

## Chapter 3: Radicals and Exponents

# Introduction

In the first chapter, we dealt with functions that could be sketched as a straight line. We observed how they intersected with each other, and how we can use algebra to describe those intersections.

In the second chapter, we learned about polynomial functions, namely quadratics, and how they can be graphed. We spent a great deal of time finding roots of a quadratic, and even learned about imaginary numbers.

In this chapter, we will learn about the inverse of a quadratic – a radical. We will learn how to play with these kinds of numbers, what *kinds* of numbers they can be, and how to express them in various ways.

Try this warm up to see if you can get all of these facts right.

a)  $3^2 =$

b)  $(-3)^2 =$

c) If  $x^2 = 9$ , then  $x =$

d)  $\sqrt{9} =$

e)  $\sqrt{x^2} =$

f)  $(\sqrt{x})^2 =$



# 1. Relations, Functions, One-to-one

We have seen several types of function in this class so far. The first type, and most straightforward was the linear function. Then we realized that if we multiplied two linear functions together, we got a quadratic function. There are also higher-degree polynomial functions that we briefly explored in the last chapter.

It turns out that functions are the life-blood of a lot of mathematics. Most of mathematics can be described using the language of functions.

But not all equations can be described as a function. It is probably a good idea to remind ourselves what exactly a function is!

## Definition 1: Function

A function is a rule that takes as input one value and outputs another value, related to the input. If  $x$  is the input of function  $f$ , and  $y$  is the output of the function, then we write

$$y = f(x)$$

It is critical to understand that for every single input, there is a single output. A function fails to be a function if you were to input a single value and get back *two* or more values.

Consider the following functions.

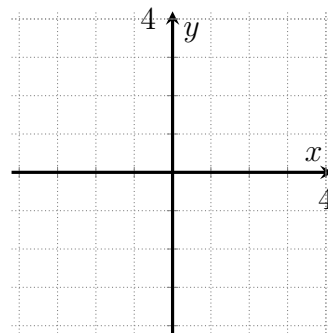
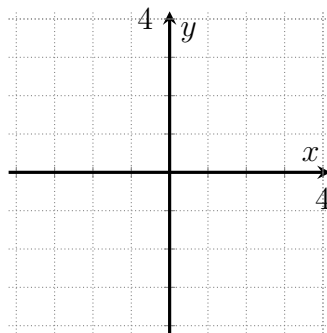
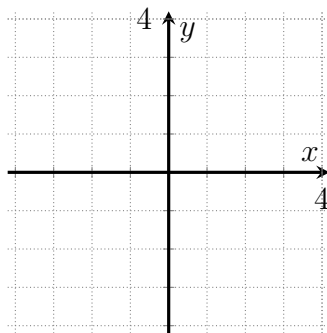
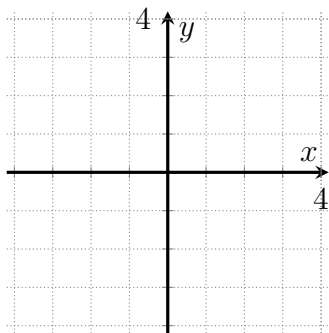
a)  $f(x) = x^2$

b)  $f(x) = (x-2)(x+1)^2$

c)  $y = x - 4$

d)  $g(x) = \sqrt{x}$

Use Desmos to help you graph the above functions below.



Some equations in math, however, are not best described as a function. Sometimes if I give you one input, you may want two outputs.

The following are not functions:

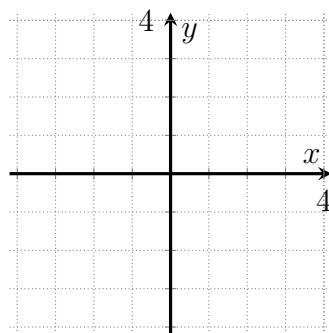
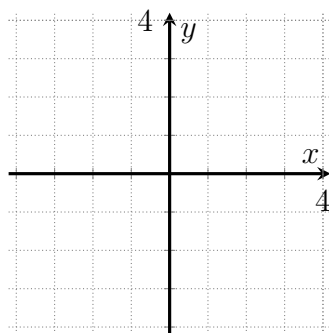
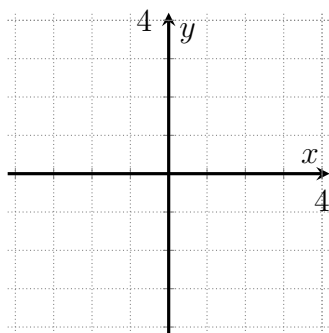
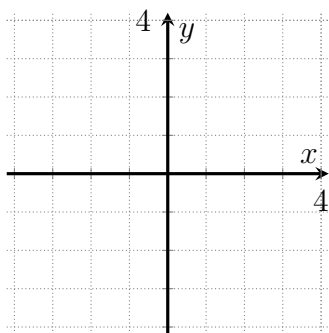
a)  $y^2 = x$

b)  $x^2 + y^2 = 1$

c)  $x^3 + y^2 = xy$

d)  $y = \pm x$

Use Desmos to help you graph these functions. Try to see if you can visualize *exactly* what makes these equations non-functions.

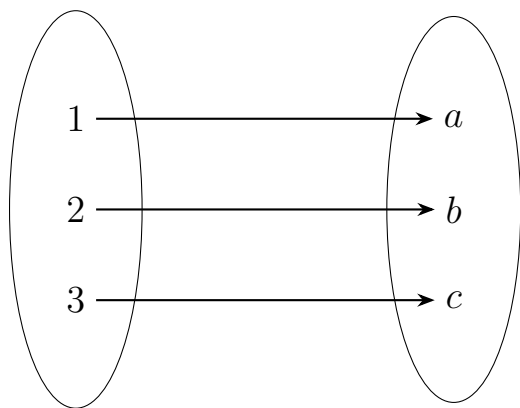


### Definition 2: Relation

An equation that has two or more variables is called a relation. Most relations are between two variables,  $x$  and  $y$ . If the relation  $R$  has a solution  $x = a$  and  $y = b$ , then the relation can be graphed by plotting  $(a, b)$ , and all other solutions to the equation. Furthermore, we can write  $aRb$  and say  $a$  is related to  $b$ .

So all of the above equations are relations because they represent a relationship between  $x$  and  $y$ . When Desmos graphs these, they are finding all possible solutions to the equation, and plotting each point, one by one.

We can also describe a relation by means of a mapping diagram.



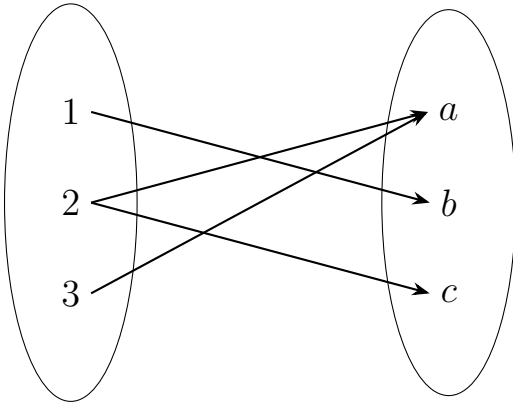
The above mapping diagram indicates that we have two sets,  $\{a, b, c\}$ , and  $\{1, 2, 3\}$ . The first set is referred to as our domain, and the second set is called the range.

### Definition 3: Domain and Range

The domain of a relation is the collection of objects that go *into* the relation, and the range is the collection of objects that come *out of* the relation.

The mapping diagram indicates that  $a$  is related to 1, (or  $a$  is 'mapped' to 1),  $b$  is related to 2, and  $c$  is related to 3. We could also write  $aR1$ ,  $bR2$ , and  $cR3$ . We can also say that the following ordered pairs describe the relation:  $(a, 1)$ ,  $(b, 2)$ , and  $(c, 3)$ .

Describe the following relation in all of the ways you know how:



Draw the mapping diagram of the relation described by the following ordered pairs:

$$(a, 4), (b, 3), (c, 0), (a, 3), (8, \alpha)$$

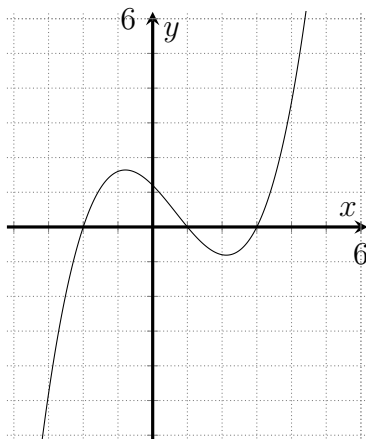
There is a quick way of determining when a graph represents a function and when it doesn't.

