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MATH 311-001 Introduction to Mathematical Analysis I

Midterm Exam (40 points)

The midterm has five problems.
Read the statements carefully and answer all the questions.
You must justify your claims to receive full credit.
Write your solutions clearly and neatly. Make sure your handwriting is readable.
Do not write arrows to connect your steps, use words.
Calculators are not allowed.

1. Give a precise statement or definition of **one** of the following:
   (a) Definition of supremum of a set.
   (b) Completeness Axiom.
   (c) Definition of convergent sequence.

2. Using the definition of convergent sequence ($\epsilon - N$ argument), prove that

$$\lim_{n\to\infty} \frac{n^2-5n}{3n^2+2} = \frac{1}{3}$$

3. Prove **one** of the following:
   (a) Let $x \in \mathbb{R}$ and $E = \{q \in \mathbb{Q} : q < x\}$. Prove that $\sup E = x$.
   (b) Let A be a non-empty, bounded above subset of the real numbers. Let B be the set $B = \{11 + a : a \in A\}$. Prove that $\sup(B) = 11 + \sup(A)$.

4. Let $\{a_n\}$ be a sequence of real numbers that converges to a nonzero number a. Prove that there is a natural number N such that

$$|a_n| > \frac{|a|}{2} \quad \text{for all } n \geq N.$$

5. For each of the following statements determine if it is true or false. If it is true, provide a short justification. If it is false, provide a counterexample (example showing the statement is false).
   (a) If $x \leq y$ and $z \in \mathbb{R}$, then $|x+z| \leq |y+z|$.
   (b) $\lim_{n\to\infty} \frac{\sin(n)}{n} = 0$.
   (c) If $\{a_n\}$ converges to a, then $\{|a_n|\}$ converges to $|a|$.
   (d) A bounded sequence is convergent.

1. I'll state (b), the Completeness Axiom:

If a subset of the real numbers is bounded above, then it has a real supremum. and non-empty -

That is,

For any $A \subseteq \mathbb{R}$, if there exists $M \in \mathbb{R}$ such that $\forall a \in A, a \leq M$, then there exists $\overset{\sup A}{\underline{\inf A}} \in \mathbb{R}$ such that all such M are greater than or equal to $\underset{\sup A.}{\inf A}$.

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2. Let $\varepsilon > 0$ be given.

By the Archimedean Property of the reals, there exists some $N \in \mathbb{N}$ such that $N > \max\{\frac{16}{\varepsilon}, 2\}$. ✓

Then, for any $n \in \mathbb{N}$ such that $n > N$, we have

$$\frac{16}{\varepsilon} < n,$$

so

$$\frac{16}{n} < \varepsilon$$

and

$$\frac{16n}{n^2} < \varepsilon.$$

Because n^2 is positive,

$$\frac{16n}{9n^2} < \varepsilon;$$

and $2 < n$, so

$$\frac{15n+2}{9n^2} < \varepsilon.$$

Also because it's positive, we can add 6 to the denominator to get $\frac{15n+2}{9n^2+6} < \varepsilon$.

This is positive, implying

$$\left|\frac{15n+2}{9n^2+6}\right| < \varepsilon,$$

and therefore

$$\left|\frac{-15n-2}{9n^2+6}\right| < \varepsilon.$$

Hence,

$$\left|\frac{3n^2-15n-3n^2-2}{3(3n^2+2)}\right| < \varepsilon.$$

And finally,

$$\left|\frac{n^2-5n}{3n^2+2} - \frac{1}{3}\right| < \varepsilon.$$ ✓

So, by definition

$$\lim_{n\to\infty} \frac{n^2-5n}{3n^2+2} = \frac{1}{3}.$$

$\square$

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3. I'll prove (a).

Any element of E is less than x, so x is an upper bound of E. ✓

Now suppose by contradiction that there exists some lesser upper bound of E called x'. Then $x' < x$ and by the density of the rationals there exists some $z \in \mathbb{Q}$ such that

$$x' < z < x. \checkmark$$

Note that z is a rational number less than x, so $z \in E$. ✓ Then, because x' is an upper bound of E, we have $x' \geq z$, which is a contradiction. ✓

Therefore, any upper bound of E is greater than or equal to the upper bound x. That is,

$$\sup E = x. \checkmark$$

$\square$

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4. The absolute value is always positive, so $\frac{|a|}{2} > 0$.

We can therefore apply the definition of convergence to find $N \in \mathbb{N}$ such that for all $n \geq N$,

$$|a_n - a| < \frac{|a|}{2}.$$

By the corrollary of the triangle inequality,

$$||a_n| - |a|| < \frac{|a|}{2}.$$

Or, equivalently,

$$-\frac{|a|}{2} < |a_n| - |a| < \frac{|a|}{2}.$$

Hence, $$|a| - \frac{|a|}{2} < |a_n| < \frac{|a|}{2} + |a|.$$

Therefore, $$\frac{|a|}{2} < |a_n| \text{ for all } n \geq N. \quad \square$$

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5. (a) False. Take $x=-1$ and $y=z=0$. Then,

$$|x+z|=|-1+0|=1\nleq 0=|0+0|=|y+z|$$

despite $x=-1\leq 0=y$.

(b) True. Consider that $-\frac{1}{n}\leq\frac{\sin(n)}{n}\leq\frac{1}{n}$, and

$$\lim_{n\to\infty}\left(-\frac{1}{n}\right)=\lim_{n\to\infty}\frac{1}{n}=0. \checkmark$$

Therefore $\lim_{n\to\infty}\frac{\sin(n)}{n}=0$ by the squeeze theorem. ✓

(c) True, and the first few sentences of my solution to problem 4 are very nearly the proof. It follows from the reverse triangle inequality.

$$||a_n|-|a||<|a_n-a|<\varepsilon.$$

(d) False. Consider the sequence $\{(-1)^n\}$, which is bounded above by 1 and below by -1 but does not converge.

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