfrak{m}_K$ then $\zeta_{\mathcal{M}} \equiv L(\mathcal{M}) \cdot \det_{\mathcal{O}_F}(\rho_{\mathcal{M}}) \pmod{\mathfrak{m}_F^d}$.

Proof. Set $\Lambda = \mathcal{O}_F/\mathfrak{m}_F^d$. Let $\mathcal{N} = \mathcal{M}/\mathfrak{m}_F^d$ and let

$$\mathcal{N}_0 \overset{i}{\underset{j}{\rightrightarrows}} \mathcal{N}_1$$

be the diagram of $\mathcal{N}$. We claim that the shtuka $\mathcal{N}$ has the following properties:

(a) $\mathrm{H}^0(\operatorname{Spec}(\mathcal{O}_F/\mathfrak{m}_F) \times X, \mathcal{N}_0) = 0$ and $\mathrm{H}^0(\operatorname{Spec}(\mathcal{O}_F/\mathfrak{m}_F) \times X, \mathcal{N}_1) = 0$,
(b) $\mathcal{N}(\mathcal{O}_F/\mathfrak{m}_F \otimes R)$ is nilpotent,
(c) $\mathcal{N}(\Lambda \otimes \mathcal{O}_K/\mathfrak{e}^{2d})$ is linear,
(d) $\mathfrak{e}^d \cdot \mathrm{H}^1(\Lambda \otimes \mathcal{O}_K, \nabla\mathcal{N}) = 0$.

The property (a) follows from the assumption (1), the property (b) holds since $\mathcal{M}$ is elliptic and (c) is a consequence of the assumption (2). Finally, one gets (d) by applying Lemma 5.8.1 to the shtuka $\mathcal{M}(\mathcal{O}_F \widehat{\otimes} \mathcal{O}_K)/\mathfrak{m}_F^d = \mathcal{N}(\Lambda \otimes \mathcal{O}_K)$.

Now we apply Theorem 6.5.2 to $\mathcal{N}$ with $\mathfrak{a} = \mathfrak{e}^d$ and conclude that the following is true:

(i) The artinian regulator $\rho_{\mathcal{N}}$ is defined for $\mathcal{N}$.
(ii) The ζ-isomorphism $\zeta_{\mathcal{N}}$ is defined for $\mathcal{N}$.
(iii) $\zeta_{\mathcal{N}} = L(\mathcal{N}) \cdot \det_\Lambda(\rho_{\mathcal{N}})$.

The congruence $L(\mathcal{M}) \equiv L(\mathcal{N}) \pmod{\mathfrak{m}_F^d}$ holds by construction. Moreover $\rho_{\mathcal{N}} \equiv \rho_{\mathcal{M}} \pmod{\mathfrak{m}_F^d}$ by Lemma 6.8.10 and $\zeta_{\mathcal{N}} \equiv \zeta_{\mathcal{M}} \pmod{\mathfrak{m}_F^d}$ by Lemma 6.8.12. So the result follows. ☐

Lemma 6.10.2. *Let $\mathcal{M}$ be an elliptic shtuka given by a diagram $\mathcal{M}_0 \rightrightarrows \mathcal{M}_1$. Suppose that $\mathrm{H}^0(\mathcal{M}_0/\mathfrak{m}_F) = 0$ and $\mathrm{H}^0(\mathcal{M}_1/\mathfrak{m}_F) = 0$. If $\alpha \in \mathcal{O}_F^\times$ is the unique element such that $\zeta_{\mathcal{M}} = \alpha \cdot \det_{\mathcal{O}_F}(\rho_{\mathcal{M}})$ then $\zeta_{\mathfrak{e}\mathcal{M}} = \alpha \cdot \det_{\mathcal{O}_F}(\rho_{\mathfrak{e}\mathcal{M}})$.*

Proof. The square

$$\begin{tikzcd} F \otimes_{\mathcal{O}_F} \det_{\mathcal{O}_F} \mathrm{R}\Gamma(\mathfrak{e}\mathcal{M}) \arrow[r, "\sim"] \arrow[d, "\det_{\mathcal{O}_F}(\rho_{\mathfrak{e}\mathcal{M}})"', "\wr"] & F \otimes_{\mathcal{O}_F} \det_{\mathcal{O}_F} \mathrm{R}\Gamma(\mathcal{M}) \arrow[d, "\det_{\mathcal{O}_F}(\rho_{\mathcal{M}})", "\wr"'] \\ F \otimes_{\mathcal{O}_F} \det_{\mathcal{O}_F} \mathrm{R}\Gamma(\nabla\mathfrak{e}\mathcal{M}) \arrow[r, "\sim"] & F \otimes_{\mathcal{O}_F} \det_{\mathcal{O}_F} \mathrm{R}\Gamma(\nabla\mathcal{M}) \end{tikzcd}$$

is commutative by naturality of ρ. At the same time Lemma 6.8.14 shows that the natural square

$$\begin{tikzcd} F \otimes_{\mathcal{O}_F} \det_{\mathcal{O}_F} \mathrm{R}\Gamma(\mathfrak{e}\mathcal{M}) \arrow[r, "\sim"] \arrow[d, "\zeta_{\mathfrak{e}\mathcal{M}}"', "\wr"] & F \otimes_{\mathcal{O}_F} \det_{\mathcal{O}_F} \mathrm{R}\Gamma(\mathcal{M}) \arrow[d, "\zeta_{\mathcal{M}}", "\wr"'] \\ F \otimes_{\mathcal{O}_F} \det_{\mathcal{O}_F} \mathrm{R}\Gamma(\nabla\mathfrak{e}\mathcal{M}) \arrow[r, "\sim"] & F \otimes_{\mathcal{O}_F} \det_{\mathcal{O}_F} \mathrm{R}\Gamma(\nabla\mathcal{M}) \end{tikzcd}$$

is commutative. So we get the result. ☐

Finally we are ready to prove the trace formula for regulators of elliptic shtukas.

Theorem 6.10.3. *Let $\mathcal{M}$ be an elliptic shtuka given by a diagram $\mathcal{M}_0 \rightrightarrows \mathcal{M}_1$. Suppose that $\mathrm{H}^0(\mathcal{M}_0/\mathfrak{m}_F) = 0$ and $\mathrm{H}^0(\mathcal{M}_1/\mathfrak{m}_F) = 0$. If the ramification ideal $\mathfrak{e}$ is contained in $\mathfrak{m}_K$ then*

$$\zeta_{\mathcal{M}} = L(\mathcal{M}) \cdot \det_{\mathcal{O}_F}(\rho_{\mathcal{M}}).$$

This theorem should certainly hold without the assumption on on the cohomology of sheaves underlying $\mathcal{M}$. The bottleneck which prevents us from establishing this more natural version of the trace formula is Lemma 6.6.1. A more general variant of this lemma is needed. The proof of such a lemma appears to be too messy at the moment. Alas, the determinant theory of [17] is not too user-friendly. Nevertheless Theorem 6.10.3 is still enough to prove the class number formula for Drinfeld modules.

Proof of Theorem 6.10.3. Let $\alpha \in \mathcal{O}_F^\times$ be the unique element such that $\zeta_{\mathcal{M}} = \alpha \cdot \det_{\mathcal{O}_F}(\rho_{\mathcal{M}})$. We will show that $\alpha \equiv L(\mathcal{M}) \pmod{\mathfrak{m}_F^d}$ for every $d \geqslant 1$.

Observe that $L(\mathcal{M}) = L(\mathfrak{e}\mathcal{M})$. Indeed the invariant L depends only on the restriction of a shtuka to $\mathcal{O}_F \otimes R$ and $\mathcal{M}(\mathcal{O}_F \otimes R) = \mathfrak{e}\mathcal{M}(\mathcal{O}_F \otimes R)$ by construction. At the same time Lemma 6.10.2 shows that $\zeta_{\mathfrak{e}\mathcal{M}} = \alpha \cdot \det_{\mathcal{O}_F}(\rho_{\mathfrak{e}\mathcal{M}})$. Furthemore the condition on the cohomology of sheaves underlying $\mathcal{M}$ is preserved under twists by the ideal $\mathfrak{e}$. We are thus free to replace $\mathcal{M}$ by $\mathfrak{e}^n\mathcal{M}$.

Observe that $(\mathfrak{e}\mathcal{M})(\mathcal{O}_F \mathbin{\widehat{\otimes}} \mathcal{O}_K)$ is the twist of $\mathcal{M}(\mathcal{O}_F \mathbin{\widehat{\otimes}} \mathcal{O}_K)$ by $\mathfrak{e}$ in the sense of Definition 5.7.1. So Proposition 5.7.6 implies that $(\mathfrak{e}^{2d}\mathcal{M})/\mathfrak{e}^{2d}$ is linear for all $d \geqslant 0$. Lemma 6.10.1 now implies the result. $\square$

CHAPTER 7

The motive of a Drinfeld module

We recall the notion of a Drinfeld module [8], its motive as introduced by Anderson [1], and a construction of Drinfeld [9] associating a shtuka to a Drinfeld module.

Let C be a smooth projective curve over $\mathbb{F}_q$ and $\infty \in C$ a closed point. Denote $A = \Gamma(C\backslash\{\infty\}, \mathcal{O}_C)$. Let E be a Drinfeld A-module over an $\mathbb{F}_q$-algebra B. The motive M of E is a left $A \otimes B\{\tau\}$-module which is a locally free $A \otimes B$-module. Drinfeld's construction yields a canonical shtuka on $C \times \operatorname{Spec} B$ extending M, a "compactification" of M in the direction of the coefficient curve C. In the subsequent chapters we will combine it with a compactification in the direction of the base. Both compactifications are important to the proof of the class number formula.

We should stress that all the material of this chapter is well-known, with the possible exception of Proposition 7.3.7 and the co-nilpotence property in Theorem 7.6.1. The latter two results are certainly known to experts.

All the nontrivial results of this chapter are due to Drinfeld [8, 9]. A detailed study of motives of Drinfeld modules over arbitrary, not necessarily reduced base rings can be found in the preprint [13] of Urs Hartl.

7.1. Forms of the additive group

In this section we work over a fixed $\mathbb{F}_q$-algebra B. To simplify the exposition *we assume that B is reduced.* The theory which we attempt to present here can be developed without this assumption. However it then becomes more subtle. In applications we will only need the case of reduced B.

We equip B with a τ-ring structure given by the q-th power map. The Frobenius τ and its powers are in a natural way $\mathbb{F}_q$-linear endomorphisms of the additive group scheme $\mathbb{G}_a$ over B.

Lemma 7.1.1. *Let $\operatorname{End}(\mathbb{G}_a)$ be the ring of $\mathbb{F}_q$-linear endomorphisms of $\mathbb{G}_a$. The natural map $B\{\tau\} \to \operatorname{End}(\mathbb{G}_a)$ is an isomorphism.* □

Lemma 7.1.2. *$B\{\tau\}^\times = B^\times$.*

Proof. Let $\varphi \in B\{\tau\}^\times$. If K is a B-algebra which is a field then the image of φ in $K\{\tau\}$ must be an element of $K^\times$. Therefore the constant coefficient of φ is a unit and all other coefficients are nilpotent. Since B is assumed to be reduced we conclude that $\varphi \in B^\times$. □

Recall that an $\mathbb{F}_q$-vector space scheme E is an abelian group scheme equipped with a compatible $\mathbb{F}_q$-multiplication.

Definition 7.1.3. We say that an $\mathbb{F}_q$-vector space scheme E is *a form of* $\mathbb{G}_a$ if it is Zariski-locally isomorphic to $\mathbb{G}_a$.

In the following we fix an $\mathbb{F}_q$-vector space scheme E which is a form of $\mathbb{G}_a$. Our main object of study is the motive of E which we now introduce.

Definition 7.1.4. The *motive of* E is the abelian group $M = \mathrm{Hom}(E, \mathbb{G}_a)$ of $\mathbb{F}_q$-linear group scheme morphisms from E to $\mathbb{G}_a$. The endomorphism ring of $\mathbb{G}_a$ acts on M by composition making it into a left $B\{\tau\}$-module.

Lemma 7.1.5. *The formation of the motive of E commutes with arbitrary base change.*

Proof. For a scheme Y over $X = \operatorname{Spec} B$ set $\mathcal{M}(Y) = \mathrm{Hom}(E_Y, \mathbb{G}_{a,Y})$ where E_Y denotes the pullback of E to Y. Zarski descent for morphisms of schemes implies that $\mathcal{M}$ is a sheaf on the big Zariski site of X. The abelian group $\mathcal{M}(Y)$ carries a natural action of $\Gamma(Y, \mathcal{O}_Y)$ on the left. Together these actions make $\mathcal{M}$ into an $\mathcal{O}_X$-module. The formation of $\mathcal{M}$ is functorial in E.

If $E = \mathbb{G}_a$ then $\mathcal{M}$ is the quasi-coherent sheaf defined by the left B-module $B\{\tau\}$. Since E is Zariski-locally isomorphic to $\mathbb{G}_a$ we conclude that $\mathcal{M}$ is quasi-coherent. Therefore the natural map $S \otimes_B \mathcal{M}(X) \to \mathcal{M}(\operatorname{Spec} S)$ is an isomorphism for every B-algebra S. □

Proposition 7.1.6. *There exists a unique invertible B-submodule $M^0 \subset M$ such that the map*

$$B\{\tau\} \otimes_B M^0 \to M, \quad \varphi \otimes m \mapsto \varphi \cdot m$$

is an isomorphism.

Corollary 7.1.7. *The motive M is a projective left $B\{\tau\}$-module.* □

Definition 7.1.8. We define the *degree filtration* M^* on M in the following way. For $n \geqslant 0$ we let $M^n = B\{\tau\}^n \cdot M^0$ where $B\{\tau\}^n \subset B\{\tau\}$ is the submodule of τ-polynomials of degree at most n. For $n < 0$ we set $M^n = 0$.

Proof of Proposition 7.1.6. First let us prove unicity. If $M^0, N^0 \subset M$ are invertible B-submodules such that the natural maps $B\{\tau\} \otimes_B M^0 \to M$ and $B\{\tau\} \otimes_B N^0 \to M$ are isomorphisms then we get an induced isomorphism $B\{\tau\} \otimes_B M^0 \cong B\{\tau\} \otimes_B N^0$ of left $B\{\tau\}$-modules. Now Lemma 7.1.2 implies that this isomorphism comes from a unique B-linear isomorphism $M^0 \cong N^0$ which is compatible with the inclusions $M^0 \hookrightarrow M$ and $N^0 \hookrightarrow M$. As a consequence the submodules M^0 and N^0 of M coincide.

Next let us prove the existence. If $E = \mathbb{G}_a$ then we can take for M^0 the submodule $B \cdot \tau^0 \subset B\{\tau\} = M$. For an affine open subscheme $\operatorname{Spec} S \subset \operatorname{Spec} B$ let E_S be the pullback of E to $\operatorname{Spec} S$ and let M_S be the motive of E_S. If E_S is isomorphic to $\mathbb{G}_{a,S}$ then by the remark above we have an invertible S-submodule $M_S^0 \subset M_S$ satisfying the condition of the proposition. Now the natural map $S \otimes_B M \to M_S$ is an isomorphism by Lemma 7.1.5 so the unicity part of the proposition implies that M_S^0 glue to an invertible B-submodule $M^0 \subset M$. The natural map $B\{\tau\} \otimes_B M^0 \to M$ is an isomorphism since it is so after the pullback to every affine open subscheme $\operatorname{Spec} S \subset \operatorname{Spec} B$ such that $E_S \cong \mathbb{G}_{a,S}$. □

Without the assumption that B is reduced the existence part of Proposition 7.1.6 still holds. However the submodule $M^0 \subset M$ is not unique anymore.

Lemma 7.1.9. *The degree filtration on M is stable under base change to an arbitrary* reduced *B-algebra.*

Proof. Let S be a reduced B-algebra. By Proposition 7.1.6 it is enough to show that the formation of M^0 commutes with the base change. Let S be a B-algebra and let M_S be the motive of E over S. The natural map $S\{\tau\} \otimes_B M^0 \to S \otimes_B M$ is an isomorphism by definition of M^0. Lemma 7.1.5 shows that the natural map $S \otimes_B M \to M_S$ is an isomorphism. In particular $S \otimes_B M^0$ is in a natural way an S-submodule of M_S. Now if S is reduced then Propostion 7.1.6 implies that the image of $S \otimes_B M^0$ in M_S is $(M_S)^0$. □

Proposition 7.1.10. *For every B-algebra S the map*

$$E(S) \to \mathrm{Hom}_{B\{\tau\}}(M, S), \quad e \mapsto \big(m \mapsto m(e)\big)$$

is an $\mathbb{F}_q$-linear isomorphism.

The left $B\{\tau\}$-module structure on S is given by the q-power map.

Proof. Let $\mathcal{E}$ be the presheaf on the big Zariski site of $\operatorname{Spec} B$ defined by the functor $\mathrm{Hom}_{B\{\tau\}}(M, -)$. The map above defines a morphism of presheaves $E \to \mathcal{E}$. Corollary 7.1.7 implies that $\mathcal{E}$ is a sheaf. By Lemma 7.1.5 the formation of M commutes with localization of B. We can thus reduce to the case $E = \mathbb{G}_a$ where the statement is clear. □

Definition 7.1.11. We denote $M^{\geqslant 1} = B\{\tau\}\tau \otimes_B M^0$ and $\Omega = M/M^{\geqslant 1}$.

Proposition 7.1.12. *The adjoint $\tau^* M \to M$ of the multiplication map $\tau\colon M \to M$ is injective with image $M^{\geqslant 1}$.* □

We identify Ω with the B-module of Lie algebra homomorphisms $\mathrm{Lie}_E \to \mathrm{Lie}_{\mathbb{G}_a}$. An image of a morphism $m \in \mathrm{Hom}(E, \mathbb{G}_a) = M$ in the quotient $\Omega = M/M^{\geqslant 1}$ is the induced morphism of Lie algebras $dm\colon \mathrm{Lie}_E \to \mathrm{Lie}_{\mathbb{G}_a}$.

Proposition 7.1.13. *For every B-algebra S the map*

$$\mathrm{Lie}_E(S) \to \mathrm{Hom}_B(\Omega, S), \quad \varepsilon \mapsto \big(dm \mapsto dm(\varepsilon)\big)$$

is an isomorphism of S-modules. □

7.2. Coefficient rings

Let A be an $\mathbb{F}_q$-algebra of finite type which is a Dedekind domain. To such an algebra A one can functorially associate a smooth connected projective curve C over $\mathbb{F}_q$ together with an open embedding $\operatorname{Spec} A \subset C$. We call C the compactification of $\operatorname{Spec} A$.

Definition 7.2.1. We say that A is a *coefficient ring* if the complement of $\operatorname{Spec} A$ in C consists of a single point. This point is called the point of A at infinity and is denoted ∞.

Example. $\mathbb{F}_q[t]$ is a coefficient ring. $\mathbb{F}_q[t,s]/(s^2 - t^3 + 1)$ is a coefficient ring provided q is coprime to 6.

Recall that an element $a \in A$ is called constant if it is algebraic over $\mathbb{F}_q$. Since we do not assume A to be geometrically irreducible there may be constant elements not in $\mathbb{F}_q \subset A$.

Lemma 7.2.2. *Let A be a coefficient ring. If $a \in A$ is not constant then the natural map $\mathbb{F}_q[a] \to A$ is finite flat.*

Proof. Since a is not constant the ring A has no $\mathbb{F}_q[a]$-torsion and so is flat over $\mathbb{F}_q[a]$. The only nontrivial claim is that it is finite over $\mathbb{F}_q[a]$.

Let C be the compactification of $\operatorname{Spec} A$ and let ∞ be the point in the complement of $\operatorname{Spec} A$ in C. The inclusion $\mathbb{F}_q[a] \subset A$ induces a morphism $C \to \mathbb{P}^1_{\mathbb{F}_q}$. This morphism is automatically proper. The only point of C which does not map to $\operatorname{Spec} \mathbb{F}_q[a] \subset \mathbb{P}^1_{\mathbb{F}_q}$ is ∞. Hence the preimage of $\operatorname{Spec} \mathbb{F}_q[a]$ in C is $\operatorname{Spec} A$. We conclude that the map $\operatorname{Spec} A \to \operatorname{Spec} \mathbb{F}_q[a]$ is proper. As a consequence it is finite [01WN]. □

Attached to A one has its local field at infinity F. It is the completion of the function field of C at the point ∞. One has a canonical inclusion $A \hookrightarrow F$.

Definition 7.2.3. Let A be a coefficient ring. For every nonzero $a \in A$ we define

$$\deg(a) = -\nu(a)$$

where $\nu\colon F^\times \to \mathbb{Z}$ is the normalized valuation.

Observe that $\deg(a) = 0$ if and only if a is a nonzero constant.

Lemma 7.2.4. *Let A be a coefficient ring, F the local field at infinity, k the residue field of F. If $a \in A$ is not constant then*

$$f \cdot \deg(a) = d$$

where $f = [k : \mathbb{F}_q]$, $d = [A : \mathbb{F}_q[a]]$.

Proof. Let F_0 be the local field of $\mathbb{F}_q[a]$ at infinity. By construction a^{-1} is a uniformizer of F_0. Hence $\deg(a)$ equals the ramification index e of F over F_0. Moreover f coincides with the inertia index of F over F_0. Since $ef = [F : F_0]$ we only need to prove that $[F : F_0] = d$. Both A and $\mathbb{F}_q[a]$ have a single point at infinity. Thus

$$F_0 \otimes_{\mathbb{F}_q[a]} A = F.$$

Since the inclusion $\mathbb{F}_q[a] \subset A$ is finite flat it follows that $[F : F_0] = d$. $\square$

7.3. Action of coefficient rings

We continue working over the fixed $\mathbb{F}_q$-algebra B. As before we suppose that B is reduced. Throughout this section we fix an $\mathbb{F}_q$-vector space scheme E over B which is a form of $\mathbb{G}_a$. We denote M its motive.

We assume that E is equipped with an action of a fixed coefficient ring A. In other words we are given an $\mathbb{F}_q$-algebra homomorphism $\varphi\colon A \to \mathrm{End}(E)$. The ring A acts on $M = \mathrm{Hom}(E, \mathbb{G}_a)$ on the right. As A is commutative we can view it as a left action. Thus M acquires a structure of a left $A \otimes B\{\tau\}$-module. In this section we study how the $A \otimes B$-module structure on M interacts with the degree filtration.

Lemma 7.3.1. *Assume that B is noetherian. If M^0 is an $A \otimes B$-submodule of M then M is* not *a finitely generated $A \otimes B$-module.*

Proof. Indeed in this case every $M^n \subset M$ is an $A \otimes B$-submodule. Since the quotients M^n/M^{n-1} are not zero we conclude that M contains an infinite increasing chain of $A \otimes B$-submodules. Hence it is not finitely generated. $\square$

Lemma 7.3.2. *Let $a \in A$ and let $d \geqslant 0$. The following are equivalent:*

1. *$M^0 a \subset M^d$ and the induced map $a\colon M^0 \to M^d/M^{d-1}$ is an isomorphism.*
2. *The same holds after base change to every B-algebra K which is a field.*

Proof. (1) $\Rightarrow$ (2) is a consequence of Lemma 7.1.9. (2) $\Rightarrow$ (1). Thanks to Lemma 7.1.9 we may assume that $E = \mathbb{G}_a$. In this case the action of A on E is given by a homomorphism $\varphi\colon A \to B\{\tau\}$. The condition (2) means that for every B-algebra K which is a field the polynomial $\varphi(a)$ has degree d in $K\{\tau\}$. Therefore the coefficient of $\varphi(a)$ at τ^d is a unit while the coefficient at τ^n is nilpotent for every $n > d$. By assumption of this section B is reduced. Hence $\varphi(a)$ is of degree d with top coefficient a unit. $\square$

Lemma 7.3.3. *Assume that $A = \mathbb{F}_q[t]$. Let $r \geqslant 1$. The following are equivalent:*

1. *$M^0 t \subset M^r$ and the induced map $t\colon M^0 \to M^r/M^{r-1}$ is an isomorphism of B-modules.*
2. *The natural map $A \otimes M^{r-1} \to M$ is an isomorphism of $A \otimes B$-modules.*
3. *M is a locally free $A \otimes B$-module of rank r.*

Proof. Thanks to Lemma 7.1.9 we may assume that $E = \mathbb{G}_a$. In this case $M = B\{\tau\}$ and the degree filtration on M is the filtration by degree of τ-polynomials. The action of A is given by a homomorphism $\varphi\colon \mathbb{F}_q[t] \to B\{\tau\}$. We split the rest of the proof into several steps.

Step 1. *If* (1) *holds then the natural map* $A \otimes M^{r-1} \to M$ *is surjective.*

By assumption $\varphi(t)$ is of degree r with top coefficient a unit. Write $\varphi(t) = \psi + \alpha_r\tau^r$ with ψ of degree less than r. We then have $\tau^r = \alpha_r^{-1}(\varphi(t) - \psi)$. Multiplying both sides by τ^n on the left we obtain a relation

$$\tau^{r+n} = \tau^n(\alpha_r)^{-1}(\tau^n\varphi(t) - \tau^n\psi).$$

Induction over n now shows that the image of the natural map $A \otimes M^{r-1} \to M$ is the whole of M.

Step 2. *If* B *is a field then* (1) *implies* (2).

According to Step 1 the natural map $A \otimes M^{r-1} \to M$ is surjective. We need to prove that it is injective. Observe that for every nonzero $\alpha \in B$ and $n, d \geqslant 0$ the τ-polynomial $\alpha\tau^n\varphi(t^d)$ is of degree $rd + n$. Hence if $f \in A \otimes B = B[t]$ is a polynomial of degree d then $f \cdot \tau^n$ is of degree $rd + n$.

Now let $f_0, \ldots, f_{r-1} \in B[t]$. If one of the f_n is nonzero then there exists a unique $n \in \{0, \ldots, r-1\}$ such that $r \cdot \deg f_n + n$ is maximal. From the observation above we deduce that $f_n \cdot \tau^n$ is of degree $r \deg f_n + n$ while for every $m \neq n$ the element $f_m \cdot \tau^m$ is of lesser degree. We conclude that

$$f_0 \cdot 1 + f_1 \cdot \tau + \ldots + f_{r-1} \cdot \tau^{r-1} \neq 0.$$

Step 3. *If* B *is noetherian then* (1) *implies* (2).

According to Step 1 the natural map $A \otimes M^{r-1} \to M$ is surjective. We thus have a short exact sequence

$$0 \to N \to A \otimes M^{r-1} \to M \to 0. \tag{7.1}$$

Since B is noetherian it follows that N is a finitely generated $A \otimes B$-module. By construction M^{r-1} and M are flat B-modules. Therefore (7.1) is a short exact sequence of flat B-modules.

Let K be a B-algebra which is a field. Lemma 7.1.9 tells that the formation of M commutes with base change to K and that the base change preserves the degree filtration. Therefore Step 2 shows that the second arrow of (7.1) becomes an isomorphism after base change to K. Since (7.1) is a sequence of flat B-modules we conclude that $N \otimes_B K = 0$ for every K. As N is a finitely generated $A \otimes B$-module Nakayama's lemma implies that $N = 0$.

Step 4. (1) *implies* (2). Write

$$\varphi(t) = \alpha_0 + \alpha_1\tau + \ldots + \alpha_r\tau^r.$$

By assumption α_r is a unit. Let B_0 be the $\mathbb{F}_q$-subalgebra of B generated by α_i and α_r^{-1}. Let $E_0 = \mathbb{G}_{a,B_0}$ equipped with the action of $\mathbb{F}_q[t]$ given by φ. As α_r is a unit in B_0 it follows that the assumption (1) holds for E_0. Step 3 now implies that (2) holds for E_0. Lemma 7.1.9 shows that (2) holds for the base change of E_0 to B. As $E_0 \otimes_{B_0} B = E$ by construction the result follows.

Step 5. (2) *implies* (3). By construction M^{r-1} is a locally free B-module of rank r. Therefore $A \otimes M^{r-1}$ is a locally free $A \otimes B$-module of rank r.

Step 6. (3) *implies* (1). Thanks to Lemma 7.3.2 we may suppose that B is a field. If $\varphi(t)$ is of degree 0 then M^0 is an $A \otimes B$-submodule of M. Lemma 7.3.1 then shows that M is not a finitely generated $A \otimes B$-module, a contradiction. Hence $\varphi(t)$ is of positive degree d. Now Step 3 shows that M is locally free of rank d

whence $d = r$. The induced map $t\colon M^0 \to M^r/M^{r-1}$ is an isomorphism since the top coefficient of $\varphi(t)$ is not zero. □

We now return to a general coefficient ring A. Let F be the local field of A at infinity and let k be the residue field of F. We denote

$$f = [k : \mathbb{F}_q]$$

the degree of the residue field extension at infinity.

Proposition 7.3.4. *Let $r \geqslant 1$ and let $a \in A$ be a nonconstant element. The following are equivalent:*

(1) *M is a locally free $A \otimes B$-module of rank r.*
(2) *$M^0 a \subset M^{fr\deg a}$ and the induced map*
$$M^0 \xrightarrow{a} M^{fr\deg a}/M^{fr\deg a-1}$$
is an isomorphism of B-modules.

Proof. According to Lemma 7.2.2 the natural map $\mathbb{F}_q[a] \to A$ is finite flat. Hence M is locally free of rank r as an $A \otimes B$-module if and only if it is locally free of rank rd as an $\mathbb{F}_q[a] \otimes B$-module where $d = [A : \mathbb{F}_q[a]]$. Since $d = f \deg a$ by Lemma 7.2.4 the result follows from Lemma 7.3.3 applied to $t = a$. □

Assuming that the base ring B is noetherian we next show that the motive M is a finitely generated $A \otimes B$-module if and only if it is locally free. We include this result only for illustrative purposes. It will not be used in the proof of the class number formula.

Lemma 7.3.5. *Assume that $A = \mathbb{F}_q[t]$ and B is a field. If M is a finitely generated $A \otimes B$-module then it is locally free of rank $\geqslant 1$.*

Proof. If $M^0 t \subset M^0$ then Lemma 7.3.1 shows that M is not finitely generated as an $\mathbb{F}_q[t] \otimes B$-module, t contradiction. Therefore $M^0 t \subset M^n$ for some $n \geqslant 1$. Without loss of generality we may assume that $M^0 t \not\subset M^{n-1}$. In this case the induced map $t\colon M^0 \to M^n/M^{n-1}$ is nonzero. As B is t field it is an isomorphism. Lemma 7.3.3 then shows that M is t locally free $A \otimes B$-module of rank $r \geqslant 1$. □

Lemma 7.3.6. *Assume that $A = \mathbb{F}_q[t]$ and B is a DVR. Let K be the fraction field and k the residue field of B. If M is a finitely generated $A \otimes B$-module then* $\operatorname{rank}_{A\otimes K} M \otimes_B K \geqslant \operatorname{rank}_{A\otimes k} M \otimes_B k$.

Proof. The Picard group of B is trivial so by Proposition 7.1.6 we may assume that $E = \mathbb{G}_a$. In this case the A-action is given by a homomorphism $\varphi\colon \mathbb{F}_q[t] \to B\{\tau\}$. Let r_K be the degree of $\varphi(t)$ in $K\{\tau\}$ and let r_k be its degree in $k\{\tau\}$. Lemma 7.3.3 shows that $M \otimes_B K$ is a locally free $A \otimes K$-module of rank r_K while $M \otimes_B k$ is a locally free $A \otimes k$-module of rank r_k. As $r_K \geqslant r_k$ by construction the result follows. □

Proposition 7.3.7. *If B is noetherian and* $\operatorname{Spec} B$ *is connected then the following are equivalent:*

(1) *M is a finitely generated $A \otimes B$-module.*
(2) *M is a locally free $A \otimes B$-module of constant rank.*

Proof. (1) ⇒ (2). If $a \in A$ is a nonconstant element then the map $\mathbb{F}_q[a] \to A$ is finite flat by Lemma 7.2.2. Hence to deduce (2) it is enough to assume that $A = \mathbb{F}_q[t]$.

Let $r\colon \operatorname{Spec} B \to \mathbb{Z}_{\geqslant 1}$ be the function which sends a prime $\mathfrak{p} \subset B$ to the rank of $M \otimes_B \operatorname{Frac} B/\mathfrak{p}$ as an $A \otimes \operatorname{Frac} B/\mathfrak{p}$-module. We will show that r is constant.

Let $\mathfrak{p} \subset \mathfrak{q}$ be primes of B such that $\mathfrak{p} \neq \mathfrak{q}$. According to [054F] there exists a discrete valuation ring V and a morphism $B \to V$ such that the generic point of $\operatorname{Spec} V$ maps to $\mathfrak{p}$ and the closed point maps to $\mathfrak{q}$. Applying Lemma 7.3.6 to the base change of E to V we deduce that $r(\mathfrak{p}) \geqslant r(\mathfrak{q})$. Hence r is lower semi-continuous. However $A \otimes B$ is noetherian and M is a finitely generated $A \otimes B$-module. The function r is therefore also upper semi-continuous. We conclude that it is in fact constant. Let us denote this constant r.

Let K be a B-algebra which is a field. Lemma 7.3.3 shows that $(M^0 \otimes_B K)t \subset (M^r \otimes_B K)$ and the induced map $M^0 \otimes_B K \to (M^r/M^{r-1}) \otimes_B K$ is an isomorphism. Hence Lemma 7.3.2 shows that the same holds already on the level of B. Applying Lemma 7.3.3 again we conclude that M is a locally free $\mathbb{F}_q[t] \otimes B$-module of rank $r \geqslant 1$. □

7.4. Drinfeld modules

We continue working over a fixed reduced $\mathbb{F}_q$-algebra B. Let A be a coefficient ring as in Definition 7.2.1. As in the previous section f denotes the degree of the residue field extension at infinity.

Definition 7.4.1. A *Drinfeld A-module* of rank $r \geqslant 1$ over B is an $\mathbb{F}_q$-vector space scheme E over B equipped with an action of A such that:

(1) E is a form of $\mathbb{G}_a$.
(2) The motive $M = \mathrm{Hom}(E, \mathbb{G}_a)$ is a locally free $A \otimes B$-module of rank r.

Proposition 7.3.4 implies that (2) can be replaced by

(2′) There exists a nonconstant element $a \in A$ such that $M^0 a \subset M^{fr \deg a}$ and the induced map $a\colon M^0 \to M^{fr \deg a}/M^{fr \deg a - 1}$ is an isomorphism.

Using this fact it is easy to show that our definition is equivalent with Drinfeld's original definition [8]. As in [8] the rank of our Drinfeld modules is constant on $\mathrm{Spec}\, B$.

Proposition 7.4.2. *Let E be a Drinfeld A-module of rank r and let M be its motive. The degree filtration M^* has the following properties.*

(1) *M^* is exhaustive.*
(2) *M^n is a locally free B-module of rank* $\max(0, n+1)$.
(3) *For every $n \geqslant 0$ and every nonzero $a \in A$ we have $M^n a \subset M^{n+fr \deg a}$ and the induced map*

$$M^n/M^{n-1} \to M^{n+fr \deg a}/M^{n+fr \deg a - 1}$$

is an isomorphism.

Proof. (1) and (2) are immediate from the definition of the degree filtration and (3) follows from Proposition 7.3.4. □

Proposition 7.4.3. *Let E be a form of $\mathbb{G}_a$ equipped with an action of A. If B is noetherian and* $\mathrm{Spec}\, B$ *is connected then the following are equivalent:*

(1) *E is a Drinfeld A-module.*
(2) *The motive $M = \mathrm{Hom}(E, \mathbb{G}_a)$ is a finitely generated $A \otimes B$-module.*

Proof. Follows instantly from Proposition 7.3.7. □

7.5. Co-nilpotence

Before we state the main result of this chapter let us introduce an auxillary notion. Let R be a τ-ring.

Definition 7.5.1. Let M be an R-module shtuka given by a diagram

$$M_0 \underset{j}{\overset{i}{\rightrightarrows}} M_1.$$

We say that M is *co-nilpotent* if the adjoint $j^a \colon \tau^* M_0 \to M_1$ of j is an isomorphism and the compsition

$$\tau^{*n}(u) \circ \ldots \circ u, \quad u = (j^a)^{-1} \circ i,$$

is zero for $n \gg 0$.

Proposition 7.5.2. *Let M be an R-module shtuka given by a diagram*

$$M = \Big[M_0 \underset{j}{\overset{i}{\rightrightarrows}} M_1 \Big].$$

and let N be a left $R\{\tau\}$-module. If M is co-nilpotent then $\mathcal{H}\mathrm{om}_R(M, N)$ is nilpotent.

Proof. Follows directly from the definition of $\mathcal{H}\mathrm{om}$ (Definition 1.13.1). In verifying this it is convenient to identify M_1 with $\tau^* M_0$ via j^a. □

7.6. Drinfeld's construction

In this section we recall Drinfeld's construction [9] of a shtuka attached to a Drinfeld module. No originality is claimed. All nontrivial results are Drinfeld's.

Fix a coefficient ring A. Let C be the projective compactification of $\operatorname{Spec} A$ and $\infty \in C$ the closed point in the complemenet of $\operatorname{Spec} A$. Let F be the local field of C at ∞. As in the previous sections f denotes the degree of the residue field of F over $\mathbb{F}_q$. Let B be an $\mathbb{F}_q$-algebra. We equip $C \times \operatorname{Spec} B$ with an endomorphism τ which acts as the identity on C and as the q-Frobenius on B.

Fix an ample line bundle on C which corresponds to the divisor ∞. Let $\mathcal{O}(1)$ be the pullback of this bundle to $C \times \operatorname{Spec} B$.

Theorem 7.6.1. *Let E be a Drinfeld A-module of rank r over B and let M be its motive. The $A \otimes B$-module shtuka*

$$M \underset{\tau}{\overset{1}{\rightrightarrows}} M$$

extends uniquely to a shtuka

$$\mathcal{E} = \Big[\mathcal{E}_{-1} \underset{j}{\overset{i}{\rightrightarrows}} \mathcal{E}_0 \Big]$$

on $C \times \operatorname{Spec} B$ with the following properties:

(1) *$\mathcal{E}_{-1}$ and $\mathcal{E}_0$ are locally free of rank r.*
(2) *For every $n \in \mathbb{Z}$ we have*

$$\mathrm{H}^0(C \times \operatorname{Spec} B,\, \mathcal{E}_0(n)) = M^{nfr},$$
$$\mathrm{H}^0(C \times \operatorname{Spec} B,\, \mathcal{E}_{-1}(n)) = M^{nfr-1}$$

as B-submodules of M.

Moreover the shtuka $\mathcal{E}(\mathcal{O}_F/\mathfrak{m}_F \otimes B)$ is co-nilpotent.

The fact that $\mathcal{E}(\mathcal{O}_F/\mathfrak{m}_F \otimes B)$ is co-nilpotent is of fundamental importance to our study. It implies that certain shtukas we construct out of $\mathcal{E}$ are nilpotent which in turn allows us to apply the theory of Chapters 5 and 6 to Drinfeld modules.

The condition (2) can be replaced with the following pair of conditions:

(2a) The adjoint of the j-arrow of $\mathcal{E}(\mathcal{O}_F/\mathfrak{m}_F \otimes B)$ is an isomorphism.
(2b) $\chi(\mathcal{E}_{-1}) = 0$ for all points of $\operatorname{Spec} B$.

However we will use the fact that $\mathcal{E}$ satisfies (2), so we opt for the less elegant formulation.

Proof of Theorem 7.6.1. Uniqueness follows from (2). Let us prove the existence. We denote $\iota\colon \operatorname{Spec}(A\otimes B) \hookrightarrow C\times\operatorname{Spec} B$ the open immersion. By Proposition 7.4.2 the degree filtration M^* has the following properties:

- M^* is exhaustive.
- M^n is a locally free B-module of rank $\max(0, n+1)$.
- For every $n \geqslant 0$ and every nonzero $a \in A$ we have $M^n a \subset M^{n+fr\deg a}$ and the induced map
$$M^n/M^{n-1} \to M^{n+fr\deg a}/M^{n+fr\deg a-1}$$
is an isomorphism.