#### Definition 4: Vertical Line Test

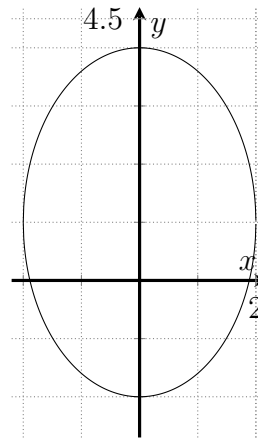
Given the graph of a relation, if there exists a vertical line that overlaps the curve of the graph more than once, then the relation is not a function. If no such vertical line exists, then the relation is a function.

Use the Vertical Line Test to determine which of these graphs represents a relation that is a function, and which is a relation that is *not* a function.

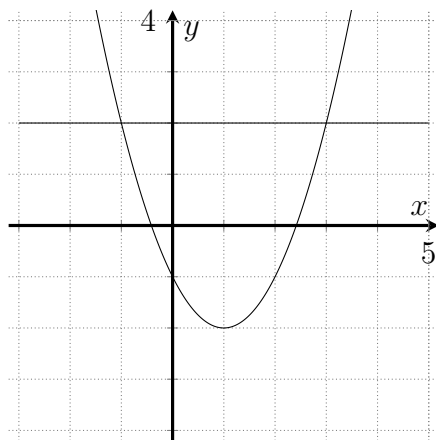
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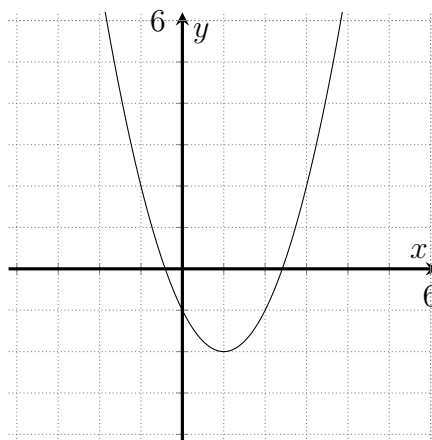
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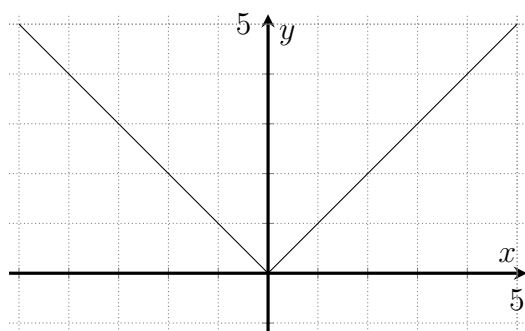
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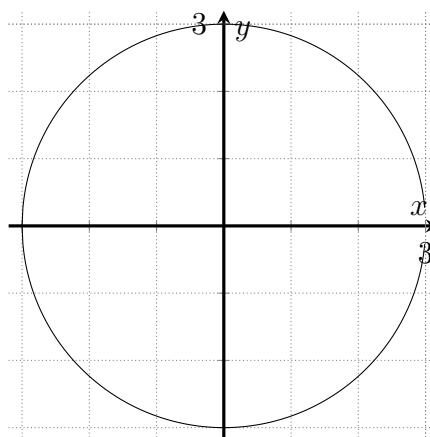
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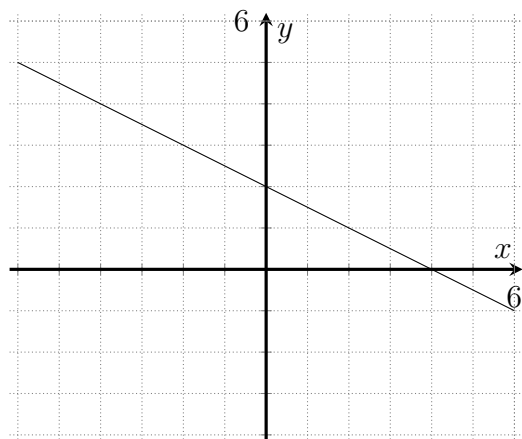
e)



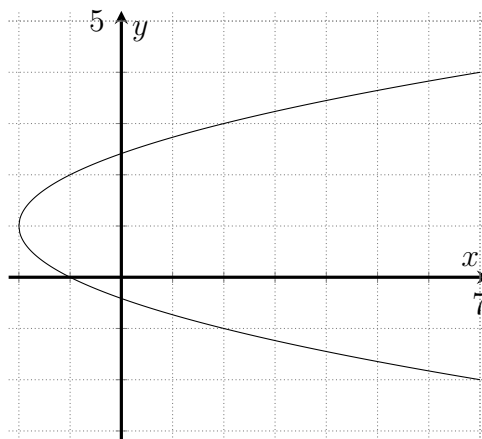
f)



g)



h)



As you may be able to sense, a function is a special type of relation. But is there something even more special? Are there types of functions that are even more exclusive? It turns out: yes. These functions are called one-to-one functions.

Recall that the feature that a relation requires is that for every one input there is exactly one output. For a function to be one to one, we require the following.

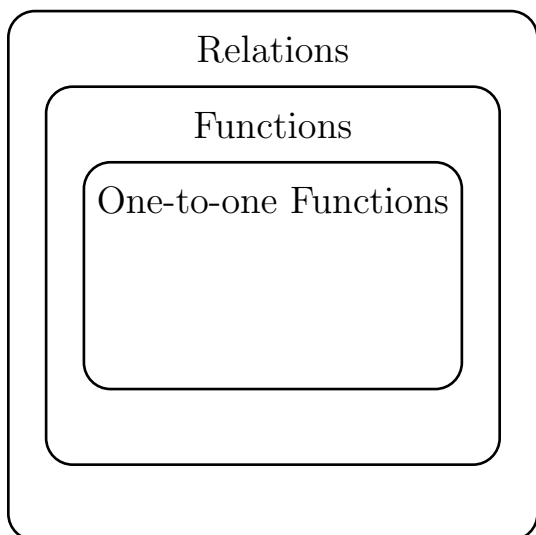
### Definition 5: One-to-one Function

A function is said to be one-to-one if for every single *output*, there is exactly one *input*. Symbolically, we can say:

$$f(a) = f(b) \text{ implies } a = b.$$

In other words, the only way that two outputs can be the same ( $f(a)$  and  $f(b)$ ), is if they are not two outputs at all – that they are indeed from the same input ( $a$  and  $b$ ).

It is useful to remember that one-to-one functions are *all* functions, and all functions are relations. But not all relations are function, and not all functions are one-to-one.



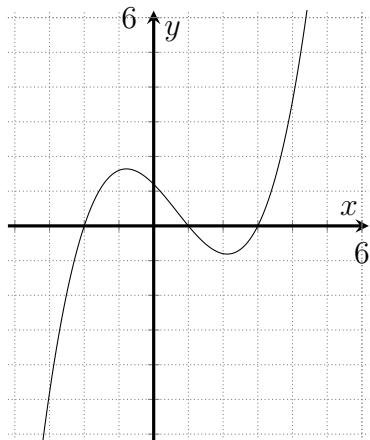
### Definition 6: Horizontal Line Test

Given the graph of a function, if there exists a horizontal line that overlaps the curve of the graph more than once, then the function is not one-to-one. If no such horizontal line exists, then the function is one-to-one.

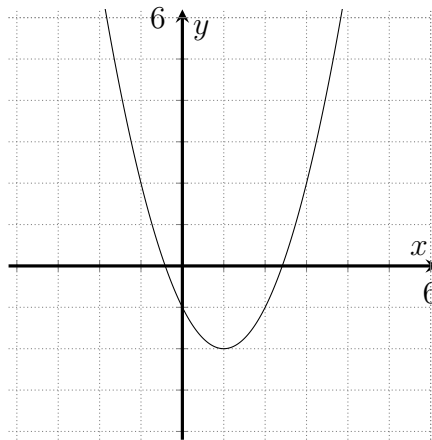
Explain why the horizontal line test works.

Use the horizontal line test to determine which of the following functions are one-to-one.

