Chapter 3: Radicals and Exponents

Introduction

In the first chapter, we dealt with functions that could be sketched as a straight line. We observed how they intersected with each other, and how we can use algebra to describe those intersections.

In the second chapter, we learned about polynomial functions, namely quadratics, and how they can be graphed. We spent a great deal of time finding roots of a quadratic, and even learned about imaginary numbers.

In this chapter, we will learn about the inverse of a quadratic – a radical. We will learn how to play with these kinds of numbers, what *kinds* of numbers they can be, and how to express them in various ways.

Try this warm up to see if you can get all of these facts right.

a) $3^2 =$

b) $(-3)^2 =$

c) If $x^2 = 9$, then $x =$

d) $\sqrt{9} =$

e) $\sqrt{x^2} =$

f) $(\sqrt{x})^2 =$

1. Relations, Functions, One-to-one

We have seen several types of function in this class so far. The first type, and most straightforward was the linear function. Then we realized that if we multiplied two linear functions together, we got a quadratic function. There are also higher-degree polynomial functoins that we briefly explored in the last chapter.

It turns out that functions are the life-blood of a lot of mathematics. Most of mathematics can be described using the language of functions.

But not all equations can be described as a function. It is probably a good idea to remind ourselves what exactly a function is!

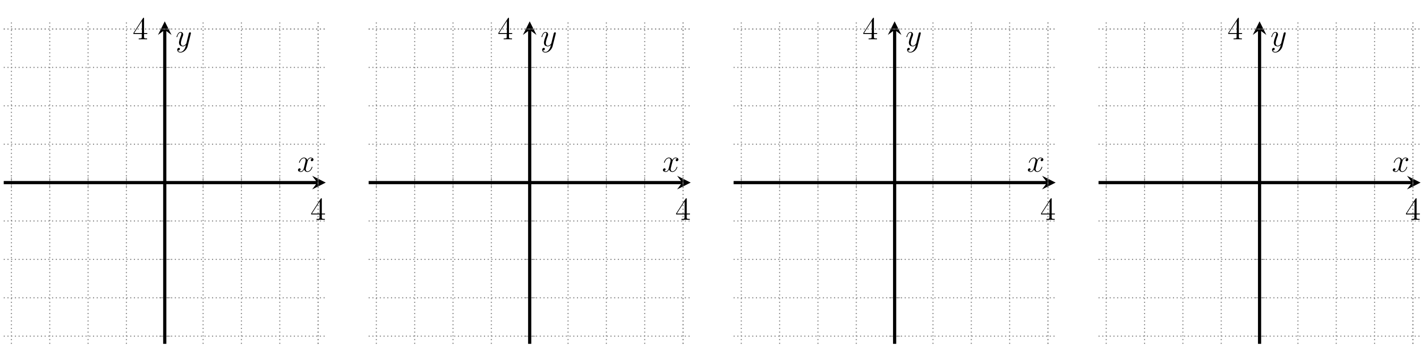
> **Definition 1: Function**
>
> A function is a rule that takes as input one value and outputs another value, related to the input. If x is the input of function f, and y is the output of the function, then we write
>
> $$y = f(x)$$

It is critical to understand that for every single input, there is a single output. A function fails to be a function if you were to input a single value and get back *two* or more values.

Consider the following functions.

a) $f(x) = x^2$　　b) $f(x) = (x-2)(x+1)^2$　　c) $y = x - 4$　　d) $g(x) = \sqrt{x}$

Use Desmos to help you graph the above functions below.



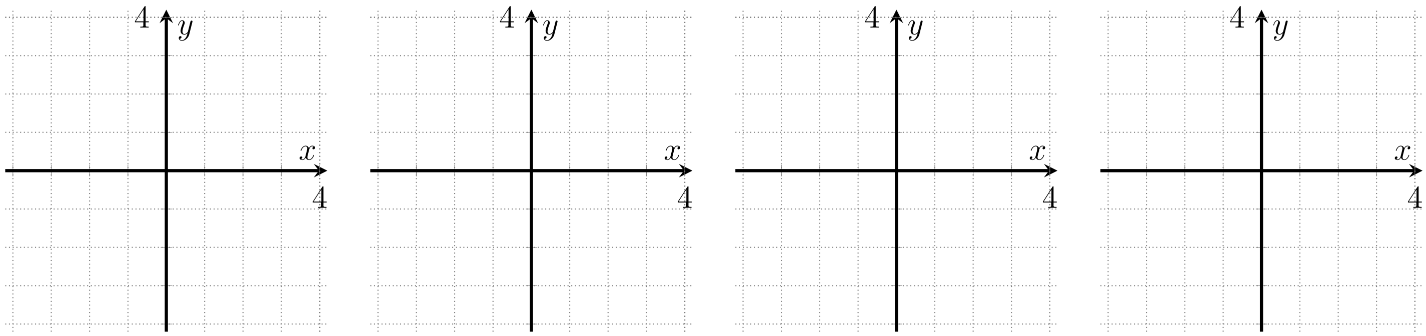
Some equations in math, however, are not best described as a function. Sometimes if I give you one input, you may want two outputs.