We can thus invoke [9, Corollary 1], [9, Proposition 3] and conclude that there exists an increasing chain
$$\ldots \subset \mathcal{E}_{-1} \subset \mathcal{E}_0 \subset \mathcal{E}_1 \subset \mathcal{E}_2 \subset \ldots$$
of locally free sheaves of rank r on $C\times\operatorname{Spec} B$ extending M such that:

(1) $\mathrm{H}^0(C\times\operatorname{Spec} B,\, \mathcal{E}_n) = M^n$.
(2) $\mathcal{E}_n(1) = \mathcal{E}_{n+fr}$.
(3) The τ-multiplication map $\tau\colon \iota_*M \to \tau_*\iota_*M$ sends $\mathcal{E}_n$ to $\mathcal{E}_{n+1}$.
(4) The map $\tau^*(\mathcal{E}_n/\mathcal{E}_{n-1}) \to \mathcal{E}_{n+1}/\mathcal{E}_n$ induced by the adjoint of τ is an isomorphism.

In particular we get a shtuka
$$\mathcal{E} = \Big[\mathcal{E}_{-1} \underset{j}{\overset{i}{\rightrightarrows}} \mathcal{E}_0\Big]$$
where $i\colon \mathcal{E}_{-1} \to \mathcal{E}_0$ is the inclusion and j is induced by τ. It satisfies all the conditions of this theorem. What remains is to prove that $\mathcal{E}(\mathcal{O}_F/\mathfrak{m}_F\otimes B)$ is conilpotent.

Let $\alpha\colon \operatorname{Spec}(\mathcal{O}_F/\mathfrak{m}_F\otimes B) \to C\times\operatorname{Spec} B$ be the closed immersion. We denote $j^a\colon \tau^*\mathcal{E}_n \to \mathcal{E}_{n+1}$ the adjoint of j. Property (4) implies that the natural map $\tau^*\mathcal{E}_{-1}/\mathcal{E}_{-fr-1} \to \mathcal{E}_0/\mathcal{E}_{-fr}$ is an isomorphism. This map coincides with $\alpha^*(j^a)$ since $\mathcal{E}_{-fr+n} = \mathcal{E}_n(1)$ by the property (2) above. The same property now implies that the composition
$$\tau^{*fr}(u)\circ\ldots\circ u, \quad u = \alpha^*(j^a)^{-1}\circ\alpha^*(i)$$
is zero. $\square$

CHAPTER 8

The motive and the Hom shtuka

Let A be a coefficient ring, F the local field of A at infinity and ω the module of Kähler differentials of A over $\mathbb{F}_q$. Let E be a Drinfeld A-module over a reduced $\mathbb{F}_q$-algebra B and let $M = \mathrm{Hom}(E, \mathbb{G}_a)$ be its motive. Rather than working with M directly we study the Hom shtuka

$$\mathcal{H}\mathrm{om}_{A\otimes B}(M,\, \omega \otimes B).$$

This chapter has three groups of results. The first is purely algebraic and deals with a Drinfeld A-module E over a reduced $\mathbb{F}_q$-algebra B. We show that

$$\mathrm{R}\Gamma(\mathcal{H}\mathrm{om}_{A\otimes B}(M,\, \omega \otimes B)) \cong E(B)[-1],$$
$$\mathrm{R}\Gamma(\nabla\, \mathcal{H}\mathrm{om}_{A\otimes B}(M,\, \omega \otimes B)) \cong \mathrm{Lie}_E(B)[-1].$$

This result is related to the formulas of Barsotti-Weil type obtained by Papanikolas-Ramachandran [19] and Taelman [23]. In some sense it is analogous to the classical Barsotti-Weil isomorphism $\mathrm{Ext}^1(h(E), \mathbb{1}) \cong E(k)$ for the motive $h(E)$ of an elliptic curve E over a field k.

The second group of results consists of two formulas. The first applies to a Drinfeld module E over a local field K. We assume that the action of A on $\mathrm{Lie}_E(K)$ extends to a continuous action of F. Under this condition there is a natural map $\exp\colon \mathrm{Lie}_E(K) \to E(K)$. We construct a quasi-isomorphism

$$\mathrm{R}\Gamma(\mathcal{H}\mathrm{om}_{A\otimes K}(M,\, \omega \mathbin{\widehat{\otimes}} K)) \cong \Big[\mathrm{Lie}_E(K) \xrightarrow{\exp} E(K)\Big].$$

This formula is inspired by the work of Anderson [1, Section 2]. Its proof relies on the fact that $\omega \mathbin{\widehat{\otimes}} K = b(F/A, K)$ is naturally the space of bounded functions $F/A \to K$ (see Definition 2.9.1).

The second formula is a variant of the first. It applies to a Drinfeld module E over an $\mathbb{F}_q$-algebra R which is discrete in a finite product of local fields K. Assuming that the action of A on $\mathrm{Lie}_E(K)$ extends to a continuous action of F we produce a quasi-isomorphism

$$\mathrm{R}\Gamma\big(\mathcal{H}\mathrm{om}_{A\otimes R}(M,\, \omega \mathbin{\widehat{\otimes}} \tfrac{K}{R})\big) \cong \Big[\mathrm{Lie}_E(K) \xrightarrow{\exp} \frac{E(K)}{E(R)}\Big].$$

This result is crucial to the proof of the class number formula.

The third group of results relates Tate modules of a Drinfeld module over a field k to the Hom shtuka $\mathcal{H}\mathrm{om}_{A\otimes k}(M,\, \omega \otimes k)$. We translate certain results of David Goss [11, Section 5.6] concerning the motive M of E to the language of Hom shtukas.

Our theorems generalize to Anderson modules [1] in place of Drinfeld modules. Their proofs extend without change. We limit our exposition to Drinfeld modules since other important parts of our theory still depend on their special properties.

Our approach relies on computations of Hochschild cohomology of A with coefficients in various (A, A)-bimodules such as the function spaces and the germ spaces. First six sections of this chapter are devoted to such computations. Section 8.7 relates Hochschild cohomology to shtuka cohomology. Save for the last section the rest of the chapter is a straightforward derivation of results on shtuka cohomology from the earlier Hochschild results.

In the case $A = \mathbb{F}_q[t]$ the shtukas isomorphic to $\mathcal{H}\mathrm{om}_{A\otimes B}(M, A\otimes B)$ have appeared in the work of Fang [10]. Generalizing an earlier construction of Taelman [25] he defined such shtukas in terms of explicit bases and matrices. His work motivated the author of this text to study Hom shtukas.

8.1. Hochschild cohomology in general

Let A be an associative unital $\mathbb{F}_q$-algebra and let M be a complex of (A, A)-bimodules. The Hochschild cohomology of A with coefficients in M is by definition the complex

$$\mathrm{RHom}_{A\otimes A^\circ}(A, M).$$

where A° is the opposite of A and the (A, A)-bimodule structure on A is the diagonal one. We denote this complex $\mathrm{RH}(A, M)$ and $\mathrm{H}^n(A, M)$ its cohomology groups.

Even though Hochschild cohomology will figure prominently in our computations, no nontrivial properties of it will be used.

8.2. Hochschild cohomology of coefficient rings

Let A be a coefficient ring in the sense of Definition 7.2.1. Given an (A, A)-bimodule M we denote $D(A, M)$ the module of M-valued derivations over $\mathbb{F}_q$, i.e. $\mathbb{F}_q$-linear maps $d\colon A \to M$ which satisfy the Leibniz identity

$$d(a_0a_1) = a_0d(a_1) + d(a_1)a_0.$$

We denote $\partial\colon M \to D(A, M)$ the map which sends an element $m \in M$ to the derivation $a \mapsto am - ma$.

Proposition 8.2.1. *For every (A, A)-bimodule M there is a natural quasi-isomorphism*

$$\mathrm{RH}(A, M) \cong \Big[M \xrightarrow{\partial} D(A, M)\Big]$$

Proof. The (A, A)-bimodule A has a canonical resolution $0 \to I \to A\otimes A \to A \to 0$. The $A\otimes A$-modules I and $A\otimes A$ are invertible, so that

$$\mathrm{RHom}_{A\otimes A}(A, M) = \Big[M \to \mathrm{Hom}_{A\otimes A}(I, M)\Big].$$

The natural map $\mathrm{Hom}_{A\otimes A}(I, M) \to D(A, M)$, $f \mapsto (a \mapsto f(a\otimes 1 - 1\otimes a))$, is an isomorphism of (A, A)-bimodules. It identifies the complex above with the complex in the statement of the lemma. □

We denote ω the module of Kähler differentials of A over $\mathbb{F}_q$. Let M be an A-module. Using the universal property of ω one easily proves the following lemma:

Lemma 8.2.2. *The map $M \to D(A, \omega\otimes_A M)$ given by the formula $m \mapsto (a \mapsto da\otimes_A m)$ is an isomorphism.* □

Lemma 8.2.3. *The sequence*

$$0 \to \omega\otimes M \xrightarrow{\partial} D(A, \omega\otimes M) \xrightarrow{-D(\mu)} D(A, \omega\otimes_A M) \to 0 \tag{8.1}$$

is exact. Here $\mu\colon \omega\otimes M \to \omega\otimes_A M$ is the contraction map.

Observe that in (8.1) we use the morphism $-D(\mu)$ instead of the more natural $D(\mu)$. With this choice of sign the statements of several results become more straightforward. One example is Proposition 8.5.1.

Proof of Lemma 8.2.3. Let $0 \to I \to A \otimes A \to A \to 0$ be the canonical resolution of the diagonal (A, A)-bimodule A. The composition with the multiplication map $A \otimes A \to A$ induces a homomorphism $\nu \colon D(A, A \otimes A) \to D(A, A)$. One easily shows that the natural sequence

$$0 \to A \otimes A \to D(A, A \otimes A) \xrightarrow{-\nu} D(A, A) \to 0 \tag{8.2}$$

is exact. The tensor product of (8.2) with $\omega \otimes M$ over $A \otimes A$ coincides with the sequence (8.1). Moreover

$$D(A, A) = \mathrm{Hom}_{A \otimes A}(I, A) = \mathrm{Hom}_A(I/I^2, A) = \mathrm{Hom}_A(\omega, A).$$

So to prove the lemma it is enough to show that

$$\mathrm{Tor}_1^{A \otimes A}(\mathrm{Hom}_A(\omega, A),\, \omega \otimes M) = 0$$

with the diagonal (A, A)-bimodule structure on $\mathrm{Hom}_A(\omega, A)$. However

$$\mathrm{Hom}_A(\omega, A) \otimes^{\mathbf{L}}_{A \otimes A} (\omega \otimes M) = \mathrm{Hom}_A(\omega, A) \otimes^{\mathbf{L}}_A \omega \otimes^{\mathbf{L}}_A M = M[0]. \quad \square$$

Proposition 8.2.4. *For every A-module M there is a natural A-linear quasi-isomorphism* $\mathrm{RH}(A, \omega \otimes M) \xrightarrow{\sim} M[-1]$. *It is the following composition:*

$$\begin{aligned}
\mathrm{RH}(A, \omega \otimes M) &\xrightarrow{\sim} \left[\omega \otimes M \xrightarrow{\partial} D(A, \omega \otimes M)\right] && \text{(Proposition 8.2.1)}\\
&\xrightarrow{\sim} D(A, \omega \otimes_A M)[-1] && \text{(Lemma 8.2.3)}\\
&\xleftarrow{\sim} M[-1] && \text{(Lemma 8.2.2)} \qquad \square
\end{aligned}$$

8.3. Vanishing

Let A be a coefficient ring and let F be the local field of A at ∞.

Lemma 8.3.1. $A \otimes_{A \otimes A} (A \mathbin{\widehat{\otimes}} F) = 0$.

Proof. Fix a nonconstant element $t \in A$. The element $1 \otimes t - t \otimes 1$ acts by zero on A. We will show that it becomes a unit in $\mathbb{F}_q[t] \mathbin{\widehat{\otimes}} F$ and a fortiori in $A \mathbin{\widehat{\otimes}} F$.

Let α be the image of t in F. Recall from Section 3.4 that $\mathbb{F}_q[t] \mathbin{\widehat{\otimes}} F$ is the ring of Tate series $F\langle t\rangle$ in variable t and with coefficients in F. The image of $1 \otimes 1 - t \otimes 1$ in $F\langle t\rangle$ is the polynomial $\alpha - t$. It is invertible since $|\alpha| > 1$. $\square$

We will use the following vanishing statement:

Proposition 8.3.2. *If M is an $A \mathbin{\widehat{\otimes}} F$-module then* $\mathrm{RH}(A, M) = 0$.

The (A, A)-bimodule structure on M is given by the natural morphism $A \otimes A \to A \mathbin{\widehat{\otimes}} F$.

Proof of Proposition 8.3.2. Indeed $A \widehat{\otimes} F$ is flat over $A \otimes A$ and the pullback of the diagonal $A \otimes A$-module A to $A \widehat{\otimes} F$ is zero. So $\mathrm{RHom}_{A \otimes A}(A, M) = \mathrm{RHom}_{A \widehat{\otimes} F}(0, M) = 0$ by extension of scalars. $\square$

8.4. Preliminaries on function spaces

As before A stands for a coefficient ring in the sense of Definition 7.2.1. From now on we denote F the local field of A at ∞ and ω the module of Kähler differentials of A over $\mathbb{F}_q$. Let $\mathrm{res} \colon \omega \otimes_A F \to \mathbb{F}_q$ be the residue map at infinity composed with the trace map to $\mathbb{F}_q$.

Let U be one of the topological A-modules F or F/A and let M be a locally compact A-module. Consider $a(U, M)$, the space of bounded locally constant $\mathbb{F}_q$-linear maps $U \to M$ as in Definition 2.10.1. The action of A on the source and the target makes $a(U, M)$ into an (A, A)-bimodule. In the following sections we study the Hochschild cohomology of $a(U, M)$ for certain A-modules M.

Besides $a(U, M)$ we will also use the space of bounded $\mathbb{F}_q$-linear maps $b(U, M)$ as in Definition 2.9.1 and the space of germs $g(U, M)$ as in Definition 2.11.1. The (A, A)-bimodule structures on such spaces are given by the action of A on the source and the target. To improve the legibility we will generally write $\mathrm{RH}(-)$ instead of $\mathrm{RH}(A, -)$.

Definition 8.4.1. We denote $\mathrm{Res}\colon \omega \otimes M \to a(F/A,\, M)$ the map given by the formula $\eta \otimes m \mapsto (x \mapsto \mathrm{res}(\eta x) m)$.

The map Res is a topological isomorphism of (A, A)-bimodules by Corollary 3.9.2. We will view it interchangeably as a map to $a(F/A, M)$, $a(F, M)$ and $b(F/A, M)$. Corollary 3.9.2 also shows that Res extends uniquely to a topological isomorphism of (A, A)-bimodules $\omega \widehat{\otimes} M \xrightarrow{\sim} b(F/A,\, M)$. We denote it Res too. The composition of Res with the natural map $b(F/A, M) \to g(F, M)$ will be denoted [Res].

Lemma 8.4.2. *The sequence*

$$0 \to \omega \otimes M \longrightarrow \omega \mathbin{\widehat{\otimes}} M \xrightarrow{[\mathrm{Res}]} g(F, M) \to 0$$

is exact.

Definition 8.4.3. We denote $\delta\colon \mathrm{RH}(g(F, M)) \to \mathrm{RH}(\omega \otimes M)[1]$ the natural morphism arising from the short exact sequence of Lemma 8.4.2.

Proof of Lemma 8.4.2. The quotient map $F \to F/A$ is a local isomorphism and so induces an isomorphism $g(F/A, M) \to g(F, M)$. Now the map Res identifies the sequence in question with the natural sequence $0 \to a(F/A, M) \to b(F/A, M) \to g(F/A, M) \to 0$ which is exact by Proposition 2.11.2. □

8.5. Vector spaces

We continue using the notation and the conventions of Section 8.4. In this section we work with a fixed finite-dimensional F-vector space V. According to Proposition 8.2.1 we have

$$\mathrm{RH}(g(F, V)) = \Big[g(F, V) \xrightarrow{\partial} D(A,\, g(F, V)) \Big].$$

One verifies easily that the map $v \mapsto (x \mapsto xv)$ defines a morphism of complexes $V[0] \to \mathrm{RH}(g(F/A, V))$.

Proposition 8.5.1. *The map $v \mapsto (x \mapsto xv)$ induces a quasi-isomorphism $V[0] \xrightarrow{\sim} \mathrm{RH}(g(F, V))$ which fits into a commutative square*

$$\begin{tikzcd} V[0] \arrow[r, equal] \arrow[d, "\wr"'] & V[0] \\ \mathrm{RH}(g(F, V)) \arrow[r, "\delta"] & \mathrm{RH}(\omega \otimes V)[1] \arrow[u, "\wr", "\text{Prp. 8.2.4}"'] \end{tikzcd}$$

Although this statement appears innocent, we do not know an easy proof. We split the proof into several lemmas.

Lemma 8.5.2. *The map* $\mathrm{Res}\colon \omega \otimes V \to a(F, V)$ *induces a quasi-isomorphism* $\mathrm{RH}(\omega \otimes V) \xrightarrow{\sim} \mathrm{RH}(a(F, V))$.

Proof. A quick inspection reveals that the natural sequence

$$0 \to a(F/A, V) \to a(F, V) \to a(A, V) \to 0$$

is exact. Moreover $a(A, V) = b(A, V)$ since A is discrete. Now Proposition 3.6.5 shows that the (A, A)-bimodule structure on $b(A, V)$ extends to an $A \mathbin{\widehat{\otimes}} F$-module structure. Hence $\mathrm{RH}(b(A, V)) = 0$ by Proposition 8.3.2 and the result follows. □

Lemma 8.5.3. *There exists a unique continuous map $\mu\colon a(F,F) \to F^*$ with the following property. If $f\colon F \to \mathbb{F}_q$ is a continuous $\mathbb{F}_q$-linear map and $\alpha \in F$ then the image of the map $x \mapsto f(x)\alpha$ under μ is the map $x \mapsto f(x\alpha)$.*

Proof. Consider the topological $\mathbb{F}_q$-vector space $F^* \otimes_{\mathrm{ic}} F$ (Definition 2.7.1). Let $\mu\colon F^* \otimes_{\mathrm{ic}} F \to F^*$ be the map which sends a tensor $f \otimes \alpha$ to the function $x \mapsto f(\alpha x)$. This map is easily shown to be continuous in the ind-tensor product topology. It induces a unique continuous map $F^* \check{\otimes} F \to F^*$ by completion. Proposition 3.6.3 identifies $F^* \check{\otimes} F$ with $a(F,F)$ and we get the result. $\square$

We denote $\rho\colon \omega \otimes_A F \xrightarrow{\sim} F^*$ the isomorphism induced by the residue pairing (Theorem 3.9.1).

Lemma 8.5.4. *The diagram*

$$\begin{tikzcd}
0 \arrow[r] & a(F,F) \arrow[r, "\partial"] & D(A,\, a(F,F)) \arrow[r, "-D(\mu)"] & D(A,\, F^*) \arrow[r] & 0 \\
0 \arrow[r] & \omega \otimes F \arrow[r, "\partial"] \arrow[u, "\mathrm{Res}"] & D(A,\, \omega \otimes F) \arrow[r, "-D(\mu)"] \arrow[u, "D(\mathrm{Res})"'] & D(A,\, \omega \otimes_A F) \arrow[r] \arrow[u, "\rho"', "\wr"] & 0
\end{tikzcd} \tag{8.3}$$

is commutative with exact rows.

Proof. Let $\eta \in \omega$ and $\alpha \in F$. By definition $\mu(\mathrm{Res}(\eta \otimes \alpha))$ is the function $x \mapsto \mathrm{res}(x\alpha\eta)$. It is the image of the element $\eta \otimes_A \alpha \in \omega \otimes_A F$ under ρ. So the right square of (8.3) is commutative. The left square commutes by definition. The bottom row is exact by Lemma 8.2.3. So Lemma 8.5.2 implies that the top row is exact. $\square$

Lemma 8.5.5. *If Λ is a finite-dimensional discrete $\mathbb{F}_q$-vector space, $f\colon F \to \Lambda$ and $g\colon \Lambda \to F$ continuous $\mathbb{F}_q$-linear maps then $\mu(g \circ f)\colon F \to \mathbb{F}_q$ is the map $x \mapsto \mathrm{tr}_\Lambda(f \circ x \circ g)$.*

Proof. If $\Lambda = \mathbb{F}_q$ then the claim is true by definition of μ. In general pick a splitting $\Lambda = \Lambda_1 \oplus \Lambda_2$. Let f_1, f_2 be the compositions of f with the projections to Λ_1, Λ_2 and let g_1, g_2 be the restrictions of g to Λ_1, Λ_2. We then have

$$\mu(g \circ f) = \mu(g_1 \circ f_1) + \mu(g_2 \circ f_2).$$

At the same time

$$\mathrm{tr}_\Lambda(f \circ x \circ g) = \mathrm{tr}_{\Lambda_1}(f_1 \circ x \circ g_1) + \mathrm{tr}_{\Lambda_2}(f_2 \circ x \circ g_2).$$

So the claim follows by induction on the dimension of Λ. $\square$

Proof of Proposition 8.5.1. First we extend the square in question to the right:

$$\begin{tikzcd}
V[0] \arrow[r, equal] \arrow[d] & V[0] \arrow[r, equal] & V[0] \\
\mathrm{RH}(g(F,V)) \arrow[r, "\delta"] & \mathrm{RH}(\omega \otimes V)[1] \arrow[r, "\sim"] \arrow[u, "\wr", "\text{Prp. 8.2.4}"'] & \mathrm{RH}(a(F,V))[1] \arrow[u, "\wr"']
\end{tikzcd}$$

Here the bottom right arrow is the natural quasi-isomorphism of Lemma 8.5.2 and the right vertical arrow is defined by commutativity. To prove the proposition it is enough to show that the outer rectangle

$$\begin{tikzcd}
V[0] \arrow[r, equal] \arrow[d] & V[0] \\
\mathrm{RH}(g(F,V)) \arrow[r] & \mathrm{RH}(a(F,V))[1] \arrow[u, "\wr"']
\end{tikzcd} \tag{8.4}$$

is commutative and the bottom map is a quasi-isomorphism.

By construction the bottom arrow in (8.4) is the map $\delta\colon \mathrm{RH}(g(F,V)) \to \mathrm{RH}(a(F,V))[1]$ arising from the short exact sequence

$$0 \to a(F,V) \to b(F,V) \to g(F,V) \to 0.$$

The (A,A)-bimodule structure on $b(F,V)$ extends to an $A \mathbin{\widehat{\otimes}} F$-module structure by Proposition 3.6.5. Proposition 8.3.2 then implies that $\mathrm{RH}(b(F,V)) = 0$. So δ is a quasi-isomorphism. What remains is to show that (8.4) is commutative.

By naturality one reduces to the case $V = F$. All the maps in (8.4) are F-linear since $a(F,V)$, $b(F,V)$ and $g(F,V)$ are (F,F)-bimodules. So to prove the commutativity it is enough to consider the element $1 \in F$.

Applying H^0 to (8.4) we get a square

$$\begin{tikzcd} F \arrow[r, equal] \arrow[d, "\wr"'] & F \\ \mathrm{H}^0(A, g(F,F)) \arrow[r, "\delta", "\sim"'] & \mathrm{H}^1(A, a(F,F)) \arrow[u, "\wr"'] \end{tikzcd}$$

The image of $1 \in F$ under the composition of the left, bottom and right arrows can be described in the following way. Pick a bounded function $f\colon F \to F$ such that $f(x) = x$ for all x in an open neighbourhood of 0. Let $D\colon A \to a(F,F)$ be the derivation given by the formula $D(a) = af - fa$. By Lemma 8.2.2 there exists a unique $\alpha \in F$ such that for all $a \in A$ we have

$$\mu(D(a)) = \rho(da \otimes_A \alpha).$$

Lemma 8.5.4 implies that the image of $1 \in F$ is the element $-\alpha \in F$. It is independent of the choice of f. Our goal is to prove that $\alpha = -1$.

Pick elements $a, b \in A$ such that $z = ab^{-1}$ is a uniformizer of F. Let k be the residue field of F so that $F = k(\!(z)\!)$. There is a well-defined element $dz \in \omega[b^{-1}]$.

First we extend $D\colon A \to a(F,F)$ to a derivation $D\colon F \to a(F,F)$ using the formula $D(x) = xf - fx$. Leibniz rule for the products $z = ab^{-1}$ and $1 = bb^{-1}$ implies that

$$D(z) = D(a)b^{-1} - zD(b)b^{-1}.$$

By assumption $\mu(D(x)) = \rho(dx \otimes_A \alpha)$ for all $x \in A$. Hence

$$\mu(D(z)) = \rho(da \otimes_A \alpha b^{-1} - db \otimes_A \alpha z b^{-1}).$$

At the same time we have an identity

$$dz \otimes_{A[b^{-1}]} \alpha = da \otimes_A \alpha b^{-1} - db \otimes_A \alpha z b^{-1}$$

in $\omega[b^{-1}] \otimes_{A[b^{-1}]} F = \omega \otimes_A F$. Therefore

$$\mu(D(z)) = \rho(dz \otimes_{A[b^{-1}]} \alpha). \tag{8.5}$$

Next we construct a suitable bounded function $f\colon F \to F$. Consider the function defined by the formula

$$f\Big(\sum_n \alpha_n z^n\Big) = \sum_{n \geqslant 0} \alpha_n z^n$$

with $\alpha_n \in k$. It is clearly bounded and satisfies $f(x) = x$ for all $x \in \mathcal{O}_F$. Moreover for every $\alpha \in k$ and $n \in \mathbb{Z}$ we have

$$D(z)\colon \alpha z^n \mapsto \begin{cases} -\alpha, & n = -1, \\ 0, & n \neq -1. \end{cases}$$

Applying Lemma 8.5.5 to the function $D(z)$, the vector space $\Lambda = k$ and the inclusion $k \hookrightarrow F$ we conclude that $\mu(D(z)) = -\rho(dz \otimes_{A[b^{-1}]} 1)$. In view of (8.5) it follows that $\alpha = -1$. $\square$

8.6. Locally compact modules

In this section we fix a finite-dimensional F-vector space V, a locally compact A-module M and a continuous A-linear map $e\colon V \to M$. We assume that e is a *local isomorphism* as in Definition 2.11.3. One may view V as the "tangent space" of M at 0 and $e\colon V \to M$ as the "exponential map" in the manner of Lie theory.

Definition 8.6.1. We denote C_e the A-module complex $\Big[V \xrightarrow{e} M\Big]$.

Our aim is to prove that $\mathrm{RH}(\omega \mathbin{\widehat{\otimes}} M) = C_e$.

Lemma 8.6.2. $\mathrm{H}^0(C_e)$ *is a finitely generated projective A-module.*

Proof. Since e is a local isomorphism it follows that $\mathrm{H}^0(C_e) = \ker e$ is a discrete A-submodule of V. As V is finite-dimensional we conclude that $\mathrm{H}^0(C_e)$ is finitely generated. It is then projective by virtue of being torsion free. ☐

In the following $\mathrm{D}(A)$ stands for the derived category of A-modules.

Lemma 8.6.3. *Let $h\colon M[-1] \to C_e$ be the map given by the identity in degree one and let C be a complex of A-modules. If $\mathrm{H}^0(C)$ is finitely generated and $\mathrm{H}^n(C) = 0$ for all $n < 0$ then the map*

$$\mathrm{Hom}_{\mathrm{D}(A)}(C_e,\, C) \xrightarrow{-\circ h} \mathrm{Hom}_{\mathrm{D}(A)}(M[-1],\, C)$$

is injective.

Proof. The map h extends to a distinguished triangle

$$M[-1] \xrightarrow{h} C_e \to V[0] \to M[0].$$

Applying $\mathrm{Hom}_{\mathrm{D}(A)}(-,\, C)$ we get an exact sequence

$$\mathrm{Hom}(V[0],\, C) \to \mathrm{Hom}(C_e,\, C) \xrightarrow{-\circ h} \mathrm{Hom}(M[-1],\, C).$$

It is thus enough to show that $\mathrm{Hom}(V[0],\, C) = 0$. Since A is of global dimension 1 and $\mathrm{H}^n(C) = 0$ for $n < 0$ there is a non-canonical quasi-isomorphism

$$C \cong \bigoplus_{n \geqslant 0} \mathrm{H}^n(C)[-n].$$

It follows that $\mathrm{Hom}(V[0], C) = \mathrm{Hom}_A(V,\, \mathrm{H}^0(C))$. However V is uniquely divisible and $\mathrm{H}^0(C)$ has no divisible elements besides zero. So the latter Hom is zero. ☐

Proposition 8.6.4. *The map $v \mapsto (x \mapsto e(xv))$ induces a quasi-isomorphism $V[0] \xrightarrow{\sim} \mathrm{RH}(g(F, M))$ which fits into a commutative square*

$$\begin{tikzcd}
V[0] \arrow[r, "e"] \arrow[d, "\wr"'] & M[0] \\
\mathrm{RH}(g(F, M)) \arrow[r, "\delta"] & \mathrm{RH}(\omega \otimes M)[1] \arrow[u, "\wr", "\text{Prp. 8.2.4}"']
\end{tikzcd}$$

Proof. The map $e\colon V \to M$ induces a morphism of short exact sequences

$$\begin{tikzcd}
0 \arrow[r] & \omega \otimes V \arrow[r] \arrow[d, "1\otimes e"] & \omega \mathbin{\widehat{\otimes}} V \arrow[r] \arrow[d, "1\widehat{\otimes} e"] & g(F, V) \arrow[r] \arrow[d, "e\circ -"] & 0 \\
0 \arrow[r] & \omega \otimes M \arrow[r] & \omega \mathbin{\widehat{\otimes}} M \arrow[r] & g(F, M) \arrow[r] & 0
\end{tikzcd}$$

As e is a local isomorphism it follows that the induced map $g(F,V) \to g(F,M)$ is an isomorphism. Taking the cohomology we get a commutative diagram

$$\begin{tikzcd}
\mathrm{RH}(g(F,V)) \arrow[r, "\delta"] \arrow[d, "e\circ-", "\wr"'] & \mathrm{RH}(\omega \otimes V)[1] \arrow[r, "\text{Prp. 8.2.4}", "\sim"'] \arrow[d, "1\otimes e"] & V[0] \arrow[d, "e"] \\
\mathrm{RH}(g(F,M)) \arrow[r, "\delta"] & \mathrm{RH}(\omega \otimes M)[1] \arrow[r, "\text{Prp. 8.2.4}", "\sim"'] & M[0]
\end{tikzcd}$$

The result now follows from Proposition 8.5.1. □

Theorem 8.6.5. *There exists a quasi-isomorphism* $\mathrm{RH}(\omega \mathbin{\widehat{\otimes}} M) \xrightarrow{\simeq} C_e$ *with the following properties.*

(1) *It is natural in* V, M, e.

(2) *It is the unique morphism in* $\mathrm{D}(A)$ *such that the square*

$$\begin{tikzcd}
\mathrm{RH}(\omega \otimes M) \arrow[r] \arrow[d, "\text{Proposition 8.2.4}"', "\wr"] & \mathrm{RH}(\omega \mathbin{\widehat{\otimes}} M) \arrow[d] \\
M[-1] \arrow[r, "\text{identity in degree 1}"] & C_e
\end{tikzcd}$$

is commutative.

(3) *It makes the square*

$$\begin{tikzcd}
\mathrm{RH}(\omega \mathbin{\widehat{\otimes}} M) \arrow[r] \arrow[d] & \mathrm{RH}(g(F,M)) \\
C_e \arrow[r, "\text{identity in degree 0}"] & V[0] \arrow[u, "\wr"', "\text{Proposition 8.6.4}"]
\end{tikzcd}$$

commutative.

Proof. Consider the diagram

$$\begin{tikzcd}
\mathrm{RH}(\omega \otimes M) \arrow[r] \arrow[d, "\text{Proposition 8.2.4}"', "\wr"] & \mathrm{RH}(\omega \mathbin{\widehat{\otimes}} M) \arrow[r] & \mathrm{RH}(g(F,M)) \arrow[r] \arrow[d, "\wr"', "\text{Proposition 8.6.4}"] & {[1]} \\
M[-1] \arrow[r, "\text{id. in deg. 1}"] & C_e \arrow[r, "\text{id. in deg. 0}"] & V[0] \arrow[r, "e"] & M[0]
\end{tikzcd}$$

Proposition 8.6.4 implies that there exists a map $\mathrm{RH}(\omega \mathbin{\widehat{\otimes}} M) \to C_e$ completing it into a morphism of distinguished triangles. Such a map is necessarily a quasi-isomorphism.

The square (2) commutes by construction. $\mathrm{H}^0(C_e)$ is finitely generated by Lemma 8.6.2. Applying Lemma 8.6.3 to $C = C_e$ we conclude that the map

$$\mathrm{Hom}_{\mathrm{D}(A)}\big(\mathrm{RH}(\omega \mathbin{\widehat{\otimes}} M), C_e\big) \to \mathrm{Hom}_{\mathrm{D}(A)}\big(\mathrm{RH}(\omega \otimes M), C_e\big)$$

given by composition with $\mathrm{RH}(\omega \otimes M) \to \mathrm{RH}(\omega \mathbin{\widehat{\otimes}} M)$ is injective. So the unicity part of (2) follows. It remains to prove naturality.

Suppose that we are given a commutative square

$$\begin{tikzcd}
V \arrow[r, "\xi"] \arrow[d, "e"'] & W \arrow[d, "f"] \\
M \arrow[r, "u"] & N
\end{tikzcd}$$

where

- W is a finite-dimensional F-vector space,
- N is a locally compact A-module,

- f is an A-linear local isomorphism,
- ξ and u are continuous and A-linear.

Let $C_u \colon C_e \to C_f$ be the morphism of complexes induced by ξ and u. Consider the diagram

$$\begin{tikzcd} \mathrm{RH}(\omega \otimes M) \arrow[r] \arrow[d, "1\otimes u"'] & \mathrm{RH}(\omega \mathbin{\widehat{\otimes}} M) \arrow[r, "\sim"] \arrow[d, "1\widehat{\otimes} u"'] & C_e \arrow[d, "C_u"] \\ \mathrm{RH}(\omega \otimes N) \arrow[r] & \mathrm{RH}(\omega \mathbin{\widehat{\otimes}} N) \arrow[r, "\sim"] & C_f \end{tikzcd}$$

We want to prove that the right square is commutative. Now $\mathrm{H}^0(C_f)$ is finitely generated by Lemma 8.6.2 so Lemma 8.6.3 implies that the map

$$\mathrm{Hom}_{\mathrm{D}(A)}\big(\mathrm{RH}(\omega \mathbin{\widehat{\otimes}} M),\, C_f\big) \to \mathrm{Hom}_{\mathrm{D}(A)}\big(\mathrm{RH}(\omega \otimes M),\, C_f\big)$$

given by composition with $\mathrm{RH}(\omega \otimes M) \to \mathrm{RH}(\omega \mathbin{\widehat{\otimes}} M)$ is injective. It is thus enough to prove that the outer rectangle is commutative. However the outer rectangle decomposes into two commutative squares

$$\begin{tikzcd} \mathrm{RH}(\omega \otimes M) \arrow[r, "\text{Prp. 8.2.4}"] \arrow[d, "1\otimes u"'] & M[-1] \arrow[r, "\text{id. in deg. 1}"] \arrow[d, "u"'] & C_e \arrow[d, "C_u"] \\ \mathrm{RH}(\omega \otimes N) \arrow[r, "\text{Prp. 8.2.4}"] & N[-1] \arrow[r, "\text{id. in deg. 1}"] & C_f \end{tikzcd}$$

by construction. □

8.7. The main lemma

The following simple lemma is our main tool to compute the cohomology of Hom shtukas associated to Drinfeld modules.

Let A and B be associative unital $\mathbb{F}_q$-algebras. For all left $A \otimes B$-modules M and N we have $\mathrm{Hom}_{A\otimes B}(M, N) = \mathrm{Hom}_{A\otimes A}(A, \mathrm{Hom}_B(M, N))$. Taking derived functors we obtain a natural map $\mathrm{RHom}_{A\otimes B}(M, N) \to \mathrm{RH}(A, \mathrm{RHom}_B(M, N))$.

Lemma 8.7.1. *For all left $A \otimes B$-modules M and N the natural map*

$$\mathrm{RHom}_{A\otimes B}(M, N) \to \mathrm{RH}(A, \mathrm{RHom}_B(M, N))$$

is a quasi-isomorphism.

In applications A will be commutative in which case the quasi-isomorphism will be A-linear.

Proof of Lemma 8.7.1. Let M be a projective left $A\otimes B$-module and N an injective left $A \otimes B$-module. We claim that $\mathrm{Hom}_B(M, N)$ is an injective (A, A)-bimodule. Indeed M is a flat A-module since it is a direct summand of a free $A \otimes B$-module and $A \otimes B$ is A-flat since B is $\mathbb{F}_q$-flat. So the functor

$$\mathrm{Hom}_{A\otimes A^\circ}(-, \mathrm{Hom}_B(M, N)) = \mathrm{Hom}_{A\otimes B}(- \otimes_A M, N).$$

is exact as the composition of exact functors $- \otimes_A M$ and $\mathrm{Hom}_{A\otimes B}(-, N)$.

Let M and N be arbitrary left $A \otimes B$-modules. Let $M^\bullet$ be a projective resolution of M and $N^\bullet$ an injective resolution of N. The argument above shows that $\mathrm{Hom}^\bullet_B(M^\bullet, N^\bullet)$ is a bounded below complex of injective (A, A)-bimodules. Hence

$$\begin{aligned} \mathrm{RHom}_{A\otimes A^\circ}(A, \mathrm{Hom}^\bullet_B(M^\bullet, N^\bullet)) &= \mathrm{Hom}_{A\otimes A^\circ}(A, \mathrm{Hom}^\bullet_B(M^\bullet, N^\bullet)) \\ &= \mathrm{Hom}^\bullet_{A\otimes B}(M^\bullet, N^\bullet) \\ &= \mathrm{RHom}_{A\otimes B}(M, N). \end{aligned}$$

Next we claim that $M^\bullet$ is a projective resolution of M as a B-module. Indeed if P is a projective left $A \otimes B$-module then it is a direct summand of a free $A \otimes$

B-module. However $A \otimes B$ is a projective B-module since $\mathrm{Hom}_B(A \otimes B, -) = \mathrm{Hom}_{\mathbb{F}_q}(A, -)$. So P is a projective B-module and $M^\bullet$ is a projective B-module resolution of M. It follows that

$$\mathrm{RHom}_B(M, N) = \mathrm{Hom}^\bullet_B(M^\bullet, N[0]) = \mathrm{Hom}^\bullet_B(M^\bullet, N^\bullet).$$

Thus $\mathrm{RHom}_{A\otimes A^\circ}(A, \mathrm{RHom}_B(M, N)) = \mathrm{RHom}_{A\otimes A^\circ}(A, \mathrm{Hom}^\bullet_B(M^\bullet, N^\bullet))$. □

More generally this lemma applies to associative unital algebras A, B over an arbitrary commutative ring k provided that A is a projective k-module and B is a flat k-module.

8.8. The Hom shtuka

Let A be a coefficient ring, E a Drinfeld A-module over a reduced[1] $\mathbb{F}_q$-algebra B and M the motive of E. As in Chapter 7 we denote $M^{\geqslant 1}$ the positive degree part of M and $\Omega = M/M^{\geqslant 1}$.

We equip $A \otimes B$ with an endomorphism τ acting as the identity on A and as the q-Frobenius on B. Note that $(A \otimes B)\{\tau\} = A \otimes (B\{\tau\})$ where the latter τ is the q-power map. The motive $M = \mathrm{Hom}(E, \mathbb{G}_a)$ carries a natural left action of $B\{\tau\}$ via $\mathbb{G}_a$ and a right action of A via E. Since A is commutative we can view M as a left $(A \otimes B)\{\tau\}$-module.

