Final Exam

Graded

Student

Max Alan Niederman

Total Points

71.75 / 80 pts

Question 1

(no title)

10 / 10 pts

- ✔ **– 0 pts** Correct
- **– 1 pt** incorrect/no answer
- **– 1 pt** minor error
- **– 2 pts** major error in calculation (e.g. cancelling out terms that may be zero)
- **– 2 pts** did not consider the possibility that the gradient of the restriction function may be zero.
- **– 5 pts** incomplete solution, but attempted Lagrange multipliers with correct f and g
- **– 7 pts** Had correct restriction/optimization functions, but no /incorrect explanation on how to use Lagrange multipliers
- **– 7 pts** attempted Lagrange multipliers with incorrect functions
- **– 9 pts** reasonable attempt
- **– 10 pts** no points

Question 2

(no title)

5.25 / 6 pts

- ✔ **+ 2 pts** Correctly computing X^ = [-2,-1,0,1,2]
- ✔ **+ 2 pts** Correctly projecting Y on span(1, X^) finding proj Y = -3/10 X^+2/5
- ✔ **+ 2 pts** Correctly replacing X^ by X-2, finding the line of best fit y = -3/10 x +1
- **+ 1 pt** Partial credit: none of above marked, but reasonable attempt
- ✔ **– 0.75 pts** Simple computation mistake
- **– 1.5 pts** Deep computation mistake or many computation mistakes
- **+ 0 pts** No credit / no attempt

Question 3

(no title) **9** / 10 pts

3.1 **(no title)** **5** / 6 pts

- **– 0 pts** Correct
- **– 0.5 pts** correct calculations, but typo in final answer
- ✔ **– 1 pt** calculation error in finding Q
- **– 1 pt** error in calculating R
- **– 2 pts** multiple calculation errors in finding Q
- **– 2 pts** conceptual error in Graham Schmit process
- **– 2 pts** incorrect/no process for finding R
- **– 5 pts** partial credit - correct definitions of Q and R / attempted Graham Schmidt
- **– 6 pts** no points

3.2 **(no title)** **4** / 4 pts

- ✔ **+ 0.5 pts** Stated $A^{-1} = R^{-1}Q^{-1}$.
- ✔ **+ 0.5 pts** Stated $Q^{-1} = Q^T$.
- ✔ **+ 0.5 pts** Showed that R^{-1} is upper triangular.
- ✔ **+ 0.5 pts** Correct diagonal entries of R^{-1} (with respect to part a)
- ✔ **+ 1.5 pts** Correct process entries for upper portion of R^{-1} (possibly with calculation errors)
- ✔ **+ 0.5 pts** Correct final answer (with respect to R, Q in part a)
- **+ 0 pts** no points

Question 4

(no title) **10** / 10 pts

4.1 **(no title)** **3** / 3 pts

✔ **– 0 pts** Correct

– 3 pts Incorrect/no response

4.2 **(no title)** **3** / 3 pts

✔ **– 0 pts** Correct

– 0.5 pts Minor arithmetic error

– 1.5 pts Multiple errors/incorrect matrix multiplication

– 2 pts Correct strategy

– 2.5 pts Some relevant work

– 3 pts Incorrect/blank

4.3 **(no title)** **4** / 4 pts

✔ **– 0 pts** Correct

– 3 pts Multiplied matrices in wrong order

– 3.5 pts Reported an answer of the form $A^n B^n$

– 4 pts Incorrect

Question 5

(no title) **14** / 14 pts

5.1 **(no title)** **3** / 3 pts

- ✔ **+ 1.5 pts** Arguing that v' is an eigenvector of eigenvalue lambda
- ✔ **+ 1.5 pts** Arguing that e_1 is an eigenvector of eigenvalue a
- **− 0.5 pts** Computation mistake or slightly incorrect presentation of answers
- **+ 0.5 pts** Reasonable work on v', but no final answer
- **+ 0.5 pts** Reasonable work on e_1, but no final answer
- **+ 0 pts** No credit / no submission

5.2 **(no title)** **4** / 4 pts

- ✔ **+ 4 pts** All correct
- **+ 1.4 pts** Found eigenvector [0,2,1] / eigenvalue 4
- **+ 1.4 pts** Found eigenvector [0,1,-2] / eigenvalue -1
- **+ 1.2 pts** Found eigenvector [1,0,0] / eigenvalue -2
- **− 0.25 pts** Small computation mistake
- **+ 0.5 pts** Reasonable attempt
- **+ 0 pts** No credit / no submission

5.3 **(no title)** **3** / 3 pts

- ✔ **+ 3 pts** Correct
- **+ 2 pts** Diagonal matrix correct, but columns of Q matrix were not normalized (so it is not orthonormal)
- **+ 1 pt** Only diagonal matrix correct
- **+ 1 pt** Roughly correct explanation of how to build Q and D with eigenvectors and eigenvalues, but no numerics
- **+ 0.5 pts** Reasonable attempt
- **− 0.5 pts** Confused Q with its transpose, or other computational mistake
- **+ 0 pts** No credit / no attempt