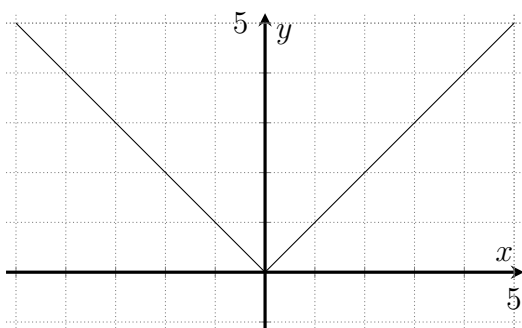
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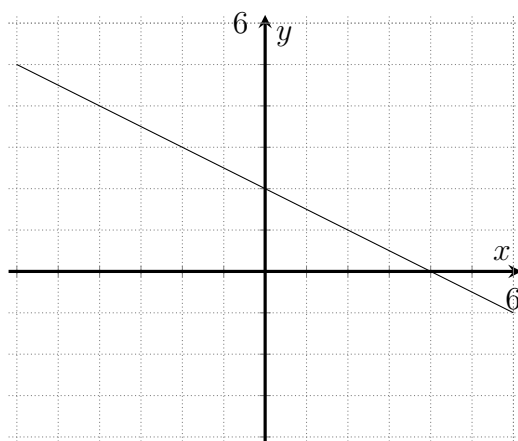
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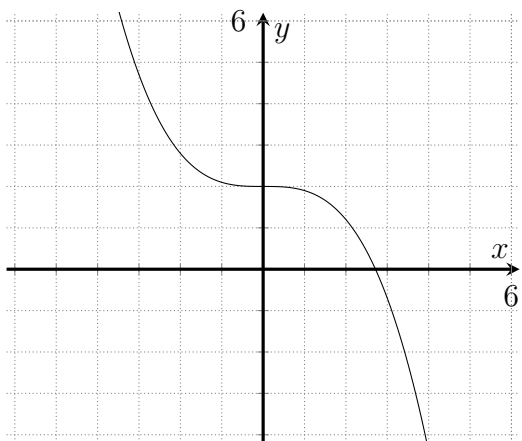
c)



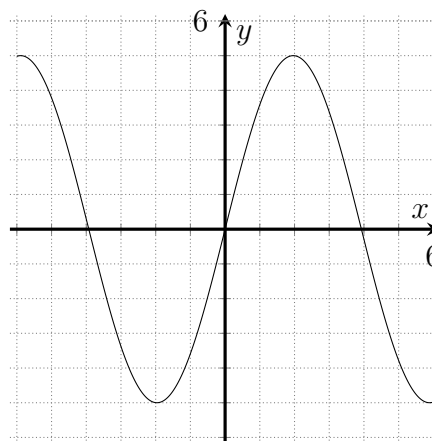
d)



e)



f)



## Exercises

**1.1.** In your own words, explain the following terms.

- a) domain      b) range      c) relation      d) function      e) one-to-one

You may want to include definitions for these items as well as for range and function.

**1.2.** What is the rule that you use to determine whether a relation (mapped or in a table) represents a function?

**1.3.** In your own words, explain what the vertical line test is used for and how it works.

**1.4.** Determine whether the following relations are functions. Justify your answer.

- a)  $A = \{(1, 2), (1, 7), (1, 8), (1, 9), (1, 10)\}$       b)  $y = -x + 3$

**1.5.** The physical education teacher calls out the first name of 6 students in a class (Aiden, Brady, Chad, Chad, Devon, Everett) and numbers them 1 through 6. If this information were written as ordered pairs (number, name), would this represent a function or a non-function? Explain.

**1.6.** State the domain and range of the following relation.

$$A = \{(1, 2)(3, 4)(5, 6)(7, 8)\}$$

**1.7.** The provincial sales tax in Manitoba is 7% of the list price. Write a relation to illustrate the relationship between the list price and the cost of the item to the consumer. State the domain and range of this relation.

**1.8.** Given the following ordered pairs, state the domain and range of the relation.

$$\{(1, 4)(2, 6)(3, 8)(4, 10)\}$$

**1.9.** Determine if the following ordered pairs represent functions or relations.

a)  $\{(0, 1), (1, 2), (2, 3), (3, 4), (4, 5)\}$

b)  $\{(2, -3), (2, 5), (2, 0), (2, 1), (2, -2)\}$

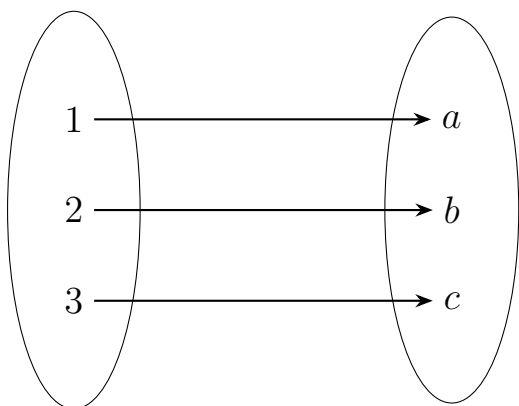
c)  $\{(-4, -1), (3, -1), (0, -1), (-1, -1), (5, -1)\}$

d)  $\{(5, 3), (4, 3), (-3, 3), (0, 5), (-3, -1)\}$

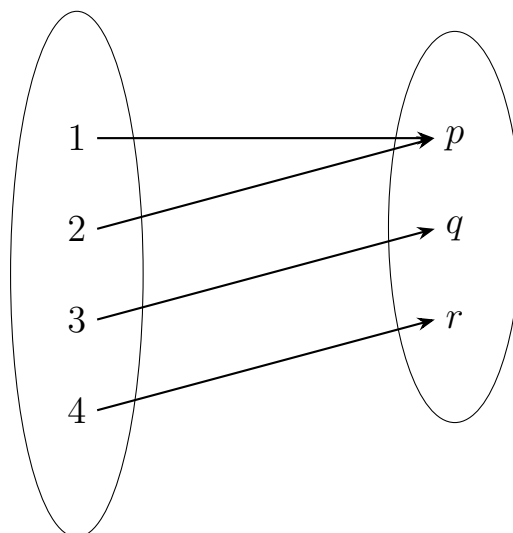
e)  $\{(-2, 4), (-1, 1), (0, 0), (1, 1), (2, 4)\}$

**1.10.** Decide whether the relations shown are one-to-one. Justify.

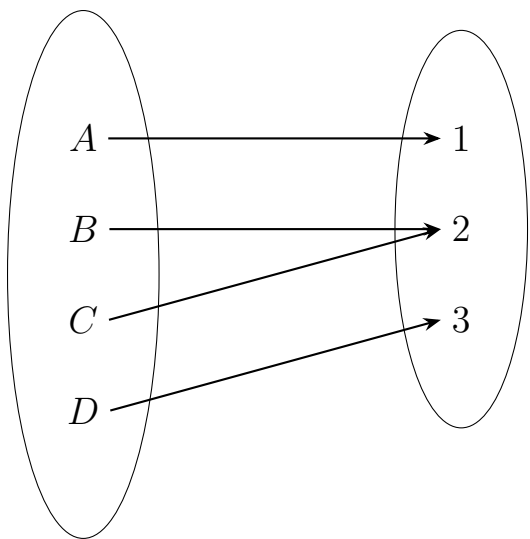
a)



b)



**1.11.** The diagram below is not one-to-one. What is the *fewest* number of domain elements you must remove to make it one-to-one? Give one possible choice.



**1.12.** Let  $f(x) = 2x + 3$ . Is  $f$  one-to-one on  $\mathbb{R}$ ? Show work.

**1.13.** Consider  $g(x) = x^2 - 4$  on the domain  $\mathbb{R}$ . Is  $g$  one-to-one? If not, give a domain restriction that makes it one-to-one and state the.

**1.14.** Is  $h(x) = |x - 2|$  one-to-one on  $\mathbb{R}$ ? If not, give a largest interval on which it is one-to-one.

**1.15.** The function  $p(x) = \frac{3x - 5}{2}$ . Explain why  $p$  is one-to-one

**1.16.** For which values of  $a$  is  $f(x) = ax + b$  one-to-one on  $\mathbb{R}$ ? Justify.

**1.17.** Determine whether the piecewise function

$$f(x) = \begin{cases} x + 1, & x < 0, \\ 1 - x, & x \geq 0 \end{cases}$$

is one-to-one on  $\mathbb{R}$ .

**1.18.** A table of values is given:

$x$	$-2$	$-1$	$0$	$1$	$2$
$f(x)$	$3$	$5$	$3$	$7$	$9$

Is  $f$  one-to-one? If not, remove the *fewest* entries to make it one-to-one and rewrite the table.



## 2. Inverse Functions

A common theme in mathematics is learning how to do something backwards. When you first learned how to add, it wasn't long before you learned how to undo adding: you learned how to subtract. After you learned how to multiply, you learned how to divide. When you solve an equation, you repeatedly undo the operations done to your variable. When you learn about the derivative in calculus, you will also learn about the antiderivative, which undoes the derivative operation.