The following are not functions:

a) $y^2 = x$　　b) $x^2 + y^2 = 1$

c) $x^3 + y^2 = xy$ d) $y = \pm x$

Use Desmos to help you graph these functions. Try to see if you can visualize *exactly* what makes these equations non-functions.

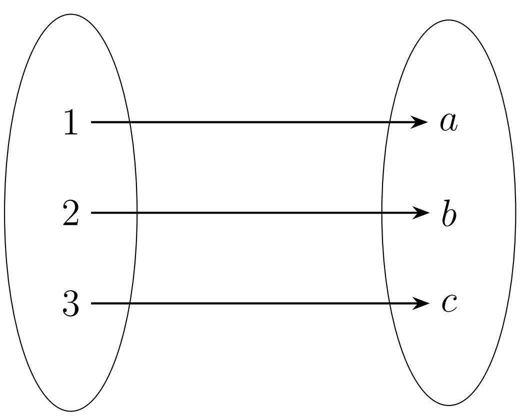


Definition 2: Relation

An equation that has two or more variables is called a relation. Most relations are between two variables, x and y. If the relation R has a solution $x = a$ and $y = b$, then the the relation can be graphed by plotting (a, b), and all other solutions to the equation. Furthermore, we can write aRb and say a is related to b.

So all of the above equations are relations because they represent a relationship between x and y. When Desmos graphs these, they are finding all possible solutions to the equation, and plotting each point, one by one.

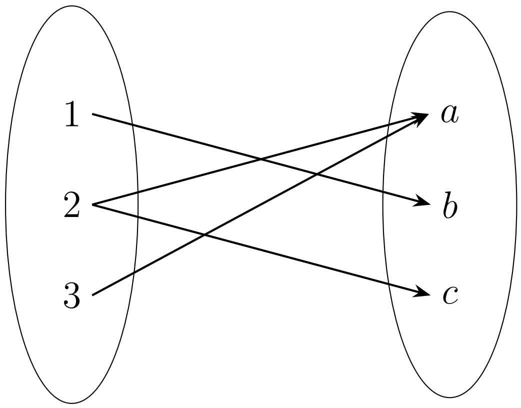
We can also describe a relation by means of a mapping diagram.



The above mapping diagram indicates that we have two sets, $\{a, b, c\}$, and $\{1, 2, 3\}$. The first set is referred to as our domain, and the second set is called the range.

The mapping diagram indicates that a is related to 1, (or a is 'mapped' to 1), b is related to 2, and c is related to 3. We could also write $aR1$, $bR2$, and $cR3$. We can also say that the following ordered pairs describe the relation: $(a, 1), (b, 2)$, and $(c, 3)$.

Describe the following relation in all of the ways you know how:



Draw the mapping diagram of the relation described by the following ordered pairs:

$$(a, 4), (b, 3), (c, 0), (a, 3), (8, \alpha)$$

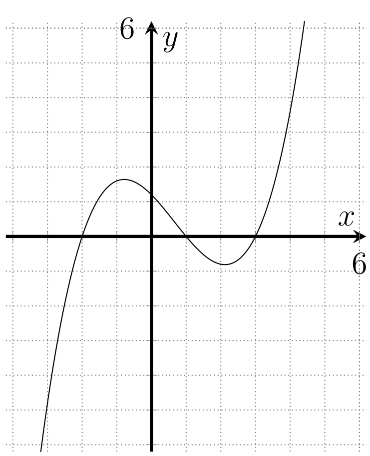
There is a quick way of determining when a graph represents a function and when it doesn't.

Definition 3: Vertical Line Test

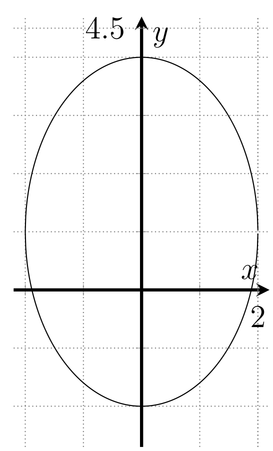
Given the graph of a relation, if there exists a vertical line that overlaps the curve of the graph more than once, then the relation is not a function. If no such vertical line exists, then the relation is a function.

Use the Vertical Line Test to determine which of these graphs represents a relation that is a function, and which is a relation that is *not* a function.

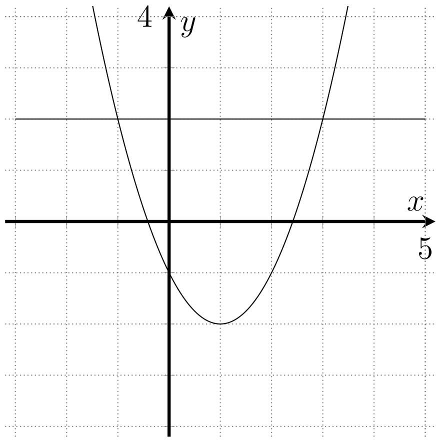
a)