5.4 **(no title)** **4** / 4 pts

- ✔ **+ 1.5 pts** Correct alpha
- ✔ **+ 2.5 pts** Correct matrix L

+ 2.5 pts Most of the work is there, but no explicit values of alpha and L presented

− 0.25 pts Small mistake on the precise choice of L or alpha

− 1 pt Computational mistakes or other problems with final answer

+ 0 pts No credit / no attempt (no partial credit given to formulas without numerical answers)

Question 6

(no title) **8.5** / 10 pts

6.1 (no title) **5** / 6 pts

✔ **+ 0.5 pts** Correctly stated that $D(g \circ f)(\pi, \pi/2) = \begin{bmatrix} 0 & 0 \end{bmatrix}$

+ 1 pt Correct *explanation* why $D(g \circ f)(\pi, \pi/2) = \begin{bmatrix} 0 & 0 \end{bmatrix}$

✔ **+ 1 pt** Correctly computed $Df(u, v)$

✔ **+ 0.5 pts** Correctly computed $Df(\pi, \pi/2)$

✔ **+ 1 pt** Correctly stated the chain rule **in the context of this problem**: $D(g \circ f)(\pi, \pi/2) = Dg(f(\pi, \pi/2))Df(\pi, \pi/2)$

Note: No credit given if ∇ was written in place of D. No credit given for statements of the chain rule without reference to the problem.

✔ **+ 1 pt** Correctly determined that $D(g \circ f)(\pi, \pi/2) = \begin{bmatrix} -7 + 2\alpha & \beta \end{bmatrix}$

✔ **+ 1 pt** Correct final answers (with correct working): $\alpha = 7/2$ and $\beta = 0$

+ 0 pts No credit

− 0.5 pts Minor error or omission

− 1 pt Several minor errors or one not-so-minor error

6.2 (no title) **1.5** / 2 pts

✔ **+ 0.5 pts** Correct computation of $f(\pi, \pi/2)$

✔ **+ 1 pt** Correct formula for the tangent plane in the context of this problem: $\nabla g(f(\pi, \pi/2)) \cdot (\mathbf{x} - f(\pi, \pi/2)) = 0$

Note: No credit for writing down the general formula with no reference to this problem

✔ **+ 0.5 pts** Correct final equation of tangent plane

+ 0 pts No credit

✔ **− 0.5 pts** Minor error or omission

− 1 pt Several minor errors or one not-so-minor error

1 this is 1

6.3 **(no title)** **2** / 2 pts

- ✔ **+ 1 pt** Correct observation **and** explanation that the desired $\mathbf{v}$ is orthogonal to $(\nabla g)(f(\pi, \pi/2))$
- ✔ **+ 1 pt** Correct **unit** vector $\mathbf{v}$ orthogonal to $(\nabla g)(f(\pi, \pi/2))$ (credit is only awarded if there is correct explanation)
- **+ 0 pts** No credit
- **− 0.5 pts** Minor error or omission
- **− 1 pt** Several minor errors or one not-so-minor error

Question 7

(no title) **9** / 10 pts

7.1 **(no title)** **1** / 2 pts

- **− 0 pts** Correct
- ✔ **− 1 pt** Correctly computed gradient but wrong critical point
- **− 2 pts** Incorrect

7.2 **(no title)** **3** / 3 pts

- ✔ **− 0 pts** Correct
- **− 2 pts** Computed Hessian correctly but wrong answer
- **− 3 pts** Incorrect

7.3 **(no title)** **5** / 5 pts

- ✔ **+ 5 pts** Correct
- **+ 1 pt** Drew ellipses centered at critical point
- **+ 2 pts** Correct eigenvectors and eigenvalues
- **+ 2 pts** Ellipses are shorter along the dominant eigenvector
- **+ 0 pts** Incorrect

Question 8

(no title) **6** / 10 pts

8.1 **(no title)** **5** / 5 pts

+ 0 pts Marked FALSE or no answer

✓ **+ 1 pt** Marked TRUE (correct answer)

+ 0 pts Marked TRUE, with very incomplete/incorrect reasoning, or minimal progress towards correct justification.

+ 1 pt Marked TRUE, with largely incorrect/incomplete justification but some correct ideas. Includes:

- Correct and complete proof of a nontrivial special case by direct calculation (e.g., proof that any 2×2 matrix A satisfying $A^\top A = 0$ must be the zero matrix), but not of the general $m \times n$ case

+ 2.5 pts Marked TRUE, with largely correct but slightly incorrect/incomplete justification.

✓ **+ 4 pts** Marked TRUE, with fully correct justification.