It may be of interest to us to find a way to undo a function. This is called the inverse.

### Definition 7: Inverse Function

The inverse of a function  $f(x)$ , if it exists, is the unique function  $f^{-1}(x)$  such that

$$f(f^{-1}(x)) = f^{-1}(f(x)) = x.$$

In other words, the inverse of  $f$  undoes what  $f$  does to  $x$

For example, the inverse of  $f(x) = \frac{1}{2}x - 2$  is  $f^{-1}(x) = 2x + 4$ . Why is that?

### How to find the inverse of a function

To find the inverse of a function, provided it exists (more on that later), we take the following steps:

1. write the function as  $y$  in terms of  $x$
2. replace  $x$  and  $y$
3. solve for  $y$
4. profit.

Find the inverse of  $f(x) = 3x - 12$  and verify that it is indeed the inverse.

## Which functions have inverses?

Not all functions have inverses. Take  $f(x) = x^2$  for example. Since  $f(2) = 4$  and  $f(-2) = 4$ , if our inverse  $f^{-1}$  *did* exist, what could  $f^{-1}(4)$  be equal to? It would have to undo both of the mappings that map to 4.

### Theorem 1: Only one-to-one functions have inverses

If a function  $f(x)$  is one-to-one, then  $f^{-1}(x)$  exists. When this happens, the domain and range of  $f$  are the range and domain of  $f^{-1}$ , respectively. In other words, the domain and range swap.

Why do you think we need  $f$  to be one-to-one in order to find its inverse?

Find the inverse of the following functions:

a)  $f(x) = 2x + 3$

b)  $g(x) = -x + 1$

c)  $h(x) = \frac{1}{2}x - 4$

d)  $p(x) = -3x - 2$

e)  $q(x) = 4x$

f)  $r(x) = -\frac{2}{3}x + 5$

## Exercises

**2.1.** Algebraically determine the equation of the inverse of each function.

a)  $f(x) = 7x$

b)  $f(x) = -3x + 4$

c)  $f(x) = \frac{x+4}{3}$

d)  $f(x) = \frac{x}{3} - 5$

e)  $f(x) = 5 - 2x$

f)  $f(x) = \frac{1}{2}(x + 6)$

**2.2.** Assume that  $f$  is a one-to-one function.

a) If  $f(2) = 7$ , find  $f^{-1}(7)$ .

b) If  $f^{-1}(3) = -1$ , find  $f(-1)$ .

c) If  $f(5) = 18$ , find  $f^{-1}(18)$ .

d) If  $f^{-1}(4) = 2$ , find  $f(2)$ .

e) If  $f(x) = 5 - 2x$ , find  $f^{-1}(3)$ .

f) If  $g(x) = x^2 + 4x$  with  $x = -2$ , find  $g^{-1}(5)$ .

**2.3.** Find the inverse function of  $f$ .

a)  $f(x) = 3x + 5$

b)  $f(x) = 7 - 5x$

c)  $f(x) = 5 - 4x^3$

d)  $f(x) = 3x^3 + 8$

e)  $f(x) = \frac{1}{x+2}$

f)  $f(x) = \frac{x-2}{x+2}$

g)  $f(x) = \frac{x}{x+4}$

h)  $f(x) = \frac{3x}{x-2}$

i)  $f(x) = \frac{2x+5}{x-7}$

$$\text{j)} \quad f(x) = \frac{4x-2}{3x+1}$$

$$\text{k)} \quad f(x) = \frac{2x+3}{1-5x}$$

$$\text{l)} \quad f(x) = \frac{3-4x}{8x-1}$$

$$\text{m)} \quad f(x) = 4 - x^2, \quad x \geq 0$$

$$\text{n)} \quad f(x) = x^2 + x, \quad x \geq -\frac{1}{2}$$

$$\text{o)} \quad f(x) = x^6, \quad x \geq 0$$

$$\text{p)} \quad f(x) = \frac{1}{x^2}, x > 0$$

$$\text{q)} \quad f(x) = \frac{2-x^3}{5}$$

$$\text{r)} \quad f(x) = (x^5 - 6)^7$$

$$\text{s)} \quad f(x) = \sqrt{5+8x}$$

$$\text{t)} \quad f(x) = 2 + \sqrt{3+x}$$

$$\text{u)} \quad f(x) = 2 + \sqrt[3]{x}$$

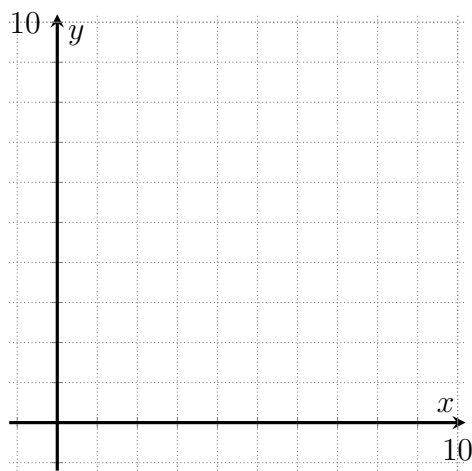
### 3. The Radical Function

The last section was a short detour to wind up talking about the radical function. In order to fully understand the radical function, we must be familiar with the following ideas:

- domain
- range
- one-to-one
- inverse

Since the function  $f(x) = x^2$  is not one-to-one, we cannot take its inverse. But what would happen if we were to *restrict* its domain to the positive real numbers only?

Graph  $y = x^2$  over the domain  $x \geq 0$ .



If we define this function, which is only defined for values  $x \geq 0$  as  $g(x)$ , then we have the very convenient fact that  $g$  is one-to-one.

How do we know this?

### Definition 8: The radical function

Let  $g(x) = x^2$  over the restricted domain  $x \geq 0$ , making  $g$  one-to-one. The function  $g^{-1}(x)$  is called the *square root* function and is in the family of radical functions. We usually write:

$$g^{-1}(x) = \sqrt{x}$$

Wait... it's taken us three chapters to define  $\sqrt{x}$ ? Yes.

Now let's revisit the blurb from the first page of this book:

a)  $3^2 =$

b)  $(-3)^2 =$

c) If  $x^2 = 9$ , then  $x =$

d)  $\sqrt{9} =$

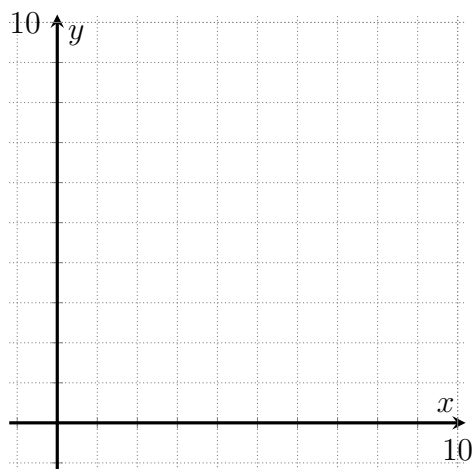
e)  $\sqrt{x^2} =$

f)  $(\sqrt{x})^2 =$

All of this is to say that if  $y = \sqrt{x}$  then  $y^2 = x$ , but just because  $y^2 = x$  does not mean that  $y = \sqrt{x}$ .

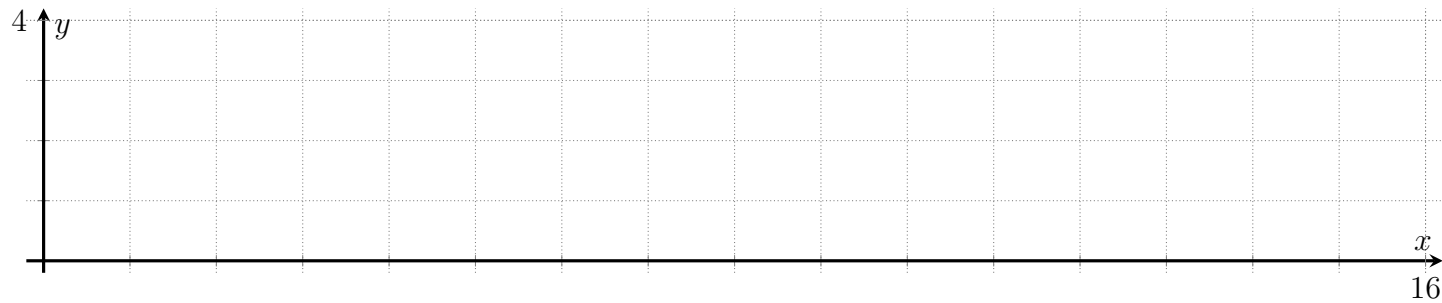
Explain the above sentence in your own words, and give an example of  $x$  and  $y$  that satisfy that sentence.

