Chapter 4: Trigonometric Functions

Introduction

The past three chapters were about functions that grow and grow and grow. Some of the functions we saw grew at a constant rate, some functions grew in a way that their slope increased at a constant rate, and some functions grew in a way that their slope changed at the same rate that the function itself was growing. Which one was which?

Trigonometric functions are completely different than that. Trigonometric functions *oscilate.*

In the final chapter of our course, we will discuss the three primary trigonometric functions, how they are defined, how we can use them to solve problems about triangles, and how they relate to circles. We will end the chapter and this course learning about other geometric figures and how to caluclate their surface area and volume.

Angles

An angle in standard position:

Initial and terminal arms:

Reference angle:

Coterminal angles:

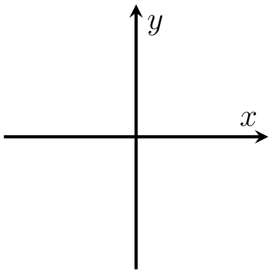
Quadrants:

Examples

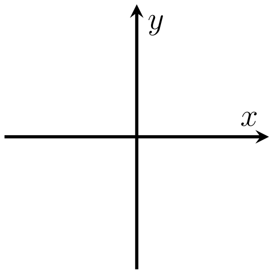
Exercises

1.1. Sketch each angle in standard position.

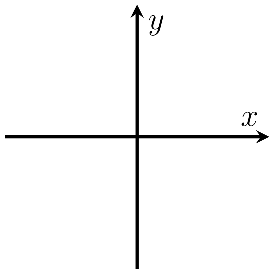
a) 100°



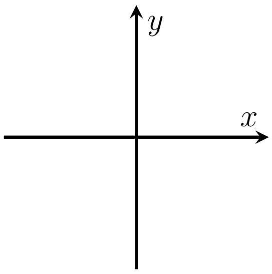
b) 80°



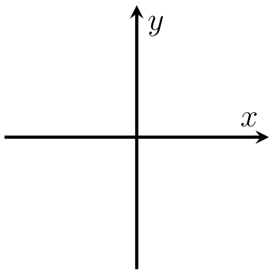
c) 200°



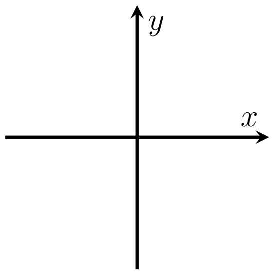
d) 320°



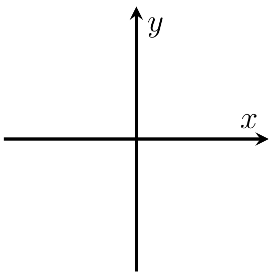
e) 270°



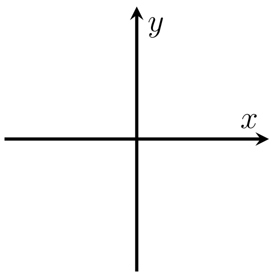
f) 450°



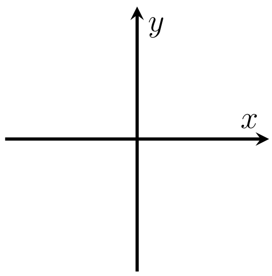
g) -45°



h) -730°



i) 3645°



1.2. In which quadrant does the terminal arm of each angle in standard position lie?

a) 48° b) 300° c) 185°

d) 75° e) 220° f) 160°

1.3. Find the reference angle.

a) 32° b) -32° c) 113° d) -113° e) 218°

f) -218° g) 304° h) -304° i) 832° j) -1213°

1.4. Determine the measure of the three other angles in standard position, $0° < \theta < 360°$, that have a reference angle of

a) $45°$

b) $60°$

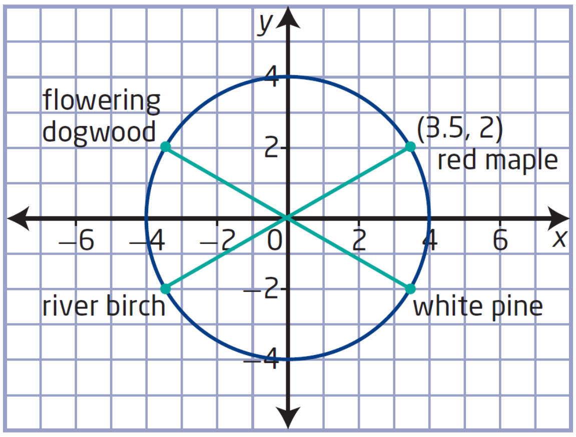
c) $30°$

d) $75°$

1.5. If quadrants II and III have the same reference angles and one of the standard position angles is $214°$, what is the other smallest positive standard position angle?

1.6. A windshield wiper has a length of 50 cm. The wiper rotates from its resting position at $30°$, in standard position, to $150°$. Determine the exact horizontal distance that the tip of the wiper travels in one swipe.

1.7. Paul and Gail decide to use a Cartesian plane to design a landscape plan for their yard. Each grid mark represents a distance of 10 m. Their home is centred at the origin. There is a red maple tree at the point $(3.5, 2)$. They will plant a flowering dogwood at a point that is a reflection in the y-axis of the position of the red maple. A white pine will be planted so that it is a reflection in the x-axis of the position of the red maple. A river birch will be planted so that it is a reflection in both the x-axis and the y-axis of the position of the red maple.



a) Determine the coordinates of the trees that Paul and Gail wish to plant.

b) Determine the angles in standard position if the lines drawn from the house to each of the trees are terminal arms. Express your answers to the nearest degree.

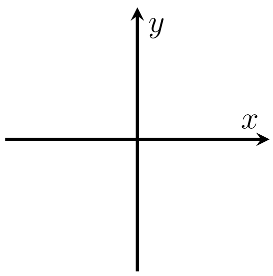
c) What is the actual distance between the red maple and the white pine?

Trigonometric Ratios

If the point (x, y) lies on the terminal arm of angle θ, then $r = \sqrt{x^2 + y^2}$, and:

$\sin\theta =$ $\qquad\qquad$ $\cos\theta =$ $\qquad\qquad$ $\tan\theta =$

The CAST rule:



The following trigonometric ratios are worth memorizing:

$\sin 0^\circ =$ $\qquad$ $\sin 30^\circ =$ $\qquad$ $\sin 45^\circ =$ $\qquad$ $\sin 60^\circ =$ $\qquad$ $\sin 90^\circ =$

$\cos 0^\circ =$ $\qquad$ $\cos 30^\circ =$ $\qquad$ $\cos 45^\circ =$ $\qquad$ $\cos 60^\circ =$ $\qquad$ $\cos 90^\circ =$

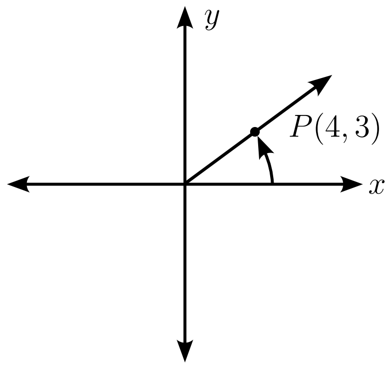
$\tan 0^\circ =$ $\qquad$ $\tan 30^\circ =$ $\qquad$ $\tan 45^\circ =$ $\qquad$ $\sin 60^\circ =$ $\qquad$ $\tan 90^\circ =$

Examples

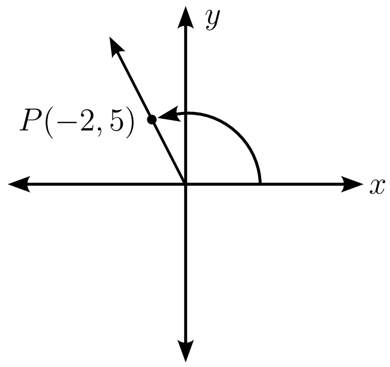
Exercises

2.1. Below are angles in standard position with a point on each terminal arm. Write the three primary trigonometric ratios (sine, cosine, and tangent) of each angle in standard position. Hint: find r using the Pythagorean Theorem, and state x and y.