− 0.5 pts Minor mistake/omission

− 1 pt Serious mistake/omission or multiple minor mistakes

8.2 **(no title)** **1** / 5 pts

+ 0 pts Marked FALSE or no answer

✓ **+ 1 pt** Marked TRUE (correct answer)

✓ **+ 0 pts** Marked TRUE, with very incomplete/incorrect reasoning, or minimal progress towards correct justification. Includes:

- Writing down eigenvalues of A but did not relate these to eigenvalues of B
- Incorrect claim that any polynomial expression (or incorrectly, "linear combination") of A must have the same characteristic polynomial as A

+ 1 pt Marked TRUE, with largely incorrect/incomplete justification but some correct ideas. Includes:

- Correct proof of the statement for one specific A, but no general proof

+ 2.5 pts Marked TRUE, with largely correct but slightly incorrect/incomplete justification.

+ 4 pts Marked TRUE, with fully correct justification.

− 0.5 pts Minor mistake/omission

− 1 pt Serious mistake/omission or multiple minor mistakes

Math 51 Final Exam — August 19, 2023

Name: Max Niederman SUNet ID: maxnie ID #: 00677 6283

- Complete the following problems. In order to receive full credit, please show all of your work and justify your answers. You do not need to simplify your answers unless specifically instructed to do so. You may use any result discussed in class or the text, but clearly state the result before using it, and verify that the hypotheses are satisfied.
- Please check that your copy of this exam contains 13 pages of exam questions, *numbered* in the upper-right, and that it is adequately stapled.

  DO NOT REMOVE ANY PAGES; if any page is missing, your exam will be considered incomplete. Incomplete exams will be assessed a 5-point penalty.
- You may use 1 piece of 8.5″ × 11″ paper (both sides) with formulas and other notes as a "reference sheet". No electronic devices, including phones, headphones, or calculation aids, are permitted for any reason.
- **You have 3 hours.** Your organizer will signal the times between which you are permitted to be writing, including anything on this cover sheet, and to have the exam booklet open. During these times, the exam and all papers must remain in the testing room. When you are finished, you must hand your exam paper to a member of teaching staff.
- Paper not provided by course staff (apart from your own reference sheet) is prohibited. If you need extra room for your answers, use one of the blank pages provided (those pages except for the one at the end are labeled at the bottom by lower-case Roman numerals, starting with "ii"), and clearly indicate that that your answer continues there. Do not unstaple or detach pages from this exam.
- It is your responsibility to look over your graded exam in a timely manner. You have until **August 25, 5 p.m.**, to resubmit your exam for any regrade considerations; consult your section leader about the exact details of the submission process.
- Please sign the following:

  "On my honor, I have neither given nor received any aid on this examination. I have furthermore abided by all other aspects of the honor code with respect to this examination."

  Signature: Max Niederman

DO NOT DETACH THIS PAGE. If you use any of this space to continue your answer, please clearly indicate the problem number(s) and **indicate on the original page of the problem that your answer continues here.**

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1. (10 points) Consider the curve defined by the equation

$$(x+1)(x^2+y^2) = 4x^2.$$

Find the rightmost point(s) (i.e., the point(s) with the largest x-coordinate) on the curve.

Remark. This curve is called *conchoid of de Sluze.*

Let $f(x,y) = x$ and $g(x,y) = (x+1)(x^2+y^2) - 4x^2$

$\nabla f = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ $\quad \nabla g = \begin{bmatrix} 2x(x+1) + x^2 + y^2 - 8x \\ 2(x+1)y \end{bmatrix}$

$= \begin{bmatrix} 3x^2 + y^2 - 6x \\ 2(x+1)y \end{bmatrix}$

$\nabla g = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$2(x+1)y = 0$

$x = -1$

$(-1+1)((-1)^2+y^2) - 4(-1)^2 = 0$

$0 - 4(-1)^2 = 0$

$-4 = 0$

Contradiction

$y = 0$

$3x^2 + (0)^2 - 6x = 0$

$3x(x-2) = 0$

If $x = 2$:

$3(4+0) - 4(4) = 0$

$12 = 16$

Contradiction

If $x = 0$:

$1(0+0) - 4(0)^2 = 0$

$0 = 0$

$(0,0)$

$\nabla f = \lambda \nabla g$

$\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \lambda \begin{bmatrix} 3x^2 + y^2 - 6x \\ 2(x+1)y \end{bmatrix}$

$\lambda \neq 0$ because otherwise $1 = 0$, so

$0 = (x+1)y$

$x = -1$

$-4 = 0$

Contradiction

$y = 0$

$(x+1)(x^2+0) - 4x^2 = 0$

$x^3 - 3x^2 = 0$

$x^2(x-3) = 0$

$(0,0)$ or $(3,0)$

$(3,0)$ is the rightmost point

2. (6 points) Find the line of best fit (that minimizes SSE (sum of square errors)) for the data

$$(0,1), \qquad (1,0), \qquad (2,1), \qquad (3,1), \qquad (4,-1).$$

The line of best fit always passes through the mean, so

$$y - \bar{y}\vec{1} \approx m(x - \bar{x}\vec{1})$$

If we let $\hat{y} = y - \bar{y}$ and $\hat{x} = x - \bar{x}$, we have that $\text{Proj}_{\hat{x}}\hat{y} = \frac{\hat{x}\cdot\hat{y}}{\hat{x}\cdot\hat{x}}\hat{x}$ is the closest vector to $\hat{y}$ in span $\hat{x}$, so $m = \frac{\hat{x}\cdot\hat{y}}{\hat{x}\cdot\hat{x}}$