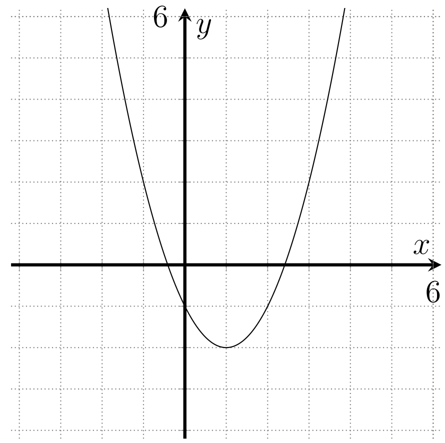
b)



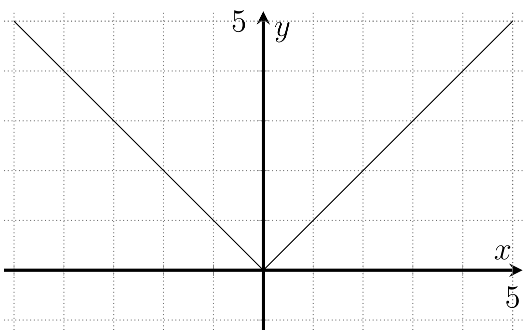
c)



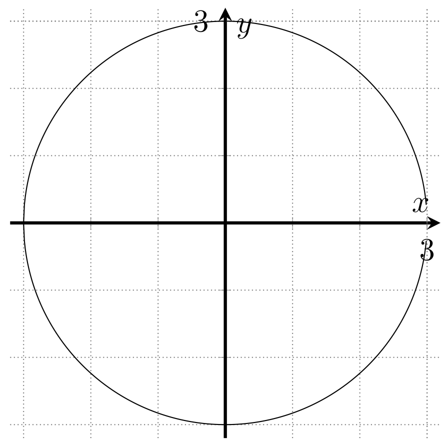
d)



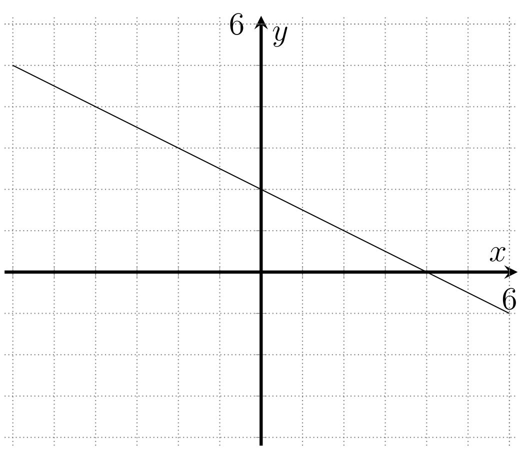
e)



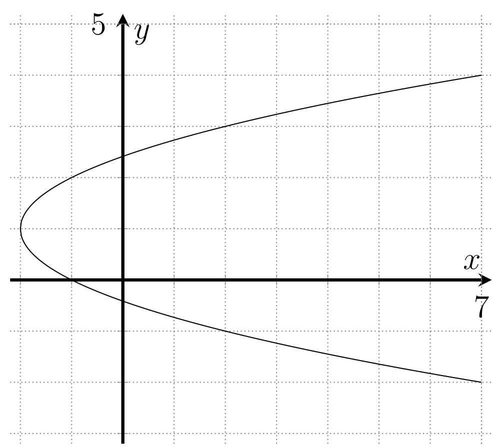
f)



g)



h)



As you may be able to sense, a function is a special type of relation. But is there something even more special? Are there types of functions are even more exclusive? It turns out: yes. These functions are called one-to-one functions.

Recall that the feature that a relation requires is that for every one input there is exactly one output. For a function to be one to one, we require the following.

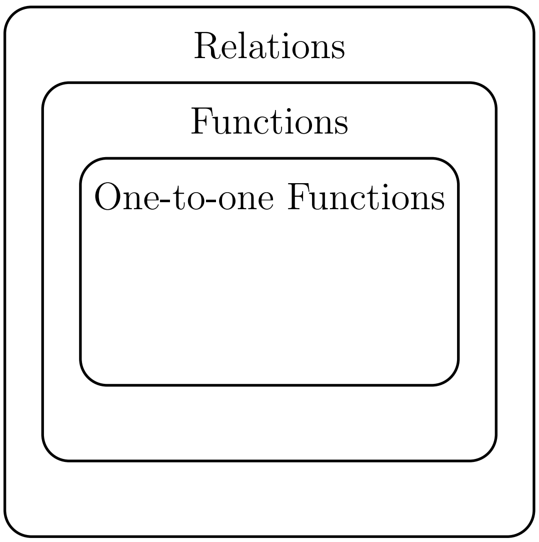
Definition 4: One-to-one Function

A function is said to be one-to-one if for every single *output*, there is exactly one *input*. Symbolically, we can say:

$$f(a) = f(b) \text{ implies } a = b.$$

In other words, the only way that two outputs can be the same ($f(a)$ and $f(b)$), is if they are not two outputs at all – that they are indeed from the same input (a and b).

It is useful to remember that one-to-one functions are *all* functions, and all functions are relations. But not all relations are function, and not all functions are one-to-one.