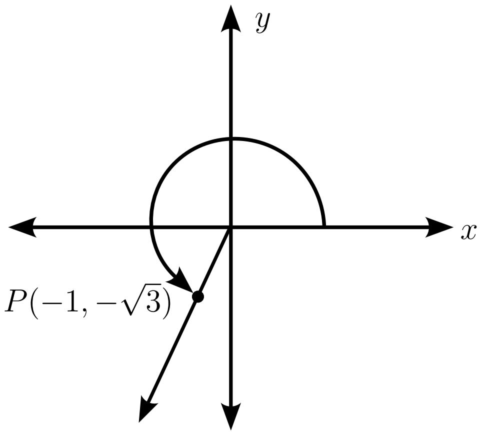
a)



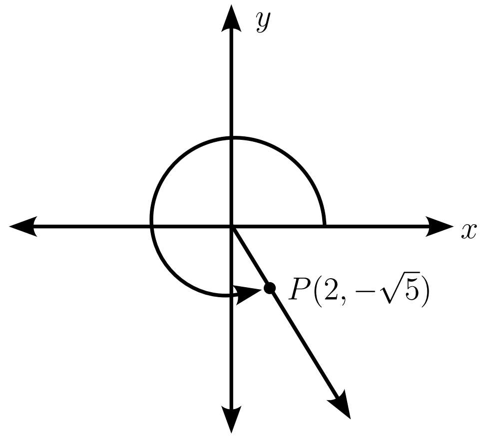
b)



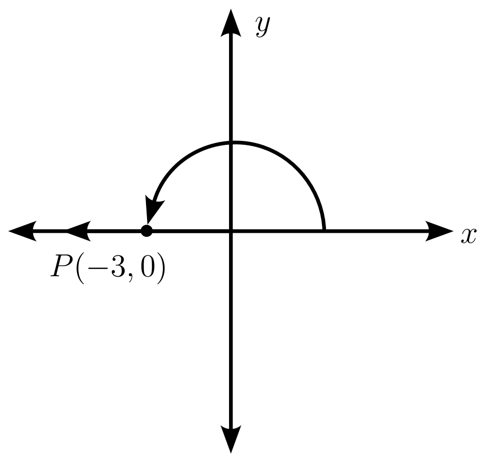
c)



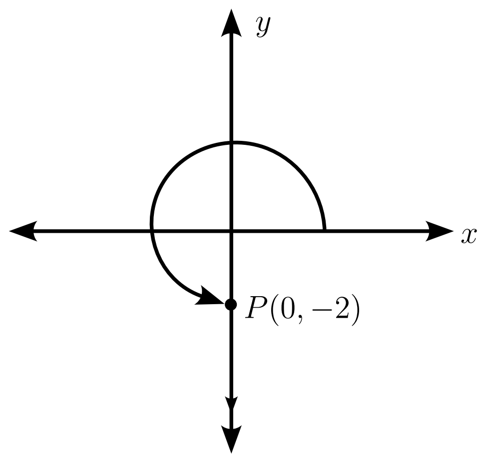
d)



e)



f)



2.2. If θ is in standard position and the given point is on the terminal side of θ, find the values of $\sin\theta, \cos\theta$, and $\tan\theta$.

a) $(3, -4)$

b) $(-12, 5)$

c) $(-7, -24)$

d) $(8, 15)$

e) $(-2\sqrt{3}, 2)$

f) $(\sqrt{2}, \sqrt{7})$

g) $(-3, -3)$

h) $(0, 4)$

i) $(-2, 0)$

j) $(-\sqrt{5}, 2)$

k) $(-5, 5)$

l) $(-2\sqrt{2}, 1)$

m) $(-40, -9)$

n) $(9, -40)$

o) $(2\sqrt{2}, 8)$

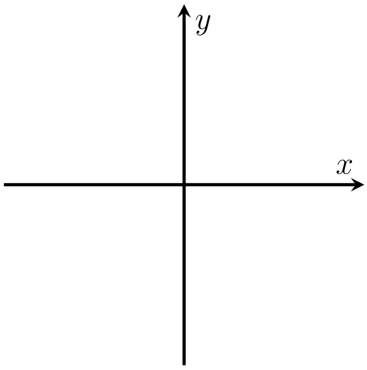
p) $(-3, \sqrt{3})$

q) $(-\sqrt{11}, -\sqrt{5})$

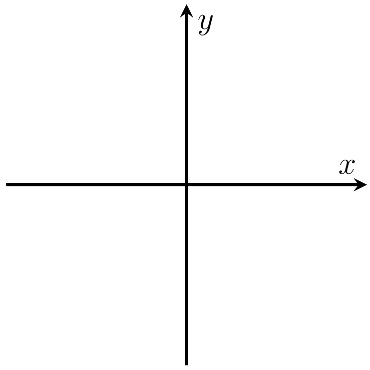
r) $(\sqrt{17}, 2\sqrt{2})$

2.3. A linear function with a domain is given. This is an equation of the terminal side of an angle in standard position. Draw the smallest positive angle θ, and find the values of $\sin\theta, \cos\theta$, and $\tan\theta$.