$$\hat{x} = \begin{bmatrix} 0-2 \\ 1-2 \\ 2-2 \\ 3-2 \\ 4-2 \end{bmatrix} = \begin{bmatrix} -2 \\ -1 \\ 0 \\ 1 \\ 2 \end{bmatrix}$$

$$m = \frac{\hat{x}\cdot\hat{y}}{\hat{x}\cdot\hat{x}}$$

$$= \frac{1}{5}\cdot\frac{-6+2+0+3-14}{4+1+0+1+4}$$

$$= \frac{1}{5}\cdot\frac{-15}{10} = -\frac{3}{10}$$

$$\hat{y} = \begin{bmatrix} 1-\frac{2}{5} \\ 0-\frac{2}{5} \\ 1-\frac{2}{5} \\ 1-\frac{2}{5} \\ -1-\frac{2}{5} \end{bmatrix} = \frac{1}{5}\begin{bmatrix} 3 \\ -2 \\ 3 \\ 3 \\ -7 \end{bmatrix}$$

So the line of best fit is:

$$y - \frac{2}{5} = -\frac{3}{10}(x-2)$$

$$y = -\frac{3}{10}x - \frac{6}{10} + \frac{2}{5}$$

$$y = -\frac{3}{10}x - \frac{1}{5}$$

3. (10 points) Let $A = \begin{bmatrix} 1 & 1 & 4 \\ -1 & 3 & 2 \\ 0 & 1 & -3 \end{bmatrix}$.

(a) (6 points) Compute a QR-decomposition for A.
Remark. The only irrational entries will involve $\sqrt{2}$.

I'll use Gram-Schmidt:

let $A = [v_1\ v_2\ v_3]$

$$w_1 = v_1 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$$

$$w_2 = v_2 - \frac{v_2 \cdot w_1}{w_1 \cdot w_1} v_1$$

$$= \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix} - \frac{-2}{2}\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}$$

$$w_3 = v_3 - \frac{v_3 \cdot w_2}{w_2 \cdot w_2} w_2 - \frac{v_3 \cdot w_1}{w_1 \cdot w_1} w_1$$

$$= \begin{bmatrix} 4 \\ 2 \\ -3 \end{bmatrix} - \frac{12}{8}\begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix} - \frac{2}{2}\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 4-3-1 \\ 2-3+1 \\ -3-\frac{3}{2}-0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -\frac{9}{2} \end{bmatrix}$$

$$w_1' = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \quad w_2' = \frac{1}{3}\begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix} \quad w_3' = \frac{2}{9}\begin{bmatrix} 0 \\ 0 \\ -\frac{9}{2} \end{bmatrix}$$

$w_1 = v_1$

$v_1 = \sqrt{2}\, w_1'$

$w_2 = v_2 + w_1$

$v_2 = -\sqrt{2}\, w_1' + 3 w_2'$

$w_3 = v_3 - \frac{3}{2} w_2 - w_1$

$v_3 = w_1 + \frac{3}{2} w_2 + w_3$

$= \sqrt{2}\, w_1' + \frac{9}{2} w_2' + \frac{9}{2} w_3'$

$$Q = [w_1'\ w_2'\ w_3'] = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{2}{3} & 0 \\ -\frac{1}{\sqrt{2}} & \frac{2}{3} & 0 \\ 0 & \frac{1}{3} & -1 \end{bmatrix}$$

$$R = \begin{bmatrix} \sqrt{2} & -\sqrt{2} & \sqrt{2} \\ 0 & 3 & \frac{9}{2} \\ 0 & 0 & \frac{9}{2} \end{bmatrix}$$

(problem continued from facing page)

(b) (4 points) Using your results from (a), write down an expression for A^{-1}. You may leave your answer as a product of two matrices.

$$A^{-1} = (QR)^{-1} = R^{-1}Q^{-1}$$

$$Q^{-1} = Q^T$$

$$= \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ \frac{2}{3} & \frac{2}{3} & \frac{1}{3} \\ 0 & 0 & -1 \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{3} & \frac{5}{9} \\ 0 & \frac{1}{3} & -\frac{1}{3} \\ 0 & 0 & \frac{2}{9} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ \frac{2}{3} & \frac{2}{3} & \frac{1}{3} \\ 0 & 0 & -1 \end{bmatrix}$$

$$R^{-1}R = I_3$$

$$\begin{bmatrix} \frac{1}{\sqrt{2}} & a & b \\ 0 & \frac{1}{3} & c \\ 0 & 0 & \frac{2}{9} \end{bmatrix} \begin{bmatrix} \sqrt{2} & -\sqrt{2} & \sqrt{2} \\ 0 & 3 & \frac{9}{2} \\ 0 & 0 & \frac{9}{2} \end{bmatrix} = I_3$$