In this section we study the sthuka $\mathcal{H}\mathrm{om}_{A\otimes B}(M, N)$ for varying left $A \otimes B\{\tau\}$-modules N. To simplify the expressions we will generally write $\mathcal{H}\mathrm{om}(M, N)$ and $\mathrm{Hom}(M, N)$ omitting the subscript $A \otimes B$.

Proposition 8.8.1. *The shtuka $\mathcal{H}\mathrm{om}(M, N)$ is represented by the diagram*

$$\mathrm{Hom}(M, N) \overset{i}{\underset{j}{\rightrightarrows}} \mathrm{Hom}(M^{\geqslant 1}, N)$$

where i is the restriction to $M^{\geqslant 1}$ and j sends an element f to the map $\tau m \mapsto \tau f(m)$.

Proof. According to Proposition 7.1.12 the adjoint $\tau^* M \to M$ of the multiplication map $\tau\colon M \to M$ is injective with image $M^{\geqslant 1}$. So the result is a consequence of Lemma 1.13.2. □

Proposition 8.8.2. *There is a natural quasi-isomorphism*

$$\mathrm{R}\Gamma(\mathcal{H}\mathrm{om}(M, N)) \xrightarrow{\sim} \mathrm{RH}(A, \mathrm{Hom}_{B\{\tau\}}(M, N)).$$

It is the following composition:

$$\begin{aligned} \mathrm{R}\Gamma(\mathcal{H}\mathrm{om}(M, N)) &\xleftarrow{\sim} \mathrm{RHom}_{A\otimes B\{\tau\}}(M, N) && \text{(Theorem 1.13.4)} \\ &\xrightarrow{\sim} \mathrm{RH}(A, \mathrm{Hom}_{B\{\tau\}}(M, N)) && \text{(Lemma 8.7.1)} \end{aligned}$$

Proof. Indeed $\mathrm{Hom}_{B\{\tau\}}(M, N) = \mathrm{RHom}_{B\{\tau\}}(M, N)$ since M is a projective left $B\{\tau\}$-module by Corollary 7.1.7. □

Consider the quotient $\Omega = M/M^{\geqslant 1}$. Observe that $\mathrm{Hom}_{A\otimes B}(\Omega, N)$ is naturally a submodule of $\mathrm{Hom}_{A\otimes B}(M, N)$.

Lemma 8.8.3. $\mathrm{Hom}_{A\otimes B}(\Omega, N) = \mathrm{H}^0(\nabla\, \mathcal{H}\mathrm{om}(M, N))$.

Proof. Indeed Proposition 8.8.1 shows that $\mathrm{H}^0(\nabla\, \mathcal{H}\mathrm{om}(M, N))$ consists of the $A\otimes B$-linear maps $f\colon M \to N$ which restrict to zero on $M^{\geqslant 1}$. □

The functor $N \mapsto \nabla\, \mathcal{H}\mathrm{om}(M, N)$ is exact. So the isomorphism of Lemma 8.8.3 induces a natural map $\mathrm{RHom}_{A\otimes B}(\Omega, N) \to \mathrm{R}\Gamma(\nabla\, \mathcal{H}\mathrm{om}(M, N))$.

[1]The assumption that B is reduced is inessential for the theory discussed in this chapter. However in Chapter 7 we defined the degree filtration on M under this assumption, so we are obliged to keep it here as well.

Lemma 8.8.4. *The natural map* $\mathrm{RHom}_{A\otimes B}(\Omega, N) \to \mathrm{R}\Gamma(\nabla\,\mathcal{H}\mathrm{om}(M, N))$ *is a quasi-isomorphism.*

Proof. The short exact sequence $0 \to M^{\geqslant 1} \to M \to \Omega \to 0$ is a projective $A\otimes B$-module resolution of Ω. Therefore

$$\mathrm{RHom}_{A\otimes B}(\Omega, N) = \Big[\mathrm{Hom}_{A\otimes B}(M, N) \xrightarrow{i} \mathrm{Hom}_{A\otimes B}(M^{\geqslant 1}, N)\Big]$$

where i is the restriction to $M^{\geqslant 1}$. Now Theorem 1.9.1 in combination with Proposition 8.8.1 shows that the complex above computes $\mathrm{R}\Gamma(\nabla\,\mathcal{H}\mathrm{om}(M, N))$. ☐

Proposition 8.8.5. *There is a natural quasi-isomorphism*

$$\mathrm{R}\Gamma(\nabla\,\mathcal{H}\mathrm{om}(M, N)) \xrightarrow{\sim} \mathrm{RH}(A,\, \mathrm{Hom}_B(\Omega, N)).$$

It is the following composition:

$$\begin{aligned} \mathrm{R}\Gamma(\nabla\,\mathcal{H}\mathrm{om}(M, N)) &\xleftarrow{\simeq} \mathrm{RHom}_{A\otimes B}(\Omega, N) && \text{(Lemma 8.8.4)} \\ &\xrightarrow{\sim} \mathrm{RH}(A,\, \mathrm{Hom}_B(\Omega, N)) && \text{(Lemma 8.7.1)} \end{aligned}$$

Proof. Ω is an invertible B-module so $\mathrm{Hom}_B(\Omega, N) = \mathrm{RHom}_B(\Omega, N)$ and the result follows. ☐

8.9. Formulas of Barsotti-Weil type

We keep the setting of the previous section. As usual ω stands for the module of Kähler differentials of A over $\mathbb{F}_q$. Given a map $m\colon E \to \mathbb{G}_a$ we denote dm the induced map from Lie_E to $\mathrm{Lie}_{\mathbb{G}_a}$. In the following S is an arbitrary B-algebra.

Lemma 8.9.1. *The natural map*

$$\omega \otimes \mathrm{Lie}_E(S) \to \mathrm{Hom}_B(\Omega,\, \omega\otimes S), \quad \eta\otimes\alpha \mapsto (dm \mapsto \eta\otimes dm(\alpha))$$

is an isomorphism of (A, S)*-bimodules.*

Proof. The natural map $\omega\otimes\mathrm{Hom}_B(\Omega, S) \to \mathrm{Hom}_B(\Omega,\, \omega\otimes S)$ is an isomorphism since Ω is a finitely generated B-module. Proposition 7.1.13 identifies $\mathrm{Hom}_B(\Omega, S)$ with $\mathrm{Lie}_E(S)$ and the result follows. ☐

Lemma 8.9.2. *The natural map*

$$\omega\otimes E(S) \to \mathrm{Hom}_{B\{\tau\}}(M,\, \omega\otimes S), \quad \eta\otimes e \mapsto (m \mapsto \eta\otimes m(e))$$

is an isomorphism of (A, A)*-bimodules.*

Proof. The motive M is a finitely generated $B\{\tau\}$-module by Proposition 7.1.6. Hence the natural map $\omega\otimes\mathrm{Hom}_{B\{\tau\}}(M, S) \to \mathrm{Hom}_{B\{\tau\}}(M,\, \omega\otimes S)$ is an isomorphism. Proposition 7.1.10 identifies $\mathrm{Hom}_{B\{\tau\}}(M, S)$ with $E(S)$ and we get the result. ☐

Combining the lemmas above with the results of Section 8.8 and the isomorphism $\mathrm{RH}(A,\, \omega\otimes N) \cong N[-1]$ of Proposition 8.2.4 we obtain the formulas of Barsotti-Weil type annouced in the introduction. In the case $A = \mathbb{F}_q[t]$ related results were obtained in [19, 23].

Theorem 8.9.3. *There is a natural* $A\otimes S$*-linear quasi-isomorphism*

$$\mathrm{R}\Gamma(\nabla\,\mathcal{H}\mathrm{om}(M,\, \omega\otimes S)) \xrightarrow{\sim} \mathrm{Lie}_E(S)[-1].$$

It is given by the composition

$$\begin{aligned} \mathrm{R}\Gamma(\nabla\,\mathcal{H}\mathrm{om}(M,\, \omega\otimes S)) &\xrightarrow{\sim} \mathrm{RH}(A,\, \mathrm{Hom}_B(\Omega,\, \omega\otimes S)) && \text{(Proposition 8.8.5)} \\ &\xleftarrow{\simeq} \mathrm{RH}(A,\, \omega\otimes\mathrm{Lie}_E(S)) && \text{(Lemma 8.9.1)} \\ &\xrightarrow{\sim} \mathrm{Lie}_E(S)[-1] && \text{(Proposition 8.2.4)} \end{aligned}$$ ☐

Theorem 8.9.4. *There is a natural A-linear quasi-isomorphism*

$$\mathrm{R\Gamma}(\mathcal{H}\mathrm{om}(M,\, \omega \otimes S)) \xrightarrow{\sim} E(S)[-1].$$

It is given by the composition

$$\begin{aligned}
\mathrm{R\Gamma}(\mathcal{H}\mathrm{om}(M,\, \omega \otimes S)) &\xrightarrow{\sim} \mathrm{RH}(A,\, \mathrm{Hom}_{B\{\tau\}}(M,\, \omega \otimes S)) && \text{(Proposition 8.8.2)}\\
&\xleftarrow{\sim} \mathrm{RH}(A,\, \omega \otimes E(S)) && \text{(Lemma 8.9.2)}\\
&\xrightarrow{\sim} E(S)[-1] && \text{(Proposition 8.2.4)} \quad \square
\end{aligned}$$

8.10. The exponential complex

As before A is a coefficient ring, F the local field of A at infinity and ω the module of Kähler differentials of A over $\mathbb{F}_q$. Let K be a finite product of local fields containing $\mathbb{F}_q$. Fix a Drinfeld A-module E over K. We denote M the motive of E and Ω the quotient $M/M^{\geqslant 1}$. We assume that the action of A on $\mathrm{Lie}_E(K)$ extends to a continuous action of F.

Definition 8.10.1. The *exponential map* of the Drinfeld module E is a map

$$\exp\colon\ \mathrm{Lie}_E(K) \to E(K)$$

satisfying the following conditions:

(1) exp is a homomorphism of A-modules,
(2) exp is an analytic function with derivative 1 at zero in the following sense. Fix an $\mathbb{F}_q$-linear isomorphism of group schemes $E \cong \mathbb{G}_a$. It identifies $E(K)$ with K while its differential identifies $\mathrm{Lie}_E(K)$ with K. We demand that the resulting map $\exp\colon K \to K$ is given by an everywhere convergent power series of the form

$$\exp(z) = z + a_1 z^q + a_2 z^{q^2} + \dots$$

with coefficients in K.

Proposition 8.10.2. *The exponential map exists, is unique and is a local isomorphism in the sense of Definition 2.11.3.*

Proof. For the existence and unicity see [7, Theorem 2.1]. The exponential map is a local isomorphism since it has nonzero derivative at zero. $\square$

Definition 8.10.3. The *exponential complex* of E is the A-module complex

$$C_{\exp} = \Big[\mathrm{Lie}_E(K) \xrightarrow{\exp} E(K)\Big].$$

Our goal is to show that $\mathrm{R\Gamma}(\mathcal{H}\mathrm{om}(M,\, \omega \mathbin{\widehat{\otimes}} K)) = C_{\exp}$. To do it we apply the results of Section 8.6 to the map $\exp\colon \mathrm{Lie}_E(K) \to E(K)$.

Lemma 8.10.4. *The map $\omega \mathbin{\widehat{\otimes}} \mathrm{Lie}_E(K) \to \mathrm{Hom}_K(\Omega,\, \omega \mathbin{\widehat{\otimes}} K)$ defined by the commutative square*

$$\begin{tikzcd}
\omega \mathbin{\widehat{\otimes}} \mathrm{Lie}_E(K) \arrow[r] \arrow[d, "\mathrm{Res}"', "\wr"] & \mathrm{Hom}_K(\Omega,\, \omega \mathbin{\widehat{\otimes}} K) \arrow[d, "\wr"', "\mathrm{Res}\circ -"] \\
b(F/A,\, \mathrm{Lie}_E(K)) \arrow[r, "f \mapsto (dm \mapsto dm \circ f)"] & \mathrm{Hom}_K(\Omega,\, b(F/A, K))
\end{tikzcd}$$

is an isomorphism of $A \mathbin{\widehat{\otimes}} K$-modules. Furthermore it is compatible with the isomorphism $\omega \otimes \mathrm{Lie}_E(K) \xrightarrow{\sim} \mathrm{Hom}_K(\Omega,\, \omega \otimes K)$ of Lemma 8.9.1.

Proof. The first claim follows since Ω is a free K-module of rank 1. The compatibility follows since Res identifies $\omega \otimes \mathrm{Lie}_E(K)$ with the subspace $a(F/A,\, \mathrm{Lie}_E(K))$ of $b(F/A,\, \mathrm{Lie}_E(K))$. $\square$

Lemma 8.10.5. *The map* $\omega \mathbin{\widehat{\otimes}} E(K) \to \mathrm{Hom}_{K\{\tau\}}(M, \omega \mathbin{\widehat{\otimes}} K)$ *defined by the commutative square*

$$\begin{tikzcd}
\omega \mathbin{\widehat{\otimes}} E(K) \arrow[r] \arrow[d, "\mathrm{Res}"', "\wr"] & \mathrm{Hom}_{K\{\tau\}}(M, \omega \mathbin{\widehat{\otimes}} K) \arrow[d, "\mathrm{Res}\circ -", "\wr"'] \\
b(F/A, E(K)) \arrow[r, "f \mapsto (m \mapsto m\circ f)"] & \mathrm{Hom}_{K\{\tau\}}(M, b(F/A, K))
\end{tikzcd}$$

is an isomorphism of (A, A)*-bimodules. Furthermore it is compatible with the isomorphism* $\omega \otimes E(K) \xrightarrow{\sim} \mathrm{Hom}_{K\{\tau\}}(M, \omega \otimes K)$ *of Lemma 8.9.2.*

Proof. Follows since M is a free $K\{\tau\}$-module of rank 1. □

Lemma 8.10.6. *The map*

$$g(F, \mathrm{Lie}_E(K)) \to \mathrm{Hom}_K(\Omega, g(F, K)), \quad f \mapsto (dm \mapsto dm \circ f)$$

is an isomorphism of $A \otimes K$*-modules.*

Proof. In view of the short exact sequence of Lemma 8.4.2 the claim follows from Lemma 8.10.4. □

Lemma 8.10.7. *The map*

$$g(F, E(K)) \to \mathrm{Hom}_{K\{\tau\}}(M, g(F, K)), \quad f \mapsto (m \mapsto m \circ f)$$

is an isomorphism of (A, A)*-bimodules.*

Proof. Follows from Lemma 8.10.5. □

Theorem 8.10.8. $\mathrm{R\Gamma}(\nabla\, \mathcal{H}\mathrm{om}(M, \omega \mathbin{\widehat{\otimes}} K)) = 0.$

Proof. In view of Lemma 8.10.4 and Proposition 8.8.5 we have

$$\mathrm{R\Gamma}(\nabla\, \mathcal{H}\mathrm{om}(M, \omega \mathbin{\widehat{\otimes}} K)) = \mathrm{RH}(\omega \mathbin{\widehat{\otimes}} \mathrm{Lie}_E(K)).$$

By assumption the action of A on $\mathrm{Lie}_E(K)$ extends to a continuous action of F. So the result follows from Proposition 8.3.2. □

Proposition 8.10.9. *The map*

$$\mathrm{Lie}_E(K) \to \mathrm{Hom}_{A\otimes K}(M, g(F, K)), \quad \alpha \mapsto (m \mapsto (x \mapsto dm(\alpha x)))$$

induces a quasi-isomorphism $\mathrm{Lie}_E(K)[0] \xrightarrow{\sim} \mathrm{R\Gamma}(\nabla\, \mathcal{H}\mathrm{om}(M, g(F, K)))$.

Proof. In view of Lemma 8.10.6 and Proposition 8.8.5 the result is a consequence of Proposition 8.5.1. □

Proposition 8.10.10. *The map*

$$\mathrm{Lie}_E(K) \to \mathrm{Hom}_{A\otimes K}(M, g(F, K)), \quad \alpha \mapsto (m \mapsto (x \mapsto m \exp(\alpha x)))$$

induces a quasi-isomorphism $\mathrm{Lie}_E(K)[0] \xrightarrow{\sim} \mathrm{R\Gamma}(\mathcal{H}\mathrm{om}(M, g(F, K)))$.

Proof. In view of Lemma 8.10.7 and Proposition 8.8.2 the result is a consequence of Proposition 8.6.4. □

We state the following theorem for the sake of completeness: it will not be used in the proof of the class number formula.

Theorem 8.10.11. *There exists a quasi-isomorphism*

$$\mathrm{R\Gamma}(\mathcal{H}\mathrm{om}(M, \omega \mathbin{\widehat{\otimes}} K)) \xrightarrow{\sim} C_{\exp}$$

with the following properties.

(1) *It is natural in* E *and* K.

(2) *It is the unique map in the derived category of A-modules such that the square*

$$\begin{tikzcd} \mathrm{R}\Gamma(\mathcal{H}\mathrm{om}(M,\, \omega \otimes K)) \arrow[r] \arrow[d, "\text{Thm. 8.9.4}"', "\wr"] & \mathrm{R}\Gamma(\mathcal{H}\mathrm{om}(M,\, \omega \mathbin{\widehat{\otimes}} K)) \arrow[d] \\ E(K)[-1] \arrow[r, "\text{identity in degree 1}"] & C_{\exp} \end{tikzcd}$$

is commutative.

(3) *It makes the square*

$$\begin{tikzcd} \mathrm{R}\Gamma(\mathcal{H}\mathrm{om}(M,\, \omega \mathbin{\widehat{\otimes}} K)) \arrow[r] \arrow[d] & \mathrm{R}\Gamma(\mathcal{H}\mathrm{om}(M,\, g(F,K))) \\ C_{\exp} \arrow[r, "\text{identity in degree 0}"] & \mathrm{Lie}_E(K)[0] \arrow[u, "\wr", "\text{Prp. 8.10.10}"'] \end{tikzcd}$$

commutative.

Proof. In view of Lemma 8.10.5 and Proposition 8.8.2 the result is a consequence of Theorem 8.6.5. □

8.11. The units complex

In this section we prove a refined variant of the formula from the previous section. It is one of the key tools in the proof of the class number formula.

Let K be a finite product of local fields and $R \subset K$ an $\mathbb{F}_q$-subalgebra. We work with a Drinfeld A-module E over R. As usual the motive of E is denoted M and Ω stands for the quotient $M/M^{\geqslant 1}$. We make the following assumptions:

(1) R is discrete in K.
(2) The action of A on $\mathrm{Lie}_E(K)$ extends to a continuous action of F.

In particular we have the exponential map $\exp\colon \mathrm{Lie}_E(K) \to E(K)$ as in Definition 8.10.1.

Definition 8.11.1. The *units complex* of E is the A-module complex

$$U_E = \Big[\, \mathrm{Lie}_E(K) \xrightarrow{\exp} \frac{E(K)}{E(R)} \,\Big].$$

In the following we denote $Q = K/R$. Our aim is to prove that

$$\mathrm{R}\Gamma(\mathcal{H}\mathrm{om}(M,\, \omega \mathbin{\widehat{\otimes}} Q)) = U_E.$$

To imporve the legibility we will write $\mathrm{Lie}_E(Q)$ for the quotient $\mathrm{Lie}_E(K)/\mathrm{Lie}_E(R)$ and $E(Q)$ for the quotient $E(K)/E(R)$.

Lemma 8.11.2. *Let V be a linearly topologized Hausdorff $\mathbb{F}_q$-vector space, $V_0 \subset V$ a discrete subspace and $V_1 = V/V_0$. Then the sequence*

$$0 \to \omega \otimes V_0 \to \omega \mathbin{\widehat{\otimes}} V \to \omega \mathbin{\widehat{\otimes}} V_1 \to 0$$

is exact.

Corollary 8.11.3. *The sequence* $0 \to \omega \otimes R \to \omega \mathbin{\widehat{\otimes}} K \to \omega \mathbin{\widehat{\otimes}} Q \to 0$ *is exact.* □

Corollary 8.11.4. *The sequences*

$$0 \to \omega \otimes \mathrm{Lie}_E(R) \to \omega \mathbin{\widehat{\otimes}} \mathrm{Lie}_E(K) \to \omega \mathbin{\widehat{\otimes}} \mathrm{Lie}_E(Q) \to 0$$
$$0 \to \omega \otimes E(R) \to \omega \mathbin{\widehat{\otimes}} E(K) \to \omega \mathbin{\widehat{\otimes}} E(Q) \to 0$$

are exact. □

Proof of Lemma 8.11.2. Let $U \subset V$ be an open $\mathbb{F}_q$-vector subspace such that $V_0 \cap U = \{0\}$. It maps isomorphically onto an open subpace $U_1 \subset V_1$. The quotients V/U and V_1/U_1 are discrete so that $\omega \mathbin{\widehat{\otimes}} (V/U) = \omega \otimes (V/U)$ and $\omega \mathbin{\widehat{\otimes}} (V_1/U_1) = \omega \otimes (V_1/U_1)$. We then have a commutative diagram

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
 & 0 \arrow[r] \arrow[d] & \omega \mathbin{\widehat{\otimes}} U \arrow[r] \arrow[d] & \omega \mathbin{\widehat{\otimes}} U_1 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \omega \otimes V_0 \arrow[r] \arrow[d] & \omega \mathbin{\widehat{\otimes}} V \arrow[r] \arrow[d] & \omega \mathbin{\widehat{\otimes}} V_1 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \omega \otimes V_0 \arrow[r] \arrow[d] & \omega \otimes V/U \arrow[r] \arrow[d] & \omega \otimes V_1/U_1 \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

Top and bottom rows and the left column are clearly exact. The middle and the right columns are exact since the open embeddings $U \to V$ and $U_1 \to V_1$ are continuously split by Lemma 2.3.1. Hence the middle row is exact. $\square$

Lemma 8.11.5. *The map $\omega \mathbin{\widehat{\otimes}} \mathrm{Lie}_E(Q) \to \mathrm{Hom}_R(\Omega, \omega \mathbin{\widehat{\otimes}} Q)$ defined by the commutative square*

$$\begin{tikzcd}
\omega \mathbin{\widehat{\otimes}} \mathrm{Lie}_E(Q) \arrow[rr] \arrow[d, "\mathrm{Res}"', "\wr"] & & \mathrm{Hom}_R(\Omega, \omega \mathbin{\widehat{\otimes}} Q) \arrow[d, "\mathrm{Res} \circ -", "\wr"'] \\
b(F/A, \mathrm{Lie}_E(Q)) \arrow[rr, "f \mapsto (dm \mapsto dm \circ f)"] & & \mathrm{Hom}_R(\Omega, b(F/A, Q))
\end{tikzcd}$$

is an isomorphism of $A \otimes R$-modules. It is compatible with the isomorphisms of Lemma 8.10.4 and of Lemma 8.9.1 for R and K.

Proof. Follows from Lemmas 8.9.1 and 8.10.4 in view of Corollary 8.11.4. $\square$

Lemma 8.11.6. *The map $\omega \mathbin{\widehat{\otimes}} E(Q) \to \mathrm{Hom}_{R\{\tau\}}(M, \omega \mathbin{\widehat{\otimes}} Q)$ defined by the commutative square*

$$\begin{tikzcd}
\omega \mathbin{\widehat{\otimes}} E(Q) \arrow[rr] \arrow[d, "\mathrm{Res}"', "\wr"] & & \mathrm{Hom}_{R\{\tau\}}(M, \omega \mathbin{\widehat{\otimes}} Q) \arrow[d, "\mathrm{Res} \circ -", "\wr"'] \\
b(F/A, E(Q)) \arrow[rr, "f \mapsto (m \mapsto m \circ f)"] & & \mathrm{Hom}_{R\{\tau\}}(M, b(F/A, Q))
\end{tikzcd}$$

is an isomorphism of (A, A)-bimodules. It is compatible with the isomorphisms of Lemma 8.10.5 and of Lemma 8.9.2 for R and K.

Proof. Follows from Lemmas 8.9.2 and 8.10.5 in view of Corollary 8.11.4. $\square$

Corollary 8.11.3 implies that the map $[\mathrm{Res}]\colon \omega \mathbin{\widehat{\otimes}} K \to g(F, K)$ factors uniquely over the natural map $\omega \mathbin{\widehat{\otimes}} K \to \omega \mathbin{\widehat{\otimes}} Q$. We thus get a map $\omega \mathbin{\widehat{\otimes}} Q \to g(F, K)$.

Theorem 8.11.7. *There exists a unique quasi-isomorphism*

$$\mathrm{R}\Gamma(\nabla\, \mathcal{H}\mathrm{om}(M, \omega \mathbin{\widehat{\otimes}} Q)) \xrightarrow{\sim} \mathrm{Lie}_E(R)[0]$$

such that the square

$$\begin{tikzcd} \mathrm{R}\Gamma(\nabla\,\mathcal{H}\mathrm{om}(M,\,\omega\,\widehat{\otimes}\,Q)) \arrow[r] \arrow[d] & \mathrm{R}\Gamma(\nabla\,\mathcal{H}\mathrm{om}(M,\,g(F,K))) \\ \mathrm{Lie}_E(R)[0] \arrow[r] & \mathrm{Lie}_E(K)[0] \arrow[u, "\wr"', "\text{Prp. 8.10.10}"] \end{tikzcd}$$

is commutative. It is natural in R, K and E.

Proof. Proposition 8.8.5 and Lemma 8.11.5 produce a natural quasi-isomorphism

$$\mathrm{R}\Gamma(\nabla\,\mathcal{H}\mathrm{om}(M,\,\omega\,\widehat{\otimes}\,Q)) = \mathrm{RH}(\omega\,\widehat{\otimes}\,\mathrm{Lie}_E(Q)).$$

Let $e\colon \mathrm{Lie}_E(K) \to \mathrm{Lie}_E(Q)$ be the natural map. Observe that $\mathrm{Lie}_E(R)[0]$ is canonically the mapping fiber of e. So the existence of the quasi-isomorphism in question follows by Theorem 8.6.5 applied to the local isomorphism e. The unicity part is then clear. □

Applying Theorem 8.9.4 to R and K we get a natural quasi-isomorphism

$$\mathrm{R}\Gamma(\mathcal{H}\mathrm{om}(M,\,\omega\otimes Q)) \xrightarrow{\sim} E(Q)[-1]. \tag{8.6}$$

Theorem 8.11.8. *There exists a quasi-isomorphism*

$$\mathrm{R}\Gamma(\mathcal{H}\mathrm{om}(M,\,\omega\,\widehat{\otimes}\,Q)) \xrightarrow{\sim} U_E$$

with the following properties:

(1) *It is natural in R, K and E.*
(2) *It is the unique map in the derived category of A-modules such that the square*

$$\begin{tikzcd} \mathrm{R}\Gamma(\mathcal{H}\mathrm{om}(M,\,\omega\otimes Q)) \arrow[r] \arrow[d, "\wr", "(8.6)"'] & \mathrm{R}\Gamma(\mathcal{H}\mathrm{om}(M,\,\omega\,\widehat{\otimes}\,Q)) \arrow[d] \\ E(Q)[-1] \arrow[r, "\text{identity in degree 1}"] & U_E \end{tikzcd}$$

is commutative.
(3) *It makes the square*

$$\begin{tikzcd} \mathrm{R}\Gamma(\mathcal{H}\mathrm{om}(M,\,\omega\,\widehat{\otimes}\,Q)) \arrow[r] \arrow[d] & \mathrm{R}\Gamma(\mathcal{H}\mathrm{om}(M,\,g(F,K))) \\ U_E \arrow[r, "\text{identity in degree 0}"] & \mathrm{Lie}_E(K)[0] \arrow[u, "\wr"', "\text{Prp. 8.10.10}"] \end{tikzcd}$$

commutative.

Proof. In view of Lemma 8.11.6 and Proposition 8.8.2 the result is a consequence of Theorem 8.6.5 applied to the local isomorphism $\exp\colon \mathrm{Lie}_E(K) \to E(Q)$. □

8.12. Tate modules and Galois action

As usual A is a coefficient ring and ω is the module of Kähler differentials of A over $\mathbb{F}_q$. We work with a Drinfeld A-module E over a field k containing $\mathbb{F}_q$. We denote M the motive of E. In this section we study the Hom shtuka $\mathcal{H}\mathrm{om}(M, \omega\otimes k)$ and its relation to the Tate modules of E.

The main results of this section are due to David Goss [11, Section 5.6]. He stated them in terms of the motive M rather than the Hom shtuka. We give a different argument which uses the Hom shtuka approach.

Let k^s be a separable closure of k and let $G_k = \mathrm{Aut}(k^s/k)$ be the Galois group. We equip k and k^s with the discrete topology. Fix a maximal ideal $\mathfrak{p}$ of A. We denote $A_\mathfrak{p}$ the completion of A at $\mathfrak{p}$ and $F_\mathfrak{p}$ the fraction field of $A_\mathfrak{p}$. Set

$\omega_\mathfrak{p} = \omega \otimes_A A_\mathfrak{p}$. The ring $A_\mathfrak{p}$, the field $F_\mathfrak{p}$ and the module $\omega_\mathfrak{p}$ are assumed to carry the $\mathfrak{p}$-adic topologies.

We equip $\omega_\mathfrak{p} \mathbin{\widehat{\otimes}} k^s$ with an endomorphism τ acting as the identity on $\omega_\mathfrak{p}$ and as the q-Frobenius on k^s. So $\omega_\mathfrak{p} \mathbin{\widehat{\otimes}} k^s$ becomes a left $A_\mathfrak{p} \mathbin{\widehat{\otimes}} k^s\{\tau\}$-module.

Proposition 8.12.1. *There exists a G_k-equivariant $A_\mathfrak{p}$-module isomorphism*

$$T_\mathfrak{p} E \xrightarrow{\sim} \operatorname{Hom}_{A\otimes k\{\tau\}}(M,\, \omega_\mathfrak{p} \mathbin{\widehat{\otimes}} k^s)$$

The Galois group G_k acts on the right hand side via k^s.

One can prove that $\operatorname{RHom}_{A\otimes k\{\tau\}}(M,\, \omega_\mathfrak{p} \mathbin{\widehat{\otimes}} k^s) = T_\mathfrak{p} E[0]$. We will only need the less precise result above.

Proof of Proposition 8.12.1. First we construct a natural isomorphism

$$c(F_\mathfrak{p}/A_\mathfrak{p},\, E(k^s)) \xrightarrow{\sim} \operatorname{Hom}_{k\{\tau\}}(M,\, \omega_\mathfrak{p} \mathbin{\widehat{\otimes}} k^s). \tag{8.7}$$

Since k is a field the motive M is a free $k\{\tau\}$-module of rank 1. Hence the map

$$c(F_\mathfrak{p}/A_\mathfrak{p},\ E(k^s)) \to \operatorname{Hom}_{k\{\tau\}}(M,\, c(F_\mathfrak{p}/A_\mathfrak{p}, k^s)), \quad f \mapsto (m \mapsto m \circ f)$$

is an isomorphism. Corollary 3.9.2 identifies $\omega_\mathfrak{p} \mathbin{\widehat{\otimes}} k^s$ with $c(F_\mathfrak{p}/A_\mathfrak{p}, k^s)$. Combining it with the isomorphism above we get (8.7).

Now (8.7) identifies $\operatorname{Hom}_{A\otimes k\{\tau\}}(M,\, \omega_\mathfrak{p} \mathbin{\widehat{\otimes}} k^s)$ with the $A_\mathfrak{p}$-module of continuous A-linear maps from $F_\mathfrak{p}/A_\mathfrak{p}$ to $E(k^s)$. As $F_\mathfrak{p}/A_\mathfrak{p}$ is discrete it follows that the latter module is

$$\operatorname{Hom}_A(F_\mathfrak{p}/A_\mathfrak{p},\, E(k^s)) = T_\mathfrak{p} E.$$

We get the result since all the isomorphisms used above are G_k-equivariant by construction. □

Lemma 8.12.2. *$\omega \otimes k$ is an $A \otimes k$-lattice in the $A_\mathfrak{p} \mathbin{\widehat{\otimes}} k^s$-module $\omega_\mathfrak{p} \mathbin{\widehat{\otimes}} k^s$.*

Proof. Follows from Lemma 3.2.4. □

As $\operatorname{Lie}_E(k)$ is a one-dimensional k-vector space the action of A on $\operatorname{Lie}_E(k)$ determines a homomorphism $A \to k$. The kernel of this homomorphism is called the *characteristic* of E.

Lemma 8.12.3. *If $\mathfrak{p}$ is different from the characteristic of E then*

$$\mathrm{R}\Gamma(\nabla\, \mathcal{H}\mathrm{om}(M,\, \omega_\mathfrak{p} \mathbin{\widehat{\otimes}} k^s) = 0.$$

Proof. Lemma 8.12.2 implies that

$$\nabla\, \mathcal{H}\mathrm{om}(M,\, \omega \otimes k) \otimes_{A\otimes k} (A_\mathfrak{p} \mathbin{\widehat{\otimes}} k^s) = \nabla\, \mathcal{H}\mathrm{om}(M,\, \omega_\mathfrak{p} \mathbin{\widehat{\otimes}} k^s).$$

At the same time Theorem 8.9.3 provides a natural quasi-isomorphism

$$\mathrm{R}\Gamma(\nabla\, \mathcal{H}\mathrm{om}(M,\, \omega \otimes k)) = \operatorname{Lie}_E(k)[-1].$$

Since $A_\mathfrak{p} \mathbin{\widehat{\otimes}} k^s$ is flat over $A \otimes k$ it is enough to prove that

$$\operatorname{Lie}_E(k) \otimes_{A\otimes k} (A_\mathfrak{p} \mathbin{\widehat{\otimes}} k^s) = 0.$$

Now $\mathfrak{p}$ is different from the characteristic of E so there exists an element of $\mathfrak{p}$ which does not act on $\operatorname{Lie}_E(k)$ by zero. It then acts by an automorphism and we conclude that $\operatorname{Lie}_E(k) \otimes_A A/\mathfrak{p} = 0$. At the same time $A_\mathfrak{p} \mathbin{\widehat{\otimes}} k^s$ is the completion of the noetherian ring $A \otimes k^s$ with respect to the ideal $\mathfrak{p} \otimes k^s$. Since $\operatorname{Lie}_E(k)$ is a finitely generated $A \otimes k$-module Nakayama's lemma shows that $\operatorname{Lie}_E(k) \otimes_{A\otimes k} (A_\mathfrak{p} \mathbin{\widehat{\otimes}} k^s)$ is zero. □

We will also need a lemma from Dieudonné-Manin theory.

Lemma 8.12.4. *Let N be a left $A_\mathfrak{p} \mathbin{\widehat{\otimes}} k^s\{\tau\}$-module which is finitely generated free as an $A_\mathfrak{p} \mathbin{\widehat{\otimes}} k^s$-module. If the adjoint $\tau^* N \to N$ of the τ-multiplication map is bijective then the natural map $N^{\tau=1} \otimes_{A_\mathfrak{p}} (A_\mathfrak{p} \mathbin{\widehat{\otimes}} k^s) \to N$ is an isomorphism.*

Proof. See [22, Proposition 4.4]. □

Proposition 8.12.5. *If $\mathfrak{p}$ is different from the characteristic of E then the natural map*

$$\mathrm{Hom}_{A\otimes k\{\tau\}}(M,\, \omega_{\mathfrak{p}} \mathbin{\widehat{\otimes}} k^s) \otimes_{A_{\mathfrak{p}}} (A_{\mathfrak{p}} \mathbin{\widehat{\otimes}} k^s) \to \mathrm{Hom}_{A\otimes k}(M,\, \omega_{\mathfrak{p}} \mathbin{\widehat{\otimes}} k^s)$$

is an isomorphism.

Proof. Consider the shtuka

$$\mathcal{H}\mathrm{om}(M,\, \omega_{\mathfrak{p}} \mathbin{\widehat{\otimes}} k^s) = \Big[\mathrm{Hom}_{A\otimes k}(M,\, \omega_{\mathfrak{p}} \mathbin{\widehat{\otimes}} k^s) \overset{i}{\underset{j}{\rightrightarrows}} \mathrm{Hom}_{A\otimes k}(\tau^* M,\, \omega_{\mathfrak{p}} \mathbin{\widehat{\otimes}} k^s)\Big].$$

Lemma 8.12.3 implies that the arrow i is bijective. We equip the $A_{\mathfrak{p}} \mathbin{\widehat{\otimes}} k^s$-module

$$H = \mathrm{Hom}_{A\otimes k}(M,\, \omega_{\mathfrak{p}} \mathbin{\widehat{\otimes}} k^s)$$

with the τ-linear endomorphism $i^{-1}j$.

The adjoint $\tau^*(\omega_{\mathfrak{p}} \mathbin{\widehat{\otimes}} k^s) \to \omega_{\mathfrak{p}} \mathbin{\widehat{\otimes}} k^s$ of the τ-multiplication map is an isomorphism. It follows from the definition of $\mathcal{H}\mathrm{om}$ that the τ-adjoint of the arrow j above is an isomorphism. We can thus apply Lemma 8.12.4 and conclude that the natural map

$$H^{i^{-1}j=1} \otimes_{A_{\mathfrak{p}}} (A_{\mathfrak{p}} \mathbin{\widehat{\otimes}} k^s) \to H$$

is an isomorphism. Proposition 1.13.3 identifies the module $\mathrm{Hom}_{A\otimes k\{\tau\}}(M,\, \omega_{\mathfrak{p}} \widehat{\otimes} k^s)$ with $\mathrm{H}^0(\mathcal{H}\mathrm{om}(M,\, \omega_{\mathfrak{p}} \mathbin{\widehat{\otimes}} k^s))$ which is equal to $H^{i^{-1}j=1}$ by Proposition 1.4.4. □

Proposition 8.12.6. *Assume that k is a finite extension of $\mathbb{F}_q$ of degree d. The shtuka*

$$\mathcal{H}\mathrm{om}(M,\, \omega \otimes k) = \Big[\mathrm{Hom}_{A\otimes k}(M,\, \omega \otimes k) \overset{i}{\underset{j}{\rightrightarrows}} \mathrm{Hom}_{A\otimes k}(\tau^* M,\, \omega \otimes k)\Big]$$

has the following properties.

1. *The arrow j is a bijection.*
2. *Let $\sigma \in G_k$ be the arithmetic Frobenius element. For every prime $\mathfrak{p}$ of A different from the characteristic of E we have an identity*

   $$\det_{A_{\mathfrak{p}}}\big(T - \sigma \,\big|\, T_{\mathfrak{p}} E\big) = \det_{A\otimes k}\big(T - (j^{-1}i)^d \,\big|\, \mathrm{Hom}_{A\otimes k}(M,\, \omega \otimes k)\big)$$

   in $A_{\mathfrak{p}} \otimes k[T]$.

Proof. (1) The endomorphism τ of $\omega \otimes k$ is an automorphism since k is a finite extension of $\mathbb{F}_q$. From the definition of $\mathcal{H}\mathrm{om}$ it follows that the arrow j is bijective.

(2) Given an $A\otimes k$-module N we will denote ψ the endomorphism of the module $\mathrm{Hom}_{A\otimes k}(M, N)$ which sends a map f to the map $m \mapsto f(\tau^d m)$.

Proposition 8.12.1 identifies $T_{\mathfrak{p}} E$ and $\mathrm{Hom}_{A\otimes k\{\tau\}}(M,\, \omega_{\mathfrak{p}} \mathbin{\widehat{\otimes}} k^s)$ on which G_k acts via k^s. So if f is an element of the latter module then $\sigma(f) = \tau^d \circ f$. As f is a homomorphism of left $A \otimes k\{\tau\}$-modules it follows that $\tau^d \circ f = \psi(f)$. We conclude that

$$\det_{A_{\mathfrak{p}}}\big(T - \sigma \,\big|\, T_{\mathfrak{p}} E\big) = \det_{A_{\mathfrak{p}}}\big(T - \psi \,\big|\, \mathrm{Hom}_{A\otimes k\{\tau\}}(M,\, \omega_{\mathfrak{p}} \mathbin{\widehat{\otimes}} k^s)\big).$$

Next, Proposition 8.12.5 implies that the latter polynomial is equal to

$$\det_{A_{\mathfrak{p}} \widehat{\otimes} k^s}\big(T - \psi \,\big|\, \mathrm{Hom}_{A\otimes k}(M,\, \omega_{\mathfrak{p}} \mathbin{\widehat{\otimes}} k^s)\big).$$

Lemma 8.12.2 identifies this polynomial with

$$\det_{A\otimes k}\big(T - \psi \,\big|\, \mathrm{Hom}_{A\otimes k}(M,\, \omega \otimes k)\big).$$

Now Proposition 8.8.1 implies that the endomorphism $j^{-1}i$ of $\mathrm{Hom}_{A\otimes k}(M,\, \omega \otimes k)$ sends a map f to the map $m \mapsto \tau^{-1}(f(\tau m))$. As τ^{-d} is the identity automorphism of k we conclude that $(j^{-1}i)^d(f) = \psi(f)$ and the result follows. □

Let $F^\natural$ be the fraction field of A and let $\omega^\natural = \omega \otimes_A F^\natural$ be the module of Kähler differentials of $F^\natural$ over $\mathbb{F}_q$. The main result of this section is the following theorem.