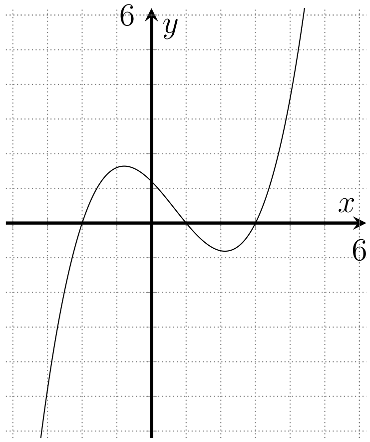
Definition 5: Horizontal Line Test

Given the graph of a function, if there exists a horizontal line that overlaps the curve of the graph more than once, then the function is not one-to-one. If no such horizontal line exists, then the function is one-to-one.

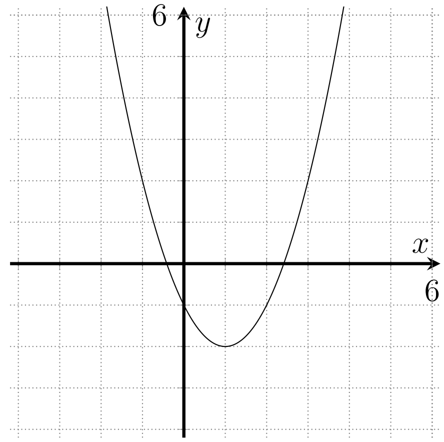
Explain why the horizontal line test works.

Use the horizontal line test to determine which of the following functions are one-to-one.

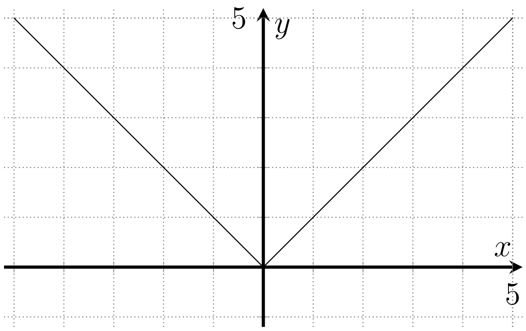
a)



b)



c)



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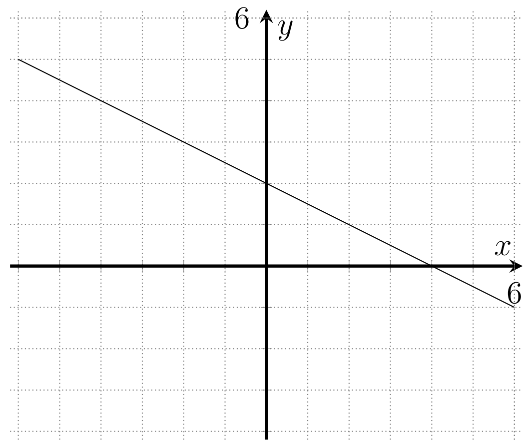
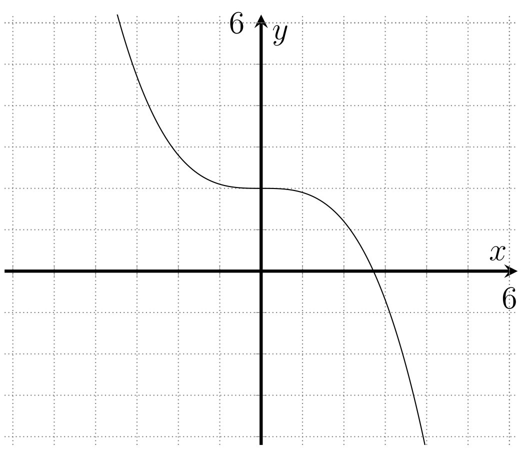
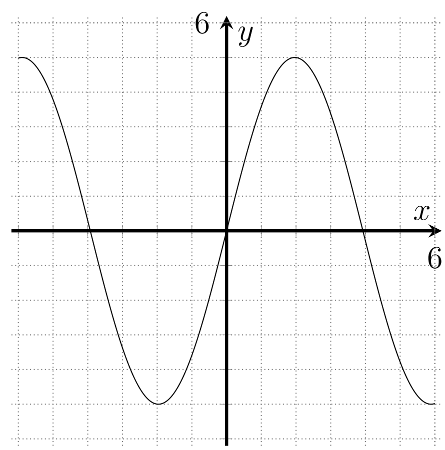


e)



f)



2. Inverse Functions

A common theme in mathematics is learning how to do something backwards. When you first learned how to add, it wasn't long before you learned how to undo adding: you learned how to subtract. After you learned how to multiply, you learned how to divide. When you solve an equation, you repeatedly undo the operations done to your variable. When you learn about the derivative in calculus, you will also learn about the antiderivative, which undoes the derivative operation.

It may be of interest to us to find a way to undo a function. This is called the inverse.

Definition 6: Inverse Function

The inverse of a function $f(x)$, if it exists, is the unique function $f^{-1}(x)$ such that

$$f(f^{-1}(x)) = f^{-1}(f(x)) = x.$$

In other words, the inverse of f undoes what f does to x

For example, the inverse of $f(x) = \frac{1}{2}x - 2$ is $f^{-1}(x) = 2x + 4$. Why is that?

How to find the inverse of a function

To find the inverse of a function, provided it exists (more on that later), we take the following steps:

1. write the function as y in terms of x
2. replace x and y
3. solve for y
4. profit.