$$-\frac{\sqrt{2}}{\sqrt{2}} + 3a = 0$$

$$a = \frac{1}{3}$$

$$\frac{9}{2} \cdot \frac{1}{3} + \frac{9}{2}c = 0$$

$$\frac{1}{3} + c = 0$$

$$c = -\frac{1}{3}$$

$$\frac{\sqrt{2}}{\sqrt{2}} + \frac{9}{2}a + \frac{9}{2}b = 0$$

$$1 + \frac{3}{2} + \frac{9}{2}b = 0$$

$$\frac{9}{2}b = \frac{5}{2} \quad b = \frac{5}{9}$$

4. (10 points) The kea are alpine parrots native to the South Island of New Zealand. The zoologists observe the following migratory patterns of the kea in three mountain ranges A, B, and C.

Every March,

- 20% of the kea in range A move to range B, 40% to range C, and the rest do not migrate;
- 50% of the kea in range B move to range C, and the rest do not migrate; and
- 30% of the kea in range C move to range A, 20% to range B, and the rest do not migrate.

Every September, for some fixed value of β,

- 40% of the kea in range A move to range C, and the rest do not migrate;
- 30% of the kea in range B move to range A, and the rest do not migrate; and
- 40% of the kea in range C move to range A, β% to range B, and the rest do not migrate.

Suppose that, in **January 2020**, there are 20,000, 10,000, and 0 kea in ranges A, B, and C, respectively; suppose also that we can safely ignore population changes due to any other factors, including births and deaths.

(a) (3 points) How many kea are there in range B in **June 2020**?

Let $M = \begin{bmatrix} 0.4 & 0 & 0.3 \\ 0.2 & 0.5 & 0.2 \\ 0.4 & 0.5 & 0.5 \end{bmatrix}$, so that $M\begin{bmatrix} A \\ B \\ C \end{bmatrix}$ contains the populations for A, B, and C after the March migration. Then range B is just the second entry of $M\begin{bmatrix} 20000 \\ 10000 \\ 0 \end{bmatrix} = \begin{bmatrix} \cdots \\ 4000+5000+0 \\ \cdots \end{bmatrix}$. So there are 9000 kea in range B.

(b) (3 points) If there are 4,500 kea in range C in **January 2021**, what is the value of β?

Let $S(\beta) = \begin{bmatrix} 0.6 & 0.3 & 0.4 \\ 0 & 0.7 & 0.01\beta \\ 0.4 & 0 & 0.6-0.01\beta \end{bmatrix}$

$$\begin{bmatrix} 0 & 0 & 1 \end{bmatrix}(S(\beta))M\begin{bmatrix} 20000 \\ 10000 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0.4 & 0 & 0.6-0.01\beta \end{bmatrix} M \begin{bmatrix} 20000 \\ 10000 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0.4 & 0 & 0.6-0.01\beta \end{bmatrix}\begin{bmatrix} 8000+0+0 \\ \cdots \\ 8000+5000+0 \end{bmatrix}$$

$$= 0.4(8000) + (0.6-0.01\beta)13000$$

$$= 3200 + 7800 - 130\beta$$

$$4500 = 11000 - 130\beta$$

$$130\beta = 6500$$

$$\beta = \frac{650}{13}.$$

(problem continued from facing page)

(c) (4 points) Let a_n, b_n, c_n represent the number of kea in mountain ranges A, B, C, respectively, in **June of the nth year after 2020.** (For example, a_1 is the number of kea in mountain range A in **June 2021**, and so on.) Write down a matrix M for which

$$M \begin{bmatrix} a_n \\ b_n \\ c_n \end{bmatrix} = \begin{bmatrix} a_{n+1} \\ b_{n+1} \\ c_{n+1} \end{bmatrix}.$$

If you wish, you may express your matrix M as a product of some fixed number of matrices (each of which contains specific numbers), but without carrying out the actual multiplication.

(*Note:* if you found the value of the constant β in part (b), you may use this value in your answer; if you did not, you may express one or more entries of your answer for M in terms of β.)

(I'll refer to the matrix I called M before as A).

M is the result of applying $S(\beta)$, which represents the September migration, Then A, which represents the March migration. Therefore,

$$M = A(S(\beta))$$

$$= \begin{bmatrix} 0.4 & 0 & 0.3 \\ 0.2 & 0.5 & 0.2 \\ 0.4 & 0.5 & 0.5 \end{bmatrix} \begin{bmatrix} 0.6 & 0.3 & 0.4 \\ 0 & 0.7 & \frac{65}{130} \\ 0.4 & 0 & 0.6 - \frac{65}{130} \end{bmatrix}$$

5. (14 points) Consider the matrix $A = \begin{bmatrix} a & 0 & 0 \\ 0 & b & c \\ 0 & d & e \end{bmatrix}$, where a, b, c, d, e are real numbers. Define the matrix $B = \begin{bmatrix} b & c \\ d & e \end{bmatrix}$.