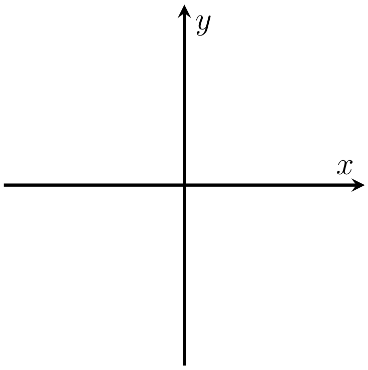
a) $y = 2x, x \leq 0$



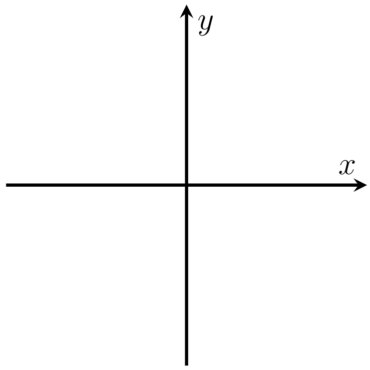
b) $y = -\frac{2}{5}x, x \geq 0$



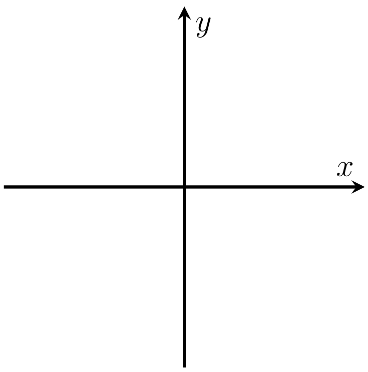
c) $y = \frac{4}{3}x, x \geq 0$



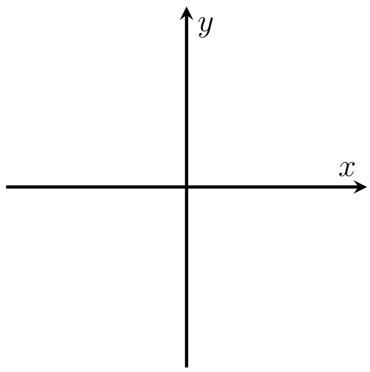
d) $y = -\frac{3}{2}x, x \leq 0$



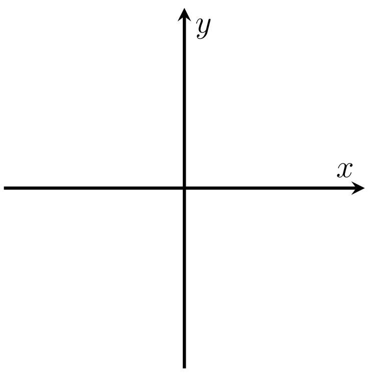
e) $y = \frac{5}{2}x, x \leq 0$



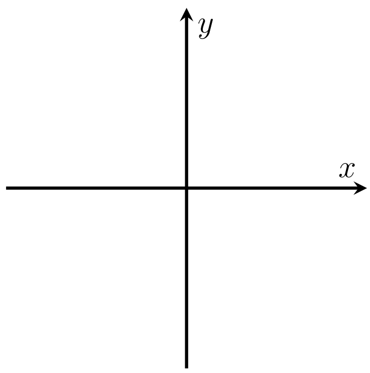
f) $y = -\frac{1}{4}x, x \geq 0$



g) $x = 0, \quad y \geq 0$



h) $y = 0, x \leq 0$



2.4. The value of one of the trigonometric functions is given, along with some additional information. Use this information to find the other two trigonometric functions of θ.

a) $\sin\theta = \frac{4}{5}, \theta$ in quadrant I

b) $\cos\theta = -\frac{12}{13}, \theta$ in quadrant II

c) $\tan\theta = \frac{7}{24}, \theta$ in quadrant III

d) $\sin\theta = -\frac{\sqrt{3}}{2}, \theta$ in quadrant IV

e) $\cos\theta = \frac{1}{2}, \tan\theta > 0$

f) $\tan\theta = \frac{2}{\sqrt{5}}, \sin\theta > 0$

g) $\sin\theta = -\frac{3}{\sqrt{10}}, \cos\theta < 0$

h) $\cos\theta = \frac{3}{\sqrt{13}}, \tan\theta < 0$

i) $\tan\theta = -1, \sin\theta < 0$

j) $\sin\theta = \frac{\sqrt{15}}{4}$

k) $\cos\theta = -\frac{\sqrt{3}}{4}$

l) $\tan\theta = -\sqrt{3}$

2.5. The value of one of the trigonometric functions is given along with some additional information. Use the trigonometric ratios to find the other two trigonometric functions of θ. Round each answer to three decimal places.

a) $\sin\theta = 0.642, \theta$ in quadrant I

b) $\cos\theta = 0.537, \theta$ in quadrant IV

c) $\tan\theta = 2, \theta$ in quadrant III

d) $\sin\theta = 0.237, \theta$ in quadrant II

e) $\cos\theta = -0.378, \sin\theta > 0$

f) $\tan\theta = -1.413, \cos\theta > 0$

g) $\sin\theta = -0.753, \tan\theta > 0$

h) $\cos\theta = -0.492, \sin\theta > 0$

i) $\tan\theta = -0.866, \sin\theta < 0$

j) $\sin\theta = -0.351$

k) $\cos\theta = 0.811$

l) $\tan\theta = 0.463$

2.6. The terminal side of angle θ passes through the intersection point of the given lines. Find the three trigonometric functions of θ.

a)
$$\begin{aligned} 2x + y &= 1 \\ -3x + y &= 6 \end{aligned}$$

b)
$$\begin{aligned} 3x + y &= 10 \\ x - 4y &= 12 \end{aligned}$$

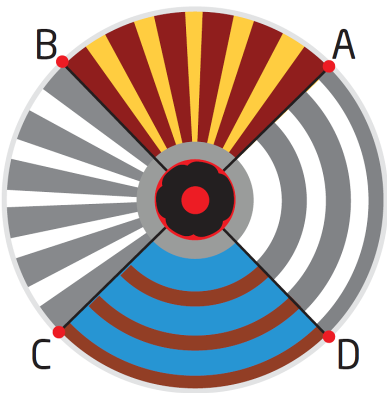
2.7. Using different values of θ, evaluate $\sin\theta$ and $\sin(-\theta)$. How does the value of $\sin(-\theta)$ compare to the value of $\sin\theta$?

2.8. Using different values of θ, evaluate $\cos\theta$ and $\cos(-\theta)$. How does the value of $\cos(-\theta)$ compare to the value of $\cos\theta$?

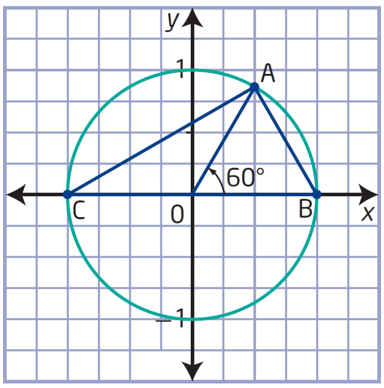
2.9. Indigenous Tourism Alberta designed a circular icon that represents both the Métis and First Nations communities of Alberta. The centre of the icon represents the collection of all peoples' perspectives and points of view relating to Indigenous history, touching every quadrant and direction.

a) Suppose the icon is placed on a coordinate plane with a reference angle of 45° for points A, B, C, and D. Determine the measure of the angles in standard position for points A, B, C, and D.

b) If the radius of the circle is 1 unit, determine the coordinates of points A, B, C, and D.



2.10. Consider an angle of 60° in standard position in a circle of radius 1 unit. Points A, B, and C lie on the circumference, as shown. Show that the lengths of the sides of $\triangle ABC$ satisfy the Pythagorean Theorem and that $\angle CAB = 90°$.



The Sine Law and Cosine Law

Solving a triangle without the Sine Law:

Solving a triangle with the Sine Law:

When you can use the Sine Law:

Proof of the Sine Law:

The Ambiguous Case:

The ambiguous case occurs when:

The Cosine Law:

When you can use the Cosine Law:

Proof of the Cosine Law:

Given	Method of Solving
ASA or AAS	1. Find the remaining angle using $\angle A + \angle B + \angle C = 180°$. 2. Find the remaining sides using the Law of Sines.
ASS	Be aware of the ambiguous case. There may be two solutions. 1. Find an angle using the Law of Sines. 2. Find the remaining angle using $\angle A + \angle B + \angle C = 180°$. 3. Find the remaining side using the Law of Sines.
SAS	1. Find the remaining side using the Law of Cosines. 2. Find the smaller of the two remaining angles using the Law of Sines. 3. Find the remaining angle using $\angle A + \angle B + \angle C = 180°$.
SSS	1. Find the largest angle using the Law of Cosines. 2. Find the remaining angle by using the Law of Sines. 3. Find the remaining angle using $\angle A + \angle B + \angle C = 180°$.