Theorem 8.12.7. *Assume that k is a finite extension of $\mathbb{F}_q$ of degree d. The $F^\natural \otimes k$-module shtuka*

$$\mathcal{H}\mathrm{om}(M,\, \omega^\natural \otimes k) = \Big[\mathrm{Hom}_{A\otimes k}(M,\, \omega^\natural \otimes k) \overset{i}{\underset{j}{\rightrightarrows}} \mathrm{Hom}_{A\otimes k}(\tau^* M,\, \omega^\natural \otimes k) \Big]$$

has the following properties.

(1) *The arrow i is an isomorphism.*

(2) *Let $\sigma^{-1} \in G_k$ be the geometric Frobenius element. For every prime $\mathfrak{p}$ of A different from the characteristic of E we have an identity*

$$\det_{A_\mathfrak{p}}\big(1 - T^d\sigma^{-1} \,\big|\, T_\mathfrak{p}E\big) = \det_{F^\natural}\big(1 - T(i^{-1}j) \,\big|\, \mathrm{Hom}_{A\otimes k}(M,\, \omega^\natural \otimes k)\big)$$

in $F_\mathfrak{p}[T]$.

In particular the polynomial $\det_{A_\mathfrak{p}}(1 - T\sigma^{-1} \mid T_\mathfrak{p}E)$ is independent of the choice of $\mathfrak{p}$ and has coefficients in the subring of $F^\sharp$ consisting of elements which are integral away from the characteristic of E and ∞. Drinfeld's construction (Theorem 7.6.1) implies that the coefficients are also integral at ∞.

Proof of Theorem 8.12.7. (1) By Theorem 8.9.3 we have $\mathrm{R}\Gamma(\nabla\,\mathcal{H}\mathrm{om}(M,\, \omega \otimes k) = \mathrm{Lie}_E(k)[-1]$. It is thus enough to prove that $\mathrm{Lie}_E(k) \otimes_A F^\natural = 0$. However $\mathrm{Lie}_E(k)$ is a torsion A-module since k is a finite extension of $\mathbb{F}_q$.

(2) We have

$$\det_{A_\mathfrak{p}}\big(1 - T^d\sigma^{-1} \,\big|\, T_\mathfrak{p}E\big) = \frac{\det_{A_\mathfrak{p}}\big(T^d - \sigma \,\big|\, T_\mathfrak{p}E\big)}{\det_{A_\mathfrak{p}}\big(-\sigma \,\big|\, T_\mathfrak{p}E\big)}.$$

Proposition 8.12.6 identifies the latter fraction with

$$\frac{\det_{A\otimes k}\big(T^d - (j^{-1}i)^d \,\big|\, \mathrm{Hom}_{A\otimes k}(M,\, \omega \otimes k)\big)}{\det_{A\otimes k}\big(-(j^{-1}i)^d \,\big|\, \mathrm{Hom}_{A\otimes k}(M,\, \omega \otimes k)\big)}$$

which is in turn equal to

$$\det_{F^\natural\otimes k}\big(1 - T^d(i^{-1}j)^d \,\big|\, \mathrm{Hom}_{A\otimes k}(M,\, \omega^\natural \otimes k)\big).$$

Now [3, Lemma 8.1.4] shows that this polynomial coincides with

$$\det_{F^\natural}\big(1 - T(i^{-1}j) \,\big|\, \mathrm{Hom}_{A\otimes k}(M,\, \omega^\natural \otimes k)\big).$$

and the result follows. ∎

CHAPTER 9

Local models

Fix a coefficient ring A as in Definition 7.2.1 and let F be the local field of A at infinity. We denote $\mathcal{O}_F \subset F$ the ring of integers and $\mathfrak{m}_F \subset \mathcal{O}_F$ the maximal ideal. Let K be a finite product of local fields containing $\mathbb{F}_q$. As usual $\mathcal{O}_K \subset K$ stands for the ring of integers and $\mathfrak{m}_K \subset \mathcal{O}_K$ denotes the Jacobson radical. The τ-ring and τ-module structures used in this chapter are as described in Section 3.8.

We study Drinfeld modules in a local situation. Namely we work with a Drinfeld A-module E over K by the means of the $F \breve{\otimes} K$-module shtuka $\mathcal{H}\mathrm{om}(M,\, a(F,K))$. Here M is the motive of E and $a(F,K)$ is the space of locally constant bounded $\mathbb{F}_q$-linear maps from F to K as in Definition 2.10.1. We introduce the notion of a local model which is an $\mathcal{O}_F \breve{\otimes} \mathcal{O}_K$-module subshtuka $\mathcal{M} \subset \mathcal{H}\mathrm{om}(M,\, a(F,K))$ with certain properties. Informally speaking, $\mathcal{M}$ compactifies $\mathcal{H}\mathrm{om}(M,\, a(F,K))$ in the direction of the coefficients $\mathcal{O}_F \subset F$ and the base $\mathcal{O}_K \subset K$. One important result of this chapter is Theorem 9.6.5 which implies that local models exist. Another important result is Theorem 9.7.7 which states that a local model is an elliptic shtuka in the sense of Chapter 5.

The constructions in this chapter are algebraic in nature and the topology on the various tensor product rings plays no essential role. Even though the (abstract) rings $\mathcal{O}_F \widehat{\otimes} \mathcal{O}_K$ and $\mathcal{O}_F \breve{\otimes} \mathcal{O}_K$ are the same we use the latter notation to stress that it is a subring of $F \breve{\otimes} K$.

9.1. Lattices

Let $R_0 \to R$ be a homomorphism of rings and M an R-module. Recall that an R_0-submodule $M_0 \subset M$ is called a lattice if the natural map $R \otimes_{R_0} M_0 \to M$ is an isomorphism (Section 2 in the chapter "Notation and conventions").

Definition 9.1.1. Let $R_0 \to R$ be a homomorphism of τ-rings, M an R-module shtuka and $M_0 \subset M$ an R_0-module subshtuka. We say that M_0 is an R_0-lattice in M if the underlying modules of M_0 are R_0-lattices in the underlying modules of M.

9.2. Reflexive sheaves

The aim of this section is to review some properties of reflexive sheaves on a scheme such as $Y = \operatorname{Spec} k[[z,\zeta]]$ with k a field and the open subscheme $U \subset Y$ which is the complement of the closed point. The main result states that every locally free sheaf on U extends uniquely to a locally free sheaf on Y. While the contents of the section is widely known, it does not seem to appear in the literature in the form which we need.

Let F be a local field containing $\mathbb{F}_q$ and let K be a finite product of local fields containing $\mathbb{F}_q$. We denote $\mathcal{O}_F \subset F$ and $\mathcal{O}_K \subset K$ the corresponding rings of integers. For the rest of this section let us fix uniformizers $z \in \mathcal{O}_F$ and $\zeta \in \mathcal{O}_K$. Let $Y = \operatorname{Spec}(\mathcal{O}_F \breve{\otimes} \mathcal{O}_K)$ and let $U = D(z) \cup D(\zeta) \subset Y$.

Lemma 9.2.1. *The ring $\mathcal{O}_F \breve{\otimes} \mathcal{O}_K$ has the following properties:*

(1) $\mathcal{O}_F \check{\otimes} \mathcal{O}_K$ *is a finite product of complete regular 2-dimensional local rings.*
(2) *The maximal ideals of* $\mathcal{O}_F \check{\otimes} \mathcal{O}_K$ *are precisely the prime ideals containing* z *and* ζ.

Proof. By Proposition 3.3.11 the natural map $\mathcal{O}_F \check{\otimes} \mathcal{O}_K \to \mathcal{O}_F \widehat{\otimes} \mathcal{O}_K$ is an isomorphism. So the result follows at once from Proposition 3.5.7. □

Recall that a coherent sheaf $\mathcal{F}$ is called *reflexive* if the natural map from $\mathcal{F}$ to its double dual $\mathcal{F}^{**}$ is an isomorphism. A locally free sheaf is automatically reflexive. We use [14] as the reference for the theory of reflexive sheaves.

Lemma 9.2.2. *If* $\mathcal{F}$ *is a reflexive sheaf on* Y *then the natural map* $\Gamma(Y, \mathcal{F}) \to \Gamma(U, \mathcal{F})$ *is an isomorphism.*

Proof. Lemma 9.2.1 implies the following two statements. First, Y is a finite disjoint union of regular schemes. Second, all the prime ideals in the complement of U are of height 2. Therefore the result follows from [14], Proposition 1.6 (i) $\Rightarrow$ (iii). □

Let $\iota \colon U \to Y$ be the open embedding.

Lemma 9.2.3. *Every coherent sheaf on* U *is globally generated.*

Proof. Let $\mathcal{F}$ be a coherent sheaf on U. The morphism ι is quasi-compact quasi-separated so $\iota_*\mathcal{F}$ is quasi-coherent. Let $f \in \mathcal{F}(D(z))$. As $\iota_*\mathcal{F}$ is quasi-coherent and Y is affine there exists an $n \gg 0$ such that fz^n lifts to a global section of $\iota_*\mathcal{F}$ or equivalently, to a global section of $\mathcal{F}$. The same argument applies to $\mathcal{F}(D(\zeta))$. We can therefore lift all the generators of $\mathcal{F}(D(z))$ and $\mathcal{F}(D(\zeta))$ to global sections of $\mathcal{F}$. □

Lemma 9.2.4. *If* $\mathcal{F}$ *is a reflexive sheaf on* U *then* $\iota_*\mathcal{F}$ *is reflexive.*

Proof. First we prove that $\iota_*\mathcal{F}$ is coherent. The sheaf $\mathcal{F}^*$ is coherent. By Lemma 9.2.3 there is a surjection $\mathcal{O}_U^n \to \mathcal{F}^*$ for $n \gg 0$. Dualizing it and taking the composition with the natural isomorphism $\mathcal{F} \cong \mathcal{F}^{**}$ we obtain an embedding $\mathcal{F} \hookrightarrow \mathcal{O}_U^n$. The sheaf $\mathcal{O}_Y$ is reflexive so Lemma 9.2.2 implies that $\iota_*\mathcal{O}_U^n = \mathcal{O}_Y^n$. Therefore $\iota_*\mathcal{F}$ embeds into $\mathcal{O}_Y^n$.

Next we have a natural commutative diagram

$$\begin{tikzcd} \Gamma(Y, \iota_*\mathcal{F}) \arrow[r] \arrow[d] & \Gamma(Y, (\iota_*\mathcal{F})^{**}) \arrow[d] \\ \Gamma(U, \mathcal{F}) \arrow[r] & \Gamma(U, \mathcal{F}^{**}). \end{tikzcd} \tag{9.1}$$

Here the horizontal arrows are induced by the natural maps $\mathcal{F} \to \mathcal{F}^{**}$ and $\iota_*\mathcal{F} \to (\iota_*\mathcal{F})^{**}$, the left vertical arrow is the restriction while the right vertical arrow is the composition

$$\Gamma(Y, (\iota_*\mathcal{F})^{**}) \to \Gamma(U, (\iota_*\mathcal{F})^{**}) \to \Gamma(U, \mathcal{F}^{**})$$

of the restriction map and the map which results from applying $\Gamma(U, -^{**})$ to the adjunction morphism $\iota^*\iota_*\mathcal{F} \to \mathcal{F}$.

Let us prove that the right vertical arrow in (9.1) is an isomorphism. By [14], Corollary 1.2 the sheaf $(\iota_*\mathcal{F})^{**}$ is reflexive. So the restriction map $\Gamma(Y, (\iota_*\mathcal{F})^{**}) \to \Gamma(U, (\iota_*\mathcal{F})^{**})$ is an isomorphism by Lemma 9.2.2. The adjunction map $\iota^*\iota_*\mathcal{F} \to \mathcal{F}$ is an isomorphism since ι is an open embedding. So our claim follows.

The right vertical arrow in (9.1) is an isomorphism by construction while the bottom horizontal arrow is an isomorphism since $\mathcal{F}$ is reflexive. It follows that the top horizontal arrow is an isomorphism whence $\iota_*\mathcal{F}$ is reflexive. □

Lemma 9.2.5. *Every reflexive sheaf on* $Y = \operatorname{Spec} \mathcal{O}_F \check{\otimes} \mathcal{O}_K$ *is locally free.*

Proof. According to Lemma 9.2.1 the ring $\mathcal{O}_F \breve{\otimes} \mathcal{O}_K$ is a finite product of regular local 2-dimensional rings. So the result follows from [14], Corollary 1.4. □

9.3. Reflexive sheaves and lattices

Using the results of the previous section we prove several technical lemmas on modules over the ring $\mathcal{O}_F \breve{\otimes} \mathcal{O}_K$.

Lemma 9.3.1. *Let $z \in \mathcal{O}_F$ and $\zeta \in \mathcal{O}_K$ be uniformizers.*

(1) *The natural map* $\operatorname{Spec}(F \breve{\otimes} \mathcal{O}_K) \to \operatorname{Spec}(\mathcal{O}_F \breve{\otimes} \mathcal{O}_K)$ *is an open embedding with image $D(z)$.*
(2) *The natural map* $\operatorname{Spec}(\mathcal{O}_F \breve{\otimes} K) \to \operatorname{Spec}(\mathcal{O}_F \breve{\otimes} \mathcal{O}_K)$ *is an open embedding with image $D(\zeta)$.*
(3) *The natural map* $\operatorname{Spec}(F \breve{\otimes} K) \to \operatorname{Spec}(\mathcal{O}_F \breve{\otimes} \mathcal{O}_K)$ *is an open embedding with image $D(z) \cap D(\zeta)$.*

Proof. Proposition 3.5.2 shows that

$$(\mathcal{O}_F \breve{\otimes} \mathcal{O}_K)[z^{-1}] = F \breve{\otimes} \mathcal{O}_K,$$
$$(\mathcal{O}_F \breve{\otimes} \mathcal{O}_K)[\zeta^{-1}] = \mathcal{O}_F \breve{\otimes} K$$

where $z \in \mathcal{O}_F$ and $\zeta \in \mathcal{O}_K$ are uniformizers. Furthermore Proposition 3.5.3 tells us that

$$(\mathcal{O}_F \breve{\otimes} \mathcal{O}_K)[(z\zeta)^{-1}] = F \breve{\otimes} K.$$

So the result follows. □

Lemma 9.3.2. *Let N be a locally free $\mathcal{O}_F \breve{\otimes} \mathcal{O}_K$-module. Consider the modules*

$$N^c = N \otimes_{\mathcal{O}_F \breve{\otimes} \mathcal{O}_K} (\mathcal{O}_F \breve{\otimes} K), \quad N^b = N \otimes_{\mathcal{O}_F \breve{\otimes} \mathcal{O}_K} (F \breve{\otimes} \mathcal{O}_K)$$

We have $N = N^c \cap N^b$ as submodules of $N \otimes_{\mathcal{O}_F \breve{\otimes} \mathcal{O}_K} (F \breve{\otimes} K)$. □

Lemma 9.3.3. *Let N be a locally free $F \breve{\otimes} K$-module of finite rank. If $N^c \subset N$ is a locally free $\mathcal{O}_F \breve{\otimes} K$-lattice and $N^b \subset N$ a locally free $F \breve{\otimes} \mathcal{O}_K$-lattice then $N^c \cap N^b$ is a locally free $\mathcal{O}_F \breve{\otimes} \mathcal{O}_K$-lattice in N^c and N^b.*

Proof. Let $z \in \mathcal{O}_F$ and $\zeta \in \mathcal{O}_K$ be uniformizers. By Lemma 9.3.1 $\operatorname{Spec}(F \breve{\otimes} \mathcal{O}_K) = D(z)$ and $\operatorname{Spec}(\mathcal{O}_F \breve{\otimes} K) = D(\zeta)$ are open subschemes in $\operatorname{Spec}(\mathcal{O}_F \breve{\otimes} \mathcal{O}_K)$ whose intersection is $\operatorname{Spec}(F \breve{\otimes} K)$. By assumption N^c and N^b restrict to the same module N on the intersection $D(z) \cap D(\zeta) = \operatorname{Spec}(F \breve{\otimes} K)$. Hence they define a locally free sheaf $\mathcal{N}$ on $D(z) \cup D(\zeta)$. Let $\iota\colon D(z) \cup D(\zeta) \to \operatorname{Spec}(\mathcal{O}_F \breve{\otimes} \mathcal{O}_K)$ be the embedding map. Lemmas 9.2.4 and 9.2.5 imply that $\iota_* \mathcal{N}$ is a locally free sheaf on $\operatorname{Spec}(\mathcal{O}_F \breve{\otimes} \mathcal{O}_K)$. By construction

$$\Gamma(\operatorname{Spec}(\mathcal{O}_F \breve{\otimes} \mathcal{O}_K), \iota_* \mathcal{N}) = N^c \cap N^b$$

so the result follows. □

9.4. Hom shtukas

Fix a Drinfeld A-module E over K. The motive $M = \operatorname{Hom}(E, \mathbb{G}_a)$ of E carries a natural structure of a left $A \otimes K\{\tau\}$-module with A acting on the right via E and $K\{\tau\}$ acting on the left via $\mathbb{G}_a$. Let F be the local field of A at infinity. In this section we list some properties of the shtuka $\mathcal{H}\mathrm{om}_{A \otimes K}(M, N)$ where N is one of the function spaces

$$a(F, K), \quad b(F, K), \quad a(F/A, K), \quad b(F/A, K)$$

or the germ space $g(F, K)$. These are immediate consequences of results in Chapters 2 and 3. The τ-module structures on these spaces are as described in Section 3.8.

Lemma 9.4.1. *$\mathcal{H}\mathrm{om}(M,\, a(F,K))$ is a locally free $F \,\check{\otimes}\, K$-module shtuka and $\mathcal{H}\mathrm{om}(M,\, b(F,K))$ is a locally free $F^{\#} \,\widehat{\otimes}\, K$-module shtuka.*

Proof. The $A \otimes K$-module M is locally free of finite rank by definition. Corollary 3.9.4 tells us that the function space $a(F,K)$ is a free $F \,\check{\otimes}\, K$-module of rank 1 while $b(F,K)$ is a free $F^{\#} \,\widehat{\otimes}\, K$-module of rank 1. The result now follows from the definition of $\mathcal{H}\mathrm{om}$. □

Lemma 9.4.2. *$\mathcal{H}\mathrm{om}(M,\, a(F,K))$ is an $F \,\check{\otimes}\, K$-lattice in $\mathcal{H}\mathrm{om}(M,\, b(F,K))$.*

Proof. According to Corollary 3.9.5 the function space $a(F,K)$ is an $F \,\check{\otimes}\, K$-lattice in $b(F,K)$. The result now follows from the definition of $\mathcal{H}\mathrm{om}$ since M is a locally free $A \otimes K$-module of finite rank. □

Lemma 9.4.3. *The shtuka $\mathcal{H}\mathrm{om}(M,\, a(F/A,K))$ has the following properties:*

(1) *It is a locally free $A \otimes K$-module shtuka.*
(2) *It is an $A \otimes K$-lattice in the $F \,\check{\otimes}\, K$-module shtuka $\mathcal{H}\mathrm{om}(M, a(F,K))$.*

Proof. By definition M is a locally free $A \otimes K$-module. By Corollaries 3.9.4 and 3.9.5 the space $a(F/A,K)$ is a locally free $A \otimes K$-lattice in the $F \,\check{\otimes}\, K$-module $a(F,K)$. So (1) and (2) follow from the definition of $\mathcal{H}\mathrm{om}$. □

Lemma 9.4.4. *The shtuka $\mathcal{H}\mathrm{om}(M,\, b(F/A,K))$ has the following properties:*

(1) *It is a locally free $A \,\widehat{\otimes}\, K$-module shtuka.*
(2) *It is an $A \,\widehat{\otimes}\, K$-lattice in the $F^{\#} \,\widehat{\otimes}\, K$-module shtuka $\mathcal{H}\mathrm{om}(M, b(F,K))$.*

Proof. By definition M is a locally free $A \otimes K$-module. By Corollaries 3.9.4 and 3.9.5 the space $b(F/A,K)$ is a locally free $A \,\widehat{\otimes}\, K$-lattice in the $F^{\#} \,\widehat{\otimes}\, K$-module $b(F,K)$. So (1) and (2) follow from the definition of $\mathcal{H}\mathrm{om}$. □

Lemma 9.4.5. *The shtuka $\mathcal{H}\mathrm{om}(M,\, a(F/A,K))$ is a locally free $A \otimes K$-lattice in the $A \,\widehat{\otimes}\, K$-module shtuka $\mathcal{H}\mathrm{om}(M,\, b(F/A,K))$.*

Proof. Follows from Corollary 3.9.5. □

9.5. The notion of a local model

As before we work with a fixed Drinfeld module E over K. We denote $M = \mathrm{Hom}(E, \mathbb{G}_a)$ the motive of E as in Definition 7.1.4. Throughout the rest of the chapter we assume that the action of A on $\mathrm{Lie}_E(K)$ extends to a continuous action of F, the local field of A at infinity.

The shtuka $\mathcal{H}\mathrm{om}(M, a(F,K))$ is a locally free $F \,\check{\otimes}\, K$-module shtuka. In this chapter we will study models of this shtuka over various subrings of $F \,\check{\otimes}\, K$. In particular we will introduce the notions of

(1) the *coefficient compactification* (a model over $\mathcal{O}_F \,\check{\otimes}\, K$),
(2) a *base compactification* (a model over $F \,\check{\otimes}\, \mathcal{O}_K$),
(3) a *local model* (over $\mathcal{O}_F \,\check{\otimes}\, \mathcal{O}_K$).

Let C be the projective compactification of $\mathrm{Spec}\, A$. We denote $\mathcal{E}$ the locally free shtuka on $C \times \mathrm{Spec}\, K$ constructed in Theorem 7.6.1.

Definition 9.5.1. The *coefficient compactification* $\mathcal{M}^c \subset \mathcal{H}\mathrm{om}(M,\, a(F,K))$ is the $\mathcal{O}_F \,\check{\otimes}\, K$-subshtuka

$$\mathcal{H}\mathrm{om}_{\mathcal{O}_F \check{\otimes} K}\big(\mathcal{E}(\mathcal{O}_F \,\check{\otimes}\, K),\, a(F/\mathcal{O}_F, K)\big).$$

The superscript "c" stands for "coefficients".

Lemma 9.5.2. *$\mathcal{M}^c \subset \mathcal{H}\mathrm{om}(M,\, a(F,K))$ is a locally free $\mathcal{O}_F \,\check{\otimes}\, K$-lattice.*

Proof. By construction $\mathcal{E}(A \otimes K)$ is the shtuka given by the left $A \otimes K\{\tau\}$-module M. Hence $\mathcal{E}(\mathcal{O}_F \mathbin{\breve{\otimes}} K)$ is a locally free $\mathcal{O}_F \mathbin{\breve{\otimes}} K$-lattice in the pullback of M to $F \mathbin{\breve{\otimes}} K$. According to Corollaries 3.9.4 and 3.9.5 the function space $a(F/\mathcal{O}_F, K)$ is a locally free $\mathcal{O}_F \mathbin{\breve{\otimes}} K$-lattice in $a(F, K)$. The result now follows from the definition of $\mathcal{H}\mathrm{om}$. □

Lemma 9.5.3. *$\mathcal{M}^c(\mathcal{O}_F/\mathfrak{m}_F \otimes K)$ is nilpotent.*

Proof. By Theorem 7.6.1 the shtuka $\mathcal{E}(\mathcal{O}_F/\mathfrak{m}_F \otimes K)$ is co-nilpotent so the result follows by Proposition 7.5.2. □

By our assumption the action of A on $\mathrm{Lie}_E(K)$ extends to a continuous action of F so that we have a continuous homomorphism $F \to K$. Such a homomorphism necessarily maps $\mathcal{O}_F$ to $\mathcal{O}_K$.

Definition 9.5.4. We define the *ramification ideal* $\mathfrak{e} \subset \mathcal{O}_K$ to be the ideal generated by $\mathfrak{m}_F$ in $\mathcal{O}_K$.

Lemma 9.5.5. *The ideal $\mathfrak{e}$ is open and is contained in the Jacobson radical $\mathfrak{m}_K$.* □

Definition 9.5.6. A *base compactification* of $\mathcal{H}\mathrm{om}(M, a(F, K))$ is a locally free $F \mathbin{\breve{\otimes}} \mathcal{O}_K$-lattice $\mathcal{M}^b$ such that $\mathcal{M}^b(F \otimes \mathcal{O}_K/\mathfrak{m}_K)$ is nilpotent and $\mathcal{M}^b(F \otimes \mathcal{O}_K/\mathfrak{e})$ is linear. The superscript "b" stands for "base".

Definition 9.5.7. A *local model* of $\mathcal{H}\mathrm{om}(M, a(F, K))$ is a locally free $\mathcal{O}_F \mathbin{\breve{\otimes}} \mathcal{O}_K$-lattice $\mathcal{M}$ such that:

(1) The subshtuka $\mathcal{M}(\mathcal{O}_F \mathbin{\breve{\otimes}} K) \subset \mathcal{H}\mathrm{om}(M, a(F, K))$ coincides with $\mathcal{M}^c$.
(2) $\mathcal{M}(F \mathbin{\breve{\otimes}} \mathcal{O}_K/\mathfrak{m}_K)$ is nilpotent and $\mathcal{M}(F \mathbin{\breve{\otimes}} \mathcal{O}_K/\mathfrak{e})$ is linear.

Proposition 9.5.8. *Let $\mathcal{M} \subset \mathcal{H}\mathrm{om}(M, a(F, K))$ be an $\mathcal{O}_F \mathbin{\breve{\otimes}} \mathcal{O}_K$-subshtuka. The following are equivalent:*

(1) *$\mathcal{M}$ is a local model.*
(2) *The subshtuka $\mathcal{M}(F \mathbin{\breve{\otimes}} \mathcal{O}_K) \subset \mathcal{H}\mathrm{om}(M, a(F, K))$ is a base compactification and $\mathcal{M} = \mathcal{M}^c \cap \mathcal{M}(F \mathbin{\breve{\otimes}} \mathcal{O}_K)$*

Proof. (1) ⇒ (2). It follows directly from the definition of a local model that $\mathcal{M}(F \mathbin{\breve{\otimes}} \mathcal{O}_K)$ is a base compactification and $\mathcal{M}^c = \mathcal{M}(\mathcal{O}_F \mathbin{\breve{\otimes}} K)$. Lemma 9.3.2 implies that $\mathcal{M} = \mathcal{M}^c \cap \mathcal{M}(F \mathbin{\breve{\otimes}} \mathcal{O}_K)$. (2) ⇒ (1). Lemma 9.3.3 shows that $\mathcal{M}$ is a locally free $\mathcal{O}_F \mathbin{\breve{\otimes}} \mathcal{O}_K$-lattice in $\mathcal{M}^c$ so (1) follows. □

Lemma 9.5.9. *If $\mathcal{M} \subset \mathcal{H}\mathrm{om}(M, a(F, K))$ is a local model then the natural map $\mathcal{M}(F^\# \mathbin{\widehat{\otimes}} \mathcal{O}_K) \to \mathcal{H}\mathrm{om}(M, b(F, K))$ is an inclusion of an $F^\# \mathbin{\widehat{\otimes}} \mathcal{O}_K$-lattice.*

Proof. $\mathcal{M}(F \mathbin{\breve{\otimes}} K) = \mathcal{H}\mathrm{om}(M, a(F, K))$ by definition and $\mathcal{H}\mathrm{om}(M, a(F, K))$ is an $F \mathbin{\breve{\otimes}} K$-lattice in $\mathcal{H}\mathrm{om}(M, b(F, K))$ by Lemma 9.4.2. Therfore the natural map $\mathcal{M}(F^\# \mathbin{\widehat{\otimes}} K) \to \mathcal{H}\mathrm{om}(M, b(F, K))$ is an isomorphism. According to Proposition 3.5.1 the ring $F^\# \mathbin{\widehat{\otimes}} K$ is a localization of $F^\# \mathbin{\widehat{\otimes}} \mathcal{O}_K$ at a uniformizer of $\mathcal{O}_K$. As $\mathcal{M}$ is locally free it follows that $\mathcal{M}(F^\# \mathbin{\widehat{\otimes}} \mathcal{O}_K) \to \mathcal{M}(F^\# \mathbin{\widehat{\otimes}} K)$ is a lattice inclusion. □

The natural map $\mathcal{O}_F \mathbin{\breve{\otimes}} \mathcal{O}_K \to \mathcal{O}_F \mathbin{\widehat{\otimes}} \mathcal{O}_K$ is an isomorphism by Proposition 3.3.11. So we can view an $\mathcal{O}_F \mathbin{\breve{\otimes}} \mathcal{O}_K$-module shtuka $\mathcal{M}$ as an $\mathcal{O}_F \mathbin{\widehat{\otimes}} \mathcal{O}_K$-module shtuka. In particular the constructions of Chapter 5 apply to shtukas on $\mathcal{O}_F \mathbin{\breve{\otimes}} \mathcal{O}_K$.

Let us recall the twisting construction of Section 5.7. Let $I \subset \mathcal{O}_F \mathbin{\breve{\otimes}} \mathcal{O}_K$ be a τ-invariant ideal. Given an $\mathcal{O}_F \mathbin{\breve{\otimes}} \mathcal{O}_K$-module shtuka

$$\mathcal{M} = \left[M_0 \underset{j}{\overset{i}{\rightrightarrows}} M_1 \right]$$

we define

$$I\mathcal{M} = \left[IM_0 \overset{i}{\underset{j}{\rightrightarrows}} IM_1 \right].$$

The fact that I is an invariant ideal guarantees that the diagram on the right indeed defines a shtuka. We will use twists by the invariant ideal $\mathcal{O}_F \breve{\otimes} \mathfrak{e}$. To improve legibility we will write $\mathfrak{e}\mathcal{M}$ in place of $(\mathcal{O}_F \breve{\otimes} \mathfrak{e})\mathcal{M}$.

Proposition 9.5.10. *If $\mathcal{M}$ is a local model then $\mathfrak{e}\mathcal{M}$ is a local model.*

Proof. Indeed $(\mathfrak{e}\mathcal{M})(\mathcal{O}_F \breve{\otimes} K) = \mathcal{M}(\mathcal{O}_F \breve{\otimes} K)$ so $(\mathfrak{e}\mathcal{M})(\mathcal{O}_F \breve{\otimes} K)$ coincides with the coefficient compactification of $\mathcal{H}\mathrm{om}(M, a(F, K))$. Lemma 5.7.5 shows that $(\mathfrak{e}\mathcal{M})(F \otimes \mathcal{O}_K/\mathfrak{m}_K)$ is nilpotent while Proposition 5.7.6 implies that $(\mathfrak{e}\mathcal{M})(F \otimes \mathcal{O}_K/\mathfrak{e})$ is linear. Since $\mathfrak{e}\mathcal{M}$ is a locally free $\mathcal{O}_F \breve{\otimes} \mathcal{O}_K$-lattice in $\mathcal{H}\mathrm{om}(M, a(F, K))$ by construction we conclude that $\mathfrak{e}\mathcal{M}$ is a local model. □

Proposition 9.5.11. *If $\mathcal{M}$, $\mathcal{N}$ are local models then there exists $n \geqslant 0$ such that $\mathfrak{e}^n\mathcal{M} \subset \mathcal{N}$.*

Proof. By Proposition 9.5.8 we have

$$\mathcal{M} = \mathcal{M}^c \cap \mathcal{M}(F \breve{\otimes} \mathcal{O}_K), \quad \mathcal{N} = \mathcal{M}^c \cap \mathcal{N}(F \breve{\otimes} \mathcal{O}_K)$$

where $\mathcal{M}^c \subset \mathcal{H}\mathrm{om}(M, a(F, K))$ is the coefficient compactification. So to prove the proposition it is enough to find an integer $n \geqslant 0$ such that $(\mathfrak{e}^n\mathcal{M})(F \breve{\otimes} \mathcal{O}_K) \subset \mathcal{N}(F \breve{\otimes} \mathcal{O}_K)$. Observe that the ramification ideal ideal $\mathfrak{e}$ is contained in the Jacobson radical $\mathfrak{m}_K$ by construction. In particular it contains a power of a uniformizer of $\mathcal{O}_K$. The result follows since $\mathcal{M}(F \breve{\otimes} \mathcal{O}_K)$ and $\mathcal{N}(F \breve{\otimes} \mathcal{O}_K)$ are $F \breve{\otimes} \mathcal{O}_K$-lattices in $\mathcal{H}\mathrm{om}(M, a(F, K))$. □

9.6. Existence of base compactifications

We keep the assumptions and the notation of the previous section. In this section we prove that the $F \breve{\otimes} K$-module shtuka $\mathcal{H}\mathrm{om}(M, a(F, K))$ admits a base compactification over $F \breve{\otimes} \mathcal{O}_K$ in the sense of the preceding section. In fact we go one step further and show that the $A \otimes K$-module shtuka $\mathcal{H}\mathrm{om}(M, a(F/A, K))$ admits a base compactification over $A \otimes \mathcal{O}_K$. This result was inspired by the construction of extension by zero from the theory of Böckle-Pink [3, Section 4.5].

Lemma 9.6.1. *If N is a finitely generated reflexive module over the ring $A \widehat{\otimes} \mathcal{O}_K$ then N is locally free.*

Proof. The ring $A \otimes \mathcal{O}_K$ is noetherian, regular and of Krull dimension 2. Hence so is its completion $A \widehat{\otimes} \mathcal{O}_K$ at the Jacobson radical $\mathfrak{m}_K \subset \mathcal{O}_K$. So the result follows from [14, Corollary 1.4]. □

Lemma 9.6.2. *Every locally free $A \widehat{\otimes} K$-module admits a locally free $A \widehat{\otimes} \mathcal{O}_K$-lattice.*

Proof. Let N be a locally free $A \widehat{\otimes} K$-module. Clearly there exists a finitely generated $A \widehat{\otimes} \mathcal{O}_K$-lattice $N_0 \subset N$. A priori N_0 need not be locally free. But the double dual N_0^{**} is still a lattice in N and is a reflexive $A \widehat{\otimes} \mathcal{O}_K$-module by [14, Corollary 1.2]. Now Lemma 9.6.1 shows that N_0^{**} is a locally free $A \widehat{\otimes} \mathcal{O}_K$-module. □

Proposition 9.6.3. *Let N be a left $A \widehat{\otimes} K\{\tau\}$-module which is locally free of finite rank as an $A \widehat{\otimes} K$-module.*

(1) *There exists an $A \widehat{\otimes} \mathcal{O}_K\{\tau\}$-submodule $N_0 \subset N$ such that N_0 is a locally free $A \widehat{\otimes} \mathcal{O}_K$-lattice in the $A \widehat{\otimes} K$-module N.*
(2) *Given an open ideal $I \subset \mathcal{O}_K$ one can choose N_0 in such a way as to ensure that τ acts by zero on $N \otimes_{A \widehat{\otimes} \mathcal{O}_K} (A \otimes \mathcal{O}_K/I)$.*

Proof. (1) By Lemma 9.6.2 the $A\widehat{\otimes}K$-module N admits a locally free $A\widehat{\otimes}\mathcal{O}_K$-lattice $N_1 \subset N$. Let $\zeta \in \mathcal{O}_K$ be a uniformizer. According to Proposition 3.5.1 the ring $A \mathbin{\widehat{\otimes}} K$ is the localization of $A \mathbin{\widehat{\otimes}} \mathcal{O}_K$ at ζ. Hence $N_1[\zeta^{-1}] = N$. As a consequence there exists an $n \geqslant 0$ such that $\tau N_1 \subset \zeta^{-n} N_1$. The locally free $A \mathbin{\widehat{\otimes}} \mathcal{O}_K$-lattice $N_0 = \zeta^n N_1$ has the property that $\tau N_0 \subset \zeta^{(q-1)n} N_1$. As $q > 1$ it follows that N_0 is a left $A \mathbin{\widehat{\otimes}} \mathcal{O}_K\{\tau\}$-submodule.

(2) Without loss of generality we may assume that $I = \zeta^n \mathcal{O}_K$ for some $n \geqslant 0$. Let $N_0 \subset N$ be a left $A\widehat{\otimes}\mathcal{O}_K\{\tau\}$-submodule as in (1) and let $N_1 = \zeta^n N_0$. We have $\tau(N_1) \subset \zeta^{(q-1)n} N_1$. The result follows since $q > 1$. □

Definition 9.6.4. A *base compactification* of $\mathcal{H}\mathrm{om}(M,\, a(F/A, K))$ is a locally free $A \otimes \mathcal{O}_K$-lattice $\mathcal{M}^b$ such that $\mathcal{M}^b(A \otimes \mathcal{O}_K/\mathfrak{m}_K)$ is nilpotent and $\mathcal{M}^b(A \otimes \mathcal{O}_K/\mathfrak{e})$ is linear. Here $\mathfrak{e} \subset \mathcal{O}_K$ is the ramification ideal ideal of Definition 9.5.4.

Any base compactification of $\mathcal{H}\mathrm{om}(M,\, a(F/A, K))$ induces a base compactification of $\mathcal{H}\mathrm{om}(M,\, a(F, K))$ in the sense of Definition 9.5.6.

Theorem 9.6.5. *The shtuka* $\mathcal{H}\mathrm{om}(M,\, a(F/A, K))$ *admits a base compactification.*

Proof. The $A \otimes K$-module shtuka $\mathcal{H}\mathrm{om}(M,\, a(F/A, K))$ is locally free by Lemma 9.4.3 while the $A\widehat{\otimes}K$-module shtuka $\mathcal{H}\mathrm{om}(M,\, b(F/A, K))$ is locally free by Lemma 9.4.4. According to Lemma 9.4.5 the shtuka $\mathcal{H}\mathrm{om}(M,\, a(F/A, K))$ is an $A \otimes K$-lattice in $\mathcal{H}\mathrm{om}(M,\, b(F/A, K))$. Hence Beauville-Laszlo glueing theorem [0BP2] implies that to prove the existence of a base compactification it is enough to construct a locally free $A\widehat{\otimes}\mathcal{O}_K$-lattice $\mathcal{M} \subset \mathcal{H}\mathrm{om}(M,\, b(F/A, K))$ such that $\mathcal{M}(A\otimes\mathcal{O}_K/\mathfrak{m}_K)$ is nilpotent and $\mathcal{M}(A \otimes \mathcal{O}_K/\mathfrak{e})$ is linear.

Now consider the shtuka

$$\mathcal{H}\mathrm{om}(M,\, b(F/A, K)) = \Big[\mathrm{Hom}(M,\, b(F/A, K)) \underset{j}{\overset{i}{\rightrightarrows}} \mathrm{Hom}(\tau^* M,\, b(F/A, K))\Big].$$

Theorem 8.10.8 shows that $\mathrm{R\Gamma}(\nabla\,\mathcal{H}\mathrm{om}(M,\, b(F/A, K))) = 0$. Hence the arrow i in the diagram above is an isomorphism. If we let τ act on the $A \mathbin{\widehat{\otimes}} K$-module $\mathrm{Hom}(M,\, b(F/A, K))$ via the endomorphism $i^{-1} \circ j$ then it becomes a left $A \mathbin{\widehat{\otimes}} K\{\tau\}$-module. By construction $\mathcal{H}\mathrm{om}(M,\, b(F/A, K))$ is isomorphic to the shtuka defined by $\mathrm{Hom}(M,\, b(F/A, K))$. Therefore applying Proposition 9.6.3 to $N = \mathrm{Hom}(M,\, b(F/A, K))$ with $I = \mathfrak{e}$ we get the result. □

9.7. Local models as elliptic shtukas

We keep the notation and the assumptions of the previous section.