Find the inverse of $f(x) = 3x - 12$ and verify that it is indeed the inverse.

Which functions have inverses?

Not all functions have inverses. Take $f(x) = x^2$ for example. Since $f(2) = 4$ and $f(-2) = 4$, if our inverse f^{-1} *did* exist, what could $f^{-1}(4)$ be equal to? It would have to undo both of the mappings that map to 4.

Theorem 1: Only one-to-one functions have inverses

If a function $f(x)$ is one-to-one, then $f^{-1}(x)$ exists. When this happens, the domain and range of f are the range and domain of f^{-1}, respectively. In other words, the domain and range swap.

Why do you think we need f to be one-to-one in order to find its inverse?

Find the inverse of the following functions:

a) $f(x) = 2x + 3$

b) $g(x) = -x + 1$

c) $h(x) = \frac{1}{2}x - 4$

d) $p(x) = -3x - 2$

e) $q(x) = 4x$

f) $r(x) = -\frac{2}{3}x + 5$

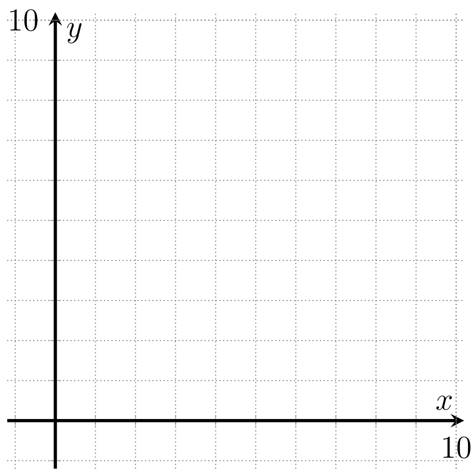
3. The Radical Function

The last section was a short detour to talk about the radical function. In order to fully understand the radical function, we must be familiar with the following ideas:

- domain
- range
- one-to-one
- inverse

Since the function $f(x) = x^2$ is not one-to-one, we cannot take its inverse. But what would happen if we were to *restrict* its domain to the positive real numbers only?

Graph $y = x^2$ over the domain $x \geq 0$.



If we define this function, which is only defined for values $x \geq 0$ as $g(x)$, then we have the very convenient fact that g is one-to-one.

How do we know this?

Definition 7: The radical function

Let $g(x) = x^2$ over the restricted domain $x \geq 0$, making g one-to-one. The function $g^{-1}(x)$ is called the *square root* function and is in the family of radical functions. We usually write:

$$g^{-1}(x) = \sqrt{x}$$

Wait... it's taken us three chapters to define $\sqrt{x}$? Yes.

Now let's revisit the blurb from the first page of this book:

a) $3^2 =$

b) $(-3)^2 =$

c) If $x^2 = 9$, then $x =$

d) $\sqrt{9} =$

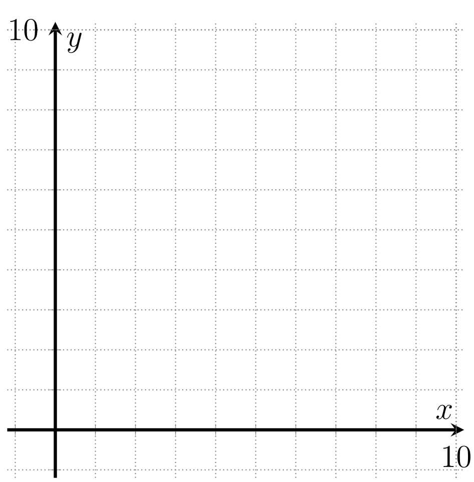
e) $\sqrt{x^2} =$

f) $(\sqrt{x})^2 =$

All of this is to say that if $y = \sqrt{x}$ then $y^2 = x$, but just because $y^2 = x$ does not mean that $y = \sqrt{x}$.

Explain the above sentence in your own words, and give an example of x and y that satisfy that sentence.

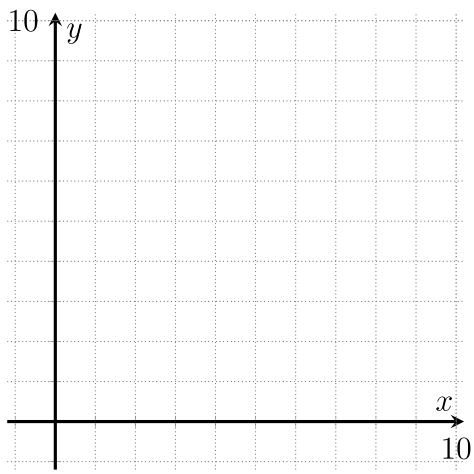
Graph the function $y = \sqrt{x}$



Use this graph to approximate $\sqrt{7}$.

4. Number Systems

Use Desmos to help you graph $y = \sqrt{x}$ as perfectly as possible. It may help to remember that the square root function is the inverse of $y = x^2$ over the restricted domain of $x \geq 0$, so its shape should be very similar.