Examples

Exercises

3.1. Is a triangle with sides measuring 10, 4, and 5 possible? Explain.

3.2. In $\triangle ABC, \angle A = 65°, \angle B = 62°$. What side of the triangle is the shortest? What side is the longest?

3.3. Explain why no triangle is possible with the given information.

a) $A = 38°$ $B = 69°$ $C = 73°$
$a = 12$ $b = 14$ $c = 13$

b) $A = 42°$ $B = 65°$ $C = 70°$
$a = 7$ $b = 11$ $c = 12$

c) $A = 39°$ $B = 46°$ $C = 95°$
$a = 5$ $b = 6$ $c = 12$

d) $A = 120°$ $B = 20°$ $C = 40°$
$a = 12$ $b = 6$ $c = 12$

3.4. Find the angle that has the same sine ratio as following, with the restriction of $0° \leq \theta \leq 180°$.

a) $\sin 10°$

b) $\sin 30°$

c) $\sin 42°$

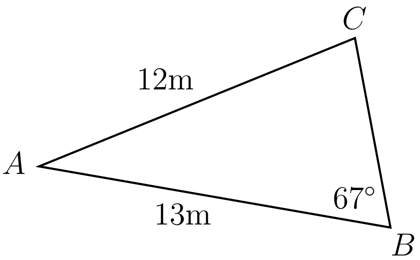
d) $\sin 71°$

e) $\sin 121°$

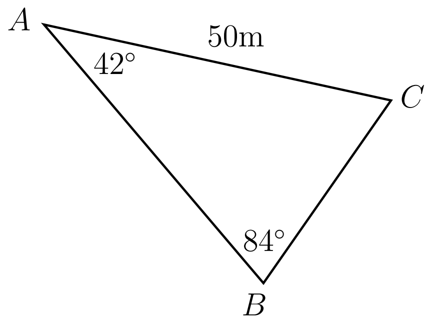
f) $\sin 137°$

3.5. Determining the lengths of all three sides and the measures of all three angles is called solving a triangle. Solve each triangle.

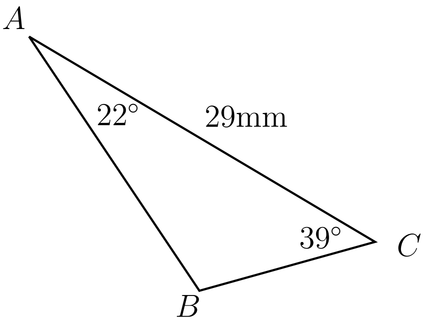
a)



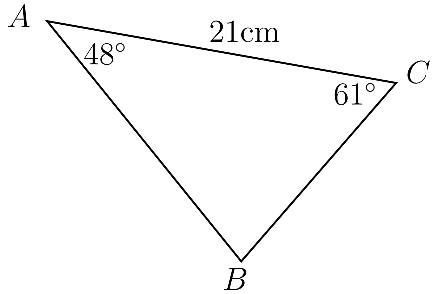
b)



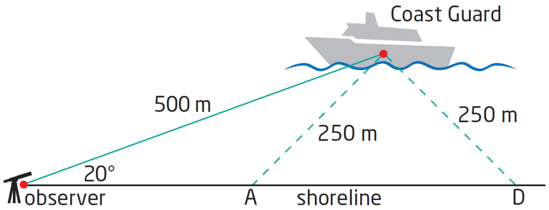
c)



d)

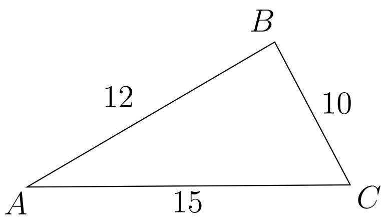


3.6. The Canadian Coast Guard Pacific Region is responsible for more than 27 000 km of coastline. The rotating spotlight from the Coast Guard ship can illuminate up to a distance of 250 m. An observer on the shore is 500 m from the ship. His line of sight to the ship makes an angle of 20° with the shoreline. What length of shoreline is illuminated by the spotlight?



3.7. Determine whether the Law of Sines or the Law of Cosines would be used to begin the solution process for each triangle.

a)



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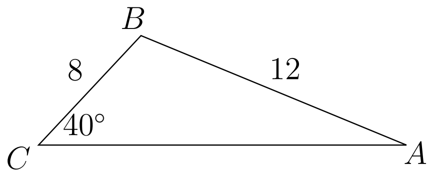
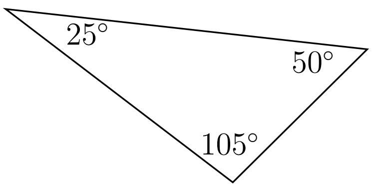
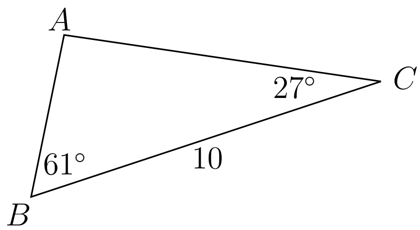


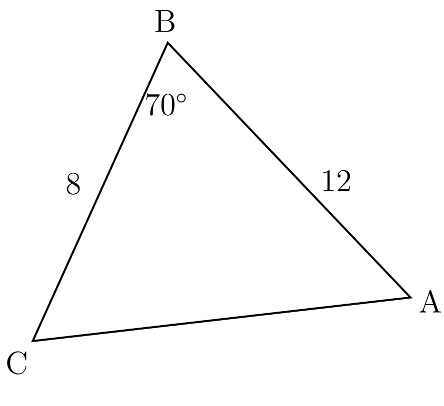
c)



d)



e)



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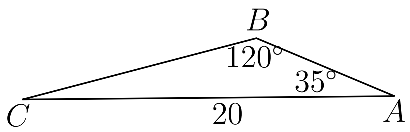


3.8. Solve the triangle using the Law of Cosines. Round answers to one decimal place. A drawing is very helpful.

a) $\angle A = 50°, b = 10, c = 15$

b) $\angle B = 36°, a = 4, c = 10$

c) $\angle C = 60°, b = 4, a = 8$

d) $a = 2, b = 3, c = 4$

e) $a = 7, b = 24, c = 25$

f) $a = 9, b = 14, c = 11$

g) $y = 4, z = 1, \angle X = 120°$

h) $x = 6, y = 7, z = 13$

i) $\angle S = 127.8°, k = 1578, d = 2654$

j) $s = 1504, q = 2365, r = 1953$

3.9. Solve each triangle using the Law of Sines. If two triangles exist, solve both completely. A drawing is very helpful.