Graph the function  $y = \sqrt{x}$



## Exercises

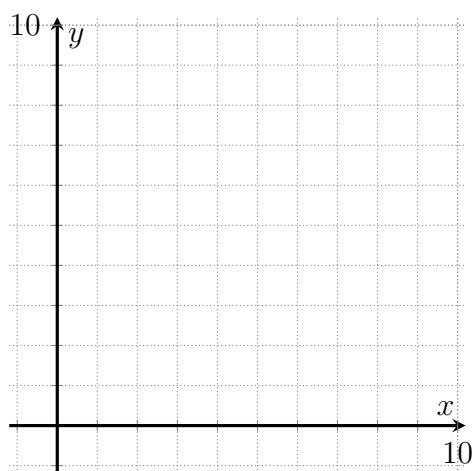
3.1. Graph the radical function  $y = \sqrt{x}$  and use it to approximate  $\sqrt{2}$ ,  $\sqrt{3}$ ,  $\dots$ , and  $\sqrt{15}$ .





## 4. Number Systems

Use Desmos to help you graph  $y = \sqrt{x}$  as perfectly as possible. It may help to remember that the square root function is the inverse of  $y = x^2$  over the restricted domain of  $x \geq 0$ , so its shape should be very similar.



When we graph the function  $y = \sqrt{x}$  we see the following point on the graph:

- $(0, 0)$  since  $0^2 = 0$
- $(1, 1)$  since  $1^2 = 1$
- $(4, 2)$  since  $2^2 = 4$
- $(9, 3)$  since  $3^2 = 9$
- and so on

But there are *so many more* points on this function. In fact, between any two different points on the graph, there are *infinitely many* points in between them. What do some of those points look like? Try seeing what happens to  $y$  when  $x = 7$ .

Are there any integers whose square root is not an integer?

### Definition 9: Rational numbers

A rational number is a number that can be written as the ratio of two integers  $a$  and  $b$ , provided that  $b \neq 0$ . The set of all rational numbers is denoted  $\mathbb{Q}$ , because  $Q$  stands for quotient. In symbols,

$$\mathbb{Q} = \left\{ \frac{a}{b} : a, b \in \mathbb{Z} \right\}$$

It turns out that the square root of an integer can be a very strange kind of number.

**Theorem 2: The irrationality of  $\sqrt{2}$**

The square root of 2 is not a rational number.

Why is this?

It turns out that there are all kinds of different numbers out there. To recap, we have the following sets of numbers:

- The Natural Numbers  $\mathbb{N} = \{0, 1, 2, 3, 4, \dots\}$ .

These are sometimes called the counting numbers. Some definitions of the natural numbers do not include zero (but they're wrong!). In this class we will use 0 as the least element. Regardless of the definition, the Natural Numbers have a smallest element, but no greatest element.

- The Integers  $\mathbb{Z} = \{\dots - 3, -2, -1, 0, 1, 2, 3, \dots\}$ .

The integers have no least element and no greatest element. It includes positive numbers, negative numbers, and zero. There are no decimal numbers (unless you include decimals that end in .0) in  $\mathbb{Z}$ .

- The Rational Numbers  $\mathbb{Q} = \{\frac{a}{b} : a, b \in \mathbb{Z}, b \neq 0\}$ .

A rational number is a value that can be expressed as the ratio of two integers, so long as we do not divide by 0. These values can be represented by fractions or decimals. When a rational number is in decimal form, it is either a repeating decimal or a terminating decimal. Conversely, all terminating decimals and repeating decimals are rational.

- Irrational Numbers

These are numbers that cannot be written as the ratio of two integers. This set can be harder to describe because irrational numbers are defined by a property that they don't have. Irrational numbers in decimal form do not terminate and do not repeat. Examples include  $\sqrt{2}$ ,  $\pi$ ,  $e$ ,  $0.10100100010000\dots$

- The Real Numbers  $\mathbb{R}$

The real numbers are all of the rational numbers and irrational numbers combined. Virtually every number you will see in high school (aside from a few novel situations) is a real number.

- The Complex Numbers  $\mathbb{C} = \{a + bi : a, b \in \mathbb{R}\}$

The complex numbers arose from the solution to an age old problem of factoring polynomials. In short, the complex numbers have a "real" part and an "imaginary" part. The imaginary part comes from the value  $i = \sqrt{-1}$ . There may be a conceptual hurdle to accepting and understanding the complex numbers, but ultimately their power is realized when you try to solve  $x^2 = -1$ .

You may notice that there isn't a "mathbb" capital letter assigned to the irrational numbers. This is because the irrational numbers are not a number system. All of the other number systems have the important property that if you add or multiply any two numbers in the system, you stay in the system. For example, the sum of two rational numbers is rational, and the product of two integers is an integer. This property is not true for irrational numbers. For example,  $\sqrt{2} \times \sqrt{2} = 2$ , and 2 is not irrational.

When all of the contents of a set  $A$  is contained in another set  $B$ , we say that  $A$  is a subset of  $B$  and write  $A \subset B$ . In the case of number systems,

$$\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R} \subset \mathbb{C}.$$

## Exercises

**4.1.** Prove that  $\sqrt{3}$  is irrational.

**4.2.** Is the sum of two irrational numbers also irrational? Why or why not?

**4.3.** Is the product of two irrational numbers also irrational? Why or why not?

**4.4.** Is an irrational taken to the power of another irrational also irrational? Why or why not?

**4.5.** Is  $\sqrt{0.16}$  rational or irrational? What about  $\sqrt{0.016}$ ? Or  $\sqrt{0.0016}$ ? Is there a pattern?

**4.6.** The roots of some quadratic functions are integers, some are rational, some are irrational, and some are complex. Make a rule based on the determinant that can identify exactly when a quadratic has each type of root.



**4.7.** Give an example of a quadratic function that has

- a) integer roots      b) rational roots      c) irrational roots      d) complex roots

4.8. Convert each number into a fraction.

- a)  $0.\overline{1}$
- b)  $0.\overline{2}$

- c)  $0.\overline{3}$
- d)  $0.\overline{4}$

- e)  $0.\overline{11}$
- f)  $0.\overline{253}$

- g)  $5.\overline{24}$
- h)  $2.01\overline{354}$

**4.9.** Explain why  $0.\overline{9} = 1$ .



## 5. Manipulating Radicals

There is a very straightforward rule that governs how we can change and alter radical expressions.

### Theorem 3: The product of radicals

For any two positive real numbers  $a$  and  $b$ ,

$$\sqrt{a} \times \sqrt{b} = \sqrt{a \times b}$$

Why is this true?

Beware of the following mistake, which is a variant of the *freshman's dream*.

$$\sqrt{a} + \sqrt{b} \neq \sqrt{a + b}$$

Whis is this not true?

## Simplifying radicals

The rule we just learned can be useful when it comes to simplifying radicals. Sometimes we have a square root that we would like to simplify, but it is not the square root of a perfect square.

Simplify  $\sqrt{12}$ .

## Exercises

5.1. Simplify the following.

a)  $\sqrt{20}$

b)  $\sqrt{72}$

c)  $\sqrt{45}$

d)  $\sqrt{24}$

e)  $\sqrt{75}$

f)  $\sqrt{125}$

g)  $\sqrt{140}$

h)  $\sqrt{128}$

i)  $\sqrt[3]{40}$

j)  $\sqrt[3]{48}$

k)  $\sqrt[3]{54}$

l)  $\sqrt[3]{135}$

m)  $\sqrt[3]{128}$

n)  $\sqrt[3]{192}$

o)  $2\sqrt[3]{27}$

p)  $-3\sqrt[3]{16}$

**5.2.** Multiply, and simplify if possible.

a)  $\sqrt[3]{4} \times \sqrt[3]{6}$

b)  $\sqrt[3]{9} \times \sqrt[3]{24}$

c)  $\sqrt[3]{5} \times \sqrt[3]{5}$

d)  $\sqrt[3]{4} \times \sqrt[3]{54}$

e)  $2\sqrt[3]{12} \times \sqrt[3]{30}$

f)  $-3\sqrt[3]{25} \times 4\sqrt[3]{75}$

g)  $2\sqrt[3]{10} \times 3\sqrt[3]{50}$

h)  $(-3\sqrt[3]{12})(-2\sqrt[3]{18})$

i)  $(-3\sqrt[3]{4})(-2\sqrt[3]{32})$

j)  $(-5\sqrt[3]{49})(2\sqrt[3]{56})$

**5.3.** A square has an area of  $150 \text{ mm}^2$ . What are the lengths of the sides of the square?

**5.4.** A cube has a volume of  $192 \text{ cm}^3$ . What are the lengths of each edge of the cube?

**5.5.** The dimensions of a rectangle are  $9\sqrt{30} \text{ cm}$  by  $4\sqrt{105} \text{ cm}$ . Calculate the area of the rectangle.