When we graph the function $y = \sqrt{x}$ we see the following point on the graph:

- $(0, 0)$ since $0^2 = 0$
- $(1, 1)$ since $1^2 = 1$
- $(4, 2)$ since $2^2 = 4$
- $(9, 3)$ since $3^2 = 9$
- and so on

But there are *so many more* points on this function. In fact, between any two different points on the graph, there are *infinitely many* points in between them. What do some of those points look like? Try seeing what happens to y when $x = 7$.
Are there any integers whose square root is not an integer?

Definition 8: Rational numbers

A rational number is a number that can be written as the ratio of two integers a and b, provided that $b \neq 0$. The set of all rational numbers is denoted $\mathbb{Q}$, because Q stands for quotient. In symbols,

$$\mathbb{Q} = \left\{ \frac{a}{b} : a, b \in \mathbb{Z} \right\}$$

It turns out that the square root of an integer can be a very strange kind of number.

Theorem 2: The irrationality of $\sqrt{2}$

The square root of 2 is not a rational number.

Why is this?
It turns out that there are all kinds of different numbers out there. To recap, we have the following sets of numbers:

5. Manipulating Radicals

There is a very straightforward rule that governs how we can change and alter radical expressions.

Theorem 3: The product of radicals

For any two positive real numbers a and b,

$$\sqrt{a} \times \sqrt{b} = \sqrt{a \times b}$$

Why is this true?

Beware of the following mistake, which is a variant of the *freshman's dream*.

$$\sqrt{a} + \sqrt{b} \neq \sqrt{a + b}$$

Whis is this not true?

Simplifying radicals

The rule we just learned can be useful when it comes to simplifying radicals. Sometimes we have a square root that we would like to simplify, but it is not the square root of a perfect square.

Simplify $\sqrt{12}$.

Exercises

5.1. Simplify the following.

a) $\sqrt{20}$

b) $\sqrt{72}$

c) $\sqrt{45}$

d) $\sqrt{24}$

e) $\sqrt{75}$

f) $\sqrt{125}$

g) $\sqrt{140}$

h) $\sqrt{128}$

i) $\sqrt[3]{40}$

j) $\sqrt[3]{48}$

k) $\sqrt[3]{54}$

l) $\sqrt[3]{135}$

m) $\sqrt[3]{128}$

n) $\sqrt[3]{192}$

o) $2\sqrt[3]{27}$

p) $-3\sqrt[3]{16}$

5.2. Multiply, and simplify if possible.

a) $\sqrt[3]{4} \times \sqrt[3]{6}$

b) $\sqrt[3]{9} \times \sqrt[3]{24}$

c) $\sqrt[3]{5} \times \sqrt[3]{5}$

d) $\sqrt[3]{4} \times \sqrt[3]{54}$

e) $2\sqrt[3]{12} \times \sqrt[3]{30}$

f) $-3\sqrt[3]{25} \times 4\sqrt[3]{75}$

g) $2\sqrt[3]{10} \times 3\sqrt[3]{50}$

h) $(-3\sqrt[3]{12})(-2\sqrt[3]{18})$

i) $(-3\sqrt[3]{4})(-2\sqrt[3]{32})$

j) $(-5\sqrt[3]{49})(2\sqrt[3]{56})$

5.3. A square has an area of 150 mm^2. What are the lengths of the sides of the square?

5.4. A cube has a volume of 192 cm^3. What are the lengths of each edge of the cube?

5.5. The dimensions of a rectangle are $9\sqrt{30}$ cm by $4\sqrt{105}$ cm. Calculate the area of the rectangle.

5.6. The dimensions of a rectangle are $5\sqrt{6}$ cm by $4\sqrt{3}$ cm. Calculate the area of the rectangle.

6. Exponential Notation

Consider the following example: $a^2 \times a^3 = (a \times a) \times (a \times a \times a) = a^5$
The exponent in the expression a^5 is the sum of the exponents in the expression $a^2 \times a^3$.
Therefore: $a^2 \times a^3 = a^{2+3} = a^5$.

Theorem 4: The Product Rule

For any numbers a and b with exponents m and n :

$$a^m \times a^n = a^{m+n}, \quad a \neq 0$$

Try it: $(-3)^4 \times (-3)^5 = (-3)^{4+5} = (-3)^9$

Theorem 5: The Quotient Rule

For any number a with exponents m and n :

$$\frac{a^m}{a^n} = a^{m-n}, \quad a \neq 0$$

Try it: $\frac{(-4)^8}{(-4)^3} = (-4)^{8-3} = (-4)^5$

Theorem 6: The Product Rule

For any numbers a and b with exponents m and n :
$(a^m)^n = a^{m \times n}$

Try it: $\left(3^5\right)^4 = 3^{5\times 4} = 3^{20}$

Theorem 7: Distributive Rule for exponents, part 1

Exponents distribute over multiplication. For any numbers a and b with exponent n :
$(ab)^n = a^n \times b^n$