a) $\angle A = 140°, \angle C = 25°, a = 20$

b) $\angle B = 38°, b = 8, a = 6$

c) $\angle C = 27°, \angle B = 46°, a = 120$

d) $\angle A = 110°, a = 24, b = 25$

e) $\angle B = 60°, b = 4\sqrt{3}, a = 8$

f) $\angle C = 41°, c = 9, a = 9$

g) $\angle A = 74°, a = 7, b = 8.1$

h) $\angle A = 58°, \angle B = 48°, b = 30.5$

i) $\angle A = 43°, \angle B = 38°, c = 17.2$

j) $\angle A = 33°, a = 27.2, b = 12.4$

k) $\angle A = 30°, a = 8, b = 10$

l) $\angle A = 58°, a = 9, b = 10$

m) $\angle A = 10°, \angle B = 60°, a = 4.5$

n) $\angle B = 10°, \angle C = 135°, c = 60$

o) $\angle C = 52°, c = 8.5, b = 12.4$

p) $\angle B = 27, b = 2, c = 5$

3.10. Solve $\triangle ABC$ using either the Law of Sines or the Law of Cosines to begin the solution.

a) $\angle A = 126°, b = 9, c = 12.2$

b) $\angle A = 28°, \angle B = 42°, c = 18.2$

c) $\angle B = 63°, b = 8, c = 10$

d) $\angle B = 41°, a = 11, c = 6$

e) $a = 12.3, b = 9.6, c = 8.9$

f) $\angle C = 38°, b = 9, c = 7$

g) $\angle C = 100°, a = 10, c = 10$

h) $\angle A = 60°, a = 2\sqrt{3}, c = 4$

i) $a = 1.32, b = 0.88, c = 0.65$

j) $\angle A = 75°, b = 4 - 2\sqrt{3}, a = \sqrt{6} - \sqrt{2}$

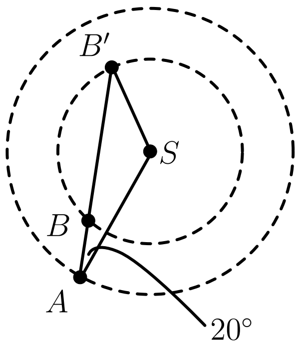
3.11. A plane is sighted by two observers 1 km apart at angles $74°$ and $78°$. How high is the plane?

3.12. A hot air balloon is flying directly between two cities that are 4 km apart. The balloonist finds that the angle of depression to one city is $38°$ and $33°$ to the other city. How high above the ground is the balloon?

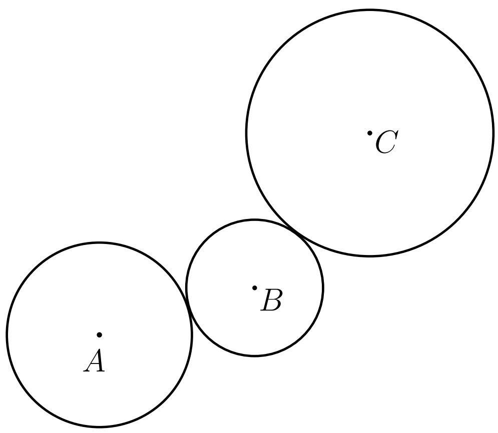
3.13. Two planes leave airport A in different directions. One plane lands at airport B, 630 km from airport A. The other plane lands at airport C some time later. If $\angle ABC = 110°$ and $\angle ACB = 40°$, how far did the second plane fly?

3.14. A plane flies 420 km from point A at a direction of 135° from due east and then travels 240 km at a direction of 240° from due east. How far is the plane from point A ?

3.15. In a solar system, the distance from the Sun (S) to planets A and B are 85 and 61 million miles respectively. When $\angle A = 20°$, how far is it from planet A to planet B and B'?

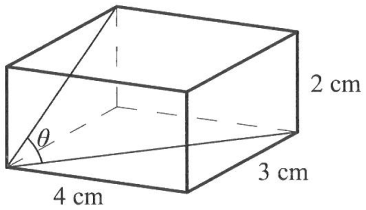


3.16. Three circles with radius $A = 4$ cm, $B = 3$ cm, and $C = 5$ cm are shown. If $\angle CAB = 35°$, how far is it from the centre of circle A to the centre of circle C?



3.17. Three circles of radius 3,5 , and 7 cm are tangent to each other. Find the largest angle formed by joining their centres.

3.18. The rectangular box has dimensions 4 cm $\times$ 3 cm $\times$ 2 cm. Find angle θ formed by a diagonal of the base, and a diagonal of the 2 cm $\times$ 3 cm side.



Volume and Surface Area

Volume of a prism:

Volume of a pyramid:

Volume of a cylinder:

Volume of a cone:

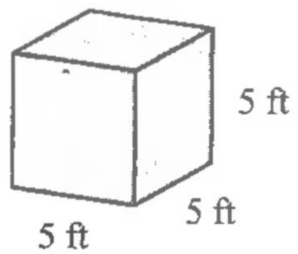
Surface area of a cone:

Examples

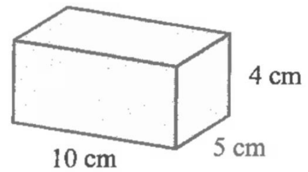
Exercises

4.1. Find the surface area and volume of each prism.

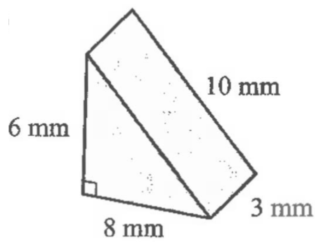
a)



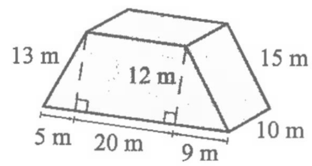
b)



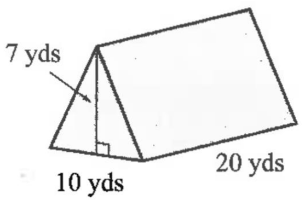
c)



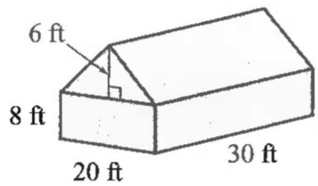
d)



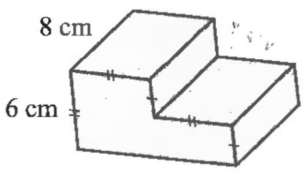
e)



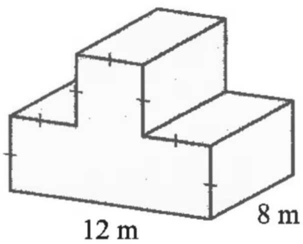
f)



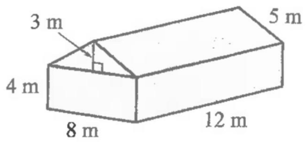
g)



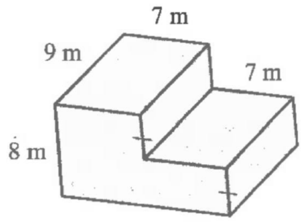
h)



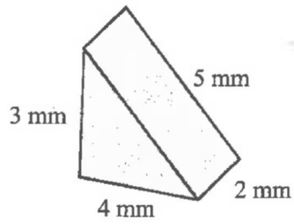
i)



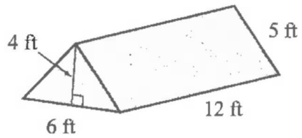
j)



k)



l)



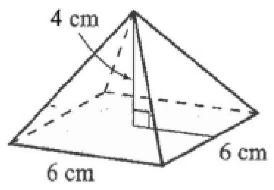
4.2. An Olympic size swimming pool is 50 m long, 21 m wide and 2 m deep. How many litres of water are needed to fill the pool? (1 $m^3 = 1000l$)

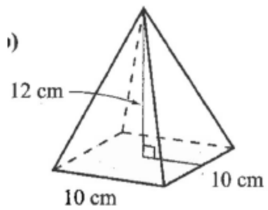
4.3. A regular hexagonal based prism has sides of 4 cm and a height of 5 cm.