By Lemma 9.4.2 the $F \mathbin{\breve{\otimes}} K$-module shtuka $\mathcal{H}\mathrm{om}(M,\, a(F, K))$ is a lattice in the $F^{\#} \mathbin{\widehat{\otimes}} K$-module shtuka $\mathcal{H}\mathrm{om}(M,\, b(F, K))$. Since $g(F, K)$ is the quotient of $b(F, K)$ by $a(F, K)$ we deduce that

$$\mathcal{H}\mathrm{om}(M,\, g(F, K)) = \tfrac{\mathcal{H}\mathrm{om}(M,\, a(F,K))(F^{\#}\widehat{\otimes}K)}{\mathcal{H}\mathrm{om}(M,\, a(F,K))(F\breve{\otimes}K)}.$$

As a result Proposition 4.1.2 provides us with natural quasi-isomorphisms

$$\mathrm{R\Gamma}_g(\mathcal{H}\mathrm{om}(M,\, a(F, K))) \xrightarrow{\sim} \mathrm{R\Gamma}(\mathcal{H}\mathrm{om}(M,\, g(F, K)))[-1],$$
$$\mathrm{R\Gamma}_g(\nabla\,\mathcal{H}\mathrm{om}(M,\, a(F, K))) \xrightarrow{\sim} \mathrm{R\Gamma}(\nabla\,\mathcal{H}\mathrm{om}(M,\, g(F, K)))[-1].$$

Definition 9.7.1. We define natural quasi-isomorphisms

$$\mathrm{R\Gamma}_g(\mathcal{H}\mathrm{om}(M,\, a(F, K))) \xrightarrow{\sim} \mathrm{Lie}_E(K)[-1],$$
$$\mathrm{R\Gamma}_g(\nabla\,\mathcal{H}\mathrm{om}(M,\, a(F, K))) \xrightarrow{\sim} \mathrm{Lie}_E(K)[-1]$$

as the compositions

$$\begin{array}{ccc} \mathrm{R\Gamma}_g(\nabla\,\mathcal{H}\mathrm{om}(M,\, a(F,K))) & & \mathrm{R\Gamma}_g(\mathcal{H}\mathrm{om}(M,\, a(F,K))) \\ {\scriptstyle \text{Proposition 4.1.2}}\big\downarrow\wr & & \wr\big\downarrow{\scriptstyle \text{Proposition 4.1.2}} \\ \mathrm{R\Gamma}(\nabla\,\mathcal{H}\mathrm{om}(M,\, g(F,K)))[-1] & & \mathrm{R\Gamma}(\mathcal{H}\mathrm{om}(M,\, g(F,K)))[-1] \\ {\scriptstyle \text{Proposition 8.10.9}}\big\downarrow\wr & & \wr\big\downarrow{\scriptstyle \text{Proposition 8.10.10}} \\ \mathrm{Lie}_E(K)[-1] & & \mathrm{Lie}_E(K)[-1] \end{array}$$

Definition 9.7.2. Let $\mathcal{M} \subset \mathcal{H}\mathrm{om}(M,\, a(F,K))$ be a local model. We define a map

$$\gamma\colon\ \mathrm{R\Gamma}(\mathcal{M}) \to \mathrm{Lie}_E(K)[-1]$$

as the composition of the following maps:

- The natural map $\mathrm{R\Gamma}(\mathcal{M}) \to \mathrm{R\Gamma}(F \check{\otimes} \mathcal{O}_K, \mathcal{M})$.
- The local germ map $\mathrm{R\Gamma}(F \check{\otimes} \mathcal{O}_K, \mathcal{M}) \xrightarrow{\sim} \mathrm{R\Gamma}_g(F \check{\otimes} K, \mathcal{M})$ of Definition 4.2.4.
- The isomorphism $\mathrm{R\Gamma}_g(F\check{\otimes}K, \mathcal{M}) = \mathrm{R\Gamma}_g(\mathcal{H}\mathrm{om}(M,\, a(F,K)))$ which results from the equality $\mathcal{M}(F \check{\otimes} K) = \mathcal{H}\mathrm{om}(M,\, a(F,K))$.
- The quasi-isomorphism $\mathrm{R\Gamma}_g(\mathcal{H}\mathrm{om}(M,\, a(F,K))) \xrightarrow{\sim} \mathrm{Lie}_E(K)[-1]$ of Definition 9.7.1.

We define a map

$$\nabla\gamma\colon\ \mathrm{R\Gamma}(\nabla\mathcal{M}) \to \mathrm{Lie}_E(K)[-1]$$

in the same way.

Observe that the map γ is $\mathcal{O}_F$-linear by construction while $\nabla\gamma$ is both $\mathcal{O}_F$-linear and $\mathcal{O}_K$-linear.

Lemma 9.7.3. *If $\mathcal{M}$ is a locally free $\mathcal{O}_F \check{\otimes} \mathcal{O}_K$-module shtuka then the natural map $F \otimes_{\mathcal{O}_F} \mathrm{R\Gamma}(\mathcal{M}) \to \mathrm{R\Gamma}(F \check{\otimes} \mathcal{O}_K,\, \mathcal{M})$ is a quasi-isomorphism.*

Proof. According to Proposition 3.5.2 the natural map $F \otimes_{\mathcal{O}_F} (\mathcal{O}_F \check{\otimes} \mathcal{O}_K) \to F \check{\otimes} \mathcal{O}_K$ is an isomorphism. The differentials in the complex computing $\mathrm{R\Gamma}(\mathcal{M})$ are $\mathcal{O}_F$-linear so the result follows. $\square$

Corollary 9.7.4. *The F-linear extensions*

$$\gamma\colon F \otimes_{\mathcal{O}_F} \mathrm{R\Gamma}(\mathcal{M}) \to \mathrm{Lie}_E(K)[-1],$$
$$\nabla\gamma\colon F \otimes_{\mathcal{O}_F} \mathrm{R\Gamma}(\nabla\mathcal{M}) \to \mathrm{Lie}_E(K)[-1]$$

of the maps γ and $\nabla\gamma$ are quasi-isomorphisms. $\square$

Proposition 9.7.5. *If $\mathcal{M} \hookrightarrow \mathcal{N}$ is an inclusion of local models then the induced maps*

$$F \otimes_{\mathcal{O}_F} \mathrm{R\Gamma}(\mathcal{M}) \to F \otimes_{\mathcal{O}_F} \mathrm{R\Gamma}(\mathcal{N}),$$
$$F \otimes_{\mathcal{O}_F} \mathrm{R\Gamma}(\nabla\mathcal{M}) \to F \otimes_{\mathcal{O}_F} \mathrm{R\Gamma}(\nabla\mathcal{N})$$

are quasi-isomorphisms.

Proof. Since γ and $\nabla\gamma$ are natural the result follows from Corollary 9.7.4. $\square$

Theorem 9.7.6. *Let $\mathcal{M}$ be a local model.*

(1) $\mathrm{H}^0(\mathcal{M}) = 0$ *and* $\mathrm{H}^1(\mathcal{M})$ *is a finitely generated free $\mathcal{O}_F$-module.*
(2) $\mathrm{H}^0(\nabla\mathcal{M}) = 0$ *and* $\mathrm{H}^1(\nabla\mathcal{M})$ *is a finitely generated free $\mathcal{O}_F$-module.*

Proof. (1) By definition $\mathcal{M}(\mathcal{O}_F/\mathfrak{m}_F \otimes K) = \mathcal{M}^c(\mathcal{O}_F/\mathfrak{m}_F \otimes K)$ where $\mathcal{M}^c \subset \mathcal{H}\mathrm{om}(M, a(F,K))$ is the coefficient compactification. Lemma 9.5.3 claims that $\mathcal{M}^c(\mathcal{O}_F/\mathfrak{m}_F \otimes K)$ is nilpotent. Hence the result follows from Theorem 5.4.3. (2) Nilpotence is preserved under linearization. So the result follows from Theorem 5.4.3 as well. □

The main result of this section is the following theorem:

Theorem 9.7.7. *A local model $\mathcal{M}$ is an elliptic shtuka of ramification ideal $\mathfrak{e}$ where $\mathfrak{e} \subset \mathcal{O}_K$ is the ideal of Definition 9.5.4.*

Proof. We verify the conditions of Definition 5.6.1 for $\mathcal{M}$.

(E0) Indeed $\mathcal{M}$ is a locally free $\mathcal{O}_F \mathbin{\widehat{\otimes}} \mathcal{O}_K$-module shtuka by definition.

(E1) By construction $\mathcal{M}(\mathcal{O}_F/\mathfrak{m}_F \otimes K) = \mathcal{M}^c(\mathcal{O}_F/\mathfrak{m}_F \otimes K)$ where $\mathcal{M}^c \subset \mathcal{H}\mathrm{om}(M, a(F,K))$ is the coefficient compactification. Thus the shtuka $\mathcal{M}(\mathcal{O}_F/\mathfrak{m}_F \otimes K)$ is nilpotent by Lemma 9.5.3.

(E2) $\mathcal{M}(F \otimes \mathcal{O}_K/\mathfrak{m}_K)$ is nilpotent by definition.

(E3) Consider the map $\mathrm{H}^1(\nabla\gamma)\colon \mathrm{H}^1(\nabla\mathcal{M}) \to \mathrm{Lie}_E(K)$. This map is $\mathcal{O}_F$-linear and $\mathcal{O}_K$-linear by construction. Corollary 9.7.4 in combination with Theorem 9.7.6 implies that the map is injective. Therefore

$$\mathfrak{m}_F \cdot \mathrm{H}^1(\nabla\mathcal{M}) = \mathfrak{e} \cdot \mathrm{H}^1(\nabla\mathcal{M})$$

by definition of the ramification ideal ideal $\mathfrak{e}$.

(E4) The shtuka $\mathcal{M}(F \otimes \mathcal{O}_K/\mathfrak{e})$ is linear by definition. As $\mathcal{M}$ is locally free it follows that $\mathcal{M}(\mathcal{O}_F \otimes \mathcal{O}_K/\mathfrak{e})$ is linear. □

Theorem 9.7.7 allows us to make the following important definition:

Definition 9.7.8. The *regulator*

$$\rho\colon \mathrm{H}^1(\mathcal{M}) \xrightarrow{\sim} \mathrm{H}^1(\nabla\mathcal{M})$$

of a local model $\mathcal{M}$ is the regulator of $\mathcal{M}$ viewed as an elliptic shtuka over $\mathcal{O}_F \mathbin{\widehat{\otimes}} \mathcal{O}_K$ (see Definition 5.14.1).

9.8. The exponential map

We keep the assumptions and the notation of the previous section. In this section we describe in detail how the maps γ and $\nabla\gamma$ act on cohomology classes and introduce the exponential map of a local model. Before we begin let us make a remark. Let $h_1, h_2 \in b(F,K)$. Recall that the expression

$$h_1 \sim h_2$$

signifies that the image of $h_1 - h_2$ in $g(F,K)$ is zero. In other words there exists an open neighbourhood $U \subset F$ of 0 such that $h_1|_U = h_2|_U$.

Proposition 9.8.1. *Let $\mathcal{M} = [\mathcal{M}_0 \overset{i}{\underset{j}{\rightrightarrows}} \mathcal{M}_1]$ be a local model and let $g \in \mathcal{M}_1$.*

(1) *There exists a unique $f \in \mathcal{M}_0(F^\# \mathbin{\widehat{\otimes}} \mathcal{O}_K)$ such that $(i-j)(f) = g$.*

(2) *Let $[g]$ be the cohomology class of g in $\mathrm{H}^1(\mathcal{M})$ and let $\alpha = \gamma[g]$. The element $\alpha \in \mathrm{Lie}_E(K)$ is uniquely characterized by the property that for every $m \in M^0$ one has*

$$f(m) \sim (x \mapsto m \exp(x\alpha)).$$

Here we view f as an element of $\mathrm{Hom}(M,\, b(F,K))$ via the natural inclusion $\mathcal{M}(F^\# \mathbin{\widehat{\otimes}} \mathcal{O}_K) \subset \mathcal{H}\mathrm{om}(M,\, b(F,K))$ of Lemma 9.5.9.

Proof. (1) is a direct consequence of Proposition 4.2.6 (1). Part (2) of this proposition implies that the image of f in $\mathrm{Hom}(M, g(F,K))$ represents the image of $[g] \in \mathrm{H}^1(\mathcal{M})$ under the local germ map

$$\mathrm{H}^1(\mathcal{M}) \xrightarrow{\sim} \mathrm{H}^0_{\mathrm{g}}(F \check{\otimes} K, \mathcal{M}) = \mathrm{H}^0(\mathcal{H}\mathrm{om}(M, g(F,K))).$$

By Proposition 8.10.10 the map

$$\mathrm{Lie}_E(K) \to \mathrm{H}^0(\mathcal{H}\mathrm{om}(M,\, g(F,K))), \quad \alpha \mapsto (m \mapsto (x \mapsto m\exp(x\alpha)))$$

is an isomorphism. Hence for every $m \in M$ we have $f(m) \sim (x \mapsto m\exp(x\alpha))$. It remains to show that if $\beta \in \mathrm{Lie}_E(K)$ is an element such that $f(m) \sim (x \mapsto m\exp(x\beta))$ for all $m \in M^0 \subset M$ then $\beta = \alpha$.

According to Proposition 7.1.6 M^0 generates M as a left $K\{\tau\}$-module. As the image of f in $\mathrm{Hom}(M, g(F,K))$ represents a class in $\mathrm{H}^0(\mathcal{H}\mathrm{om}(M, g(F,K)))$ Proposition 1.13.3 shows that the map $M \to g(F,K)$ induced by f is in fact a homomorphism of left $A \otimes K\{\tau\}$-modules. As a consequence $f(m) \sim (x \mapsto m\exp(x\beta))$ for all $m \in M$. Whence the result. □

Proposition 9.8.2. *Let $\mathcal{M} = [\mathcal{M}_0 \overset{i}{\underset{j}{\rightrightarrows}} \mathcal{M}_1]$ be a local model and let $g \in \mathcal{M}_1$.*

(1) *There exists a unique $f \in \mathcal{M}_0(F^{\#} \hat{\otimes} \mathcal{O}_K)$ such that $i(f) = g$.*
(2) *Let $[g]$ be the cohomology class of g in $\mathrm{H}^1(\nabla\mathcal{M})$ and let $\alpha = \nabla\gamma[g]$. The element $\alpha \in \mathrm{Lie}_E(K)$ is uniquely characterized by the property that for every $m \in M^0$ one has*

$$f(m) \sim (x \mapsto dm(x\alpha)).$$

Here we view f as an element of $\mathrm{Hom}(M, b(F,K))$ via the natural inclusion $\mathcal{M}(F^{\#} \hat{\hat{\otimes}} \mathcal{O}_K) \subset \mathcal{H}\mathrm{om}(M, b(F,K))$ of Lemma 9.5.9. Given $m \in M = \mathrm{Hom}(E, \mathbb{G}_a)$ we denote $dm\colon \mathrm{Lie}_E \to \mathrm{Lie}_{\mathbb{G}_a}$ the induced map of Lie algebras.

Proof. Same as the proof of Proposition 9.8.1. □

In the following it would be convenient for us to assemble the maps γ and $\nabla\gamma$ into a map acting on the cohomology of a local model.

Definition 9.8.3. Let $\mathcal{M}$ be a local model. We define the *exponential map*

$$\exp\colon F \otimes_{\mathcal{O}_F} \mathrm{H}^1(\nabla\mathcal{M}) \to F \otimes_{\mathcal{O}_F} \mathrm{H}^1(\mathcal{M})$$

as the composition $\mathrm{H}^1(\gamma) \circ \mathrm{H}^1(\nabla\gamma)^{-1}$.

Even though this exponential map is induced by the identity map on the Lie algebra, one can justify its name by looking at what it does to the auxillary elements $f \in \mathrm{Hom}(M, b(F,K))$ as described in Propositions 9.8.2 and 9.8.1.

We will show in the subsequent chapters that the exponential map is nothing but the inverse of the regulator map $\rho\colon \mathrm{H}^1(\mathcal{M}) \to \mathrm{H}^1(\nabla\mathcal{M})$ which we introduced in Definition 9.7.8. This important result appears to be neither easy nor evident. For one thing, the regulator is a purely shtuka-theoretic construct while the exponential map uses the arithmetic data of the Drinfeld module in an essential way. The only proof we have at the moment is rather technical and is based on explicit computations.

CHAPTER 10

Change of coefficients

This chapter is of a technical nature. In it we verify that the constructions of Chapter 9 are compatible with restriction of the coefficient ring A. This result will be used in Chapter 11 where we show that the regulator of a local model is the inverse of the exponential map. We will do it by reduction to an explicit computation in the case $A = \mathbb{F}_q[t]$.

10.1. Duality for Hom shtukas

In this section we describe a general duality construction for Hom shtukas. It will be used several times in the rest of the chapter. We begin with an auxillary result.

Lemma 10.1.1. *Let R be a τ-ring. If $M = [M_0 \overset{i_M}{\underset{j_M}{\rightrightarrows}} M_1]$ is an R-module shtuka and N a left $R\{\tau\}$-module then*

$$\mathcal{H}\mathrm{om}_R(M, N) = \Big[\mathrm{Hom}_R(M_1, N) \overset{i}{\underset{j}{\rightrightarrows}} \mathrm{Hom}_R(\tau^* M_0, N)\Big]$$

where

$$i(f)\colon r \otimes m \mapsto r f j_M(m),$$
$$j(f)\colon r \otimes m \mapsto r\tau \cdot f i_M(m).$$

Proof. Follows directly from the definition of $\mathcal{H}\mathrm{om}$ (Definition 1.13.1). □

If $\varphi\colon R \to S$ is a ring homomorphism, M an S-module and N an R-module then the R-modules $\mathrm{Hom}_R(M, N)$ and $\mathrm{Hom}_R(S, N)$ carry natural S-module structures: via M in the first case and via S in the second. Furthermore the natural duality map

$$\mathrm{Hom}_S(M, \mathrm{Hom}_R(S, N)) \to \mathrm{Hom}_R(M, N), \quad f \mapsto [m \mapsto f(m)(1)]$$

is an S-module isomorphism. We would like to establish an analog of this duality for a τ-ring homomorphism $\varphi\colon (R,\tau) \to (S,\sigma)$ and $\mathcal{H}\mathrm{om}$ in place of Hom. In the following it will be important to distinguish the τ-endomorphisms of R and S. We thus denote them by different letters.

Let $\varphi\colon (R,\tau) \to (S,\sigma)$ be a homomorphism of τ-rings. For every S-module M there is a natural base change map

$$\mu_M\colon \tau^* M \to \sigma^* M, \quad r \otimes m \mapsto \varphi(r) \otimes m.$$

In particular we have a base change map $\mu_S\colon \tau^* S \to \sigma^* S = S$.

Consider the commutative diagram

$$(*) \qquad \begin{array}{ccc} S & \xrightarrow{\sigma} & S \\ {\scriptstyle\varphi}\uparrow & & \uparrow{\scriptstyle\varphi} \\ R & \xrightarrow{\tau} & R \end{array}$$

For the duality statements below to work it will be necessary to assume that $(*)$ is cocartesian in the category of rings. It is cocartesian if and only if the base change map $\mu_S\colon \tau^*S \to S$ is an isomorphism.

Proposition 10.1.2. *Let $\varphi\colon (R,\tau) \to (S,\sigma)$ be a homomorphism of τ-rings and let N be a left $R\{\tau\}$-module. Consider the shtuka*

$$\mathcal{H}\mathrm{om}_R(S,N) = \Big[\mathrm{Hom}_R(S,N) \overset{i}{\underset{j}{\rightrightarrows}} \mathrm{Hom}_R(\tau^*S,N)\Big].$$

If the commutative diagram of rings $(*)$ is cocartesian then the following holds:

(1) *i is an isomorphism.*
(2) *The endomorphism $i^{-1}j$ of $\mathrm{Hom}_R(S,N)$ makes it into a left $S\{\sigma\}$-module.*
(3) *If $f \in \mathrm{Hom}_R(S,N)$ then $g = i^{-1}j(f)$ is the unique R-linear map such that*

$$g\big[\varphi(r)\sigma(s)\big] = r\tau \cdot f(s)$$

for every $r \in R$ and $s \in S$.

Proof. (1) Let $\tau^a\colon \tau^*S \to S$ be the adjoint of the τ-multiplication map of S. By definition

$$\tau^a(r \otimes s) = \varphi(r)\sigma(s).$$

At the same time

$$\mu_S(r \otimes s) = \varphi(r) \otimes s = \varphi(r)\sigma(s).$$

Thus $\tau^a = \mu_S$ is an isomorphism and we conclude that i is an isomorphism from Lemma 10.1.1.

(2) We will deduce from (3) that the endomorphism $i^{-1}j$ is σ-linear. Let us temporarily denote $e = i^{-1}j$. Let $f \in \mathrm{Hom}_R(S,N)$. If $s_1 \in S$ then the maps $e(f \cdot s_1)$ and $e(f) \cdot \sigma(s_1)$ satisfy

$$\begin{aligned} e(f \cdot s_1)\big[\varphi(r)\sigma(s)\big] &= r\tau \cdot f(s_1 s) \\ e(f) \cdot \sigma(s_1)\big[\varphi(r)\sigma(s)\big] &= e(f)\big[\sigma(s_1)\varphi(r)\sigma(s)\big] = r\tau \cdot f(s_1 s) \end{aligned}$$

for every $r \in R$, $s \in S$. Thus e is σ-linear.

(3) Let $\tau_N^a\colon \tau^*N \to N$ be the adjoint of the τ-multiplication map. If $f \in \mathrm{Hom}_R(S,N)$ then according to Lemma 10.1.1

$$j(f)\colon r \otimes s \mapsto r\tau \cdot f(s)$$

for every $r \in R$ and $s \in S$. As we saw in (1) the map i is given by the composition with the base change map $\mu_S\colon r \otimes s \mapsto \varphi(r)\sigma(s)$. The result is now clear. $\square$

Proposition 10.1.3. *Let $\varphi\colon (R,\tau) \to (S,\sigma)$ be a homomorphism of τ-rings. Let*

$$M = \Big[M_0 \overset{i_M}{\underset{j_M}{\rightrightarrows}} M_1\Big]$$

be an S-module shtuka and N a left $R\{\tau\}$-module. If the commutative diagram of rings $(*)$ is cocartesian then the maps

$$\mathrm{Hom}_S(M_1, \mathrm{Hom}_R(S,N)) \xrightarrow{\quad\text{duality}\quad} \mathrm{Hom}_R(M_1,N),$$

$$\mathrm{Hom}_S(\sigma^*M_0, \mathrm{Hom}_R(S,N)) \xrightarrow{(\mu_{M_0})^* \circ\, \text{duality}} \mathrm{Hom}_R(\tau^*M_0,N)$$

define an isomorphism

$$\mathcal{H}\mathrm{om}_S(M, \mathrm{Hom}_R(S,N)) \cong \mathcal{H}\mathrm{om}_R(M,N)$$

of R-module shtukas. The left $S\{\sigma\}$-module structure on $\mathrm{Hom}_R(S,N)$ is as constructed in Proposition 10.1.2.

Proof. Denote the duality maps

$$\eta_0\colon \operatorname{Hom}_S(M_1, \operatorname{Hom}_R(S,N)) \to \operatorname{Hom}_R(M,N),$$
$$\eta_1\colon \operatorname{Hom}_S(\sigma^* M_0, \operatorname{Hom}_R(S,N)) \to \operatorname{Hom}_R(\tau^* M_0, N).$$

If $(*)$ is cocartesian then the base change map μ_{M_0} is an isomorphism. The inverse of μ_{M_0} is given by the formula

$$\varphi(r)\sigma(s)\otimes m \mapsto r\otimes sm.$$

Thus η_0 and η_1 define an isomorphism of R-module shtukas provided they form a morphism of shtukas. Let us show that it is indeed the case.

Let i_S, j_S be the arrows of $\mathcal{H}\mathrm{om}_S(M, \operatorname{Hom}_R(S,N))$ and let i_R, j_R be the arrows of $\mathcal{H}\mathrm{om}_R(M,N)$. If $f \in \operatorname{Hom}_S(M_1, \operatorname{Hom}_R(S,N))$ then

$$\eta_0(f)\colon m \mapsto f(m)(1).$$

So Lemma 10.1.1 implies that

$$j_R(\eta_0(f))\colon r\otimes m \mapsto r\tau\cdot[f i_M(m)](1).$$

By the same Lemma

$$j_S(f)\colon s\otimes m\mapsto s\sigma\cdot f i_M(m).$$

If $g \in \operatorname{Hom}_S(\sigma^* M_0, \operatorname{Hom}_R(S,N))$ then

$$\eta_1(g)\colon r\otimes m\mapsto g(\varphi(r)\otimes m)(1).$$

Therefore

$$\begin{aligned}\eta_1(j_S(f))\colon r\otimes m &\mapsto [j_S(f)(\varphi(r)\otimes m)](1)\\ &= [\varphi(r)\sigma\cdot f i_M(m)](1)\\ &= [\sigma\cdot f i_M(m)](\varphi(r)).\end{aligned}$$

According to Proposition 10.1.2

$$\sigma\cdot f i_M(m)\colon \varphi(r)\mapsto r\tau\cdot[f i_M(m)](1).$$

Hence

$$\eta_1(j_S(f))\colon r\otimes m\mapsto r\tau\cdot[f i_M(m)](1).$$

We conclude that $\eta_1(j_S(f)) = j_R(\eta_0(f))$. It is easy to see that $\eta_1(i_S(f)) = i_R(\eta_0(f))$. □

10.2. Tensor products

Lemma 10.2.1. *Let T be a locally compact $\mathbb{F}_q$-algebra and let $S' \subset S$ be an extension of locally compact $\mathbb{F}_q$-algebras. If S is finitely generated free as a topological S'-module then the natural map $S\otimes_{S'}(S'\mathbin{\check{\otimes}} T)\to S\mathbin{\check{\otimes}} T$ is an isomorphism.*

Proof. We rewrite the natural map in question as

$$(S\otimes_{\mathrm{ic}} T)\otimes_{(S'\otimes_{\mathrm{ic}}T)}(S'\mathbin{\check{\otimes}} T)\to S\mathbin{\check{\otimes}} T.$$

By assumption S is a finitely generated free topological S'-module. Therefore $S\otimes_{\mathrm{ic}} T$ is a finitely generated free topological $S'\otimes_{\mathrm{ic}} T$-module. The result now follows from Lemma 3.2.4. □

Lemma 10.2.2. *Let T be a locally compact $\mathbb{F}_q$-algebra and let $S' \subset S$ be an extension of locally compact $\mathbb{F}_q$-algebras. If S is locally free of finite rank as an S'-module without topology then the natural map $S\otimes_{S'}((S')^{\#}\mathbin{\widehat{\otimes}} T)\to S^{\#}\mathbin{\widehat{\otimes}} T$ is an isomorphism.*

Proof. We rewrite the natural map in question as

$$(S^\# \otimes_{\mathrm{c}} T) \otimes_{((S')^\# \otimes_{\mathrm{c}} T)} ((S')^\# \mathbin{\widehat{\otimes}} T) \to S^\# \mathbin{\widehat{\otimes}} T.$$

By assumption S is locally free of finite rank as an S'-module without topology. Therefore $S^\# \otimes_{\mathrm{c}} T$ is a topological direct summand of a finitely generated free $(S')^\# \otimes_{\mathrm{c}} T$-module. The result now follows from Lemma 3.2.4. □

10.3. The setting

We now start with the main part of this chapter. The setting is as follows. Fix a coefficient ring A as in Definition 7.2.1. As usual we denote F the local field of A at infinity, $\mathcal{O}_F \subset F$ the ring of integers and $\mathfrak{m}_F \subset \mathcal{O}_F$ the maximal ideal. Let K be a finite product of local fields containing $\mathbb{F}_q$. Fix a Drinfeld A-module E over K and let $M = \mathrm{Hom}(E, \mathbb{G}_a)$ be its motive. Throughout the chapter we assume that the action of A on $\mathrm{Lie}_E(K)$ extends to a continuous action of F. This is the context in which we defined and studied local models in Chapter 9.

Fix an $\mathbb{F}_q$-subalgebra $A' \subset A$ such that A is finite flat over A'. Note that A' is itself a coefficient ring. We denote F' the local field of A' at infinity, $\mathcal{O}_{F'} \subset F'$ the ring of integers and $\mathfrak{m}_{F'} \subset \mathcal{O}_{F'}$ the maximal ideal.

Lemma 10.3.1. *The motive of E viewed as a Drinfeld A'-module is M with its natural left $A' \otimes K\{\tau\}$-module structure.* □

10.4. Hom shtukas

Lemma 10.4.1. *The commutative square of rings*

$$\begin{tikzcd} A \otimes K \arrow[r] & F \mathbin{\check{\otimes}} K \\ A' \otimes K \arrow[r] \arrow[u] & F' \mathbin{\check{\otimes}} K \arrow[u] \end{tikzcd}$$

is cocartesian.

Proof. Indeed $A \otimes_{A'} F' = F$ so

$$A \otimes_{A'} (F' \mathbin{\check{\otimes}} K) = (A \otimes_{A'} F') \otimes_{F'} (F' \mathbin{\check{\otimes}} K) = F \otimes_{F'} (F' \mathbin{\check{\otimes}} K).$$

Now Lemma 10.2.1 shows that $F \otimes_{F'} (F' \mathbin{\check{\otimes}} K) = F \mathbin{\check{\otimes}} K$ and the result follows. □

Corollary 10.4.2. *The natural map $M \otimes_{A' \otimes K} (F' \mathbin{\check{\otimes}} K) \to M \otimes_{A \otimes K} (F \mathbin{\check{\otimes}} K)$ is an isomorphism of left $F' \mathbin{\check{\otimes}} K\{\tau\}$-modules.* □

Lemma 10.4.3. *The commutative square of rings*

$$\begin{tikzcd} F \mathbin{\check{\otimes}} K \arrow[r, "\tau"] & F \mathbin{\check{\otimes}} K \\ F' \mathbin{\check{\otimes}} K \arrow[r, "\tau"] \arrow[u] & F' \mathbin{\check{\otimes}} K \arrow[u] \end{tikzcd}$$

is cocartesian.

Proof. Immediate from Lemma 10.2.1. □

We are thus in position to apply the duality machinery of Section 10.1 to the τ-ring homomorphism $F' \mathbin{\check{\otimes}} K \to F \mathbin{\check{\otimes}} K$. Proposition 10.1.2 equips the $F \mathbin{\check{\otimes}} K$-module

$$\mathrm{Hom}_{F' \check{\otimes} K}(F \mathbin{\check{\otimes}} K, a(F', K))$$

with the structure of a left $F \mathbin{\check{\otimes}} K\{\tau\}$-module.

Lemma 10.4.4. *The natural map*

$$\mathrm{Hom}_{F'\breve{\otimes}K}(F \breve{\otimes} K,\, a(F',K)) \to \mathrm{Hom}_{F'}(F,\, a(F',K))$$

is an isomorphism of $F \breve{\otimes} K$-modules.

Proof. Follows since $F \otimes_{F'} (F' \breve{\otimes} K) = F \breve{\otimes} K$ by Lemma 10.2.1. $\square$

So we get a left $F \breve{\otimes} K\{\tau\}$-module structure on $\mathrm{Hom}_{F'}(F,\, a(F',K))$.

Lemma 10.4.5. *If $g \in \mathrm{Hom}_{F'}(F,\, a(F',K))$ then $\tau \cdot g$ maps $x \in F$ to $\tau(g(x))$.* $\square$

Lemma 10.4.6. *The map*

$$a(F,K) \to \mathrm{Hom}_{F'}(F,\, a(F',K)), \quad f \mapsto [x \mapsto (y \mapsto f(yx))]$$

is an isomorphism of left $F \breve{\otimes} K\{\tau\}$-modules.

Proof. For a finite-dimensional F'-vector space V let α_V be the map

$$\alpha_V \colon a(V,K) \to \mathrm{Hom}_{F'}(V,\, a(F',K)), \quad f \mapsto [v \mapsto (y \mapsto f(yv))].$$

It is clearly an $F' \breve{\otimes} K$-linear isomorphism if V is of dimension 1. The map α_V is also natural in V. Hence α_V is an $F' \breve{\otimes} K$-linear isomorphism for any finite-dimensional F'-vector space V and in particular for $V = F$.

The map α_F is also F-linear. By Lemma 10.2.1 we have $F \otimes_{F'} (F' \breve{\otimes} K) = F \breve{\otimes} K$. Hence α_F is $F \breve{\otimes} K$-linear. A simple computation shows that it commutes witht the action of τ. So we get the result. $\square$

Proposition 10.4.7. *The restriction map $a(F,K) \to a(F',K)$ induces an isomorphism*

$$\mathcal{H}\mathrm{om}_{A\otimes K}(M,\, a(F,K)) \cong \mathcal{H}\mathrm{om}_{A'\otimes K}(M,\, a(F',K))$$

of $F' \breve{\otimes} K$-module shtukas.

Proof. We have

$$\mathcal{H}\mathrm{om}_{A\otimes K}(M,\, a(F,K)) = \mathcal{H}\mathrm{om}_{F\breve{\otimes}K}(M \otimes_{A\otimes K} (F \breve{\otimes} K),\, a(F,K)).$$

Lemma 10.4.6 gives us a natural isomorphism

$$\begin{aligned} &\mathcal{H}\mathrm{om}_{F\breve{\otimes}K}(M \otimes_{A\otimes K} (F \breve{\otimes} K),\, a(F,K)) \\ \cong\, &\mathcal{H}\mathrm{om}_{F\breve{\otimes}K}(M \otimes_{A\otimes K} (F \breve{\otimes} K),\, \mathrm{Hom}_{F'}(F, a(F',K))). \end{aligned}$$

In view of Lemma 10.4.3 we can apply Proposition 10.1.3 to get an isomorphism

$$\begin{aligned} &\mathcal{H}\mathrm{om}_{F\breve{\otimes}K}(M \otimes_{A\otimes K} (F \breve{\otimes} K),\, \mathrm{Hom}_{F'}(F, a(F',K))) \\ \cong\, &\mathcal{H}\mathrm{om}_{F'\breve{\otimes}K}(M \otimes_{A\otimes K} (F \breve{\otimes} K),\, a(F',K)) \end{aligned}$$

of $F' \breve{\otimes} K$-module shtukas. Corollary 10.4.2 identifies $M \otimes_{A\otimes K} (F \breve{\otimes} K)$ with $M \otimes_{A'\otimes K} (F' \breve{\otimes} K)$. So we get an isomorphism

$$\mathcal{H}\mathrm{om}_{A\otimes K}(M,\, a(F,K)) \xrightarrow{\sim} \mathcal{H}\mathrm{om}_{A'\otimes K}(M,\, a(F',K))$$

of $F' \breve{\otimes} K$-module shtukas. A straightforward computation shows that this isomorphism is induced by the restriction map $a(F,K) \to a(F',K)$. $\square$

10.5. Coefficient compactifications

We denote C the projective compactification of $\operatorname{Spec} A$ and C' the projective compactification of A'. Let $\rho\colon C \times \operatorname{Spec} K \to C' \times \operatorname{Spec} K$ be the map induced by the inclusion $A' \subset A$.

Lemma 10.5.1. *The commutative square of schemes*

$$\begin{tikzcd} \operatorname{Spec} \mathcal{O}_F \breve{\otimes} K \arrow[r] \arrow[d] & C \times X \arrow[d, "\rho"] \\ \operatorname{Spec} \mathcal{O}_{F'} \breve{\otimes} K \arrow[r] & C' \times X \end{tikzcd}$$

is cartesian.

Proof. Lemma 10.2.1 implies that the square

$$\begin{tikzcd} \mathcal{O}_F \otimes K \arrow[r] & \mathcal{O}_F \breve{\otimes} K \\ \mathcal{O}_{F'} \otimes K \arrow[r] \arrow[u] & \mathcal{O}_{F'} \breve{\otimes} K \arrow[u] \end{tikzcd}$$

is cocartesian. Whence the result. □

Corollary 10.5.2. *For every quasi-coherent sheaf $\mathcal{E}$ on $C \times X$ the base change map*

$$(\rho_* \mathcal{E})(\mathcal{O}_{F'} \breve{\otimes} K) \to \mathcal{E}(\mathcal{O}_F \breve{\otimes} K)$$

is an isomorphism of $\mathcal{O}_{F'} \breve{\otimes} K$-modules. □

Lemma 10.5.3. *If $\mathcal{E}$ is the shtuka on $C \times K$ which corresponds to the left $A \otimes K\{\tau\}$-module M by Theorem 7.6.1 then $\rho_* \mathcal{E}$ is the shtuka on $C' \times K$ which corresponds to M viewed as a left $A' \otimes K\{\tau\}$-module.*

Proof. Suppose that $\mathcal{E}$ is given by the diagram

$$\Big[\mathcal{E}_{-1} \underset{j}{\overset{i}{\rightrightarrows}} \mathcal{E}_0\Big] \subset \Big[M \underset{\tau}{\overset{1}{\rightrightarrows}} M\Big].$$

Let f be the degree of the residue field of F over $\mathbb{F}_q$ and let r be the rank of M as an $A \otimes K$-module. By Theorem 7.6.1 the sheaves $\mathcal{E}_{-1}$, $\mathcal{E}_0$ are locally free of rank r and have the following property:

$$\begin{aligned} \mathrm{H}^0(C \times \operatorname{Spec} K,\, \mathcal{E}_0(n)) &= M^{frn}, \\ \mathrm{H}^0(C \times \operatorname{Spec} K,\, \mathcal{E}_{-1}(n)) &= M^{frn-1}. \end{aligned}$$

Let $d = [F : F']$. Observe that $f = df'$ where f' is the degree of the residue field of F' over $\mathbb{F}_q$. The morphism $\rho\colon C \times K \to C' \times K$ is finite flat of degree d. Hence $\rho_* \mathcal{E}_{-1}$, $\rho_* \mathcal{E}_0$ are locally free of rank dr and

$$\begin{aligned} \mathrm{H}^0(C \times \operatorname{Spec} K,\, \rho_* \mathcal{E}_0(n)) &= M^{f'drn}, \\ \mathrm{H}^0(C \times \operatorname{Spec} K,\, \rho_* \mathcal{E}_{-1}(n)) &= M^{f'drn-1}. \end{aligned}$$

The unicity part of Theorem 7.6.1 now implies the result. □

We next study the function space $a(F/\mathcal{O}_F, K)$.

Lemma 10.5.4. *The commutative square of rings*

$$\begin{array}{ccc} \mathcal{O}_F \check{\otimes} K & \xrightarrow{\tau} & \mathcal{O}_F \check{\otimes} K \\ \uparrow & & \uparrow \\ \mathcal{O}_{F'} \check{\otimes} K & \xrightarrow{\tau} & \mathcal{O}_{F'} \check{\otimes} K \end{array}$$

is cocartesian.

Proof. Immediate from Lemma 10.2.1. □

So we can apply the duality constructions of Section 10.1 to the τ-ring homomorphism $\mathcal{O}_{F'} \check{\otimes} K \to \mathcal{O}_F \check{\otimes} K$. Proposition 10.1.2 equips the $\mathcal{O}_F \check{\otimes} K$-module

$$\mathrm{Hom}_{\mathcal{O}_{F'} \check{\otimes} K}(\mathcal{O}_F \check{\otimes} K,\, a(F'/\mathcal{O}_{F'}, K))$$

with the structure of a left $\mathcal{O}_F \check{\otimes} K\{\tau\}$-module.