Try it: $(3x)^3 = 3^3 \times x^3 = 27x^3$

Theorem 8: Distributive Rule for exponents, part 2

Exponents distribute over division. For any numbers a and $b, b \neq 0$, with exponent n :

$$\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$$

Try it: $\left(\frac{2}{x}\right)^3 = \frac{2^3}{x^3} = \frac{8}{x^3}$

Theorem 9: Negative Exponents, part 1

For any number $a, a \neq 0$, with exponent n :

$$a^{-n} = \frac{1}{a^n}$$

Try it: $2^{-3} = \frac{1}{2^3} = \frac{1}{8}$

Theorem 10: Negative Exponents, part 2

or any non-zero numbers a and b, with exponents m and n :
$\frac{a^{-m}}{b^{-n}} = \frac{b^n}{a^m}$ and $\left(\frac{a}{b}\right)^{-m} = \left(\frac{b}{a}\right)^m$

Try it: $\left(\frac{2}{3}\right)^{-3} = \left(\frac{3}{2}\right)^3 = \frac{27}{8}$

Rational Exponents: $a^{\frac{1}{n}}$

Consider the square root example: $\sqrt{2} \times \sqrt{2} = 2$.
Now consider the exponent rule example: $2^{\frac{1}{2}} \times 2^{\frac{1}{2}} = 2^{\frac{1}{2}+\frac{1}{2}} = 2^1 = 2$.

Since $\sqrt{2} \times \sqrt{2}$ and $2^{\frac{1}{2}} \times 2^{\frac{1}{2}}$ equal 2, $\sqrt{2}$ should equal $2^{\frac{1}{2}}$.

Theorem 11: Rational Exponents, part 1

For any non-negative real number a and any positive integer n.

$$a^{\frac{1}{n}} = \sqrt[n]{a}$$

Try it: $x^{\frac{1}{2}} = \sqrt{x}$

Theorem 12: Rational Exponents, part 2

For any non-negative real number a and any positive integer n,

$$a^{\frac{m}{n}} = \sqrt[n]{a^m}$$

Try it: $4^{\frac{3}{2}} = (2^2)^{\frac{3}{2}} = 2^{\left(2\times\frac{3}{2}\right)} = 2^3 = 8$

Exercises

6.1. Simplify. Express without brackets or negative exponents.

Summary of Exponent Rules

For any integers m and n :		
Exponent of 1	$a^1 = a$	$3^1 = 3$
Exponent of 0	$a^0 = 1, \quad a \neq 0$	$(-5)^0 = 1$
Product Rule	$a^m \times a^n = a^{m+n}, \quad a \neq 0$	$2^3 \times 2^4 = 2^{3+4} = 2^7$
Quotient Rule	$\frac{a^m}{a^n} = a^{m-n}, \quad a \neq 0$	$\frac{3^5}{3^3} = 3^{5-3} = 3^2$
Power Rules	$(a^m)^n = a^{m \times n}$ $(ab)^n = a^n \times b^n$ $\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$	$\left(2^3\right)^4 = 2^{3\times 4} = 2^{12}$ $(2x)^3 = 2^3 \times x^3$ $\left(\frac{2}{3}\right)^4 = \frac{2^4}{3^4}$
Negative Exponents	$a^{-n} = \frac{1}{a^n}$ $\left(\frac{a}{b}\right)^{-n} = \left(\frac{b}{a}\right)^n$ $\frac{a^{-m}}{b^{-n}} = \frac{b^n}{a^m}$	$2^{-3} = \frac{1}{2^3}$ $\left(\frac{3}{4}\right)^{-2} = \left(\frac{4}{3}\right)^2$ $\frac{2^{-3}}{3^{-4}} = \frac{3^4}{2^3}$
Rational Exponents	$\sqrt[n]{a} = a^{\frac{1}{n}}$ $\sqrt[n]{a^m} = a^{\frac{m}{n}}$	$\sqrt[3]{5} = 5^{\frac{1}{3}}$ $\sqrt[4]{5^3} = 5^{\frac{3}{4}}$