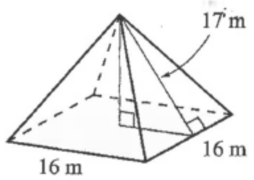
a) What is the surface area of the prism? b) What is the volume of the prism?

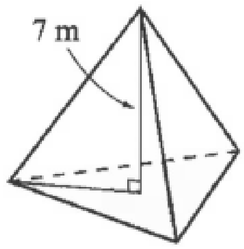
4.4. Find the volume of a right triangular prism with height $2(x + 1)$. The base is an isosceles right triangle with a hypotenuse of $\sqrt{2x}$.

4.5.

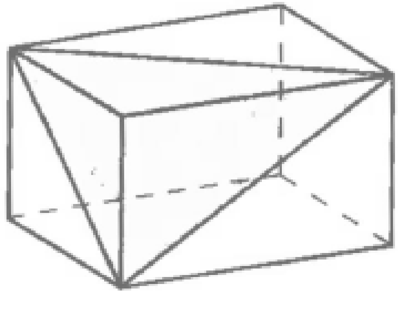
a) 

b) 

c) 

d) 

4.6. What is the ratio of the volume of the pyramid to the volume of the rectangular solid?



4.7. A regular hexagonal pyramid has base edges of length 1 cm, and lateral edges of length 2 cm.

a) What is the volume of the pyramid?

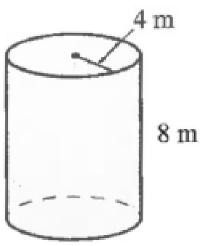
b) What is the surface area of the pyramid?

4.8. If you double all the dimensions of a pyramid, what does it do to the surface area? Volume?

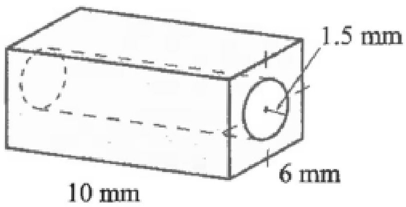
4.9. A pyramid has a rectangular base with length twice the width. If the height is 6 cm and the volume is 256 cm^3, what are the dimensions of the rectangular base?

4.10. Calculate the surface area and volume of each figure.

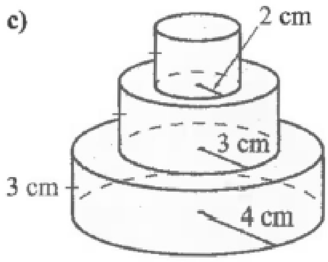
a)



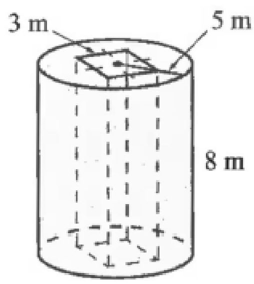
b)



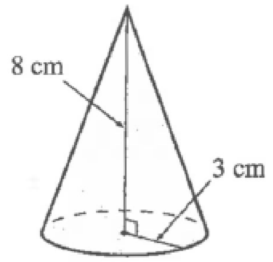
c)



d)



e)



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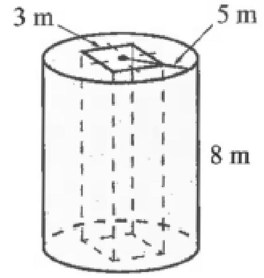

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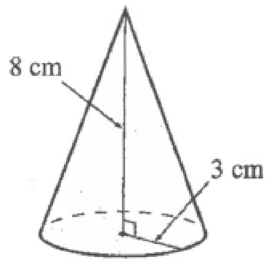

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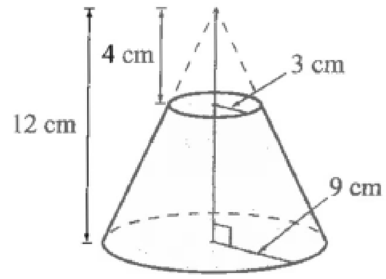
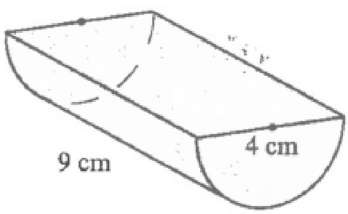
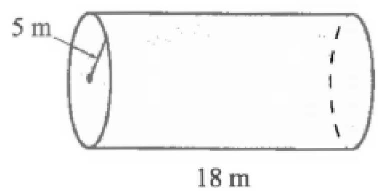


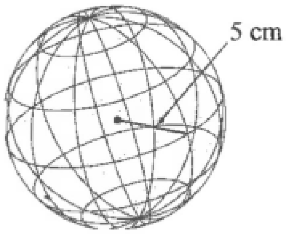
i)



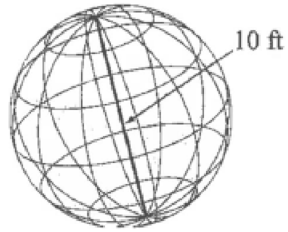
j)



k)

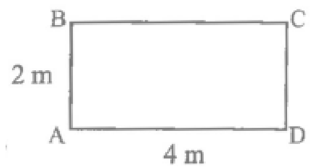


l)



4.11. Soup is sold in two can sizes. The can is cylindrical in shape. The height of can A is twice the height of can B. The radius of can B is twice the radius of can A. Can B costs twice as much as can A. Which can is the better buy?

4.12. If rectangle ABCD is is revolved about the line AD, what is the volume of the space through which it moves?



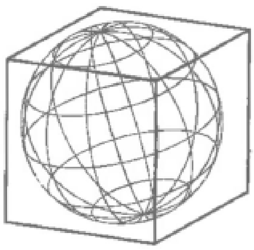
4.13. A cylindrical block of cheese has a 60° slice removed. If the volume of the removed slice is $18\pi \text{cm}^3$, what is the height of the block of cheese?

4.14. A hemispherical room requires four cans of paint to paint the floor. How many cans of paint are required to paint the walls?

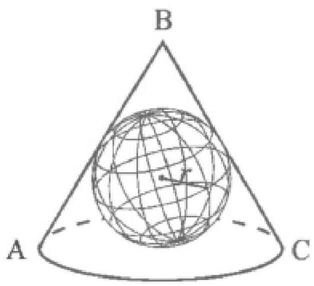
4.15. If a sphere and a cone have the same radius and volume, what must be the height of the cone in terms of the radius?

4.16. A sphere with a radius of 4 cm has a 45° section removed. What is the remaining surface area of the sphere?

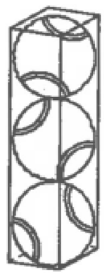
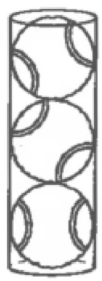
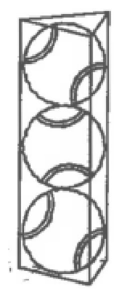
4.17. A sphere fits exactly into a cube. What is the ratio of the surface area of the sphere to the surface area of the cube?