Lemma 10.5.5. *The natural map*

$$\mathrm{Hom}_{\mathcal{O}_{F'} \check{\otimes} K}(\mathcal{O}_F \check{\otimes} K,\, a(F'/\mathcal{O}_{F'}, K)) \to \mathrm{Hom}_{\mathcal{O}_{F'}}(\mathcal{O}_F,\, a(F'/\mathcal{O}_{F'}, K))$$

is an isomorphism of $\mathcal{O}_F \check{\otimes} K$-modules. □

So we get a left $\mathcal{O}_F \check{\otimes} K\{\tau\}$-module structure on $\mathrm{Hom}_{\mathcal{O}_{F'}}(\mathcal{O}_F,\, a(F'/\mathcal{O}_{F'}, K))$.

Lemma 10.5.6. *If $g \in \mathrm{Hom}_{\mathcal{O}_{F'}}(\mathcal{O}_F, a(F'/\mathcal{O}_{F'}, K))$ then $\tau \cdot g$ maps $x \in \mathcal{O}_F$ to $\tau(g(x))$.* □

Lemma 10.5.7. *The map*

$$a(F/\mathcal{O}_F, K) \to \mathrm{Hom}_{\mathcal{O}_{F'}}(\mathcal{O}_F,\, a(F'/\mathcal{O}_{F'}, K)), \quad f \mapsto [x \mapsto (y \mapsto f(yx))]$$

is an isomorphism of left $\mathcal{O}_F \check{\otimes} K\{\tau\}$-modules.

Proof. We view $a(F/\mathcal{O}_F, K)$ as a subspace of $a(F, K)$ consisting of functions which vanish on $\mathcal{O}_F$, and similarly for $a(F'/\mathcal{O}_{F'}, K)$. Note that

$$\mathrm{Hom}_{F'}(F, a(F, K)) = \mathrm{Hom}_{\mathcal{O}_{F'}}(\mathcal{O}_F,\, a(F', K)).$$

We can thus identify

$$\mathrm{Hom}_{\mathcal{O}_{F'}}(\mathcal{O}_F,\, a(F'/\mathcal{O}_{F'}, K))$$

with a submodule of $\mathrm{Hom}_{F'}(F,\, a(F, K))$. In view of this remark the result follows from Lemma 10.4.6. □

Proposition 10.5.8. *The restriction isomorphism of Proposition 10.4.7 identifies the coefficient compactification of $\mathcal{H}\mathrm{om}_{A \otimes K}(M, a(F, K))$ with the coefficient compactification of $\mathcal{H}\mathrm{om}_{A' \otimes K}(M, a(F', K))$.*

Proof. Lemma 10.5.7 gives us a natural isomorphism

$$\mathcal{H}\mathrm{om}_{\mathcal{O}_F \check{\otimes} K}(\mathcal{E}(\mathcal{O}_F \check{\otimes} K),\, a(F/\mathcal{O}_F, K)) \xrightarrow{\sim}$$
$$\mathcal{H}\mathrm{om}_{\mathcal{O}_F \check{\otimes} K}(\mathcal{E}(\mathcal{O}_F \check{\otimes} K),\, \mathrm{Hom}_{\mathcal{O}_{F'}}(\mathcal{O}_F, a(F'/\mathcal{O}_{F'}, K))).$$

In view of Lemma 10.5.4 we can apply Proposition 10.1.3 and identify the shtuka above with

$$\mathcal{H}\mathrm{om}_{\mathcal{O}_{F'} \check{\otimes} K}(\mathcal{E}(\mathcal{O}_F \check{\otimes} K),\, a(F'/\mathcal{O}_{F'}, K))$$

It is easy to see that the resulting isomorphism

$$\mathcal{H}\mathrm{om}_{\mathcal{O}_F \check{\otimes} K}(\mathcal{E}(\mathcal{O}_F \check{\otimes} K),\, a(F/\mathcal{O}_F, K)) \xrightarrow{\sim}$$
$$\mathcal{H}\mathrm{om}_{\mathcal{O}_{F'} \check{\otimes} K}(\mathcal{E}(\mathcal{O}_F \check{\otimes} K),\, a(F'/\mathcal{O}_{F'}, K))$$

is induced by the restriction map $a(F/\mathcal{O}_F, K) \to a(F'/\mathcal{O}_{F'}, K)$. Now Corollary 10.5.2 implies that the natural map $\rho_*\mathcal{E}(\mathcal{O}_{F'} \breve{\otimes} K) \to \mathcal{E}(\mathcal{O}_F \breve{\otimes} K)$ is an isomorphism. Lemma 10.5.3 implies that the shtuka

$$\mathcal{H}\mathrm{om}_{\mathcal{O}_{F'} \breve{\otimes} K}(\rho_*\mathcal{E}(\mathcal{O}_{F'} \breve{\otimes} K),\, a(F'/\mathcal{O}_{F'}, K))$$

is the coefficient compactification of $\mathcal{H}\mathrm{om}_{A' \otimes K}(M,\, a(F', K))$. Whence the result. $\square$

10.6. Base compactifications

Lemma 10.6.1. *The commutative square of rings*

$$\begin{array}{ccc} F \breve{\otimes} \mathcal{O}_K & \longrightarrow & F \breve{\otimes} K \\ \uparrow & & \uparrow \\ F' \breve{\otimes} \mathcal{O}_K & \longrightarrow & F' \breve{\otimes} K \end{array}$$

is cocartesian.

Proof. Follows from Lemma 10.2.1. $\square$

Lemma 10.6.2. *For every open ideal $I \subset \mathcal{O}_K$ the commutative square of rings*

$$\begin{array}{ccc} F \breve{\otimes} \mathcal{O}_K & \longrightarrow & F \otimes \mathcal{O}_K/I \\ \uparrow & & \uparrow \\ F' \breve{\otimes} \mathcal{O}_K & \longrightarrow & F' \otimes \mathcal{O}_K/I \end{array}$$

is cocartesian.

Proof. Indeed Lemma 10.2.1 implies that the square

$$\begin{array}{ccc} F \otimes \mathcal{O}_K & \longrightarrow & F \breve{\otimes} \mathcal{O}_K \\ \uparrow & & \uparrow \\ F' \otimes \mathcal{O}_K & \longrightarrow & F' \breve{\otimes} \mathcal{O}_K \end{array}$$

is cocartesian. Whence the result. $\square$

According to our assumptions the action of A on $\mathrm{Lie}_E(K)$ extends to a continuous action of F so that we have a continuous homomorphism $F \to K$. Recall that the *ramification ideal* $\mathfrak{e}$ is the ideal generated by $\mathfrak{m}_F$ in $\mathcal{O}_K$ (Definition 9.5.4). In the same manner we get a ramification ideal $\mathfrak{e}' \subset \mathcal{O}_K$ for the coefficient subring $A' \subset A$.

Proposition 10.6.3. *Let $\mathcal{M} \subset \mathcal{H}\mathrm{om}_{A\otimes K}(M,\, a(F, K))$ be a base compactification. If $\mathcal{M}(F \otimes \mathcal{O}_K/\mathfrak{e}')$ is linear then the image of $\mathcal{M}$ under the restriction isomorphism of Proposition 10.4.7 is a base compactification of $\mathcal{H}\mathrm{om}_{A'\otimes K}(M,\, a(F', K))$.*

Proof. Let $\mathcal{M}'$ be the image of $\mathcal{M}$ in $\mathcal{H}\mathrm{om}_{A'\otimes K}(M,\, a(F', K))$. The shtuka $\mathcal{M}$ is an $F \breve{\otimes} \mathcal{O}_K$-lattice in $\mathcal{H}\mathrm{om}_{A\otimes K}(M,\, a(F, K))$ so Lemma 10.6.1 implies that $\mathcal{M}'$ is an $F' \breve{\otimes} \mathcal{O}_K$-lattice in $\mathcal{H}\mathrm{om}_{A'\otimes K}(M, a(F', K))$. According to Lemma 10.6.2 we have natural isomorphisms

$$\mathcal{M}'(F' \otimes \mathcal{O}_K/\mathfrak{m}) \cong \mathcal{M}(F \otimes \mathcal{O}_K/\mathfrak{m}),$$
$$\mathcal{M}'(F' \otimes \mathcal{O}_K/\mathfrak{e}') \cong \mathcal{M}(F \otimes \mathcal{O}_K/\mathfrak{e}).$$

By definition $\mathcal{M}(F \otimes \mathcal{O}_K/\mathfrak{m})$ is nilpotent while $\mathcal{M}(F \otimes \mathcal{O}_K/\mathfrak{e}')$ is linear by assumption. It follows that $\mathcal{M}'$ is a base compactification. $\square$

10.7. Local models

Proposition 10.7.1. *Let $\mathcal{M} \subset \mathcal{H}\mathrm{om}_{A\otimes K}(M,\, a(F,K))$ be a local model. If $\mathcal{M}(F \otimes \mathcal{O}_K/\mathfrak{e}')$ is linear then the image of $\mathcal{M}$ under the restriction isomorphism of Proposition 10.4.7 is a local model of the shtuka $\mathcal{H}\mathrm{om}_{A'\otimes K}(M,\, a(F',K))$.*

Proof. By Proposition 9.5.8 the local model $\mathcal{M}$ is the intersection of the coefficient compactification $\mathcal{M}^c$ and a base compactification $\mathcal{M}^b$. Proposition 10.5.8 claims that $\mathcal{M}^c$ is mapped isomorphically onto the coefficient compactification of $\mathcal{H}\mathrm{om}_{A'\otimes K}(M,\, a(F',K))$. The image of $\mathcal{M}^b$ in $\mathcal{H}\mathrm{om}_{A'\otimes K}(M,\, a(F',K))$ is a base compactification by Propostion 10.6.3. According to Proposition 9.5.8 their intersection is a local model. □

Proposition 10.7.2. *Let $\mathcal{M} \subset \mathcal{H}\mathrm{om}_{A\otimes K}(M,\, a(F,K))$ be a local model such that $\mathcal{M}(F \otimes \mathcal{O}_K/\mathfrak{e}')$ is linear. Let $\mathcal{M}' \subset \mathcal{H}\mathrm{om}_{A'\otimes K}(M,\, a(F',K))$ be the image of $\mathcal{M}$ under the restriction isomorphism of Proposition 10.4.7. The diagram*

$$\begin{tikzcd}
\mathrm{H}^1(\mathcal{M}) \arrow[r, "\rho"] \arrow[d, "res."', "\wr"] & \mathrm{H}^1(\nabla\mathcal{M}) \arrow[d, "res.", "\wr"'] \\
\mathrm{H}^1(\mathcal{M}') \arrow[r, "\rho'"] & \mathrm{H}^1(\nabla\mathcal{M}')
\end{tikzcd}$$

is commutative. Here ρ is the regulator of $\mathcal{M}$ and ρ' is the regulator of $\mathcal{M}'$.

Proof. Recall that the natural maps

$$\mathcal{O}_F \mathbin{\check{\otimes}} \mathcal{O}_K \to \mathcal{O}_F \mathbin{\widehat{\otimes}} \mathcal{O}_K,$$
$$\mathcal{O}_{F'} \mathbin{\check{\otimes}} \mathcal{O}_K \to \mathcal{O}_{F'} \mathbin{\widehat{\otimes}} \mathcal{O}_K$$

are isomorphisms by Proposition 2.7.8. We need to prove that the regulator of $\mathcal{M}$ viewed as an elliptic $\mathcal{O}_F \mathbin{\widehat{\otimes}} \mathcal{O}_K$-shtuka of ramification ideal $\mathfrak{e}$ coincides with the regulator of $\mathcal{M}$ viewed as an elliptic $\mathcal{O}_{F'} \mathbin{\widehat{\otimes}} \mathcal{O}_K$-shtuka of ramification ideal $\mathfrak{e}'$. The field extension F/F' is totally ramified of degree d. As a consequence $\mathfrak{e}' = \mathfrak{e}^d$. The result now follows from Theorem 5.14.4. □

Next we prove that the exponential maps of local models are stable under restriction of coefficients.

Lemma 10.7.3. *The restriction map $b(F,K) \to b(F',K)$ induces an isomorphism*

$$\mathcal{H}\mathrm{om}_{A\otimes K}(M,\, b(F,K)) \cong \mathcal{H}\mathrm{om}_{A'\otimes K}(M,\, b(F',K))$$

of $F^{\#} \mathbin{\widehat{\otimes}} K$-module shtukas.

Proof. The argument is the same as in Proposition 10.4.7 save for the fact that one needs to use Lemma 10.2.2 in place of Lemma 10.2.1. □

Lemma 10.7.4. *The commutative square of rings*

$$\begin{tikzcd}
\mathcal{O}_F \mathbin{\check{\otimes}} \mathcal{O}_K \arrow[r] & F^{\#} \mathbin{\widehat{\otimes}} \mathcal{O}_K \\
\mathcal{O}_{F'} \mathbin{\check{\otimes}} \mathcal{O}_K \arrow[r] \arrow[u] & (F')^{\#} \mathbin{\widehat{\otimes}} \mathcal{O}_K \arrow[u]
\end{tikzcd}$$

is cocartesian.

Proof. Lemma 10.2.1 implies that the square

$$\begin{tikzcd} \mathcal{O}_F \otimes \mathcal{O}_K \arrow[r] & \mathcal{O}_F \check{\otimes} \mathcal{O}_K \\ \mathcal{O}_{F'} \otimes \mathcal{O}_K \arrow[r] \arrow[u] & \mathcal{O}_{F'} \check{\otimes} \mathcal{O}_K \arrow[u] \end{tikzcd}$$

is cocartesian. At the same time the square

$$\begin{tikzcd} F \otimes \mathcal{O}_K \arrow[r] & F^{\#} \widehat{\otimes} \mathcal{O}_K \\ F' \otimes \mathcal{O}_K \arrow[r] \arrow[u] & (F')^{\#} \widehat{\otimes} \mathcal{O}_K \arrow[u] \end{tikzcd}$$

is cocartesian by Lemma 10.2.2. Whence the result. □

Proposition 10.7.5. *Let $\mathcal{M} \subset \mathcal{H}\mathrm{om}_{A\otimes K}(M,\, a(F, K))$ be a local model such that $\mathcal{M}(F \otimes \mathcal{O}_K/\mathfrak{c}')$ is linear. Let $\mathcal{M}' \subset \mathcal{H}\mathrm{om}_{A'\otimes K}(M,\, a(F', K))$ be the image of $\mathcal{M}$ under the restriction isomorphism of Proposition 10.4.7. The diagram*

$$\begin{tikzcd} F \otimes_{\mathcal{O}_F} \mathrm{H}^1(\nabla\mathcal{M}) \arrow[r, "\exp"] \arrow[d, "res."', "\wr"] & F \otimes_{\mathcal{O}_F} \mathrm{H}^1(\mathcal{M}) \arrow[d, "\wr"', "res."] \\ F' \otimes_{\mathcal{O}_{F'}} \mathrm{H}^1(\nabla\mathcal{M}') \arrow[r, "\exp'"] & F' \otimes_{\mathcal{O}'_F} \mathrm{H}^1(\mathcal{M}') \end{tikzcd}$$

is commutative. Here exp *is the exponential map of $\mathcal{M}$ and* exp$'$ *is the exponential map of $\mathcal{M}'$.*

Proof. Suppose that $\mathcal{M}$ is given by a diagram

$$\Big[\mathcal{M}_0 \overset{i}{\underset{j}{\rightrightarrows}} \mathcal{M}_1\Big].$$

By definition the exponential map of $\mathcal{M}$ is the composition $\mathrm{H}^1(\gamma) \circ \mathrm{H}^1(\nabla\gamma)^{-1}$ of the maps

$$\mathrm{H}^1(\gamma)\colon F \otimes_{\mathcal{O}_F} \mathrm{H}^1(\mathcal{M}) \to \mathrm{Lie}_E(K),$$
$$\mathrm{H}^1(\nabla\gamma)\colon F \otimes_{\mathcal{O}_F} \mathrm{H}^1(\nabla\mathcal{M}) \to \mathrm{Lie}_E(K)$$

of Definition 9.7.2. Similarly the exponential map of $\mathcal{M}'$ is the composition of the maps

$$\mathrm{H}^1(\gamma')\colon F' \otimes_{\mathcal{O}_{F'}} \mathrm{H}^1(\mathcal{M}) \to \mathrm{Lie}_E(K),$$
$$\mathrm{H}^1(\nabla\gamma')\colon F' \otimes_{\mathcal{O}_{F'}} \mathrm{H}^1(\nabla\mathcal{M}) \to \mathrm{Lie}_E(K).$$

To prove the proposition it is enough to show that $\mathrm{H}^1(\gamma)$ and $\mathrm{H}^1(\nabla\gamma)$ are compatible with the corresponding maps of $\mathcal{M}'$.

Let $c \in \mathrm{H}^1(\mathcal{M})$ be a cohomology class and let α be the image of c under $\mathrm{H}^1(\gamma)$. Let $g \in \mathcal{M}_1$ be an element representing c. According to Proposition 9.8.1 there exists a unique element

$$f \in \mathcal{M}_0(F^{\#} \widehat{\otimes} \mathcal{O}_K) \subset \mathcal{H}\mathrm{om}_{A\otimes K}(M,\, b(F, K))$$

such that $(i - j)(f) = g$. The element $\alpha \in \mathrm{Lie}_E(K)$ is characterized by the fact that for all x in an open neighbourhood of 0 in F and for all $m \in M^0$ we have $f(m)(x) = m \exp(x\alpha)$.

Lemma 10.7.3 identifies $\mathcal{H}\mathrm{om}_{A\otimes K}(M,\, b(F,K))$ with $\mathcal{H}\mathrm{om}_{A'\otimes K}(M,\, b(F',K))$ while Lemma 10.7.4 implies that the natural map $\mathcal{M}'((F')^{\#}\widehat{\otimes}\mathcal{O}_K) \to \mathcal{M}(F^{\#}\widehat{\otimes}\mathcal{O}_K)$ is an isomorphism. Using Proposition 9.8.1 we conclude that the square

$$\begin{tikzcd}
\mathrm{H}^1(\mathcal{M}) \arrow[r, "\mathrm{H}^1(\gamma)"] \arrow[d, "\mathrm{res.}"', "\wr"] & \mathrm{Lie}_E(K) \arrow[d, equal] \\
\mathrm{H}^1(\mathcal{M}') \arrow[r, "\mathrm{H}^1(\gamma')"] & \mathrm{Lie}_E(K)
\end{tikzcd}$$

is commutative. The same argument shows that the square

$$\begin{tikzcd}
\mathrm{H}^1(\nabla\mathcal{M}) \arrow[r, "\mathrm{H}^1(\nabla\gamma)"] \arrow[d, "\mathrm{res.}"', "\wr"] & \mathrm{Lie}_E(K) \arrow[d, equal] \\
\mathrm{H}^1(\nabla\mathcal{M}') \arrow[r, "\mathrm{H}^1(\nabla\gamma')"] & \mathrm{Lie}_E(K)
\end{tikzcd}$$

is commutative. So we get the result. □

CHAPTER 11

Regulators of local models

Let E be a Drinfeld A-module over K, a finite product of local fields containing $\mathbb{F}_q$. As in Chapter 9 we assume that the action of A on $\mathrm{Lie}_E(K)$ extends to a continuous action of F. In that chapter we introduced the notion of a local model $\mathcal{M}$, its regulator $\rho\colon \mathrm{H}^1(\mathcal{M}) \xrightarrow{\sim} \mathrm{H}^1(\nabla\mathcal{M})$ and its exponential map $\exp\colon F \otimes_{\mathcal{O}_F} \mathrm{H}^1(\nabla\mathcal{M}) \xrightarrow{\sim} F \otimes_{\mathcal{O}_F} \mathrm{H}^1(\mathcal{M})$. Our goal is to show that the square

$$\begin{tikzcd}
\mathrm{H}^1(\nabla\mathcal{M}) \arrow[r] & F \otimes_{\mathcal{O}_F} \mathrm{H}^1(\nabla\mathcal{M}) \arrow[d, "\exp", "\wr"'] \\
\mathrm{H}^1(\mathcal{M}) \arrow[u, "\rho", "\wr"'] \arrow[r] & F \otimes_{\mathcal{O}_F} \mathrm{H}^1(\mathcal{M})
\end{tikzcd}$$

commutes. Unfortunately the only proof we have proceeds by reduction to the case $A = \mathbb{F}_q[t]$ (using the results of Chapter 10) and a brute force computation. On the positive side, this chapter makes the abstract machinery of Chapter 9 more explicit.

11.1. The setting

From now on we consider the special case $A = \mathbb{F}_q[t]$. As usual $F = \mathbb{F}_q(\!(t^{-1})\!)$ stands for the local field of A at infinity and $\mathcal{O}_F = \mathbb{F}_q[[t^{-1}]]$ denotes the ring of integers of F. Let K be a local field containing $\mathbb{F}_q$. As before its ring of integers is denoted $\mathcal{O}_K$ and $\mathfrak{m}_K \subset \mathcal{O}_K$ stands for the maximal ideal. We also fix a norm on K such that $|\zeta| = q^{-1}$ for a uniformizer $\zeta \in K$.

Fix a Drinfeld A-module E of rank r over K. Let $\varphi\colon A \to K\{\tau\}$ be the ring homomorphism determined by the action of A on E. We define the elements $\theta, \alpha_1, \ldots, \alpha_r \in K$ via the equation

$$\varphi(t) = \theta + \alpha_1\tau + \ldots + \alpha_r\tau^r.$$

As in Chapter 9 we assume that the action of A on $\mathrm{Lie}_E(K)$ extends to a continuous action of F.

Lemma 11.1.1. *The following are equivalent:*

(1) *The action of A on $\mathrm{Lie}_E(K)$ extends to a continuous action of F.*
(2) $|\theta| > 1$. □

According to Definition 9.5.4 the ramification ideal $\mathfrak{e} \subset \mathcal{O}_K$ is the ideal generated by $\mathfrak{m}_F$ under the homomorphism $F \to K$ induced by the action of A on $\mathrm{Lie}_E(K)$.

Lemma 11.1.2. $\mathfrak{e} = \theta^{-1}\mathcal{O}_K$. □

Throughout this chapter $M = \mathrm{Hom}(E, \mathbb{G}_a)$ stands for the motive of E. Without loss of generality we assume that the underlying group scheme of E is $\mathbb{G}_a$ so that we can identify M with $K\{\tau\}$.

Lemma 11.1.3. *The motive $M = K\{\tau\}$ has the following properties.*

(1) *M is a free $A \otimes K$-module of rank r with a basis $1, \ldots, \tau^{r-1}$.*
(2) *$M^{\geqslant 1}$ is a free $A \otimes K$-module of rank r with a basis $\tau, \ldots, \tau^r$.*

(3) *We have a relation*

$$\tau^r = \alpha_r^{-1}\big((t-\theta)\cdot 1 - \alpha_1\tau - \alpha_2\tau^2 - \ldots - \alpha_{r-1}\tau^{r-1}\big).$$

in the $A \otimes K$-module M.

Proof. (1) follows from Lemma 7.3.3 (2). In view of (1) the result (2) is a corollary of Proposition 7.1.12. (3) is a consequence of (1). □

11.2. Coefficient compactification

Lemma 11.2.1. *Let $k \geqslant 1$ be an integer.*

(1) *Let $f \in \mathrm{Hom}(M,\, a(F,K))$. The following are equivalent:*
  (a) *$f(\tau^n)$ vanishes on $\mathcal{O}_F$ for all $n \leqslant kr$.*
  (b) *$f(1), \ldots, f(\tau^{r-1})$ vanish on $t^{k-1}\mathcal{O}_F$ and furthermore $f(1)$ vanishes on $t^k\mathcal{O}_F$.*

(2) *Let $g \in \mathrm{Hom}(M^{\geqslant 1},\, a(F,K))$. The following are equivalent:*
  (a) *$g(\tau^n)$ vanishes on $\mathcal{O}_F$ for all $n \leqslant kr$.*
  (b) *$g(\tau), \ldots, g(\tau^r)$ vanish on $t^{k-1}\mathcal{O}_F$.*

Proof. (2) From the relation

$$\tau^n t = \theta^{q^n}\tau^n + \alpha_1^{q^n}\tau^{n+1} + \ldots + \alpha_r^{q^n}\tau^{n+r}.$$

one concludes that if $g(\tau^n), \ldots, g(\tau^{n+r-1})$ vanish on an open neighbourhood $U \subset F$ of 0 then the following are equivalent:

(i) $g(\tau^{n+r})$ vanishes on U.
(ii) $g(\tau^n)$ vanishes on tU.

With (i) one gets (a) ⇒ (b) and (ii) implies (b) ⇒ (a). The argument for (1) is the same as for (2). □

Proposition 11.2.2. *The $\mathcal{O}_F \check{\otimes} K$-modules $\mathcal{M}_0^c$, $\mathcal{M}_1^c$ in the coefficient compactification*

$$\Big[\mathcal{M}_0^c \underset{j}{\overset{i}{\rightrightarrows}} \mathcal{M}_1^c\Big] \subset \mathcal{H}\mathrm{om}(M,\, a(F,K)).$$

admit the following description:

$$\mathcal{M}_0^c = \big\{f\colon M \to a(F,K) \;\mid\; f(\tau), \ldots, f(\tau^{r-1}) \text{ vanish on } t^{-1}\mathcal{O}_F \text{ and } f(1) \text{ vanishes on } \mathcal{O}_F\big\},$$

$$\mathcal{M}_1^c = \big\{g\colon M \to a(F,K) \;\mid\; g(\tau), \ldots, g(\tau^r) \text{ vanish on } t^{-1}\mathcal{O}_F\big\}.$$

Proof. Let C be the compactification of $\operatorname{Spec} A$. We denote

$$\mathcal{E} = \Big[\mathcal{E}_{-1} \underset{j}{\overset{i}{\rightrightarrows}} \mathcal{E}_0\Big] \subset \Big[M \underset{\tau}{\overset{1}{\rightrightarrows}} M\Big]$$

the shtuka on $C \times \operatorname{Spec} K$ produced by Theorem 7.6.1. By definition $\mathcal{M}_0^c$ consists of those $f\colon M \to a(F,K)$ which send the submodule

$$\mathcal{E}_0(\mathcal{O}_F \check{\otimes} K) \subset M \otimes_{A\otimes K} (F \check{\otimes} K)$$

to $a(F/\mathcal{O}_F, K) \subset a(F,K)$.

Take $k \gg 0$ such that $\mathcal{E}_0(k)$ is globally generated. Theorem 7.6.1 implies that the $\mathcal{O}_F \check{\otimes} K$-submodule

$$\mathcal{E}_0(k)(\mathcal{O}_F \check{\otimes} K) \subset M \otimes_{A\otimes K} (F \check{\otimes} K)$$

is generated by $\mathrm{H}^0(C \times \operatorname{Spec} K,\, \mathcal{E}_0(k)) = M^{kr}$. By Lemma 11.2.1 (1) a morphism $f\colon M \to a(F,K)$ sends this submodule to $a(F/\mathcal{O}_K, K)$ if and only if $f(1)$ vanishes on $t^k\mathcal{O}_F$ and $f(\tau), \ldots, f(\tau^{r-1})$ vanish on $t^{k-1}\mathcal{O}_F$. However

$$\mathcal{E}_0(\mathcal{O}_F \mathbin{\check{\otimes}} K) = t^{-k}\mathcal{E}_0(k)(\mathcal{O}_F \mathbin{\check{\otimes}} K)$$

so we get the result for $\mathcal{M}_0^{\mathrm{c}}$. The case of $\mathcal{M}_1^{\mathrm{c}}$ is handled in a similar manner. □

11.3. Explicit models

We equip the space $b(F,K)$ and its subspace $a(F,K)$ with the sup-norm induced by the norm on K.

Lemma 11.3.1. *Let $\mu\colon F \to \mathbb{F}_q$ be a nonzero continuous $\mathbb{F}_q$-linear map.*

(1) *μ generates $a(F,K)$ as an $F \mathbin{\check{\otimes}} K$-module.*
(2) *For every $\beta \in K^\times$ the subspace $\{f : |f| \leqslant |\beta|\} \subset a(F,K)$ is a free $F \mathbin{\check{\otimes}} \mathcal{O}_K$-submodule generated by $\beta \cdot \mu$.*

Proof. (1) Let F^* be the continuous $\mathbb{F}_q$-linear dual of F. According to Theorem 3.9.1 the topological F-module F^* is free of rank one. So a nonzero function $\mu\colon F \to \mathbb{F}_q$ generates F^* as an F-module. Now Corollary 3.9.3 implies that F^* is an F-lattice in the $F \mathbin{\check{\otimes}} K$-module $a(F,K)$ so we get the result.

(2) Without loss of generality we assume that $\beta = 1$. In this case the subspace of $a(F,K)$ in question is $a(F,\mathcal{O}_K)$. By Corollary 3.9.3 the space F^* is an F-lattice in the $F \mathbin{\check{\otimes}} \mathcal{O}_K$-module $a(F,\mathcal{O}_K)$. Whence the result. □

Definition 11.3.2. Let $b, b_1, \ldots, b_{r-1} \in q^{\mathbb{Z}}$ be real numbers. We introduce conditions on $A \otimes K$-linear maps $f\colon M \to b(F,K)$ and $g\colon M^{\geqslant 1} \to b(F,K)$:

$$(\mathrm{EM}_0) \qquad |f(1)| \leqslant |\alpha_r\theta^{-1}|b, \quad |f(\tau)| \leqslant b_1, \quad \ldots \quad |f(\tau^{r-1})| \leqslant b_{r-1},$$

$$(\mathrm{EM}_1) \qquad |g(\tau)| \leqslant b_1, \quad \ldots \quad |g(\tau^{r-1})| \leqslant b_{r-1}, \quad |g(\tau^r)| \leqslant b.$$

Proposition 11.3.3. *Let $b, b_1 \ldots b_{r-1} \in q^{\mathbb{Z}}$ be real numbers. Consider the $F \mathbin{\check{\otimes}} \mathcal{O}_K$-submodules*

$$\mathcal{M}_0 = \big\{\ f \in \operatorname{Hom}(M, a(F,K)) \mid f \text{ satisfies } (\mathrm{EM}_0)\ \big\},$$
$$\mathcal{M}_1 = \big\{\ g \in \operatorname{Hom}(M^{\geqslant 1}, a(F,K)) \mid g \text{ satisfies } (\mathrm{EM}_1)\ \big\}.$$

If $b, b_1 \ldots b_{r-1}$ satisfy the inequalities

$$(\mathrm{EP}_1) \qquad \Big|\frac{\alpha_n}{\alpha_r}\Big|\frac{b_n}{b} \leqslant 1, \quad n \in \{1, \ldots, r-1\}$$

$$(\mathrm{EP}_2) \qquad \Big|\frac{\alpha_r^q}{\theta^q}\Big|\frac{b^q}{b_1} \leqslant |\theta^{-1}|; \quad \frac{b_{r-1}^q}{b} \leqslant |\theta^{-1}|; \quad \frac{b_n^q}{b_{n+1}} \leqslant |\theta^{-1}|, \quad n \in \{1, \ldots, r-2\}$$

then $\mathcal{M}_0$, $\mathcal{M}_1$ define an $F \mathbin{\check{\otimes}} \mathcal{O}_K$-subshtuka

$$\mathcal{M} = [\mathcal{M}_0 \rightrightarrows \mathcal{M}_1] \subset \mathcal{H}\mathrm{om}(M,\, a(F,K))$$

which is a base compactification in the sense of Definition 9.5.6.

Proof. It will be convenient for us to work with bases of various modules. Our first goal is to construct bases for $\operatorname{Hom}(M, a(F,K))$ and $\operatorname{Hom}(M^{\geqslant 1}, a(F,K))$.

According to Lemma 11.1.3 the elements $1, \tau, \ldots, \tau^{r-1}$ form an $A \otimes K$-basis of M. Similarly the elements $\tau, \ldots, \tau^r$ form a basis of $M^{\geqslant 1}$. Fix a nonzero continuous $\mathbb{F}_q$-linear map $\mu\colon F \to \mathbb{F}_q$. Lemma 11.3.1 (1) shows that μ generates $a(F,K)$ as an $F \mathbin{\check{\otimes}} K$-module. We thus get $F \mathbin{\check{\otimes}} K$-module bases of $\operatorname{Hom}(M, a(F,K))$, $\operatorname{Hom}(M^{\geqslant 1}, a(F,K))$ which are dual to the aforementioned bases of M.

Let $\beta, \beta_1 \dots \beta_{r-1} \in K^\times$ be such that $|\beta| = b, |\beta_1| = b_1 \dots |\beta_{r-1}| = b_{r-1}$. Lemma 11.3.1 (2) implies that the modules $\mathcal{M}_0$, $\mathcal{M}_1$ have $F \breve{\otimes} \mathcal{O}_K$-bases given by matrices

$$\begin{pmatrix} \alpha_r\theta^{-1}\beta & 0 & & 0 \\ 0 & \beta_1 & & 0 \\ & & \ddots & \\ 0 & 0 & & \beta_{r-1} \end{pmatrix}, \quad \begin{pmatrix} \beta_1 & & 0 & 0 \\ & \ddots & & \\ 0 & & \beta_{r-1} & 0 \\ 0 & & 0 & \beta \end{pmatrix}.$$

with respect to the fixed bases of $\mathrm{Hom}(M, a(F,K))$ and $\mathrm{Hom}(M^{\geqslant 1}, a(F,K))$. We conclude that $\mathcal{M}_0$, $\mathcal{M}_1$ are $F \breve{\otimes} \mathcal{O}_K$-lattices in these modules.

Let i and j be the arrows of the shtuka

$$\mathcal{H}\mathrm{om}(M,\, a(F,K)) = \Big[\mathrm{Hom}(M, a(F,K)) \underset{j}{\overset{i}{\rightrightarrows}} \mathrm{Hom}(M^{\geqslant 1}, a(F,K))\Big].$$

According to Proposition 8.8.1 the map i is the restriction to $M^{\geqslant 1}$. Hence the matrix of the map i in the fixed bases of $\mathcal{H}\mathrm{om}(M,\, a(F,K))$ is

$$\begin{pmatrix} 0 & 1 & & 0 \\ & & \ddots & \\ 0 & 0 & & 1 \\ \frac{t-\theta}{\alpha_r} & -\frac{\alpha_1}{\alpha_r} & & -\frac{\alpha_{r-1}}{\alpha_r} \end{pmatrix}$$

Rewriting it in the bases of $\mathcal{M}_0$, $\mathcal{M}_1$ gives

$$\begin{pmatrix} 0 & 1 & & 0 \\ & & \ddots & \\ 0 & 0 & & 1 \\ t\theta^{-1} - 1 & -\frac{\alpha_1\beta_1}{\alpha_r\beta} & & -\frac{\alpha_{r-1}\beta_{r-1}}{\alpha_r\beta} \end{pmatrix}$$

The assumptions $|\theta| > 1$ and (EP$_1$) imply that the bottom row is in $F \breve{\otimes} \mathcal{O}_K$ and therefore $i(\mathcal{M}_0) \subset \mathcal{M}_1$. The determinant of the matrix is $(-1)^r(1 - \theta^{-1}t)$. As $|\theta| > 1$ it reduces to $(-1)^r$ modulo $F \breve{\otimes} \mathfrak{m}_K$ and in particular i is an isomorphism modulo $F \breve{\otimes} \mathfrak{m}_K$.

According to Proposition 8.8.1 the map j sends $f \in \mathrm{Hom}(M, a(F,K))$ to the map

$$\tau m \mapsto \tau f(m).$$

Hence the matrix of j in the bases of $\mathcal{M}_0$, $\mathcal{M}_1$ is

$$\begin{pmatrix} \frac{\alpha_r^q\beta^q}{\theta^q\beta_1} & 0 & & 0 & 0 \\ 0 & \frac{\beta_1^q}{\beta_2} & & 0 & 0 \\ & & \ddots & & \\ 0 & 0 & & \frac{\beta_{r-2}^q}{\beta_{r-1}} & 0 \\ 0 & 0 & & 0 & \frac{\beta_{r-1}^q}{\beta} \end{pmatrix}$$

In our case the ramification ideal $\mathfrak{e}$ is the ideal $\theta^{-1}\mathcal{O}_K$. Hence the assumptions (EP$_2$) imply that the matrix above lies in $\mathfrak{e}$ whence $j(\mathcal{M}_0) \subset \mathcal{M}_1$ and j reduces to zero modulo $\mathfrak{e}$. Above we demonstrated that i reduces to an isomorphism modulo $\mathfrak{m}_K$. Hence $\mathcal{M}$ is a base compactification as claimed. □

Proposition 11.3.4. *There exist $b, b_1, \dots, b_{r-1} \in q^{\mathbb{Z}}$ satisfying* (EP$_1$) *and* (EP$_2$).

Proof. A direct verification shows that for all $\varepsilon \in q^{\mathbb{Z}}$ small enough the real numbers

$$b = \varepsilon^2,\ b_1 = \varepsilon^3, \dots, b_{r-1} = \varepsilon^3$$

satisfy (EP$_1$) and (EP$_2$). □

Definition 11.3.5. Let $b, b_1, \ldots, b_{r-1} \in q^{\mathbb{Z}}$ be real numbers satisfying (EP$_1$), (EP$_2$). The *explicit model* $\mathcal{M}$ of parameters $b, b_1 \ldots b_{r-1}$ is the subshtuka in $\mathcal{H}\mathrm{om}(M,\, a(F,K))$ defined as the intersection

$$\mathcal{M} = \mathcal{M}^c \cap \mathcal{M}^b$$

where $\mathcal{M}^c$ is the coefficient compactification of Definition 9.5.1 and $\mathcal{M}^b$ is the subshtuka of $\mathcal{H}\mathrm{om}(M,\, a(F,K))$ desribed in Proposition 11.3.3. The number b will be called the *leading parameter*.

It follows from Proposition 9.5.8 that an explicit model is a local model in the sense of Definition 9.5.7. In particular we have the twists $\mathfrak{e}^n\mathcal{M}$ of a local model as in Proposition 9.5.10.

Proposition 11.3.6. *If $\mathcal{M}$ is an explicit model of parameters $b, b_1, \ldots, b_{r-1}$ then $\mathfrak{e}\mathcal{M}$ is an explicit model of parameters $b|\theta^{-1}|, b_1|\theta^{-1}|, \ldots, b_{r-1}|\theta^{-1}|$.*

Proof. Follows since $\mathfrak{e} = \theta^{-1}\mathcal{O}_K$. □

Next we prove a technical statement on explicit models as lattices in the shtuka $\mathcal{H}\mathrm{om}(M,\, b(F,K))$.

Lemma 11.3.7. *For every $\beta \in K^\times$ the subspace $\{f \in a(F,K) : |f| \leqslant |\beta|\}$ is an $F \breve{\otimes} \mathcal{O}_K$-lattice in the $F^\# \widehat{\otimes} \mathcal{O}_K$-module $\{f \in b(F,K) : |f| \leqslant |\beta|\}$.*

Proof. Without loss of generality we assume that $\beta = 1$. In this case the spaces in question are $a(F, \mathcal{O}_K)$ and $b(F, \mathcal{O}_K)$ so the result follows from Corollary 3.9.5. □

If $\mathcal{M} \subset \mathcal{H}\mathrm{om}(M,\, a(F,K))$ is a local model then Lemma 9.5.9 shows that the natural map

$$\mathcal{M}(F^\# \widehat{\otimes} \mathcal{O}_K) \to \mathcal{H}\mathrm{om}(M,\, b(F,K))$$

is an inclusion of an $F^\# \widehat{\otimes} \mathcal{O}_K$-lattice. In particular we can view $\mathcal{M}(F^\# \widehat{\otimes} \mathcal{O}_K)$ as a subshtuka of $\mathcal{H}\mathrm{om}(M,\, b(F,K))$.