(a) (3 points) Suppose $\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$ is an eigenvector for B with eigenvalue λ. Show that $\mathbf{v}' = \begin{bmatrix} 0 \\ v_1 \\ v_2 \end{bmatrix}$ and $\mathbf{e}_1$ are eigenvectors for A, and find their respective eigenvalues.

$A e_1 = \begin{bmatrix} a \\ 0 \\ 0 \end{bmatrix}$

$= a \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

$= a e_1$

e_1 has eigenvalue a.

$Bv = \lambda v$

$\begin{bmatrix} b & c \\ d & e \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \lambda \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$

$bv_1 + cv_2 = \lambda v_1$

$dv_1 + ev_2 = \lambda v_2$

$0a + 0v_1 + 0v_2 = 0$

$0 + bv_1 + cv_2 = \lambda v_1$

$0 + dv_1 + ev_2 = \lambda v_2$

$\begin{bmatrix} a & 0 & 0 \\ 0 & b & c \\ 0 & d & e \end{bmatrix} \begin{bmatrix} 0 \\ v_1 \\ v_2 \end{bmatrix} = \lambda \begin{bmatrix} 0 \\ v_1 \\ v_2 \end{bmatrix}$

$A \begin{bmatrix} 0 \\ v_1 \\ v_2 \end{bmatrix} = \lambda \begin{bmatrix} 0 \\ v_1 \\ v_2 \end{bmatrix}$

$\begin{bmatrix} 0 \\ v_1 \\ v_2 \end{bmatrix}$ has eigenvalue λ.

For parts (b) – (d) of this problem, consider the matrix $A = \begin{bmatrix} -2 & 0 & 0 \\ 0 & 3 & 2 \\ 0 & 2 & 0 \end{bmatrix}$.

(b) (4 points) Using the results of part (a), find all the eigenvalues of A and give an eigenvector associated with each eigenvalue. Make sure to verify your answers.

First, we have e_1 as an eigenvector with eigenvalue -2.

$\det\left(\begin{bmatrix} 3 & 2 \\ 2 & 0 \end{bmatrix} - \lambda I\right)$

$= \lambda^2 - 3\lambda - 4$

$= (\lambda - 4)(\lambda + 1)$

$\lambda_2 = 4$ and $\lambda_3 = -1$ are the other eigenvalues

$\begin{bmatrix} -1 & 2 \\ 2 & -4 \end{bmatrix} v_2 = 0$

$v_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

$\begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}$ is an eigenvector with eigenvalue 4.

$\begin{bmatrix} 4 & 2 \\ 2 & 1 \end{bmatrix} v_3 = 0$

$v_3 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$

$\begin{bmatrix} 0 \\ 1 \\ -2 \end{bmatrix}$ is an eigenvector with eigenvalue -1.

(problem continued from facing page)

(c) (3 points) Find a matrix Q and a diagonal matrix D for which $A = QDQ^{\mathsf{T}}$.

A is symmetric, so its eigenvectors form a basis. We construct Q to be the orthonormal which converts that basis to the standard ~~one:~~

$$Q = \left[e_1 \ \frac{1}{\sqrt{5}}\begin{bmatrix}0\\2\\1\end{bmatrix} \ \frac{1}{\sqrt{5}}\begin{bmatrix}0\\1\\-2\end{bmatrix} \right] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \\ 0 & \frac{1}{\sqrt{5}} & -\frac{2}{\sqrt{5}} \end{bmatrix}$$

Then the entries of D are the eigenvalues:

$$D = \begin{bmatrix} -2 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

(d) (4 points) Find a scalar α and a 3×3 matrix L, whose entries are all real numbers (i.e., no n in the entries), for which

$$A^n \approx \alpha^n L$$

for large n.

For large n, the effect of A^n on x is dominated by the projection of x onto the principal eigenvector; in this case $\frac{1}{\sqrt{5}}\begin{bmatrix}0\\2\\1\end{bmatrix}$. Therefore, $A^n \approx \lambda_2^n \operatorname{Proj}_{\frac{1}{\sqrt{5}}\begin{bmatrix}0\\2\\1\end{bmatrix}}$

$\alpha = \lambda_2 = 4$

$$L = \frac{\frac{1}{\sqrt{5}}\begin{bmatrix}0\\2\\1\end{bmatrix} \frac{1}{\sqrt{5}}\begin{bmatrix}0\\2\\1\end{bmatrix}^T}{\frac{1}{\sqrt{5}}\begin{bmatrix}0\\2\\1\end{bmatrix} \cdot \frac{1}{\sqrt{5}}\begin{bmatrix}0\\2\\1\end{bmatrix}} = \frac{1}{5}\begin{bmatrix}0\\2\\1\end{bmatrix}\begin{bmatrix}0\\2\\1\end{bmatrix}^T$$

$$= \frac{1}{5}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 4 & 2 \\ 0 & 2 & 1 \end{bmatrix}$$

6. (10 points) Let $\mathbf{f}: \mathbf{R}^2 \to \mathbf{R}^3$ be given by

$$\mathbf{f}(u,v) = \begin{bmatrix} (7+\cos v)\sin u \\ 2u + \sin v \\ (7+\cos v)\cos u \end{bmatrix}.$$

The set $\mathbf{f}(\mathbf{R}^2)$ (i.e. image of $\mathbf{R}^2$ under $\mathbf{f}$) describes a surface in $\mathbf{R}^3$ and is in fact the level set of a function $g: \mathbf{R}^3 \to \mathbf{R}$ at level 13. In particular, this means that $g(\mathbf{f}(u,v)) = 13$ for all $(u,v) \in \mathbf{R}^2$.