4.18. A sphere of radius r is inscribed in a cone with diameter $AC = AB = BC$. Find the ratio of the volume of the sphere to the volume of the cone.



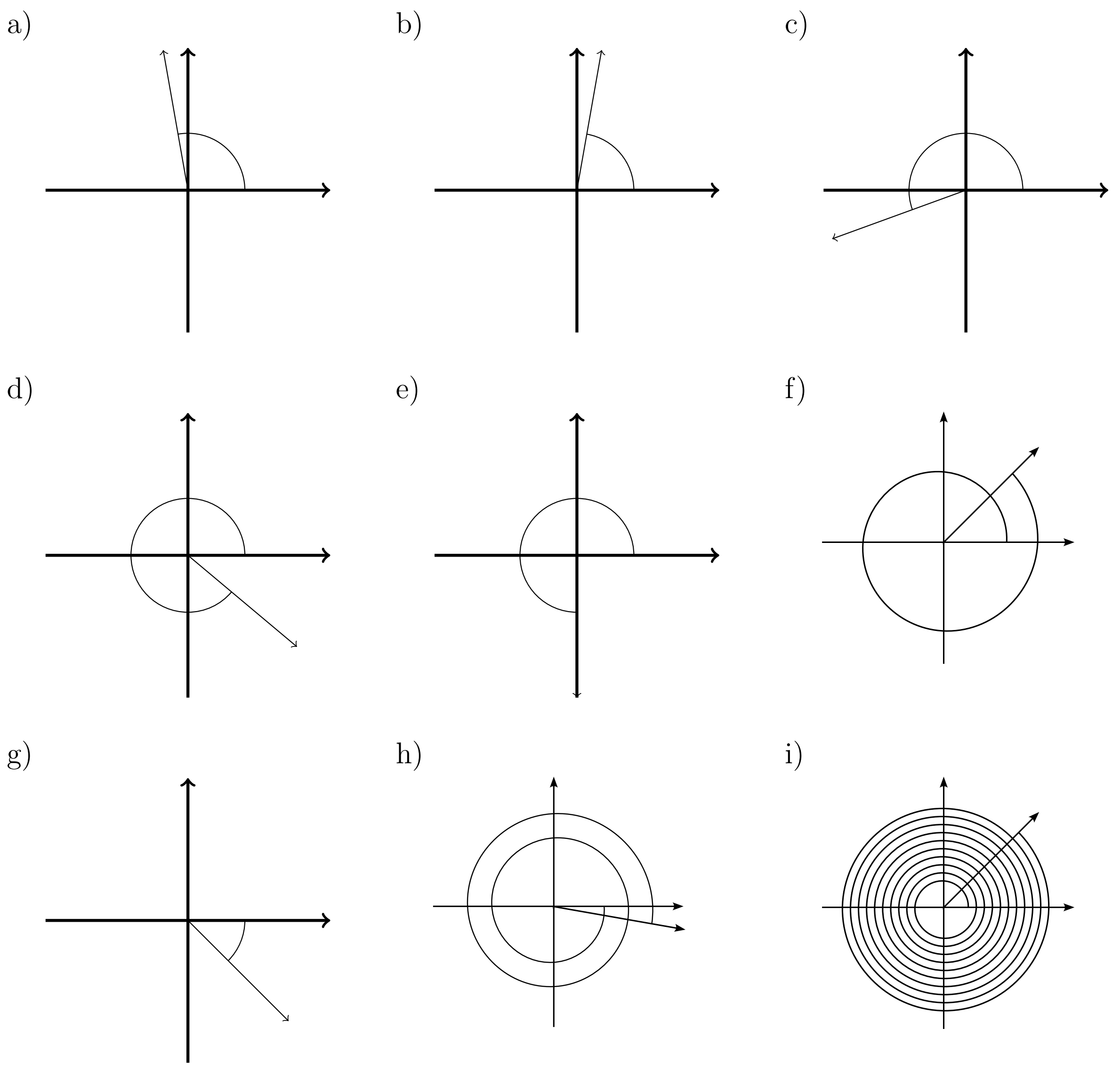
4.19. Three tennis balls are packed tightly in different containers. What is the ratio of the volume of the tennis balls to the volume of the box in figures a, b and c?

a)  b)  c) 

Selected Solutions.

Section 1 Solutions

1.1.



1.2.

a) $48°$ is is quadrant I. b) $300°$ is in quadrant IV. c) $185°$ is in quadrant III.

d) $75°$ is in quadrant I. e) $220°$ is in quadrant III. f) $160°$ is in quadrant II.

1.3.

a) $32°$ b) $32°$ c) $67°$ d) $67°$ e) $38°$

f) $38°$ g) $56°$ h) $56°$ i) $68°$ j) $47°$

1.4.

a) $135°, 225°$, and $315°$.

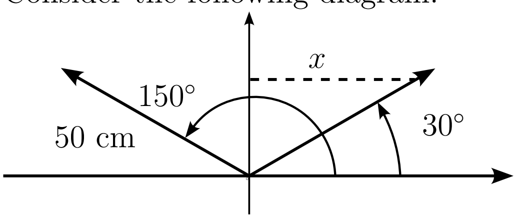
b) $120°, 240°$, and $300°$.

c) $150°, 210°$, and $330°$.

d) $105°, 255°$, and $285°$.

1.5. $146°$

1.6. Consider the following diagram.



$$\begin{aligned} \cos 30° &= \frac{x}{50} \\ \frac{\sqrt{3}}{2} &= \frac{x}{50} \\ x &= 25\sqrt{3} \end{aligned}$$

By symmetry, the horizontal distance that the tip of the wiper travels in one swipe will be $2x$, or $50\sqrt{3}$ cm.

1.7.

a) The coordinates of the other three trees are found using symmetries of the diagram: flowering dogwood $(-3.5, 2)$, river birch $(-3.5, -2)$, white pine $(3.5, -2)$.

b) For the red maple,

$$\begin{aligned} \tan\theta &= \frac{2}{3.5} \\ \theta &= \tan^{-1}\left(\frac{2}{3.5}\right) \\ \theta &= 29.744\ldots \end{aligned}$$

The angle in standard position for the red maple is $30°$, to the nearest degree. Then, the angle in standard position for the flowering dogwood is $180° - 30°$ or $150°$, to the nearest degree. The angle in standard position for the river birch is $180° + 30°$ or $210°$, to the nearest degree. The angle in standard position for the white pine is $360° - 30°$ or $330°$, to the nearest degree.

c) On the grid, there are 4 vertical units of distance between the red maple and the white pine. Since each grid mark represents 10 m, the distance between these two trees is 40 m.