a) $\left(2^4\right)^2$

b) $\left(5^3\right)^{-2}$

c) $\left(3^{-4}\right)^{-2}$

d) $\left(-3x^{-2}\right)^0$

e) $(2x)^3$

f) $\left(3x^{-4}\right)^2$

g) $\left(2a^{-4}\right)^3$

h) $\left(3x^4y^{-2}\right)^4$

i) $\left(-4a^{-3}b^{-2}\right)^2$

j) $\left(-2^{-3}x^{-2}y\right)^3$

6.2. Simplify. Express without brackets or negative exponents.

a) $\dfrac{3^4 \times 3^7}{3^5}$

b) $\dfrac{2^5}{2^4 \times 2^3}$

c) $\dfrac{4^{-3} \times 4}{4^{-1}}$

d) $\dfrac{5^4 \times 5^{-2}}{5^3 \times 5^{-1}}$

e) $\dfrac{7^0 \times 7^{-3}}{7 \times 7^{-2}}$

f) $\dfrac{11^2 \times 11^3}{11^{-1}}$

g) $\dfrac{3\left(x^3\right)^2}{x^{-2}}$

h) $\dfrac{\left(3x^2\right)^{-3}}{x^3}$

i) $\left(2a^2b^{-4}c^{-5}\right)^3$

j) $\left(\dfrac{2a^2}{3b^4}\right)^{-3}$

6.3. Simplify. Express without brackets or negative exponents.

a) $\dfrac{(2a^2b^3)^{-2} \times (4ab^{-1})^3}{(a^3b)^{-4}}$

b) $\dfrac{(x^5y^2)^{-2} \times (x^2y^{-2})^3}{x^{-1}y^{-2}}$

c) $\dfrac{(5m^{-1}n^2)^2 \times (2m^{-2}n^{-3})^3}{(2m^3n^2)^{-1}}$

d) $\dfrac{(3a^{-2}b^3)^2 \times (3a^{-1}b^{-4})^{-1}}{(3a^2b^{-2})^{-3}}$

e) $\dfrac{(3^{-1}x^{-2}y)^{-1} \times (5x^2y^4)^{-2}}{(4x^{-2}y^{-3})^2}$

f) $\dfrac{(3^{-1}a^{-1}b^{-2})^{-2} \times (4a^{-3}b^4)^{-2}}{(3a^{-3}b^{-4})^2}$

g) $\left(\dfrac{4^{-2}x^2y^{-3}}{x^{-2}y}\right)^3 \left(\dfrac{8^{-1}x^{-3}y}{x^3y^{-1}}\right)^{-2}$

h) $\left(\dfrac{9ab^{-1}}{8a^{-2}b^2}\right)^{-2} \left(\dfrac{3a^{-2}b^2}{2a^2b^{-1}}\right)^3$

i) $\dfrac{(2x^{-1}y^2)\,(4x^2y^{-3})^{-2}}{(12x^2y^2)}$

j) $\left[\dfrac{(5x^{-3}y^4)^{-2}\,(6x^2y^{-5})}{15x^2y^{-4}}\right]^{-2}$

6.4. Simplify. Write your answer as a radical.

a) $2^{\frac{1}{4}} \times 2^{\frac{5}{4}}$

b) $3^{\frac{2}{3}} \times 3^{\frac{7}{3}}$

c) $4^{\frac{1}{4}} \times 4^{-\frac{3}{4}}$

d) $5^{-\frac{2}{3}} \times 5^{-\frac{1}{3}}$

e) $\dfrac{6^{\frac{3}{4}}}{6^{\frac{5}{4}}}$

f) $\dfrac{7^{\frac{2}{5}}}{7^{-\frac{1}{5}}}$

g) $\dfrac{8^{-\frac{2}{7}} \times 8^{\frac{4}{7}}}{8^{-\frac{3}{7}}}$

h) $\dfrac{9^{\frac{3}{5}}}{9^{\frac{2}{5}} \times 9^{-\frac{4}{5}}}$

i) $a^{\frac{3}{4}} \times a^{\frac{5}{4}}$

j) $b^{\frac{5}{6}} \times b^{-\frac{1}{3}}$

k) $\dfrac{c^{\frac{2}{3}}}{c^{\frac{5}{6}}}$

7. Logarithms

This chapter was predicated on the notion of finding the inverse of the quadratic function, and seeing what follows from it. What if we were looking for a different kind of inverse?

When finding the inverse of x^2, we knew the exponent and wanted to find the base. What if we knew the base but wanted to find the exponent?

If $2^x = 64$, what is x?

If $5^x = 125$, what is x?

Definition 9: The logarithm

The logarithm has a base, just like an exponent. We say that if $a^b = c$ then $\log_a c = b$. In other words, the logarithm undoes the exponent.

Try it: $\log_2 64$

$\log_5 125$

$\log_3 81$

Which logarithm is bigger?

a) $\log_2 1$ or $\log_4 2$

b) $\log_3\left(\frac{1}{9}\right)$ or $\log_9\left(\frac{1}{3}\right)$

Log Laws

Just like exponents, logarithms have several laws that govern how they behave. Here are a few of them.

Theorem 13: Change of Base

To change the base of a logarithm, we can perform the following operation:

$$\log_a b = \frac{\log_c b}{\log_c a}$$

This can be helpful because most scientific calculators only have one logarithm: base 10. In fact, in high school, logarithms that have no indicated base are assumed to be base 10. It's useful to know, however that in different contexts, logarithms with no indicated base can have different meanings. For example, in computer science, logarithms have an assumed base of 2.

Use a scientific calculator to find the following logarithms.

a) $\log_4 64$

b) $\log_{\frac{2}{3}} \frac{8}{27}$

c) $\log_{\sqrt{2}} 2$

d) $\log 100$

Theorem 14: Logarithm Product Law

Given real numbers m and n with $m > 0$ and $n > 0$,

$$\log_b(m \times n) = \log_b n + \log_b n$$

In other words, logarithm of a product is the sum of logarithms.

How does this compare to the product law for exponents?

Expand each logarithm using the product law.

a) $\log(xy)$ b) $\log(5x)$ c) $\log(3(x+1))$ d) $\log(10x^2y)$

Condense each sum into a single logarithm.

a) $\log 3 + \log 4$ b) $\log \frac{2}{3} + \log \frac{3}{4}$ c) $\log x^2 + \log x^3$ d) $\log(x+1)+\log(x-2)$

Theorem 15: Logarithm Quotient Law

Given positive real numbers m and n,

$$\log_b \left(\frac{m}{n}\right) = \log_b m - \log_b n$$

In other words, the logarithm of a quotient is a difference of logarithms.

How does this compare to the quotient law for exponents?

Selected Solutions.