(a) (6 points) Given that

$$(\nabla g)\,(\mathbf{f}(\pi, \pi/2)) = \begin{bmatrix} 1 \\ \alpha \\ \beta \end{bmatrix},$$

find the values of α and β, by using the Multivariate Chain Rule.

$$Dg\left(f\left(\pi, \tfrac{\pi}{2}\right)\right) Df\left(\pi, \tfrac{\pi}{2}\right) = D(g \circ f)$$
$$= 0$$

$$\begin{bmatrix} 1 & \alpha & \beta \end{bmatrix} Df\left(\pi, \tfrac{\pi}{2}\right) = 0$$

$$1\,0(1) + 0(\alpha) + 1\beta = 0$$
$$\underline{\beta = 0}$$

$$Df = \begin{bmatrix} (7+\cos v)\cos u & -\sin u \sin v \\ 2 & \cos v \\ -(7+\cos v)\sin u & -\cos u \sin v \end{bmatrix}$$

$$-7(1) + 2(\alpha) = 0$$
$$2\alpha - 7 = 0$$
$$\underline{\alpha = \tfrac{7}{2}}$$

$$(Df)\left(\pi, \tfrac{\pi}{2}\right) = \begin{bmatrix} -7 & 0 \\ 2 & 0 \\ 0 & 1 \end{bmatrix}$$

(b) (2 points) Find the equation of the tangent plane to the surface $\mathbf{f}(\mathbf{R}^2)$ at the point $\mathbf{f}(\pi, \pi/2)$. You may leave your answer in terms of α and β if you have not found them in part (a).

$\nabla g(f(\pi, \frac{\pi}{2})) = \begin{bmatrix} 1 \\ \frac{7}{2} \\ 0 \end{bmatrix}$ is normal to the plane, so an equation for it is

$$\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix} - f(\pi, \tfrac{\pi}{2})\right) \cdot \begin{bmatrix} 1 \\ \frac{7}{2} \\ 0 \end{bmatrix} = 0$$

$$x + \frac{7}{2}y = (7 + \cos\frac{\pi}{2})\sin\pi + \frac{7}{2}(2\pi + \sin\frac{\pi}{2})$$

$$x + \frac{7}{2}y = 7\pi$$

(c) (2 points) Give, with justification, a unit vector $\mathbf{v} \in \mathbf{R}^3$ for which the *absolute value* of the change in g at the point $\mathbf{f}(\pi, \pi/2)$ in the direction of $\mathbf{v}$ is minimized. You may leave your answer in terms of α and β if you have not found them in part (a).

This is just any unit vector orthogonal to $\nabla g(f(\pi, \frac{\pi}{2})) = \begin{bmatrix} 1 \\ \frac{7}{2} \\ 0 \end{bmatrix}$

One obvious vector meeting that requirement is $v = e_3$

7. (10 points) Consider the function

$$f(x, y) = 17x^2 - 8xy + 2y^2 - 10x - 4y - 2.$$

(a) (2 points) Show that f has a unique critical point.

$$\nabla f = \begin{bmatrix} 34x - 8y - 10 \\ -8x + 4y - 4 \end{bmatrix} = 0$$

$$-8x + 4y = 4$$

$$y = -2x + 4$$

$$34x - 8(-2x + 4) = 10$$

$$50x - 32 = 10$$

$$x = \frac{42}{50} = \frac{21}{25}$$

$$y = -\frac{42}{25} + 4 = \frac{58}{25}$$

(b) (3 points) By analyzing the Hessian at the critical point, determine whether it is a local maximum, local minimum, or neither.

$$(Hf) = \begin{bmatrix} 34 & -8 \\ -8 & 4 \end{bmatrix}$$

$$\det \begin{bmatrix} 34 & -8 \\ -8 & 4 \end{bmatrix} = 136 - 64 > 0$$

$$tr \begin{bmatrix} 34 & -8 \\ -8 & 4 \end{bmatrix} = 34 + 4 > 0$$

Local minimum

(problem continued from facing page)

(c) (5 points) Sketch an approximate contour plot of f at the critical point on the coordinate plane provided below.

Sketch qualitatively correct level sets, including justification in terms of the eigenvalues and eigenvectors.