**5.6.** The dimensions of a rectangle are  $5\sqrt{6} \text{ cm}$  by  $4\sqrt{3} \text{ cm}$ . Calculate the area of the rectangle.



## 6. Exponential Notation

Consider the following example:  $a^2 \times a^3 = (a \times a) \times (a \times a \times a) = a^5$

The exponent in the expression  $a^5$  is the sum of the exponents in the expression  $a^2 \times a^3$ .

Therefore:  $a^2 \times a^3 = a^{2+3} = a^5$ .

### Theorem 4: The Product Rule

For any numbers  $a$  and  $b$  with exponents  $m$  and  $n$  :

$$a^m \times a^n = a^{m+n}, \quad a \neq 0$$

Try it:  $(-3)^4 \times (-3)^5$

### Theorem 5: The Quotient Rule

For any number  $a$  with exponents  $m$  and  $n$  :

$$\frac{a^m}{a^n} = a^{m-n}, \quad a \neq 0$$

Try it:  $\frac{(-4)^8}{(-4)^3}$

### Theorem 6: The Product Rule

For any numbers  $a$  and  $b$  with exponents  $m$  and  $n$  :

$$(a^m)^n = a^{m \times n}$$

Try it:  $(3^5)^4$

### Theorem 7: Distributive Rule for exponents, part 1

Exponents distribute over multiplication. For any numbers  $a$  and  $b$  with exponent  $n$  :

$$(ab)^n = a^n \times b^n$$

Try it:  $(3x)^3$

### Theorem 8: Distributive Rule for exponents, part 2

Exponents distribute over division. For any numbers  $a$  and  $b, b \neq 0$ , with exponent  $n$  :

$$\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$$

Try it:  $\left(\frac{2}{x}\right)^3$

### Theorem 9: Negative Exponents, part 1

For any number  $a, a \neq 0$ , with exponent  $n$  :

$$a^{-n} = \frac{1}{a^n}$$

Try it:  $2^{-3}$

### Theorem 10: Negative Exponents, part 2

or any non-zero numbers  $a$  and  $b$ , with exponents  $m$  and  $n$ ,

$$\frac{a^{-m}}{b^{-n}} = \frac{b^n}{a^m} \text{ and } \left(\frac{a}{b}\right)^{-m} = \left(\frac{b}{a}\right)^m$$

Try it:  $\left(\frac{2}{3}\right)^{-3}$

## Rational Exponents: $a^{\frac{1}{n}}$

Consider the square root example:  $\sqrt{2} \times \sqrt{2} = 2$ .

Now consider the exponent rule example:  $2^{\frac{1}{2}} \times 2^{\frac{1}{2}} = 2^{\frac{1}{2} + \frac{1}{2}} = 2^1 = 2$ .

Since  $\sqrt{2} \times \sqrt{2}$  and  $2^{\frac{1}{2}} \times 2^{\frac{1}{2}}$  equal 2,  $\sqrt{2}$  should equal  $2^{\frac{1}{2}}$ .

### Theorem 11: Rational Exponents, part 1

For any non-negative real number  $a$  and any positive integer  $n$ .

$$a^{\frac{1}{n}} = \sqrt[n]{a}$$

Try it:  $x^{\frac{1}{2}}$

### Theorem 12: Rational Exponents, part 2

For any non-negative real number  $a$  and any positive integer  $n$ ,

$$a^{\frac{m}{n}} = \sqrt[n]{a^m}$$

Try it:  $4^{\frac{3}{2}}$

## Summary of Exponent Rules

For any integers $m$ and $n$ :		
Exponent of 1	$a^1 = a$	$3^1 = 3$
Exponent of 0	$a^0 = 1, \quad a \neq 0$	$(-5)^0 = 1$
Product Rule	$a^m \times a^n = a^{m+n}, \quad a \neq 0$	$2^3 \times 2^4 = 2^{3+4} = 2^7$
Quotient Rule	$\frac{a^m}{a^n} = a^{m-n}, \quad a \neq 0$	$\frac{3^5}{3^3} = 3^{5-3} = 3^2$
Power Rules	$(a^m)^n = a^{m \times n}$ $(ab)^n = a^n \times b^n$ $\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$	$(2^3)^4 = 2^{3 \times 4} = 2^{12}$ $(2x)^3 = 2^3 \times x^3$ $\left(\frac{2}{3}\right)^4 = \frac{2^4}{3^4}$
Negative Exponents	$a^{-n} = \frac{1}{a^n}$ $\left(\frac{a}{b}\right)^{-n} = \left(\frac{b}{a}\right)^n$ $\frac{a^{-m}}{b^{-n}} = \frac{b^n}{a^m}$	$2^{-3} = \frac{1}{2^3}$ $\left(\frac{3}{4}\right)^{-2} = \left(\frac{4}{3}\right)^2$ $\frac{2^{-3}}{3^{-4}} = \frac{3^4}{2^3}$
Rational Exponents	$\sqrt[n]{a} = a^{\frac{1}{n}}$ $\sqrt[n]{a^m} = a^{\frac{m}{n}}$	$\sqrt[3]{5} = 5^{\frac{1}{3}}$ $\sqrt[4]{5^3} = 5^{\frac{3}{4}}$

## Exercises

**6.1.** Simplify. Express without brackets or negative exponents.

a)  $(2^4)^2$

b)  $(5^3)^{-2}$

c)  $(3^{-4})^{-2}$

d)  $(-3x^{-2})^0$

e)  $(2x)^3$

f)  $(3x^{-4})^2$

g)  $(2a^{-4})^3$

h)  $(3x^4y^{-2})^4$

i)  $(-4a^{-3}b^{-2})^2$

j)  $(-2^{-3}x^{-2}y)^3$

**6.2.** Simplify. Express without brackets or negative exponents.

a)  $\frac{3^4 \times 3^7}{3^5}$

b)  $\frac{2^5}{2^4 \times 2^3}$

c)  $\frac{4^{-3} \times 4}{4^{-1}}$

d)  $\frac{5^4 \times 5^{-2}}{5^3 \times 5^{-1}}$

e)  $\frac{7^0 \times 7^{-3}}{7 \times 7^{-2}}$

f)  $\frac{11^2 \times 11^3}{11^{-1}}$

$$\text{g)} \quad \frac{3(x^3)^2}{x^{-2}}$$

$$\text{h)} \quad \frac{(3x^2)^{-3}}{x^3}$$

$$\text{i)} \quad (2a^2b^{-4}c^{-5})^3$$

$$\text{j)} \quad \left(\frac{2a^2}{3b^4}\right)^{-3}$$

**6.3.** Simplify. Express without brackets or negative exponents.

$$\text{a)} \quad \frac{(2a^2b^3)^{-2} \times (4ab^{-1})^3}{(a^3b)^{-4}}$$

$$\text{b)} \quad \frac{(x^5y^2)^{-2} \times (x^2y^{-2})^3}{x^{-1}y^{-2}}$$

$$\text{c)} \quad \frac{(5m^{-1}n^2)^2 \times (2m^{-2}n^{-3})^3}{(2m^3n^2)^{-1}}$$

$$\text{d)} \quad \frac{(3a^{-2}b^3)^2 \times (3a^{-1}b^{-4})^{-1}}{(3a^2b^{-2})^{-3}}$$

$$\text{e)} \quad \frac{(3^{-1}x^{-2}y)^{-1} \times (5x^2y^4)^{-2}}{(4x^{-2}y^{-3})^2}$$

$$\text{f)} \quad \frac{(3^{-1}a^{-1}b^{-2})^{-2} \times (4a^{-3}b^4)^{-2}}{(3a^{-3}b^{-4})^2}$$

$$\text{g)} \quad \left( \frac{4^{-2}x^2y^{-3}}{x^{-2}y} \right)^3 \left( \frac{8^{-1}x^{-3}y}{x^3y^{-1}} \right)^{-2}$$

$$\text{h)} \quad \left( \frac{9ab^{-1}}{8a^{-2}b^2} \right)^{-2} \left( \frac{3a^{-2}b^2}{2a^2b^{-1}} \right)^3$$

$$\text{i)} \quad \frac{(2x^{-1}y^2)(4x^2y^{-3})^{-2}}{(12x^2y^2)}$$

$$\text{j)} \quad \left[ \frac{(5x^{-3}y^4)^{-2}(6x^2y^{-5})}{15x^2y^{-4}} \right]^{-2}$$