Lemma 11.3.8. *If $\mathcal{M} = [\mathcal{M}_0 \rightrightarrows \mathcal{M}_1]$ is an explicit model then*

$$\mathcal{M}_0(F^\# \widehat{\otimes} \mathcal{O}_K) = \big\{\ f \in \mathrm{Hom}(M,\, b(F,K)) \mid f \text{ satisfies } (\mathrm{EM}_0)\ \big\},$$
$$\mathcal{M}_1(F^\# \widehat{\otimes} \mathcal{O}_K) = \big\{\ g \in \mathrm{Hom}(M^{\geqslant 1},\, b(F,K)) \mid g \text{ satisfies } (\mathrm{EM}_1)\ \big\}.$$

Proof. Follows immediately from Lemma 11.3.7. □

11.4. Exponential maps of Drinfeld modules

As we identified the underlying group scheme of the Drinfeld module E with $\mathbb{G}_a$ we can view the exponential map of E as a function $K \to K$. The map $\exp\colon K \to K$ is a local isomorphism of topological $\mathbb{F}_q$-vector spaces. In the following we will need a slightly more precise version of this result.

Lemma 11.4.1. *There exists a constant $B_{\exp} \in \mathbb{R}$ such that*

(1) $0 < B_{\exp} \leqslant \frac{1}{q}$.
(2) *If $|z| \leqslant B_{\exp}$ then $|\exp z - z| \leqslant |z|^q$. In particular $|\exp z| = |z|$.*

Proof. Indeed $\exp z$ is given by an everywhere convergent power series

$$\exp z = z + a_1 z^q + a_2 z^{q^2} + \ldots$$

Therefore $\exp z - z = z^q(a_1 + a_2 z^q + \ldots)$. The power series in brackets also converges everywhere and defines a continuous function from K to K. Hence there exists a nonzero constant $B_{\exp}$ such that as soon as $|z| \leqslant B_{\exp}$ the values of this function have norm less than or equal to 1. Without loss of generality we may assume that $B_{\exp} \leqslant \frac{1}{q}$. We thus get (1) and (2). □

Let $B_{\exp}$ be a constant as in Lemma 11.4.1 and let $U \subset K$ be the ball of radius $B_{\exp}$ around 0. Lemma 11.4.1 (2) implies that $\exp(U) = U$ and the induced map $\exp\colon U \to U$ is an isomorphism of topological $\mathbb{F}_q$-vector spaces.

Definition 11.4.2. In the following we denote $\log\colon U \to U$ the inverse of exp on U. We call it the *logarithmic map* of the Drinfeld module E.

Denote $\varphi_t\colon K \to K$ the map given by the action of t on $K = E(K)$. In other words $\varphi_t(z) = \theta z + \alpha_1 z^q + \ldots + \alpha_r z^{q^r}$.

Lemma 11.4.3. *Let $z \in K$. If $|\varphi_t(z)| \leqslant B_{\exp}$ and $|z| \leqslant B_{\exp}$ then $\log(\varphi_t(z)) = \theta \log(z)$.* □

11.5. Exponential maps of explicit models

Let $\mathcal{M}$ be an explicit model. In Section 9.8 we introduced the exponential map of $\mathcal{M}$:

$$\exp\colon F \otimes_{\mathcal{O}_F} \mathrm{H}^1(\nabla\mathcal{M}) \to F \otimes_{\mathcal{O}_F} \mathrm{H}^1(\mathcal{M}).$$

In this section we will describe this map on explicit representatives of cohomology classes.

Definition 11.5.1. Let $h \in b(F, K)$. We define $A \otimes K$-linear maps

$$\begin{aligned} \nabla g(h)\colon M^{\geqslant 1} &\to b(F, K), \\ g(h)\colon M^{\geqslant 1} &\to b(F, K) \end{aligned}$$

by prescribing them on the basis $\tau, \ldots, \tau^r$ of $M^{\geqslant 1}$ as follows:

$$\begin{aligned} \nabla g(h)\colon \tau^1, \ldots, \tau^{r-1} &\mapsto 0, \\ \tau^r &\mapsto \alpha_r^{-1}(ht - \theta h), \\ g(h)\colon \tau^1, \ldots, \tau^{r-1} &\mapsto 0, \\ \tau^r &\mapsto \alpha_r^{-1} \exp\circ(ht - \theta h). \end{aligned}$$

where $\exp\colon K \to K$ is the exponential map of the Drinfeld module E. Note that $g(h)(\tau^r)$ maps $x \in F$ to $\alpha_r^{-1}\exp(h(tx) - \theta h(x))$ in K.

We define an $A \otimes K$-linear map $\nabla f(h)\colon M \to b(F, K)$ by prescribing it on the basis $1, \ldots, \tau^{r-1}$ of M as follows:

$$\nabla f(h)\colon 1 \mapsto h, \quad \tau, \ldots, \tau^{r-1} \mapsto 0.$$

We will use $g(h)$ and $\nabla g(h)$ as representatives for classes in the cohomology groups $\mathrm{H}^1(\mathcal{M})$ and $\mathrm{H}^1(\nabla\mathcal{M})$ respectively.

Lemma 11.5.2. *Let i, j be the arrows of the shtuka*

$$\mathcal{H}\mathrm{om}(M,\, b(F, K)) = \Big[\mathrm{Hom}(M, b(F, K)) \overset{i}{\underset{j}{\rightrightarrows}} \mathrm{Hom}(M^{\geqslant 1}, b(F, K))\Big].$$

If $h \in b(F, K)$ then $i(\nabla f(h)) = \nabla g(h)$.

Proof. According to Proposition 8.8.1 the map $i(\nabla f(h))$ is the restriction of $\nabla f(h)$ to $M^{\geqslant 1}$. Applying $\nabla f(h)$ to the relation

$$\tau^r = \alpha_r^{-1}\big((t - \theta) \cdot 1 - \alpha_1\tau - \alpha_2\tau^2 - \ldots - \alpha_{r-1}\tau^{r-1}\big)$$

we conclude that $\nabla f(h)(\tau^r) = \alpha_r^{-1}(ht - \theta h)$. □

Lemma 11.5.3. *Let $\mathcal{M}$ be an explicit model given by a diagram*

$$\mathcal{M}_0 \underset{j}{\overset{i}{\rightrightarrows}} \mathcal{M}_1$$

and let $c \in \mathrm{H}^1(\nabla\mathcal{M})$ be a cohomology class. There exists a function $h \in b(F,K)$ such that the following holds:

(1) *$\nabla g(h)$ belongs to $\mathcal{M}_1$ and represents c in $\mathrm{H}^1(\nabla\mathcal{M}) = \operatorname{coker}(i)$.*
(2) *$\nabla f(h) \in \mathcal{M}_0(F^{\#} \mathbin{\widehat{\otimes}} \mathcal{O}_K)$.*

Proof. Let $g \in \mathcal{M}_1 \subset \operatorname{Hom}(M^{\geqslant 1}, a(F,K))$ be a representative of the cohomology class c. According to Proposition 9.8.2 there exists a unique $f \in \mathcal{M}_0(F^{\#} \mathbin{\widehat{\otimes}} \mathcal{O}_K) \subset \operatorname{Hom}(M, b(F,K))$ such that $i(f) = g$. Set $h = f(1)$.

Since f is an element of $\mathcal{M}_0(F^{\#} \mathbin{\widehat{\otimes}} \mathcal{O}_K)$ Lemma 11.3.8 shows that the function h satisfies $|h| \leqslant |\alpha_r \theta^{-1}| b$. The same lemma then implies that $\nabla f(h) \in \mathcal{M}_0(F^{\#} \mathbin{\widehat{\otimes}} \mathcal{O}_K)$ which is the claim (2). In order to deduce (1) it is enough to show that the difference

$$\delta = f - \nabla f(h)$$

is an element of $\mathcal{M}_0$. Indeed $i(\nabla f(h)) = \nabla g(h)$ by Lemma 11.5.2 so $\nabla g(h) = g - i(\delta)$ is an element of $\mathcal{M}_1$ which represents the same cohomology class as g.

According to Proposition 8.8.1 the map $i(f) = g$ is the restriction of f to $M^{\geqslant 1}$. Therefore the difference δ acts on the generators of M as follows:

$$\delta \colon 1 \mapsto 0, \ \tau^n \mapsto g(\tau^n), \ n \in \{1, \ldots, r-1\}.$$

Let $\mathcal{M}^c = [\mathcal{M}_0^c \rightrightarrows \mathcal{M}_1^c]$ be the coefficient compactification. By definition of an explicit model

$$\mathcal{M}_0 = \mathcal{M}_0^c \cap \mathcal{M}_0(F \mathbin{\check{\otimes}} \mathcal{O}_K).$$

We first prove that $\delta \in \mathcal{M}_0^c$. As $g \in \mathcal{M}_1^c$ Proposition 11.2.2 shows that the functions $g(\tau), \ldots, g(\tau^r)$ vanish on $t^{-1}\mathcal{O}_F$. Since δ maps 1 to 0 the same Proposition implies that $\delta \in \mathcal{M}_0^c$.

Now we prove that $\delta \in \mathcal{M}_0(F \mathbin{\check{\otimes}} \mathcal{O}_K)$. By definition g is an element of $\mathcal{M}_1$ so (EM_1) implies that

$$|\delta(\tau^n)| = |g(\tau^n)| \leqslant b_n, \quad n \in \{1, \ldots, r-1\}$$

where b_n are the parameters of $\mathcal{M}$. Moreover $\delta(1) = 0$ so $\delta \in \mathcal{M}_0(F \mathbin{\check{\otimes}} \mathcal{O}_K)$ by (EM_0). ☐

In the following let us fix a constant $B_{\exp}$ satisfying the assumptions of Lemma 11.4.1.

Lemma 11.5.4. *Let $\mathcal{M}$ be an explicit model given by a diagram*

$$\mathcal{M}_0 \underset{j}{\overset{i}{\rightrightarrows}} \mathcal{M}_1.$$

Let $h \in b(F,K)$ be a function such that

$$\nabla g(h), g(h) \in \mathcal{M}_1, \quad \nabla f(h) \in \mathcal{M}_0(F^{\#} \mathbin{\widehat{\otimes}} \mathcal{O}_K).$$

Assume that the leading parameter b of $\mathcal{M}$ satisfies $|\alpha_r| b \leqslant B_{\exp}$. If an element $f \in \mathcal{M}_0(F^{\#} \mathbin{\widehat{\otimes}} \mathcal{O}_K)$ satisfies $(i-j)(f) = g(h)$ then $f(1) = \exp h$.

Proof. Set $h_1 = f(1)$. As before we denote $\varphi_t \colon K \to K$ the action of t on $K = E(K)$. We have

$$\varphi_t(z) = \theta z + \alpha_1 z^q + \ldots + \alpha_r z^{q^r}.$$

We split the proof in several steps.

Step 1. *$h_1 \circ t - \varphi_t \circ h_1 = \exp \circ (ht - \theta h)$ in $b(F, K)$.*

According to Proposition 8.8.1 the map $(i - j)(f)$ satisfies

$$(i - j)(f)\colon \tau^{n+1} \mapsto f(\tau^{n+1}) - \tau f(\tau^n)$$

for all $n \geqslant 0$. Comparing with the definition of $g(h)$ we deduce that

$$f(\tau^n) = \tau^n f(1) = \tau^n h_1 \tag{11.1}$$

for all $n \in \{0, \dots, r - 1\}$ and that

$$f(\tau^r) = \tau^r h_1 + \alpha_r^{-1} \exp(ht - \theta h). \tag{11.2}$$

Appling the $A \otimes K$-linear map f to the relation

$$1 \cdot t - \theta \cdot 1 = \alpha_1 \tau + \ldots + \alpha_r \tau^r$$

in M, we obtain what we need.

Step 2. *The function $\varphi_t \circ h_1$ in $b(F, K)$ satisfies $|\varphi_t \circ h_1| \leqslant |\alpha_r| b$.*

We prove it estimating the expression

$$\varphi_t \circ h_1 = \theta h_1 + \alpha_1 \tau h_1 + \ldots + \alpha_r \tau^r h_1$$

term by term. As $f \in \mathcal{M}_0(F^{\#} \mathbin{\widehat{\otimes}} \mathcal{O}_K)$ Lemma 11.3.8 implies that $|h_1| \leqslant |\alpha_r \theta^{-1} b|$. Hence

$$|\theta h_1| \leqslant |\alpha_r| b. \tag{11.3}$$

Next let $b_1, \dots, b_{r-1}$ be the parameters of $\mathcal{M}$ and let $n \in \{1, \dots, r-1\}$. According to (11.1) we have $f(\tau^n) = \tau^n h_1$. The map f belongs to $\mathcal{M}_0(F^{\#} \mathbin{\widehat{\otimes}} \mathcal{O}_K)$. Hence Lemma 11.3.8 shows that

$$|f(\tau^n)| \leqslant b_n.$$

However the conditions (EP_1) imply that

$$|\alpha_n| b_n \leqslant |\alpha_r| b.$$

One thus obtains the inequality

$$|\alpha_n \tau^n h_1| \leqslant |\alpha_n| b_n \leqslant |\alpha_r| b. \tag{11.4}$$

It remains to estimate $\alpha_r \tau^r h_1$. The equation (11.2) implies that

$$\tau^r h_1 = f(\tau^r) - g(h)(\tau^r).$$

The element f belongs to $\mathcal{M}_0(F^{\#} \mathbin{\widehat{\otimes}} \mathcal{O}_K)$ by assumption. Hence $i(f) \in \mathcal{M}_1(F^{\#} \mathbin{\widehat{\otimes}} \mathcal{O}_K)$. According to Proposition 8.8.1 the map $i(f)$ is the restriction of f to $M^{\geqslant 1}$. Therefore Lemma 11.3.8 implies that $|f(\tau^r)| \leqslant b$. As $g(h) \in \mathcal{M}_1$ we deduce that $|g(h)(\tau^r)| \leqslant b$. Hence

$$|\tau^r h_1| \leqslant b. \tag{11.5}$$

Combining (11.3), (11.4), (11.5) we get the inequality $|\varphi_t \circ h_1| \leqslant |\alpha_r| b$.

Step 3. *$\log h_1 \in b(F, K)$ is well-defined.*

The logarithmic map is defined on the ball of radius $B_{\exp}$ around 0 (see Definition 11.4.2). The function h_1 satisfies $|h_1| \leqslant |\alpha_r \theta^{-1}| b$ because $f \in \mathcal{M}_0(F^{\#} \mathbin{\widehat{\otimes}} \mathcal{O}_K)$. Since $|\theta| > 1$ and $|\alpha_r| b \leqslant B_{\exp}$ by assumption we conclude that $\log h_1$ is well-defined.

Step 4. $\nabla g(\log h_1) = \nabla g(h)$.

According to Step 1 we have

$$h_1 \circ t - \varphi_t \circ h_1 = \exp \circ (ht - \theta h). \tag{11.6}$$

We would like to apply log to this equation. Step 2 shows that

$$|\varphi_t \circ h_1| \leqslant |\alpha_r| b \leqslant B_{\exp}$$

so $\log \varphi_t h_1$ is well-defined. Moreover $\log h_1$ is well-defined by Step 3. Now $f(h) \in \mathcal{M}_0(F^\# \widehat{\otimes} \mathcal{O}_K)$ by assumption so Lemma 11.3.8 shows that

$$|h| \leqslant |\alpha_r \theta^{-1}| b \leqslant |\theta^{-1}| B_{\exp}.$$

According to Lemma 11.4.1 we have $|\exp(z)| = |z|$ provided $|z| \leqslant B_{\exp}$. Therefore

$$|\exp \circ (ht - \theta h)| \leqslant |ht - \theta h| \leqslant B_{\exp}$$

and we conclude that $\log(\exp \circ (ht - \theta h))$ is well-defined. Applying log to (11.6) we get

$$\log h_1 t - \log \varphi_t h_1 = ht - \theta h.$$

Lemma 11.4.3 shows that $\log(\varphi_t(z)) = \theta \log z$ provided $\varphi_t(z)$ and z are in the domain of definition of log. As a consequence $\log \varphi_t h_1 = \theta \log h_1$ and

$$\log h_1 t - \theta \log h_1 = ht - \theta h.$$

It follows that $\nabla g(\log h_1) = \nabla g(h)$ by definition of ∇g and g.

Step 5. $\nabla f(\log h_1) \in \mathcal{M}_0(F^\# \widehat{\otimes} \mathcal{O}_K)$.

From Lemma 11.4.1 it follows that $|\log z| = |z|$ for all $z \in K$ satisfying $|z| \leqslant B_{\exp}$. Thus $|\log h_1| = |h_1| \leqslant |\alpha_r \theta^{-1}| b$. Applying Lemma 11.3.8 we conclude that $\nabla f(\log h_1) \in \mathcal{M}_0(F^\# \widehat{\otimes} \mathcal{O}_K)$.

Step 6. $h_1 = \exp h$. According to Lemma 11.5.2

$$i(\nabla f(h)) = \nabla g(h),$$
$$i(\nabla f(\log h_1)) = \nabla g(\log h_1).$$

Furthermore $\nabla g(\log h_1) = g(h)$ by Step 4. Now $\nabla f(h)$ belongs to $\mathcal{M}_0(F^\# \widehat{\otimes} \mathcal{O}_K)$ by assumption while $\nabla f(\log h_1) \in \mathcal{M}_0(F^\# \widehat{\otimes} \mathcal{O}_K)$ by Step 5. Hence the unicity part of Proposition 9.8.1 implies that $\nabla f(h) = \nabla f(\log h_1)$. By definition of ∇f it means that $h = \log h_1$. Therefore $h_1 = \exp h$. □

Lemma 11.5.5. *Let $\mathcal{M} = [\mathcal{M}_0 \rightrightarrows \mathcal{M}_1]$ be an explicit model with leading parameter b satisfying $|\alpha_r| b \leqslant B_{\exp}$. If $h \in b(F, K)$ is a function such that $\nabla g(h) \in \mathcal{M}_1(F \breve{\otimes} \mathcal{O}_K)$ then*

$$\nabla g(h) - g(h) \in I\mathcal{M}_1(F \breve{\otimes} \mathcal{O}_K)$$

where $I = \{x \in K : |x| \leqslant (|\alpha_r| b)^{q-1}\} \subset \mathcal{O}_K$.

Proof. The fact that $\nabla g(h)$ is an element of $\mathcal{M}_1(F \breve{\otimes} \mathcal{O}_K)$ implies a bound

$$|ht - \theta h| \leqslant |\alpha_r| b.$$

According to Lemma 11.4.1 $|\exp z - z| \leqslant |z|^q$ as soon as $|z| \leqslant B_{\exp}$. As $|\alpha_r| b \leqslant B_{\exp}$ we conclude that

$$|\exp(ht - \theta h) - (ht - \theta h)| \leqslant |ht - \theta h|^q \leqslant (|\alpha_r| b)^q.$$

Therefore

$$|\nabla g(h)(\tau^r) - g(h)(\tau^r)| \leqslant (|\alpha_r| b)^{q-1} \cdot b$$

which implies that $\nabla g(h) - g(h) \in I\mathcal{M}_1(F \breve{\otimes} \mathcal{O}_K)$. □

Proposition 11.5.6. *Let $\mathcal{M} = [\mathcal{M}_0 \rightrightarrows \mathcal{M}_1]$ be an explicit model with leading parameter b. If $|\alpha_r|b \leqslant B_{\exp}$ then for every cohomology class $c \in \mathrm{H}^1(\nabla\mathcal{M})$ there exists $h \in b(F,K)$ such that the following holds.*

(1) *$\nabla g(h)$ belongs to $\mathcal{M}_1$ and represents c.*
(2) *$g(h)$ belongs to $\mathcal{M}_1$.*
(3) *$\exp[\nabla g(h)] = [g(h)]$ in $F \otimes_{\mathcal{O}_F} \mathrm{H}^1(\mathcal{M})$.*
(4) *$\nabla g(h) - g(h) \in I\mathcal{M}_1$ where $I = \{x \in K : |x| \leqslant (|\alpha_r|b)^{q-1}\} \subset \mathcal{O}_K$.*

Here the brackets $[\,]$ denote cohomology classes and exp *is the exponential map of $\mathcal{M}$ as in Definition 9.8.3.*

Proof. According to Lemma 11.5.3 there exists an $h \in b(F,K)$ such that $\nabla g(h) \in \mathcal{M}_1$ represents the class c and $\nabla f(h) \in \mathcal{M}_0(F^\# \mathbin{\widehat{\otimes}} \mathcal{O}_K)$.

We claim that $\nabla g(h) - g(h) \in I\mathcal{M}_1$. To prove it we consider the coefficient compactification $\mathcal{M}^c = [\mathcal{M}_0^c \rightrightarrows \mathcal{M}_1^c]$. As $\nabla g(h) \in \mathcal{M}_1^c$ Proposition 11.2.2 implies that $g(h) \in \mathcal{M}_1^c$. Lemma 11.5.5 shows that

$$\nabla g(h) - g(h) \in I\mathcal{M}_1(F \mathbin{\check{\otimes}} \mathcal{O}_K)$$

By definition of an explicit model

$$\mathcal{M}_1^c \cap I\mathcal{M}_1(F \mathbin{\check{\otimes}} \mathcal{O}_K) = I\mathcal{M}_1.$$

Hence $\nabla g(h) - g(h) \in I\mathcal{M}_1$ and $g(h) \in \mathcal{M}_1$.

It remains to prove that $\exp[\nabla g(h)] = [g(h)]$. Consider the isomorphisms

$$\gamma\colon F \otimes_{\mathcal{O}_F} \mathrm{R}\Gamma(\mathcal{M}) \xrightarrow{\sim} \mathrm{Lie}_E(K)[-1],$$
$$\nabla\gamma\colon F \otimes_{\mathcal{O}_F} \mathrm{R}\Gamma(\nabla\mathcal{M}) \xrightarrow{\sim} \mathrm{Lie}_E(K)[-1]$$

of Definition 9.7.2. By Definition 9.8.3 the exponential map of $\mathcal{M}$ is the composition

$$\mathrm{H}^1(\gamma) \circ \mathrm{H}^1(\nabla\gamma)^{-1}.$$

Let $\alpha \in \mathrm{Lie}_E(K)$ be such that $\mathrm{H}^1(\nabla\gamma)(c) = \alpha$. Proposition 9.8.2 shows that the element α is characterized by the following property:

$$\nabla f(h)(1) = h \sim (x \mapsto x\alpha).$$

Let i and j be the arrows of $\mathcal{M}$. According to Proposition 9.8.1 there exists a unique $f \in \mathcal{M}_0(F^\# \mathbin{\widehat{\otimes}} \mathcal{O}_K)$ such that $(i-j)f = g(h)$. Lemma 11.5.4 tells us that $f(1) = \exp h$. Hence

$$f(1) \sim (x \mapsto \exp(x\alpha)).$$

Thus $\mathrm{H}^1(\gamma)([g(h)]) = \alpha$ by Proposition 9.8.1. We conclude that $\exp[\nabla g(h)] = [g(h)]$. □

11.6. Regulators of explicit models

Let $\mathcal{M}$ be a local model. By Theorem 9.7.6 the cohomology modules $\mathrm{H}^1(\mathcal{M})$, $\mathrm{H}^1(\nabla\mathcal{M})$ are free $\mathcal{O}_F$-modules of finite rank.

Lemma 11.6.1. *Let $\mathcal{M}$ be an explicit model with leading parameter b. If $|\alpha_r|b \leqslant B_{\exp}$ then the exponential map* $\exp\colon F \otimes_{\mathcal{O}_F} \mathrm{H}^1(\nabla\mathcal{M}) \to F \otimes_{\mathcal{O}_F} \mathrm{H}^1(\mathcal{M})$ *sends the $\mathcal{O}_F$-submodule $\mathrm{H}^1(\nabla\mathcal{M})$ to $\mathrm{H}^1(\mathcal{M})$.*

Proof. Immediate from Proposition 11.5.6. □

As in Section 5.7 we denote $\mathcal{M}/\mathfrak{e}^n$ the quotient $\mathcal{M}/(\mathfrak{e}^n\mathcal{M})$.

Lemma 11.6.2. *Let $\mathcal{M}$ be an explicit model with leading parameter b satisfying $|\alpha_r|b \leqslant B_{\exp}$. Let*
$$I = \{x \in K : |x| \leqslant (|\alpha_r|b)^{q-1}\} \subset \mathcal{O}_K.$$
Let $n \geqslant 0$. If $I \subset \mathfrak{e}^n$ and $\mathcal{M}/\mathfrak{e}^n$ is linear then the following square is commutative:
$$\begin{tikzcd} \mathrm{H}^1(\nabla\mathcal{M}) \arrow[r] \arrow[d, "\exp"] & \mathrm{H}^1(\nabla\mathcal{M}/\mathfrak{e}^n) \arrow[d, "1"] \\ \mathrm{H}^1(\mathcal{M}) \arrow[r] & \mathrm{H}^1(\mathcal{M}/\mathfrak{e}^n). \end{tikzcd}$$
Here the horizontal arrows are induced by reduction modulo $\mathfrak{e}^n$ and the right vertical arrow comes from the identity of the shtukas $\nabla\mathcal{M}/\mathfrak{e}^n$ and $\mathcal{M}/\mathfrak{e}^n$.

Proof. Let $\mathcal{M} = [\mathcal{M}_0 \rightrightarrows \mathcal{M}_1]$ and let $c \in \mathrm{H}^1(\nabla\mathcal{M})$ be a cohomology class. According to Proposition 11.5.6 there exists a function $h \in b(F, K)$ such that
$$\nabla g(h), g(h) \in \mathcal{M}_1,$$
$$\nabla g(h) - g(h) \in I\mathcal{M}_1,$$
$$[\nabla g(h)] = c,$$
$$\exp[\nabla g(h)] = [g(h)].$$
Here the brackets [ ] denote cohomology classes. We get the result since the images of $\nabla g(h)$ and $g(h)$ in $\mathcal{M}_1/\mathfrak{e}^n$ are the same. □

Theorem 11.6.3. *Let $\mathcal{M}$ be an explicit model with leading parameter b. If $|\alpha_r|b \leqslant B_{\exp}$ then the exponential map $\exp\colon \mathrm{H}^1(\nabla\mathcal{M}) \to \mathrm{H}^1(\mathcal{M})$ is the inverse of the regulator $\rho\colon \mathrm{H}^1(\mathcal{M}) \to \mathrm{H}^1(\nabla\mathcal{M})$.*

Proof. Let $n \geqslant 0$. By Proposition 11.3.6 the twist $\mathfrak{e}^n\mathcal{M}$ is an explicit model with leading parameter $b|\theta^{-n}|$. Consider the ideal
$$I_n = \{x \in K : |x| \leqslant (|\alpha_r\theta^{-n}|b)^{q-1}\} \subset \mathcal{O}_K.$$
Since $|\alpha_r|b \leqslant B_{\exp} < 1$ and $\mathfrak{e} = \theta^{-1}\mathcal{O}_K$ we conclude that $I_n \subset \mathfrak{e}^n$. The shtuka $(\mathfrak{e}^n\mathcal{M})/\mathfrak{e}^n$ is linear by Proposition 5.7.6. So Lemma 11.6.2 implies that
$$\begin{tikzcd} \mathrm{H}^1(\nabla\mathfrak{e}^n\mathcal{M}) \arrow[r] \arrow[d, "\exp"] & \mathrm{H}^1(\nabla(\mathfrak{e}^n\mathcal{M})/\mathfrak{e}^n) \arrow[d, "1"] \\ \mathrm{H}^1(\mathfrak{e}^n\mathcal{M}) \arrow[r] & \mathrm{H}^1((\mathfrak{e}^n\mathcal{M})/\mathfrak{e}^n). \end{tikzcd}$$
Now Theorem 5.14.4 shows that exp is the inverse of the regulator map. □

11.7. Regulators in general

In this section we let A be an arbitrary coefficient ring and K a finite product of local fields containing $\mathbb{F}_q$. As before we fix a Drinfeld A-module E over K with motive M. We assume that the action of A on $\mathrm{Lie}_E(K)$ extends to a continuous action of F.

We are finally ready to prove the following result:

Theorem 11.7.1. *If $\mathcal{M} \subset \mathcal{H}\mathrm{om}(M, a(F, K))$ is a local model then the diagram*
$$\begin{tikzcd} \mathrm{H}^1(\nabla\mathcal{M}) \arrow[r] & F \otimes_{\mathcal{O}_F} \mathrm{H}^1(\nabla\mathcal{M}) \arrow[d, "\wr"', "\exp"] \\ \mathrm{H}^1(\mathcal{M}) \arrow[r] \arrow[u, "\rho", "\wr"'] & F \otimes_{\mathcal{O}_F} \mathrm{H}^1(\mathcal{M}) \end{tikzcd}$$
is commutative.

Proof. We split the proof into several steps.

Step 1. *Let $\mathcal{N} \subset \mathcal{M}$ be an inclusion of local models. The theorem holds for $\mathcal{N}$ if and only if it holds for $\mathcal{M}$.* Indeed Proposition 9.7.5 shows that the maps

$$F \otimes_{\mathcal{O}_F} \mathrm{R}\Gamma(\mathcal{N}) \to F \otimes_{\mathcal{O}_F} \mathrm{R}\Gamma(\mathcal{M}),$$
$$F \otimes_{\mathcal{O}_F} \mathrm{R}\Gamma(\mathcal{N}) \to F \otimes_{\mathcal{O}_F} \mathrm{R}\Gamma(\nabla\mathcal{M})$$

are quasi-isomorphisms. The result follows since both ρ and exp are natural transformations of functors on the category of local models.

Step 2. *The theorem holds in the case $A = \mathbb{F}_q[t]$.*

Without loss of generality we assume that K is a single local field. Let $\mathcal{M}$ be a local model. According to Proposition 11.3.4 there exists an explicit model $\mathcal{N}$ with some parameters b, $b_1, \ldots, b_{r-1}$. Proposition 9.5.11 shows that the local model $\mathfrak{e}^n\mathcal{N}$ is a subshtuka of $\mathcal{M}$ for all $n \gg 0$. By Proposition 11.3.6 the twist $\mathfrak{e}^n\mathcal{N}$ is an explicit model with leading parameter $b|\theta^{-n}|$. So taking $n \gg 0$ we can ensure that $|\alpha_r|b \leqslant B_{\mathrm{exp}}$. In this situation Theorem 11.6.3 shows that the result holds for $\mathfrak{e}^n\mathcal{N}$. Step 1 implies that it holds for $\mathcal{M}$ as well.

Next let us fix a nonconstant element $a \in A$. We denote $A' = \mathbb{F}_q[a]$ and F' the local field of A' at infinity and $\mathfrak{e}' \subset \mathcal{O}_K$ the ideal generated by $\mathfrak{m}_{F'}$ under the map $F' \to K$ given by the action of A' on $\mathrm{Lie}_E(K)$.

Step 3. *If $\mathcal{M}$ is a local model such that $\mathcal{M}(F \otimes \mathcal{O}_K/\mathfrak{e}')$ is linear then the theorem holds for $\mathcal{M}$.*

Let $\mathcal{M}'$ be $\mathcal{M}$ viewed as an $\mathcal{O}_{F'}\check{\otimes}\mathcal{O}_K$-module shtuka. By Proposition 10.7.1 the shtuka $\mathcal{M}'$ is a local model of the Drinfeld A'-module E. Proposition 10.7.2 shows that the regulator of $\mathcal{M}'$ is compatible with the regulator of $\mathcal{M}$ while Proposition 10.7.5 does the same for the exponential map. Hence the theorem holds for $\mathcal{M}$ with the coefficient ring A if and only if it holds for $\mathcal{M}'$ with the coefficient ring A'. Applying Step 2 to $\mathcal{M}'$ we get the result.

Step 4. *The theorem holds for an arbitrary local model.*

Let $\mathcal{M}$ be a local model. The ramification ideal $\mathfrak{e}'$ is a power of $\mathfrak{e}$ by construction. Hence Proposition 9.5.10 demonstrates that $\mathfrak{e}'\mathcal{M}$ is a local model. Proposition 5.7.6 shows that $(\mathfrak{e}'\mathcal{M})(F \otimes \mathcal{O}_K/\mathfrak{e}')$ is linear. Therefore the theorem holds for $\mathfrak{e}'\mathcal{M}$ by Step 3. As $\mathfrak{e}'\mathcal{M} \subset \mathcal{M}$ Step 1 implies that the theorem holds for $\mathcal{M}$. $\square$

CHAPTER 12

Global models and the class number formula

In this chapter we introduce global shtuka models of a Drinfeld module E over a Dedekind domain of finite type over $\mathbb{F}_q$ and use them to derive the class number formula for Drinfeld modules.

Remark. The notation of this chapter differs from the notation used in the introduction. Namely we write F in place of F_∞ and K in place of K_∞.

12.1. The setting

Fix a coefficient ring A. As before we denote F the local field of A at infinity, $\mathcal{O}_F \subset F$ the ring of integers and $\mathfrak{m}_F \subset \mathcal{O}_F$ the maximal ideal. We denote C the compactification of $\operatorname{Spec} A$ and Ω_C the sheaf of Kähler differentials of C over $\mathbb{F}_q$. We use the following notation:

$$\begin{array}{llll} C^\circ &= \operatorname{Spec} A & \omega &= \Gamma(C^\circ, \Omega_C), \\ D &= \operatorname{Spec} \mathcal{O}_F & \omega_{\mathcal{O}_F} &= \Gamma(D, \Omega_C), \\ D^\circ &= \operatorname{Spec} F & \omega_F &= \Gamma(D^\circ, \Omega_C). \end{array}$$

Let R be a Dedekind domain of finite type over $\mathbb{F}_q$. We denote Y the spectrum of R and X the projective curve over $\mathbb{F}_q$ which compactifies Y.

Let E be a Drinfeld A-module over R. As usual $M = \operatorname{Hom}(E, \mathbb{G}_a)$ stands for the motive of E. The action of A on Lie_E determines a homomorphism $A \to R$. We assume that it is *finite flat*.

The F-algebra $K = R \otimes_A F$ is a finite product of local fields. Note that it is not necessarily étale. As before we denote $\mathcal{O}_K \subset K$ the ring of integers and $\mathfrak{m}_K \subset \mathcal{O}_K$ the Jacobson radical. Let $\mathfrak{e}$ be the ideal generated by $\mathfrak{m}_F$ in $\mathcal{O}_K$ under the natural map $F \to K$. We call $\mathfrak{e}$ the *ramification ideal*.

12.2. The notion of a global model

To the Drinfeld module E with motive M we associate the $A \otimes R$-module shtuka $\mathcal{H}\mathrm{om}(M, \omega \otimes R)$. In order to prove the class number formula for E we will extend it to a locally free shtuka on $C \times X$. In this section we introduce the appropriate notion of such an extension.

Let $\pi \colon C \times Y \to C$ be the projection map. By construction $\pi^*\Omega_C$ is the sheaf of Kähler differentials of $C \times Y$ over Y. We introduce the shtuka

$$\Omega_{C,Y} = \Big[\pi^*\Omega_C \underset{j}{\overset{1}{\rightrightarrows}} \pi^*\Omega_C\Big]$$

where j is the τ-adjoint of the natural isomorphism $\tau^*\pi^*\Omega_C \cong \pi^*\Omega_C$ induced by the equality $\pi \circ \tau = \pi$.

Definition 12.2.1. We set $\mathcal{M}^c = \underline{\mathcal{H}\mathrm{om}}(\mathcal{E}, \Omega_{C,Y})$ where $\underline{\mathcal{H}\mathrm{om}}$ is the sheaf Hom shtuka of Definition 1.14.1 and $\mathcal{E}$ is the shtuka on $C \times Y$ constructed in Theorem 7.6.1.

Lemma 12.2.2. $\mathcal{M}^c(A \otimes R) = \mathcal{H}\mathrm{om}(M, \omega \otimes R)$.

Proof. Indeed we have

$$\mathcal{E}(A \otimes R) = \Big[M \overset{1}{\underset{\tau}{\rightrightarrows}} M\Big], \quad \Omega_{C,Y}(A \otimes R) = \Big[\omega \otimes R \overset{1}{\underset{\tau}{\rightrightarrows}} \omega \otimes R\Big]$$

by construction. □

Definition 12.2.3. A *global model* of $\mathcal{H}\mathrm{om}(M, \omega \otimes R)$ is a shtuka on $C \times X$ which has the following properties:

(1) $\mathcal{M}$ is a locally free shtuka extending $\mathcal{M}^c$ on $C \times Y$.
(2) $\mathcal{M}(A \otimes \mathcal{O}_K/\mathfrak{m}_K)$ is nilpotent and $\mathcal{M}(A \otimes \mathcal{O}_K/\mathfrak{e})$ is linear.

Proposition 12.2.4. *If $\mathcal{M}$ is a global model then $\mathcal{M}(\mathcal{O}_F \breve{\otimes} \mathcal{O}_K)$ is a local model in the sense of Definition 9.5.7.*

Proof. Corollary 3.9.5 implies that $\omega \otimes R$ is an $A \otimes R$-lattice in the $F \breve{\otimes} K$-module $a(F, K)$. Hence $\mathcal{M}(F \breve{\otimes} K) = \mathcal{H}\mathrm{om}(M, a(F, K))$. The shtuka $\mathcal{M}(\mathcal{O}_F \breve{\otimes} \mathcal{O}_K)$ is a locally free $\mathcal{O}_F \breve{\otimes} \mathcal{O}_K$-lattice in $\mathcal{M}(F \breve{\otimes} K)$. To prove that $\mathcal{M}(\mathcal{O}_F \breve{\otimes} \mathcal{O}_K)$ is a local model we need to show the following:

(1) $\mathcal{M}(\mathcal{O}_F \breve{\otimes} K)$ is the coefficient compactification in the sense of Definition 9.5.1.
(2) $\mathcal{M}(F \otimes \mathcal{O}_K/\mathfrak{m}_K)$ is nilpotent and $\mathcal{M}(F \otimes \mathcal{O}_K/\mathfrak{e})$ is linear.

Corollary 3.9.5 implies that $\omega_{\mathcal{O}_F} \otimes R$ is an $\mathcal{O}_F \otimes R$-lattice in the $\mathcal{O}_F \breve{\otimes} K$-module $a(F/\mathcal{O}_F, K)$. We conclude that $\Omega_{C,Y}(\mathcal{O}_F \breve{\otimes} K) = a(F/\mathcal{O}_F, K)$. Hence

$$\mathcal{M}(\mathcal{O}_F \breve{\otimes} K) = \mathcal{H}\mathrm{om}_{\mathcal{O}_F \breve{\otimes} K}\big(\mathcal{E}(\mathcal{O}_F \breve{\otimes} K),\, a(F/\mathcal{O}_F, K)\big)$$

is the coefficient compactification as in (1). The property (2) follows directly from the definition of a global model. □

Proposition 12.2.5. *If $\mathcal{M}$ is a global model then the restriction of $\mathcal{M}$ to $D \times X$ is an elliptic shtuka of ramification ideal $\mathfrak{e}$ in the sense of Definition 6.8.1.*

Proof. We need to prove the following:

(1) $\mathcal{M}(\mathcal{O}_F/\mathfrak{m}_F \otimes R)$ is nilpotent.
(2) $\mathcal{M}(\mathcal{O}_F \breve{\otimes} \mathcal{O}_K)$ is an elliptic shtuka of ramification ideal $\mathfrak{e}$.

By Theorem 7.6.1 the shtuka $\mathcal{E}(\mathcal{O}_F/\mathfrak{m}_F \otimes R)$ is co-nilpotent. So (1) follows from Proposition 7.5.2. The shtuka $\mathcal{M}(\mathcal{O}_F \breve{\otimes} \mathcal{O}_K)$ is a local model by Proposition 12.2.4 so (2) is a corollary of Theorem 9.7.7. □

12.3. Existence of global models

Lemma 12.3.1. *There exists a shtuka $\mathcal{M}$ on $C^\circ \times X$ with the following properties:*

(1) *$\mathcal{M}$ is a locally free shtuka extending $\mathcal{H}\mathrm{om}(M, \omega \otimes R)$ on $C^\circ \times Y$.*
(2) *$\mathcal{M}(A \otimes \mathcal{O}_K/\mathfrak{m}_K)$ is nilpotent and $\mathcal{M}(A \otimes \mathcal{O}_K/\mathfrak{e})$ is linear.*

Proof. By Theorem 9.6.5 the shtuka $\mathcal{H}\mathrm{om}(M, a(F/A, K))$ admits an $A \otimes \mathcal{O}_K$-lattice $\mathcal{M}^b$ with the following properties:

(1) $\mathcal{M}^b$ is locally free.
(2) $\mathcal{M}^b(A \otimes \mathcal{O}_K/\mathfrak{m}_K)$ is nilpotent and $\mathcal{M}^b(A \otimes \mathcal{O}_K/\mathfrak{e})$ is linear.