$\det((Hf) - \lambda I)$

$= \lambda^2 - 38\lambda + 72$

$= (\lambda - 36)(\lambda - 2)$

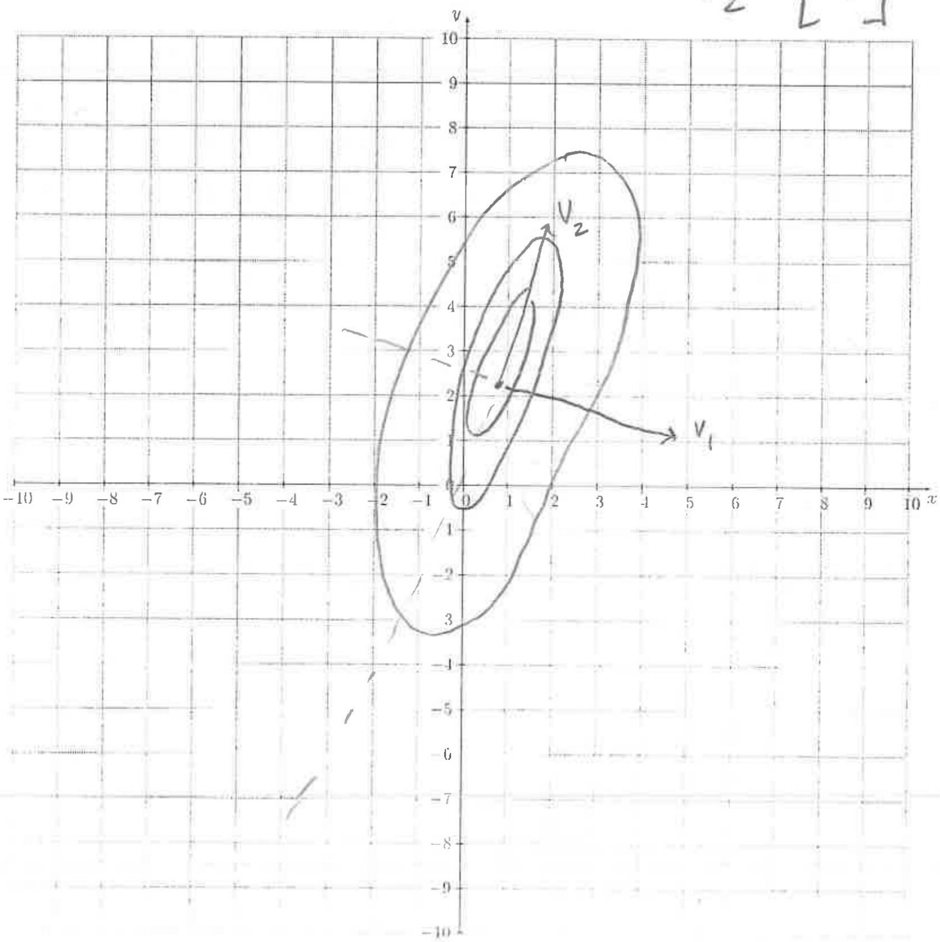
$\begin{bmatrix} -2 & -8 \\ -8 & -32 \end{bmatrix} v_1 = 0$

$v_1 = \begin{bmatrix} 4 \\ -1 \end{bmatrix}$

$\begin{bmatrix} 32 & -8 \\ -8 & 2 \end{bmatrix} v_2 = 0$

$v_2 = \begin{bmatrix} 1 \\ 4 \end{bmatrix}$

$36t_1^2 + 2t_2^2$



8. (10 points) For each of the following statements, circle either TRUE (meaning, "always true") or FALSE (meaning, "not always true"), and briefly and convincingly justify your answer. 1 point for the correct choice, and the rest for convincing justification.

(a) (5 points) Suppose A is an $m \times n$ matrix with the property $A^{\intercal}A = 0$. Then, A must be the zero matrix.

Circle one, and justify below: TRUE FALSE

The ii-th entry of A^TA is the dot product of the i-th row of A^T, which is the i-th column of A, with the i-th column of A. But the only vector orthogonal to itself is the zero vector, so each column of A must be the zero vector. Therefore, A is the zero matrix.

(b) (5 points) If a 2×2 matrix A has characteristic polynomial $P_A(\lambda) = \lambda^2 - \lambda - 2$, then the characteristic polynomial $P_B(\lambda)$ of $B = A^3 - 4A^2 + 7I_2$ must also be $P_B(\lambda) = \lambda^2 - \lambda - 2$.

Circle one, and justify below: ~~TRUE~~ FALSE

$$\begin{array}{r} \lambda - 3 \\ \lambda^2 - \lambda - 2 \,\overline{)\, \lambda^3 - 4\lambda^2 + 7} \\ -\lambda^3 + \lambda^2 + 2\lambda \\ \hline -3\lambda^2 + 2\lambda + 7 \\ +3\lambda^2 - 3\lambda - 2 \\ \hline -\lambda + 5 \end{array}$$

DO NOT DETACH THIS PAGE. If you use any of this space to continue your answer, please clearly indicate the problem number(s) and **indicate on the original page of the problem that your answer continues here**.