**6.4.** Simplify. Write your answer as a radical.

a)  $2^{\frac{1}{4}} \times 2^{\frac{5}{4}}$

b)  $3^{\frac{2}{3}} \times 3^{\frac{7}{3}}$

c)  $4^{\frac{1}{4}} \times 4^{-\frac{3}{4}}$

d)  $5^{-\frac{2}{3}} \times 5^{-\frac{1}{3}}$

e)  $\frac{6^{\frac{3}{4}}}{6^{\frac{5}{4}}}$

f)  $\frac{7^{\frac{2}{5}}}{7^{-\frac{1}{5}}}$

g)  $\frac{8^{-\frac{2}{7}} \times 8^{\frac{4}{7}}}{8^{-\frac{3}{7}}}$

h)  $\frac{9^{\frac{3}{5}}}{9^{\frac{2}{5}} \times 9^{-\frac{4}{5}}}$

i)  $a^{\frac{3}{4}} \times a^{\frac{5}{4}}$

j)  $b^{\frac{5}{6}} \times b^{-\frac{1}{3}}$



# 7. Logarithms

This chapter was predicated on the notion of finding the inverse of the quadratic function, and seeing what follows from it. What if we were looking for a different kind of inverse?

When finding the inverse of  $x^2$ , we knew the exponent and wanted to find the base. What if we knew the base but wanted to find the exponent?

If  $2^x = 64$ , what is  $x$ ?

If  $5^x = 125$ , what is  $x$ ?

## Definition 10: The logarithm

The logarithm has a base, just like an exponent. We say that if  $a^b = c$  then  $\log_a c = b$ . In other words, the logarithm undoes the exponent.

Try it:

a)  $\log_2 64$

b)  $\log_5 125$

c)  $\log_3 81$

Which logarithm is bigger?

a)  $\log_2 1$  or  $\log_4 2$

b)  $\log_3 \left(\frac{1}{9}\right)$  or  $\log_9 \left(\frac{1}{3}\right)$

## Log Laws

Just like exponents, logarithms have several laws that govern how they behave. Here are a few of them.

### Theorem 13: Change of Base

To change the base of a logarithm, we can perform the following operation:

$$\log_a b = \frac{\log_c b}{\log_c a}$$

This can be helpful because most scientific calculators only have one logarithm: base 10. In fact, in high school, logarithms that have no indicated base are assumed to be base 10. It's useful to know, however that in different contexts, logarithms with no indicated base can have different meanings. For example, in computer science, logarithms have an assumed base of 2.

Use a scientific calculator to find the following logarithms.

- a)  $\log_4 64$                       b)  $\log_{\frac{2}{3}} \frac{8}{27}$                       c)  $\log_{\sqrt{2}} 2$                       d)  $\log 100$

### Theorem 14: Logarithm Product Law

Given real numbers  $m$  and  $n$  with  $m > 0$  and  $n > 0$ ,

$$\log_b(m \times n) = \log_b m + \log_b n$$

In other words, the logarithm of a product is the sum of logarithms.

How does this compare to the product law for exponents?

Expand each logarithm using the product law.

- a)  $\log(xy)$                       b)  $\log(5x)$                       c)  $\log(3(x+1))$                       d)  $\log(10x^2y)$

Condense each sum into a single logarithm.

- a)  $\log 3 + \log 4$                       b)  $\log \frac{2}{3} + \log \frac{3}{4}$                       c)  $\log x^2 + \log x^3$                       d)  $\log(x+1) + \log(x-2)$

### Theorem 15: Logarithm Quotient Law

Given positive real numbers  $m$  and  $n$ ,

$$\log_b \left( \frac{m}{n} \right) = \log_b m - \log_b n$$

In other words, the logarithm of a quotient is a difference of logarithms.

How does this compare to the quotient law for exponents?

Expand into several logarithms.

a)  $\log \left( \frac{x}{y} \right)$

b)  $\log(x - y)$

c)  $\log \left( \frac{x+1}{100} \right)$

d)  $\log_3 \left( \frac{x}{3(x+1)} \right)$

Condense into a single logarithm.

a)  $\log 12 - \log 4$

b)  $\log \frac{1}{3} - \log 2$

c)  $\log x^5 - \log x^2$

d)  $\log 2 + \log x - \log(x + 3)$

### Theorem 16: Power Law

For positive real numbers  $m$ ,  $n$ , and  $b$ ,

$$\log_b(m^n) = n \log_b m$$

Expand the following logarithms.

a)  $\log x^2$

b)  $(\log x)^2$

c)  $\log x^3 + \log x^4$

d)  $\log x^{a+1}$

Condense into a single logarithm.

a)  $3 \log x$

b)  $2 \log(x - 1)$

c)  $3 \log(2x^2)$

d)  $5 \log x - 3 \log x$

### Theorem 17: Other Logarithm Rules

It is useful to know the following rules about logarithms.

$\log_b x$  has the domain  $x > 0$

$$\log_b 1 = 0$$

$$\log_b b = 1$$

$$b^{\log_b x} = x$$

$$\log_b b^x = x$$

## Exercises

**7.1.** If  $10^k = 4$ , then  $10^{1+2k} =$

**7.2.** If  $3^a = k$ , then  $\log_3 k^4 =$

**7.3.** If  $\log_b 4 = k$ , then  $\log_b 16 =$

**7.4.** If  $\log_2 a = h$ , then  $\log_4 a =$

**7.5.** If  $\log_b h = 3$  and  $\log_b k = 4$ , then  $\log_b \left(\frac{1}{hk}\right) =$

**7.6.** If  $\log_h 4 = 2$  and  $\log_g k = 2$ , then  $\log_2(hk) =$

**7.7.** Write  $\log x + 1$  as a single logarithm.

**7.8.** Write  $3 + \log_2 x$  as a single logarithm.





# Selected Solutions.

**1.6.** The domain is all possible  $x$ -values.

$$D = \{1, 3, 5, 7\}$$

$$R = \{2, 4, 6, 8\}$$

**1.7.** Some ordered pairs for this relation are  $(\$10.00, \$10.70)$ ,  $(\$20.00, \$21.40)$ ,  $(\$0.70, \$0.75)$ . The possible list price and cost pairs are endless. To simplify, you could specify the relationship using a formula with variables to represent the relationship. The cost is the list price plus 7% of the list price.

$$C = L + 0.07L$$

or

$$C = 1.07L$$

Where  $C$  is the cost to the consumer (price after tax) and  $L$  is the list price. The domain of this relation is the set of all possible list prices, and the range is the resulting cost to the consumer when tax is added to the list price. These values will be a subset of positive real numbers.

From the above example it is obvious that the domain (the list price of an item) affects the range (the total cost). You can think of it in terms of input and output. The  $x$ -value that is put into the relation determines the  $y$ -value that comes out. Organized as a table of values, it may look like this:

Input	Output
$L$	$C$
\$10.00	\$10.70
\$20.00	\$21.40
\$0.70	\$0.75

**1.8.** Domain:  $\{1, 2, 3, 4\}$ , range:  $\{4, 6, 8, 10\}$ .

**1.9.**

- a) This is a function and a relation because each  $x$ -value has only one possible  $y$ -value.
- b) This is only a relation because the input of 2 has a variety of possible outputs.
- c) This is a function and a relation because each element in the domain has only one possible value for its range.
- d) This relation is a non-function. The input of -3 has two possible outputs: 3 and -1.
- e) This represents a relation and a function. The value of 1 is the output in two ordered pairs  $(-1, 1)$  and  $(1, 1)$ , but each input has only one possible output so it still fits the definition of a function.

### 1.10.

- a) Each output is hit by exactly one input, and no output repeats. **Yes**, it is one-to-one.
- b) Outputs repeat:  $p$  is the image of both 1 and 2, so different inputs share an output. **Not** one-to-one.

**1.11.** Output 2 has two preimages (B and C). Remove *one* of  $\{B, C\}$ . After removing one, each output has at most one preimage, so it becomes one-to-one.