Using the Beauville-Laszlo Theorem [0BP2] we glue the $A \otimes R$-module shtuka $\mathcal{H}\mathrm{om}(M, \omega \otimes R)$ to the $A \mathbin{\widehat{\otimes}} \mathcal{O}_K$-module shtuka $\mathcal{M}^b(A \mathbin{\widehat{\otimes}} \mathcal{O}_K)$ over $A \mathbin{\widehat{\otimes}} K$ and obtain the desired shtuka $\mathcal{M}$ on $C^\circ \times X$. □

Let U denote the union of open subschemes $C \times Y$ and $C^\circ \times X$ of $C \times X$ and let $\iota\colon U \to C \times X$ be the open immersion. Note that the complement of U in $C \times X$ consists of finitely many points.

Lemma 12.3.2. *If $\mathcal{F}$ is a locally free sheaf on U then $\iota_*\mathcal{F}$ is a locally free sheaf on $C \times X$.*

Proof. Consider the cartesian square of schemes

$$\begin{tikzcd} V \arrow[r, "\kappa"] \arrow[d, "g"'] & \operatorname{Spec} \mathcal{O}_F \check{\otimes} \mathcal{O}_K \arrow[d, "f"] \\ U \arrow[r, "\iota"] & C \times X \end{tikzcd}$$

where V is the complement of the closed points of $\operatorname{Spec} \mathcal{O}_F \check{\otimes} \mathcal{O}_K$ and g is the natural morphism. Observe that $f^*\iota_*\mathcal{F} = \kappa_* g^*\mathcal{F}$. Now Lemmas 9.2.4 and 9.2.5 imply that $\kappa_* g^* \mathcal{F}$ is a locally free sheaf on $\operatorname{Spec} \mathcal{O}_F \check{\otimes} \mathcal{O}_K$. Whence $\iota_*\mathcal{F}$ is locally free. □

Proposition 12.3.3. *The shtuka $\mathcal{H}\mathrm{om}(M, \omega \otimes R)$ admits a global model.*

Proof. Let $\mathcal{M}^b$ be a shtuka constructed in Lemma 12.3.1. The shtukas $\mathcal{M}^c$ and $\mathcal{M}^b$ restrict to $\mathcal{H}\mathrm{om}(M, \omega \otimes R)$ on $C^\circ \times Y$. We thus obtain a shtuka $\mathcal{M}$ on the union U of $C \times Y$ and $C^\circ \times X$. By Lemma 12.3.2 the shtuka $\iota_*\mathcal{M}$ on $C \times X$ is locally free. It is therefore a global model. □

12.4. Cohomology of the Hom shtukas

Definition 12.4.1. The *complex of units* of the Drinfeld module E is the A-module complex

$$U_E = \Big[\operatorname{Lie}_E(K) \xrightarrow{\exp} \frac{E(K)}{E(R)} \Big]$$

where $\exp\colon \operatorname{Lie}_E(K) \to E(K)$ is the exponential map.

Our goal is to construct quasi-isomorphisms

$$\mathrm{R}\Gamma_{\mathrm{c}}(\mathcal{H}\mathrm{om}(M, \omega \otimes R)) \xrightarrow{\sim} U_E[-1],$$
$$\mathrm{R}\Gamma_{\mathrm{c}}(\nabla\, \mathcal{H}\mathrm{om}(M, \omega \otimes R)) \xrightarrow{\sim} \operatorname{Lie}_E(R)[-1]$$

where $\mathrm{R}\Gamma_{\mathrm{c}}$ is the compactly supported cohomology of shtukas on $A \otimes R$ (Definition 4.4.1). In the next section we will use these quasi-isomorphisms to study the cohomology of global models.

We first study the germ cohomology of shtukas on $A \otimes K$. To improve legibility we will write $\mathrm{R}\Gamma_g(-)$ in place of $\mathrm{R}\Gamma_g(A \otimes K, -)$. The module $\omega \otimes K$ is an $A \otimes K$-lattice in the $A \mathbin{\widehat{\otimes}} K$-module $\omega \mathbin{\widehat{\otimes}} K$. Moreover the sequence

$$0 \to \omega \otimes M \longrightarrow \omega \mathbin{\widehat{\otimes}} M \xrightarrow{[\mathrm{Res}]} g(F, M) \to 0$$

is exact by Lemma 8.4.2. We thus get a natural identification

$$g(F, K) = \tfrac{(A \widehat{\otimes} K) \otimes_{A \otimes K} (\omega \otimes K)}{\omega \otimes K}.$$

As a result Proposition 4.1.2 provides us with natural quasi-isomorphisms

$$\mathrm{R}\Gamma_g(\mathcal{H}\mathrm{om}(M, \omega \otimes K)) \xrightarrow{\sim} \mathrm{R}\Gamma(\mathcal{H}\mathrm{om}(M, g(F, K)))[-1],$$
$$\mathrm{R}\Gamma_g(\nabla\, \mathcal{H}\mathrm{om}(M, \omega \otimes K)) \xrightarrow{\sim} \mathrm{R}\Gamma(\nabla\, \mathcal{H}\mathrm{om}(M, g(F, K)))[-1].$$

Definition 12.4.2. We define natural quasi-isomorphisms

$$\mathrm{R}\Gamma_g(\mathcal{H}\mathrm{om}(M, \omega \otimes K)) \xrightarrow{\sim} \operatorname{Lie}_E(K)[-1],$$
$$\mathrm{R}\Gamma_g(\nabla\, \mathcal{H}\mathrm{om}(M, \omega \otimes K)) \xrightarrow{\sim} \operatorname{Lie}_E(K)[-1]$$

as the compositions

$$\begin{tikzcd}
\mathrm{R}\Gamma_g(\mathcal{H}\mathrm{om}(M,\,\omega\otimes K)) \arrow[d, "\wr", "\text{Proposition 4.1.2}"'] & \mathrm{R}\Gamma_g(\nabla\,\mathcal{H}\mathrm{om}(M,\,\omega\otimes K)) \arrow[d, "\wr"', "\text{Proposition 4.1.2}"] \\
\mathrm{R}\Gamma(\mathcal{H}\mathrm{om}(M,\,g(F,K)))[-1] \arrow[d, "\wr", "\text{Proposition 8.10.10}"'] & \mathrm{R}\Gamma(\nabla\,\mathcal{H}\mathrm{om}(M,\,g(F,K)))[-1] \arrow[d, "\wr"', "\text{Proposition 8.10.9}"] \\
\mathrm{Lie}_E(K)[-1] & \mathrm{Lie}_E(K)[-1]
\end{tikzcd}$$

Let $Q = K/R$. By Corollary 8.11.3 the natural sequence

$$0 \to \omega\otimes R \to \omega\mathbin{\widehat{\otimes}} K \to \omega\mathbin{\widehat{\otimes}} Q \to 0$$

is exact. Moreover $\omega\otimes R$ is an $A\otimes R$-lattice in the $A\mathbin{\widehat{\otimes}} K$-module $\omega\mathbin{\widehat{\otimes}} K$. We thus get a natural identification

$$\omega\mathbin{\widehat{\otimes}} Q = \tfrac{(A\widehat{\otimes}K)\otimes_{A\otimes R}(\omega\otimes R)}{\omega\otimes R}.$$

As a result Proposition 4.4.2 provides us with natural quasi-isomorphisms

$$\mathrm{R}\Gamma_{\mathrm{c}}(\mathcal{H}\mathrm{om}(M,\,\omega\otimes R)) \xrightarrow{\sim} \mathrm{R}\Gamma(\mathcal{H}\mathrm{om}(M,\,\omega\mathbin{\widehat{\otimes}} Q))[-1],$$
$$\mathrm{R}\Gamma_{\mathrm{c}}(\nabla\,\mathcal{H}\mathrm{om}(M,\,\omega\otimes R)) \xrightarrow{\sim} \mathrm{R}\Gamma(\nabla\,\mathcal{H}\mathrm{om}(M,\,\omega\mathbin{\widehat{\otimes}} Q))[-1].$$

Definition 12.4.3. We define natural quasi-isomorphisms

$$\mathrm{R}\Gamma_{\mathrm{c}}(\mathcal{H}\mathrm{om}(M,\,\omega\otimes R)) \xrightarrow{\sim} U_E[-1],$$
$$\mathrm{R}\Gamma_{\mathrm{c}}(\nabla\,\mathcal{H}\mathrm{om}(M,\,\omega\otimes R)) \xrightarrow{\sim} \mathrm{Lie}_E(R)[-1]$$

as the compositions

$$\begin{tikzcd}
\mathrm{R}\Gamma_{\mathrm{c}}(\mathcal{H}\mathrm{om}(M,\,\omega\otimes R)) \arrow[d, "\wr", "\text{Proposition 4.4.2}"'] & \mathrm{R}\Gamma_{\mathrm{c}}(\nabla\,\mathcal{H}\mathrm{om}(M,\,\omega\otimes R)) \arrow[d, "\wr"', "\text{Proposition 4.4.2}"] \\
\mathrm{R}\Gamma(\mathcal{H}\mathrm{om}(M,\,\omega\mathbin{\widehat{\otimes}} Q))[-1] \arrow[d, "\wr", "\text{Theorem 8.11.8}"'] & \mathrm{R}\Gamma(\nabla\,\mathcal{H}\mathrm{om}(M,\,\omega\mathbin{\widehat{\otimes}} Q))[-1] \arrow[d, "\wr"', "\text{Theorem 8.11.7}"] \\
U_E[-1] & \mathrm{Lie}_E(R)[-1]
\end{tikzcd}$$

Theorems 8.11.8 and 8.11.7 imply that these quasi-isomorphisms are natural in E.

For every quasi-coherent shtuka $\mathcal{M}$ on $A\otimes R$ we have a natural morphism $\mathrm{R}\Gamma_{\mathrm{c}}(A\otimes R,\,\mathcal{M}) \to \mathrm{R}\Gamma_g(A\otimes K,\,\mathcal{M})$ (see Definition 4.4.3).

Proposition 12.4.4. *The square*

$$\begin{tikzcd}
\mathrm{R}\Gamma_{\mathrm{c}}(\mathcal{H}\mathrm{om}(M,\,\omega\otimes R)) \arrow[r] \arrow[d, "\wr", "\text{Definition 12.4.3}"'] & \mathrm{R}\Gamma_g(\mathcal{H}\mathrm{om}(M,\,\omega\otimes K)) \arrow[d, "\wr"', "\text{Definition 12.4.2}"] \\
U_E[-1] \arrow[r, "\text{identity in degree 1}"] & \mathrm{Lie}_E(K)[-1]
\end{tikzcd}$$

is commutative.

Proof. For every locally free $A\otimes R$-module shtuka $\mathcal{M}$ the square

$$\begin{tikzcd}
\mathrm{R}\Gamma_{\mathrm{c}}(A\otimes R,\,\mathcal{M}) \arrow[r] \arrow[d, "\wr", "\text{Proposition 4.4.2}"'] & \mathrm{R}\Gamma_g(A\otimes K,\,\mathcal{M}) \arrow[d, "\wr"', "\text{Proposition 4.1.2}"] \\
\mathrm{R}\Gamma\Big(A\otimes R,\,\frac{\mathcal{M}(A\widehat{\otimes}K)}{\mathcal{M}(A\otimes R)}\Big)[-1] \arrow[r] & \mathrm{R}\Gamma\Big(A\otimes K,\,\frac{\mathcal{M}(A\widehat{\otimes}K)}{\mathcal{M}(A\otimes K)}\Big)[-1]
\end{tikzcd}$$

is commutative by construction. So the result follows from Theorem 8.11.8 (3). $\square$

Proposition 12.4.5. *The square*

$$\begin{tikzcd} \mathrm{R}\Gamma_{\mathrm{c}}(\nabla\,\mathcal{H}\mathrm{om}(M,\,\omega\otimes R)) \arrow[r] \arrow[d, "\wr"', "\text{Definition 12.4.3}"] & \mathrm{R}\Gamma_{g}(\nabla\,\mathcal{H}\mathrm{om}(M,\,\omega\otimes K)) \arrow[d, "\wr", "\text{Definition 12.4.2}"'] \\ \mathrm{Lie}_E(R)[-1] \arrow[r, "\text{embedding in degree 1}"] & \mathrm{Lie}_E(K)[-1] \end{tikzcd}$$

is commutative.

Proof. Same as the proof of Proposition 12.4.4. $\square$

12.5. Cohomology of global models

We work with a fixed global model $\mathcal{M}$.

Lemma 12.5.1. *The natural maps*

$$\mathrm{R}\Gamma_{\mathrm{c}}(C^\circ\times Y,\,\mathcal{M})\to\mathrm{R}\Gamma(C^\circ\times X,\,\mathcal{M}),$$
$$\mathrm{R}\Gamma_{\mathrm{c}}(C^\circ\times Y,\,\nabla\mathcal{M})\to\mathrm{R}\Gamma(C^\circ\times X,\,\nabla\mathcal{M})$$

are quasi-isomorphisms.

Proof. The shtuka $\mathcal{M}(A\otimes\mathcal{O}_K/\mathfrak{m}_K)$ is nilpotent so the claim for $\mathrm{R}\Gamma_{\mathrm{c}}(C^\circ\times Y,\,\mathcal{M})$ follows by Proposition 4.4.5. The same applies to $\nabla\mathcal{M}$. $\square$

Definition 12.5.2. We define natural quasi-isomorphisms

$$\mathrm{R}\Gamma(C^\circ\times X,\,\mathcal{M})\xrightarrow{\sim}U_E[-1],$$
$$\mathrm{R}\Gamma(C^\circ\times X,\,\nabla\mathcal{M})\xrightarrow{\sim}\mathrm{Lie}_E(R)[-1]$$

as the compositions

$$\begin{tikzcd} \mathrm{R}\Gamma(C^\circ\times X,\,\mathcal{M}) & \mathrm{R}\Gamma(C^\circ\times X,\,\nabla\mathcal{M}) \\ \mathrm{R}\Gamma_{\mathrm{c}}(C^\circ\times Y,\,\mathcal{M}) \arrow[u, "\wr"', "\text{Proposition 4.4.5}"] \arrow[d, "\wr"', "\text{Definition 12.4.3}"] & \mathrm{R}\Gamma_{g}(C^\circ\times Y,\,\nabla\mathcal{M}) \arrow[u, "\wr", "\text{Proposition 4.4.5}"'] \arrow[d, "\wr", "\text{Definition 12.4.3}"'] \\ U_E[-1] & \mathrm{Lie}_E(R)[-1] \end{tikzcd}$$

These quasi-isomorphisms are natural in $\mathcal{M}$ and E by construction.

According to Proposition 12.2.4 the shtuka $\mathcal{M}(\mathcal{O}_F\mathbin{\check{\otimes}}\mathcal{O}_K)$ is a local model. So Lemma 9.7.3 shows that the maps of Definition 9.7.2 induce F-linear quasi-isomorphisms

$$\gamma\colon\ \mathrm{R}\Gamma(F\mathbin{\check{\otimes}}\mathcal{O}_K,\,\mathcal{M})\xrightarrow{\sim}\mathrm{Lie}_E(K)[-1],$$
$$\nabla\gamma\colon\ \mathrm{R}\Gamma(F\mathbin{\check{\otimes}}\mathcal{O}_K,\,\nabla\mathcal{M})\xrightarrow{\sim}\mathrm{Lie}_E(K)[-1]$$

Proposition 12.5.3. *The square*

$$\begin{tikzcd} \mathrm{R}\Gamma(C^\circ\times X,\,\mathcal{M}) \arrow[r, "\text{pullback}"] \arrow[d, "\wr"', "\text{Definition 12.5.2}"] & \mathrm{R}\Gamma(F\mathbin{\check{\otimes}}\mathcal{O}_K,\,\mathcal{M}) \arrow[d, "\wr"', "\gamma"] \\ U_E[-1] \arrow[r, "\text{identity in degree 1}"] & \mathrm{Lie}_E(K)[-1] \end{tikzcd}$$

is commutative.

Proof. We verify it in several steps.

Step 1. Consider the square

$$\begin{tikzcd} U_E[-1] \arrow[r, "\text{identity in degree 1}"] & \mathrm{Lie}_E(K)[-1] \\ \mathrm{R}\Gamma(C^\circ \times X,\, \mathcal{M}) \arrow[u, "\text{Definition 12.5.2}", "\wr"'] \arrow[r, "\text{global}"] & \mathrm{R}\Gamma_g(A \otimes K,\, \mathcal{M}) \arrow[u, "\wr", "\text{Definition 12.4.2}"'] \end{tikzcd}$$

where the arrow labelled "global" is the global germ map

$$\mathrm{R}\Gamma(A \times X, \mathcal{M}) \xleftarrow{\simeq} \mathrm{R}\Gamma_{\mathrm{c}}(A \otimes R, \mathcal{M}) \to \mathrm{R}\Gamma_g(A \otimes K, \mathcal{M})$$

of Definition 4.4.6. Proposition 12.4.4 implies that this square is commutative.

Step 2. Consider the diagram

$$\begin{tikzcd} \mathrm{R}\Gamma(C^\circ \times X,\, \mathcal{M}) \arrow[r, "\text{global}"] \arrow[d] & \mathrm{R}\Gamma_g(A \otimes K,\, \mathcal{M}) \arrow[d] \\ \mathrm{R}\Gamma(D^\circ \times X,\, \mathcal{M}) \arrow[r, "\text{global}"] \arrow[d] & \mathrm{R}\Gamma_g(F \,\breve{\otimes}\, K,\, \mathcal{M}) \arrow[d, equal] \\ \mathrm{R}\Gamma(F \,\breve{\otimes}\, \mathcal{O}_K,\, \mathcal{M}) \arrow[r, "\text{local}"] & \mathrm{R}\Gamma_g(F \,\breve{\otimes}\, K,\, \mathcal{M}) \end{tikzcd}$$

where the arrow labelled "local" is the local germ map of Definition 4.2.4 and the unlabelled arrows are the pullback morphisms. The top square of this diagram commutes by naturality of the global germ map. The commutativity of the bottom square follows from Theorem 4.5.1.

Step 3. Combining Step 1 and Step 2 we get a commutative diagram

$$\begin{tikzcd} U_E[-1] \arrow[r, "\text{identity in degree 1}"] & \mathrm{Lie}_E(K)[-1] \\ \mathrm{R}\Gamma(C^\circ \times X,\, \mathcal{M}) \arrow[u, "\text{Definition 12.5.2}", "\wr"'] \arrow[r, "\text{global}"] \arrow[d] & \mathrm{R}\Gamma_g(A \otimes K,\, \mathcal{M}) \arrow[u, "\wr", "\text{Definition 12.4.2}"'] \arrow[d] \\ \mathrm{R}\Gamma(F \,\breve{\otimes}\, \mathcal{O}_K,\, \mathcal{M}) \arrow[r, "\text{local}"] & \mathrm{R}\Gamma_g(F \,\breve{\otimes}\, K,\, \mathcal{M}). \end{tikzcd}$$

Step 4. We claim that the arrow $\mathrm{R}\Gamma_g(A \otimes K, \mathcal{M}) \to \mathrm{R}\Gamma_g(F \,\breve{\otimes}\, K, \mathcal{M})$ is a quasi-isomorphism and that the square

$$\begin{tikzcd} \mathrm{R}\Gamma(F \,\breve{\otimes}\, \mathcal{O}_K,\, \mathcal{M}) \arrow[r, "\gamma"] \arrow[d, "\text{local}"'] & \mathrm{Lie}_E(K)[-1] \\ \mathrm{R}\Gamma_g(F \,\breve{\otimes}\, K,\, \mathcal{M}) & \mathrm{R}\Gamma_g(A \otimes K,\, \mathcal{M}) \arrow[l, "\sim"'] \arrow[u, "\wr", "\text{Definition 12.4.2}"'] \end{tikzcd}$$

is commutative. Together with Step 3 this claim immediately implies the theorem.

To prove this claim we first observe that the square

$$\begin{tikzcd} \mathrm{R}\Gamma_g(A \otimes K,\, \mathcal{M}) \arrow[r, "\text{Def. 12.4.2}", "\sim"'] \arrow[d] & \mathrm{Lie}_E(K)[-1] \arrow[d, equal] \\ \mathrm{R}\Gamma_g(F \,\breve{\otimes}\, K,\, \mathcal{M}) \arrow[r, "\text{Def. 9.7.1}", "\sim"'] & \mathrm{Lie}_E(K)[-1] \end{tikzcd}$$

is commutative by construction. By definition the quasi-isomorphism γ is the composition

$$\mathrm{R}\Gamma(F \breve{\otimes} \mathcal{O}_K,\, \mathcal{M}) \xrightarrow{\text{local}} \mathrm{R}\Gamma_g(F \breve{\otimes} K,\, \mathcal{M}) \xrightarrow{\text{Def. 9.7.1}} \mathrm{Lie}_E(K)[-1].$$

Hence the claim and the proposition follow. □

Proposition 12.5.4. *The square*

$$\begin{tikzcd} \mathrm{R}\Gamma(C^\circ \times X,\, \nabla\mathcal{M}) \arrow[r, "\text{pullback}"] \arrow[d, "\wr"', "\text{Definition 12.5.2}"'] & \mathrm{R}\Gamma(F \breve{\otimes} \mathcal{O}_K,\, \nabla\mathcal{M}) \arrow[d, "\wr"', "\nabla\gamma"] \\ \mathrm{Lie}_E(R)[-1] \arrow[r, "\text{embedding in degree 1}"] & \mathrm{Lie}_E(K)[-1] \end{tikzcd}$$

is commutative.

Proof. Same as the proof of Proposition 12.5.3. □

12.6. Regulators

Definition 12.6.1. The *arithmetic regulator*

$$\rho_E\colon F \otimes_A U_E \to \mathrm{Lie}_E(K)[0]$$

of E is the F-linear extension of the morphism $U_E \to \mathrm{Lie}_E(K)[0]$ given by the identity in degree zero.

Taelman [24] demonstrated that ρ_E is a quasi-isomorphism. This will also follow from the results below.

We work with a fixed global model $\mathcal{M}$. By Proposition 12.2.5 the restriction of $\mathcal{M}$ to $D \times X$ is an elliptic shtuka of ramification ideal $\mathfrak{e}$. We can thus make the following definition.

Definition 12.6.2. The *regulator*

$$\rho\colon\ \mathrm{R}\Gamma(D^\circ \times X,\, \mathcal{M}) \xrightarrow{\sim} \mathrm{R}\Gamma(D^\circ \times X,\, \nabla\mathcal{M})$$

of a global model $\mathcal{M}$ is the F-linear extension of the regulator $\rho\colon\ \mathrm{R}\Gamma(D\times X,\, \mathcal{M}) \xrightarrow{\sim} \mathrm{R}\Gamma(D \times X,\, \nabla\mathcal{M})$ of the elliptic shtuka $\mathcal{M}|_{D\times X}$.

The F-linear extensions of the maps of Definition 12.5.2 give us the quasi-isomorphisms

$$\mathrm{R}\Gamma(D^\circ \times X,\, \mathcal{M}) \xrightarrow{\sim} F \otimes_A U_E[-1], \tag{12.1}$$

$$\mathrm{R}\Gamma(D^\circ \times X,\, \nabla\mathcal{M}) \xrightarrow{\sim} \mathrm{Lie}_E(K)[-1]. \tag{12.2}$$

Theorem 12.6.3. *The square*

$$\begin{tikzcd} \mathrm{R}\Gamma(D^\circ \times X,\, \mathcal{M}) \arrow[r, "\rho"] \arrow[d, "\wr"', "(12.1)"'] & \mathrm{R}\Gamma(D^\circ \times X,\, \nabla\mathcal{M}) \arrow[d, "\wr"', "(12.2)"] \\ F \otimes_A U_E[-1] \arrow[r, "{\rho_E[-1]}"] & \mathrm{Lie}_E(K)[-1] \end{tikzcd}$$

is commutative.

Proof. According to Proposition 12.2.4 the shtuka $\mathcal{M}(\mathcal{O}_F \breve{\otimes} \mathcal{O}_K)$ is a local model of $\mathcal{H}\mathrm{om}(M, a(F,K))$. So it is an elliptic shtuka of ramification ideal $\mathfrak{e}$ by Theorem 9.7.7. As such it has a regulator

$$\mathrm{R}\Gamma(\mathcal{O}_F \breve{\otimes} \mathcal{O}_K,\, \mathcal{M}) \to \mathrm{R}\Gamma(\mathcal{O}_F \breve{\otimes} \mathcal{O}_K,\, \nabla\mathcal{M}).$$

By Lemma 9.7.3 its F-linear extension can be identified with a quasi-isomorphism

$$\breve{\rho}\colon\ \mathrm{R}\Gamma(F \breve{\otimes} \mathcal{O}_K,\, \mathcal{M}) \to \mathrm{R}\Gamma(F \breve{\otimes} \mathcal{O}_K,\, \nabla\mathcal{M}).$$

Theorem 6.8.7 shows that the natural square

$$\begin{tikzcd}
\mathrm{R}\Gamma(D^\circ \times X,\, \mathcal{M}) \arrow[r, "\rho"] \arrow[d] & \mathrm{R}\Gamma(D^\circ \times X,\, \nabla\mathcal{M}) \arrow[d] \\
\mathrm{R}\Gamma(F \check{\otimes} \mathcal{O}_K,\, \mathcal{M}) \arrow[r, "\check{\rho}"] & \mathrm{R}\Gamma(F \check{\otimes} \mathcal{O}_K,\, \nabla\mathcal{M})
\end{tikzcd}$$

is commutative. The vertical arrows in it are quasi-isomorphisms by Proposition 6.8.3. Now consider the diagram

$$\begin{tikzcd}
 & F \otimes_A \mathrm{R}\Gamma(C^\circ \times X,\, \mathcal{M}) \arrow[r, "\sim"] \arrow[d] & F \otimes_A U_E[-1] \arrow[d, "\rho_E", "\wr"'] \\
\mathrm{R}\Gamma(D^\circ \times X,\, \mathcal{M}) \arrow[r, "\sim"] \arrow[d, "\rho"] & \mathrm{R}\Gamma(F \check{\otimes} \mathcal{O}_K,\, \mathcal{M}) \arrow[r, "\gamma", "\sim"'] \arrow[d, "\check{\rho}"'] & \mathrm{Lie}_E(K)[-1] \arrow[d, equal] \\
\mathrm{R}\Gamma(D^\circ \times X,\, \nabla\mathcal{M}) \arrow[r, "\sim"] & \mathrm{R}\Gamma(F \check{\otimes} \mathcal{O}_K,\, \nabla\mathcal{M}) \arrow[r, "\nabla\gamma", "\sim"'] & \mathrm{Lie}_E(K)[-1] \\
 & F \otimes_A \mathrm{R}\Gamma(C^\circ \times X,\, \nabla\mathcal{M}) \arrow[r, "\sim"] \arrow[u] & F \otimes_A \mathrm{Lie}_E(R)[-1] \arrow[u, "\wr"']
\end{tikzcd}$$

The top and bottom squares are induced by the commutative squares of Proposition 12.5.3 and 12.5.4 respectively. The middle square commutes by Theorem 11.7.1. The square

$$\begin{tikzcd}
\mathrm{R}\Gamma(D^\circ \times X,\, \mathcal{M}) \arrow[r, "(12.1)", "\sim"'] \arrow[d, "\wr"'] & F \otimes_A U_E[-1] \\
\mathrm{R}\Gamma(F \check{\otimes} \mathcal{O}_K,\, \mathcal{M}) & F \otimes_A \mathrm{R}\Gamma(C^\circ \times X,\, \mathcal{M}) \arrow[l, "\sim"'] \arrow[u, "\wr"', "\text{Def. 12.5.2}"]
\end{tikzcd}$$

commutes by construction. An analogous observation applies to (12.2). We thus get the result. □

12.7. Euler products

As before we denote $F^\natural$ the fraction field of A and $\omega^\natural$ the module of Kähler differentials of $F^\natural$ over $\mathbb{F}_q$. Given a prime $\mathfrak{p}$ of A we denote $A_\mathfrak{p}$ the $\mathfrak{p}$-adic completion of A.

Let $\mathfrak{m}$ be a maximal ideal of R. We denote $E_\mathfrak{m}$ the pullback of E to $R/\mathfrak{m}$. For each $\mathfrak{m}$ we fix a separable closure of the residue field $R/\mathfrak{m}$. With this choice we have for every prime $\mathfrak{p}$ of A the $A_\mathfrak{p}$-adic Tate module $T_\mathfrak{p} E_\mathfrak{m}$. Given a maximal ideal $\mathfrak{m}$ of R and a prime $\mathfrak{p}$ of A different from $\mu^{-1}\mathfrak{m}$ we define

$$P_\mathfrak{m}(T) = \det\nolimits_{A_\mathfrak{p}}\big(1 - T\sigma_\mathfrak{m}^{-1} \,\big|\, T_\mathfrak{p} E_\mathfrak{m}\big) \in A_\mathfrak{p}[T]$$

where $\sigma_\mathfrak{m}^{-1}$ is the geometric Frobenius element at $\mathfrak{m}$.

Proposition 12.7.1. *The characteristic polynomial $P_\mathfrak{m}$ has coefficients in $F^\natural$ and is independent of the choice of $\mathfrak{p}$.*

Proof. Theorem 8.12.7 implies that

$$P_\mathfrak{m}(T^d) = \det\nolimits_{F^\natural}\big(1 - T(i^{-1}j) \,\big|\, \mathrm{Hom}_{A\otimes R}(M,\, \omega^\natural \otimes R/\mathfrak{m})\big)$$

where i and j are the arrows of the shtuka $\mathcal{H}\mathrm{om}(M,\, \omega^\natural \otimes R/\mathfrak{m})$. □

Definition 12.7.2. We define a formal product $L(E^*, 0) \in F$ as follows:

$$L(E^*, 0) = \prod_{\mathfrak{m}} P_{\mathfrak{m}}(1)^{-1}$$

where $\mathfrak{m} \subset R$ ranges over the maximal ideals.

In a moment we will see that this product converges.

Lemma 12.7.3. *If $\mathcal{M}$ is a global model then $\mathcal{M}(\mathcal{O}_F/\mathfrak{m}_F \otimes R)$ is nilpotent.*

Proof. Indeed the restriction of $\mathcal{M}$ to $D \times X$ is an elliptic shtuka by Proposition 12.2.5. □

Hence the L-invariants of Definitions 6.9.2 and 6.9.4 make sense for $\mathcal{M}$.

Proposition 12.7.4. *Let $\mathcal{M}$ be a global model. For every maximal ideal $\mathfrak{m}$ we have an equality $L(\mathcal{M}(\mathcal{O}_F \otimes R/\mathfrak{m})) = P_{\mathfrak{m}}(1)$ in F.*

Proof. Suppose that $\mathcal{M}$ is given by the diagram

$$\mathcal{M}_0 \overset{i}{\underset{j}{\rightrightarrows}} \mathcal{M}_1.$$

By definition $L(\mathcal{M}(\mathcal{O}_F \otimes R/\mathfrak{m}))$ is the determinant

$$\det_{\mathcal{O}_F}\big(1 - i^{-1}j \,\big|\, \mathcal{M}_0(\mathcal{O}_F \otimes R/\mathfrak{m})\big).$$

As $\mathcal{M}$ is a global model the shtuka $\mathcal{M}(\mathcal{O}_F \otimes R/\mathfrak{m})$ is a locally free $\mathcal{O}_F \otimes R/\mathfrak{m}$-lattice in the $F \otimes R/\mathfrak{m}$-module shtuka $\mathcal{H}\mathrm{om}(M,\, a(F, R/\mathfrak{m}))$. Hence the determinant above is equal to

$$\det_F\big(1 - i^{-1}j \,\big|\, \mathrm{Hom}_{A\otimes R}(M,\, a(F, R/\mathfrak{m}))\big).$$

Next $\omega^\natural \otimes R/\mathfrak{m}$ is an $F^\natural \otimes R/\mathfrak{m}$-lattice in the $F \otimes R/\mathfrak{m}$-module $a(F, R/\mathfrak{m})$ so the latter determinant coincides with

$$\det_{F^\natural}\big(1 - i^{-1}j \,\big|\, \mathrm{Hom}_{A\otimes R}(M,\, \omega^\natural \otimes R/\mathfrak{m})\big).$$

This determinant is equal to $P_{\mathfrak{m}}(1)$ by Theorem 8.12.7. □

Definition 12.7.5. Let $\mathcal{M}$ be a global model. We define $L(\mathcal{M}) = L(\mathcal{M}(\mathcal{O}_F \otimes R))$.

Proposition 12.7.6. *The formal product $L(E^*, 0)$ has the following properties.*

(1) *$L(E^*, 0)$ converges to an element of $\mathcal{O}_F$.*
(2) *$L(E^*, 0) \equiv 1 \pmod{\mathfrak{m}_F}$.*
(3) *For every global model $\mathcal{M}$ we have $L(\mathcal{M}) = L(E^*, 0)$.*

Proof. Let $\mathcal{M}$ be a global model. By definition

$$L(\mathcal{M}) = L(\mathcal{M}(\mathcal{O}_F \otimes R)) = \prod_{\mathfrak{m}} L(\mathcal{M}(\mathcal{O}_F \otimes R/\mathfrak{m}))^{-1}$$

where $\mathfrak{m}$ runs over all the maximal ideals. The product defining $L(\mathcal{M})$ converges by Lemma 6.9.3. Proposition 12.7.4 now implies that $L(\mathcal{M}) = L(E^*, 0)$ and we get (3). Proposition 6.9.6 shows that $L(\mathcal{M}) \equiv 1 \pmod{\mathfrak{m}_F}$. Since global models exist by Proposition 12.3.3 we get (1) and (2). □

12.8. Trace formula

Definition 12.8.1. Let $\mathcal{M}$ be a locally free shtuka on $C \times X$ given by a diagram

$$\mathcal{M}_0 \underset{j}{\overset{i}{\rightrightarrows}} \mathcal{M}_1.$$

The *twist* $\mathfrak{e}\mathcal{M}$ is the shtuka

$$\mathcal{I}\mathcal{M}_0 \underset{j}{\overset{i}{\rightrightarrows}} \mathcal{I}\mathcal{M}_1$$

where $\mathcal{I} \subset \mathcal{O}_{C\times X}$ is the pullback to $C \times X$ of the unique ideal sheaf $\mathcal{I}_0 \subset \mathcal{O}_X$ satisfying $\mathcal{I}_0(\operatorname{Spec} R) = R$ and $\mathcal{I}_0(\operatorname{Spec} \mathcal{O}_K) = \mathfrak{e}$.

Proposition 12.8.2. *If $\mathcal{M}$ is a global model then $\mathfrak{e}\mathcal{M}$ is a global model.*

Proof. The restrictions of $\mathcal{M}$ and $\mathfrak{e}\mathcal{M}$ to $C \times Y$ coincide by construction. Moreover the same argument as in the proof of Proposition 5.7.7 shows that $\mathfrak{e}\mathcal{M}(A \otimes \mathcal{O}_K/\mathfrak{m}_K)$ is nilpotent and $\mathfrak{e}\mathcal{M}(A \otimes \mathcal{O}_K/\mathfrak{e})$ is linear. Hence $\mathfrak{e}\mathcal{M}$ is a global model. □

Definition 12.8.3. The *ζ-isomorphism* of a global model $\mathcal{M}$ is the ζ-isomorphism

$$\zeta\colon \det_A \mathrm{R\Gamma}(C^\circ \times X, \mathcal{M}) \xrightarrow{\sim} \det_A \mathrm{R\Gamma}(C^\circ \times X, \nabla\mathcal{M})$$

of the restriction of $\mathcal{M}$ to $C^\circ \times X$.

Theorem 12.8.4. *Let $\mathcal{M}$ be a global model. For all $n \gg 0$ we have*

$$\zeta_{\mathfrak{e}^n\mathcal{M}} = L(\mathfrak{e}^n\mathcal{M}) \cdot \det_F(\rho_{\mathfrak{e}^n\mathcal{M}}).$$

This formula should hold for $n = 0$ as well. See the remark below Theorem 6.10.3.

Proof of Theorem 12.8.4. Let $\mathcal{M}_0$ and $\mathcal{M}_1$ be the underlying sheaves of $\mathcal{M}$. The ideal sheaf $\mathcal{I}$ above is anti-ample relative to C. Hence for all $n \gg 0$ and $* \in \{0, 1\}$ we have

$$\mathrm{H}^0(\operatorname{Spec}(\mathcal{O}_F/\mathfrak{m}_F) \times X, \mathcal{I}^n\mathcal{M}_*) = 0.$$

We can thus apply Theorem 6.10.3 to the restriction of $\mathfrak{e}^n\mathcal{M}$ to $D \times X$. The ζ-isomorphisms are compatible with the change of coefficients by Proposition 4.8.2. Hence Theorem 6.10.3 implies the result. □

12.9. The class number formula

Recall that the complex of units of the Drinfeld module E is

$$U_E = \Big[\operatorname{Lie}_E(K) \xrightarrow{\exp} \frac{E(K)}{E(R)}\Big].$$

The arithmetic regulator $\rho_E\colon F \otimes_A U_E \to \operatorname{Lie}_E(K)[0]$ is the F-linear extension of the morphism $U_E \to \operatorname{Lie}_E(K)[0]$ given by the identity in degree zero. Taelman [24] observed that U_E is a perfect A-module complex and ρ_E is a quasi-isomorphism.

By construction $\operatorname{Lie}_E(K) = F \otimes_A \operatorname{Lie}_E(R)$ so the one-dimensional F-vector space $\det_F \operatorname{Lie}_E(K)$ contains a canonical A-lattice $\det_A \operatorname{Lie}_E(R)$.

Theorem 12.9.1. *The image of $\det_A U_E$ under $\det_F(\rho_E)$ is*

$$L(E^*, 0) \cdot \det_A \operatorname{Lie}_E(R).$$

Proof. Proposition 12.3.3 associates a global shtuka model $\mathcal{M}$ with the Drinfeld module E. The model comes equipped with a ζ-isomorphism

$$\zeta\colon \det_A \mathrm{R\Gamma}(C^\circ \times X, \mathcal{M}) \xrightarrow{\sim} \det_A \mathrm{R\Gamma}(C^\circ \times X, \nabla\mathcal{M})$$

and a regulator $\rho\colon \mathrm{R\Gamma}(D^\circ \times X, \mathcal{M}) \xrightarrow{\sim} \mathrm{R\Gamma}(D^\circ \times X, \nabla\mathcal{M})$.

Theorem 12.8.4 shows that after replacing $\mathcal{M}$ with a twist $\mathfrak{e}^n\mathcal{M}$ by a high enough power of $\mathfrak{e}^n$ we have

$$\zeta = L(\mathcal{M}) \cdot \det_F(\rho). \tag{12.3}$$

Definition 12.5.2 provides us with quasi-isomorphisms

$$\mathrm{R}\Gamma(C^\circ \times X,\, \mathcal{M}) \xrightarrow{\sim} U_E[-1],$$
$$\mathrm{R}\Gamma(C^\circ \times X,\, \nabla\mathcal{M}) \xrightarrow{\sim} \mathrm{Lie}_E(R)[-1].$$

Hence ζ induces an A-module isomorphism

$$\det_A(U_E[-1]) \xrightarrow{\sim} \det_A(\mathrm{Lie}_E(R)[-1]).$$

Thanks to Theorem 12.6.3 we know that under the quasi-isomorphisms above the regulator ρ matches with the shifted arithmetic regulator

$$\rho_E[-1]\colon F \otimes_A U_E[-1] \to \mathrm{Lie}_E(K)[-1].$$

Moreover $L(\mathcal{M}) = L(E^*, 0)$ by Proposition 12.7.6. As $\det_F(\rho_E[-1])$ is the inverse of the dual of $\det_F(\rho_E)$ we conclude that (12.3) implies the theorem. $\square$

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The references of the form $[wxyz]$ point to tags in the Stacks project [27] as explained in the chapter "Notation and conventions".

Index of terms

Index of symbols