**1.12.**  $f(x_1) = f(x_2) \Rightarrow 2x_1 + 3 = 2x_2 + 3 \Rightarrow x_1 = x_2$ . Therefore  $f$  is **one-to-one** (its graph also passes the horizontal line test).

**1.13.**  $g$  is **not** one-to-one since  $g(2) = g(-2) = 0$ . Restrict to  $x \geq 0$  (or  $x \leq 0$ ) to make it one-to-one.

**1.14.** Not one-to-one ( $h(1) = h(3) = 1$ ). On  $[2, \infty)$  it is one-to-one.

**1.15.** Linear with nonzero slope  $\frac{3}{2} \Rightarrow$  strictly increasing  $\Rightarrow$  one-to-one.

**1.16.** If  $a = 0$ ,  $f(x) = b$  is constant (not one-to-one). If  $a \neq 0$ , it's strictly monotonic, so **one-to-one**. Thus:  $a \neq 0$ .

**1.17.** No. For example,  $f(-1) = 0$  and  $f(1) = 0$ , so two different inputs give the same output.  $\Rightarrow$  not one-to-one.

**1.18.** Not one-to-one since  $f(-2) = f(0) = 3$ . Remove either  $(-2, 3)$  or  $(0, 3)$ . One possible one-to-one table:  $\{(-1, 5), (0, 3), (1, 7), (2, 9)\}$ .

## Section 2 Solutions

### 2.1.

a)

$$\begin{aligned}f(x) &= 7x \\y &= 7x \\x &= 7y \\y &= \frac{1}{7}x \\f^{-1}(x) &= \frac{1}{7}x\end{aligned}$$

b)

$$\begin{aligned}f(x) &= -3x + 4 \\y &= -3x + 4 \\x &= -3y + 4 \\x - 4 &= -3y \\y &= -\frac{1}{3}(x - 4) \\f^{-1}(x) &= -\frac{1}{3}(x - 4)\end{aligned}$$

c)

$$\begin{aligned}
 f(x) &= \frac{x+4}{3} \\
 y &= \frac{x+4}{3} \\
 x &= \frac{y+4}{3} \\
 3x &= y+4 \\
 y &= 3x-4 \\
 f^{-1}(x) &= 3x-4
 \end{aligned}$$

d)

$$\begin{aligned}
 f(x) &= \frac{x}{3} - 5 \\
 y &= \frac{x}{3} - 5 \\
 x &= \frac{y}{3} - 5 \\
 x+5 &= \frac{y}{3} \\
 y &= 3(x+5) \\
 f^{-1}(x) &= 3(x+5)
 \end{aligned}$$

e)

$$\begin{aligned}
 f(x) &= 5-2x \\
 y &= 5-2x \\
 x &= 5-2y \\
 x-5 &= -2y \\
 y &= -\frac{1}{2}(x-5) \\
 f^{-1}(x) &= -\frac{1}{2}(x-5)
 \end{aligned}$$

f)

$$\begin{aligned}
 f(x) &= \frac{1}{2}(x+6) \\
 y &= \frac{1}{2}(x+6) \\
 x &= \frac{1}{2}(y+6) \\
 2x &= y+6 \\
 y &= 2x-6 \\
 f^{-1}(x) &= 2x-6
 \end{aligned}$$

### 2.3.

a)  $f^{-1}(x) = \frac{1}{3}x - \frac{5}{3}$

c)  $f^{-1}(x) = \sqrt[3]{\frac{1}{4}(5-x)}$

e)  $f^{-1}(x) = \frac{1}{x} - 2$

g)  $f^{-1}(x) = \frac{4x}{1-x}$

i)  $f^{-1}(x) = \frac{7x+5}{x-2}$

k)  $f^{-1}(x) = \frac{x-3}{5x+2}$

m)  $f^{-1}(x) = \sqrt{4-x}, x \leq 4$

o)  $f^{-1}(x) = \sqrt[6]{x}, x \geq 0$

q)  $f^{-1}(x) = \sqrt[3]{2-5x}$

s)  $f^{-1}(x) = \frac{x^2-5}{8}, x \geq 0$

t)  $f^{-1}(x) = (x-2)^3$

## Section 3 Solutions

**3.1.** Use a calculator to verify your answers.

## Section 4 Solutions

**4.1.** See the proof that the square root of 2 is irrational, and compare to that.

**4.2.** No. Consider  $x = \sqrt{2}$  and  $y = 1 - \sqrt{2}$ .  $x + y = 1$  which is rational.

**4.3.** No. Consider  $x = \sqrt{2}$  and  $y = \sqrt{2}$ . Then  $x \times y = 2$ .

**4.4.** No. Consider  $x = (\sqrt{2})^{\sqrt{2}}$ . If  $x$  is rational, then we are done. However, if  $x$  is irrational, then consider  $x^{\sqrt{2}} = (\sqrt{2}^{\sqrt{2}})^{\sqrt{2}} = \sqrt{2}^2 = 2$ , which is rational.

## Section 5 Solutions

**5.1.**

- |                    |                   |                    |                   |
|--------------------|-------------------|--------------------|-------------------|
| a) $2\sqrt{5}$     | b) $6\sqrt{2}$    | c) $3\sqrt{5}$     | d) $2\sqrt{6}$    |
| e) $5\sqrt{3}$     | f) $5\sqrt{5}$    | g) $2\sqrt{35}$    | h) $8\sqrt{2}$    |
| i) $-4\sqrt{5}$    | j) $2\sqrt[3]{5}$ | k) $2\sqrt[3]{6}$  | l) $3\sqrt[3]{2}$ |
| m) $3\sqrt[3]{5}$  | n) $4\sqrt[3]{2}$ | o) $4\sqrt[3]{3}$  | p) 6              |
| q) $-6\sqrt[3]{2}$ | r) 2              | s) $-3\sqrt[3]{2}$ |                   |

**5.3.**  $5\sqrt{6}$  mm

**5.4.**  $4\sqrt[3]{3}$  cm

**5.5.**  $540\sqrt{14}$  cm<sup>2</sup>

**5.6.**  $60\sqrt{2}$  cm<sup>2</sup>

## Section 6 Solutions

**6.1.**

- |                        |                          |                       |                           |
|------------------------|--------------------------|-----------------------|---------------------------|
| a) $2^8$               | b) $\frac{1}{5^6}$       | c) $3^8$              | d) 1                      |
| e) $8x^3$              | f) $\frac{9}{x^8}$       | g) $\frac{8}{a^{12}}$ | h) $\frac{81x^{16}}{y^8}$ |
| i) $\frac{16}{a^6b^4}$ | j) $-\frac{y^3}{512x^6}$ |                       |                           |

**6.2.**

- |                                |                            |                  |                      |
|--------------------------------|----------------------------|------------------|----------------------|
| a) $3^6$                       | b) $\frac{1}{2^2}$         | c) $\frac{1}{4}$ | d) 1                 |
| e) $\frac{1}{7^2}$             | f) $11^6$                  | g) $3x^8$        | h) $\frac{1}{27x^9}$ |
| i) $\frac{8a^6}{b^{12}c^{15}}$ | j) $\frac{27b^{12}}{8a^6}$ |                  |                      |

**6.3.**

- |                           |                                  |                              |                              |
|---------------------------|----------------------------------|------------------------------|------------------------------|
| a) $\frac{16a^{11}}{b^5}$ | b) $\frac{1}{x^3y^8}$            | c) $\frac{400}{m^5n^3}$      | d) $81a^3b^4$                |
| e) $\frac{3x^2}{400y^3}$  | f) $\frac{a^{14}b^4}{16}$        | g) $\frac{x^{24}}{64y^{16}}$ | h) $\frac{8b^{15}}{3a^{18}}$ |
| i) $\frac{y^6}{96x^7}$    | j) $\frac{15625y^{18}}{4x^{12}}$ |                              |                              |

**6.4.**

a)  $2^{\frac{3}{2}}$

b)  $3^3$

c)  $\frac{1}{2}$

d)  $\frac{1}{5}$

e)  $\frac{1}{6^{\frac{1}{2}}}$

f)  $7^{\frac{3}{5}}$

g)  $2^{\frac{15}{7}}$

h)  $9$

i)  $a^2$

j)  $b^{\frac{1}{2}}$

## Section 7 Solutions

**7.1.**  $160$

**7.2.**  $4a$

**7.3.**  $2k$

**7.4.**  $\frac{h}{2}$

**7.5.**  $-7$

**7.6.**  $7$

**7.7.**  $\log(10x)$

**7.8.**  $\log_2(8x)$