Section 2 Solutions

2.1.

a) $\sin\theta = \frac{3}{5}, \cos\theta = \frac{4}{5}, \tan\theta = \frac{3}{4}$

b) $\sin\theta = \frac{5}{\sqrt{29}}, \cos\theta = -\frac{2}{\sqrt{29}}, \tan\theta = -\frac{5}{2}$

c) $\sin\theta = -\frac{\sqrt{3}}{2}, \cos\theta = -\frac{1}{2}, \tan\theta = \sqrt{3}$

d) $\sin\theta = -\frac{\sqrt{5}}{3}, \cos\theta = \frac{2}{3}, \tan\theta = -\frac{\sqrt{5}}{2}$

e) $\sin\theta = 0, \cos\theta = -1, \tan\theta = 0$

f) $\sin\theta = -1, \cos\theta = 0, \tan\theta$ is undefined

2.2.

a) $\sin\theta = -\frac{4}{5}, \cos\theta = \frac{3}{5}, \tan\theta = -\frac{4}{3}$

b) $\sin\theta = \frac{5}{13}, \cos\theta = -\frac{12}{13}, \tan\theta = -\frac{5}{12}$

c) $\sin\theta = -\frac{24}{25}, \cos\theta = -\frac{7}{25}, \tan\theta = \frac{24}{7}$

d) $\sin\theta = \frac{15}{17}, \cos\theta = \frac{8}{17}, \tan\theta = \frac{15}{8}$

e) $\sin\theta = \frac{1}{2}, \cos\theta = -\frac{\sqrt{3}}{2}, \tan\theta = -\frac{1}{\sqrt{3}}$

f) $\sin\theta = \frac{\sqrt{7}}{3}, \cos\theta = \frac{\sqrt{2}}{3}, \tan\theta = \frac{\sqrt{7}}{\sqrt{2}}$

g) $\sin\theta = -\frac{1}{\sqrt{2}}, \cos\theta = -\frac{1}{\sqrt{2}}, \tan\theta = 1$

h) $\sin\theta = 1, \cos\theta = 0, \tan\theta$ is undefined

i) $\sin\theta = 0, \cos\theta = -1, \tan\theta = 0$

j) $\sin\theta = \frac{2}{3}, \cos\theta = -\frac{\sqrt{5}}{3}, \tan\theta = -\frac{2}{\sqrt{5}}$

k) $\sin\theta = \frac{1}{\sqrt{2}}, \cos\theta = -\frac{1}{\sqrt{2}}, \tan\theta = -1$

l) $\sin\theta = \frac{1}{3}, \cos\theta = -\frac{2\sqrt{2}}{3}, \tan\theta = -\frac{1}{2\sqrt{2}}$

m) $\sin\theta = -\frac{9}{41}, \cos\theta = -\frac{40}{41}, \tan\theta = \frac{9}{40}$

n) $\sin\theta = -\frac{40}{41}, \cos\theta = \frac{9}{41}, \tan\theta = -\frac{40}{9}$

o) $\sin\theta = \frac{2\sqrt{2}}{3}, \cos\theta = \frac{1}{3}, \tan\theta = 2\sqrt{2}$

p) $\sin\theta = \frac{1}{2}, \cos\theta = -\frac{\sqrt{3}}{2}, \tan\theta = \frac{-\sqrt{3}}{3}$

q) $\sin\theta = -\frac{\sqrt{5}}{4}, \cos\theta = -\frac{\sqrt{11}}{4}, \tan\theta = \frac{\sqrt{55}}{11}$

r) $\sin\theta = \frac{2\sqrt{2}}{5}, \cos\theta = \frac{\sqrt{17}}{5}, \tan\theta = \frac{2\sqrt{34}}{17}$

2.3.

a) $\sin\theta = -\frac{2}{\sqrt{5}}, \cos\theta = -\frac{1}{\sqrt{5}}, \tan\theta = 2$

b) $\sin\theta = -\frac{2}{\sqrt{29}}, \cos\theta = \frac{5}{\sqrt{29}}, \tan\theta = -\frac{2}{5}$

c) $\sin\theta = \frac{4}{5}, \cos\theta = \frac{3}{5}, \tan\theta = \frac{4}{3}$

d) $\sin\theta = \frac{3}{\sqrt{13}}, \cos\theta = -\frac{2}{\sqrt{13}}, \tan\theta = -\frac{3}{2}$

e) $\sin\theta = -\frac{5}{\sqrt{29}}, \cos\theta = -\frac{2}{\sqrt{29}}, \tan\theta = \frac{5}{2}$

f) $\sin\theta = -\frac{1}{\sqrt{17}}, \cos\theta = \frac{4}{\sqrt{17}}, \tan\theta = -\frac{1}{4}$

g) $\sin\theta = 1, \cos\theta = 0, \tan\theta = \infty$

h) $\sin\theta = 0, \cos\theta = -1, \tan\theta = 0$

2.4.

a) $\cos\theta = \frac{3}{5}, \tan\theta = \frac{4}{3}$

b) $\sin\theta = \frac{5}{13}, \tan\theta = -\frac{5}{12}$

c) $\sin\theta = -\frac{7}{25}, \cos\theta = -\frac{24}{25}$

d) $\cos\theta = \frac{1}{2}, \tan\theta = -\sqrt{3}$

e) $\sin\theta = \frac{\sqrt{3}}{2}, \tan\theta = \sqrt{3}$

f) $\sin\theta = \frac{2}{3}, \cos\theta = \frac{\sqrt{5}}{3}$

g) $\cos\theta = -\frac{1}{\sqrt{10}}, \tan\theta = 3$

h) $\sin\theta = -\frac{2}{\sqrt{13}}, \tan\theta = -\frac{2}{3}$

i) $\sin\theta = -\frac{1}{\sqrt{2}}, \cos\theta = \frac{1}{\sqrt{2}}$

j) $\cos\theta = \pm\frac{1}{4}, \tan\theta = \pm\sqrt{15}$

k) $\sin\theta = \pm\frac{\sqrt{13}}{4}, \tan\theta = \pm\frac{\sqrt{39}}{3}$

l) $\sin\theta = \pm\frac{\sqrt{3}}{2}, \cos\theta = \pm\frac{1}{2}$

2.5.

a) $\cos\theta = 0.767, \tan\theta = 0.837$

b) $\sin\theta = -0.844, \tan\theta = -1.571$

c) $\sin\theta = -0.894, \cos\theta = -0.447$

d) $\cos\theta = -0.972, \tan\theta = -0.244$

e) $\sin\theta = 0.926, \tan\theta = -2.450$

f) $\sin\theta = -0.816, \cos\theta = 0.578$

g) $\cos\theta = -0.658, \tan\theta = 1.144$

h) $\sin\theta = 0.871, \tan\theta = -1.770$

i) $\sin\theta = -0.655, \cos\theta = 0.756$

j) $\cos\theta = \pm 0.936, \tan\theta = \pm 0.375$

k) $\sin\theta = \pm 0.585, \tan\theta = \pm 0.721$

l) $\sin\theta = \pm 0.420, \cos\theta = \pm 0.907$

2.7. $\sin(-\theta) = -\sin\theta$

2.8. $\cos(-\theta) = \cos\theta$

2.9.

a) The measure for $\angle A$ is $45°$, for $\angle B$ is $135°$, for $\angle C$ is $225°$, and for $\angle D$ is $315°$.

b) Point A is on the terminal arm of $45°$: $A\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$. B is on the terminal arm of $135°$: $B\left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$. C is on the terminal arm of $225°$: $C\left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$. D is on the terminal arm of $315°$: $D\left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$.

2.10. Since they are both radii, $OA = OB = 1$. Therefore, $\triangle OAB$ is isosceles with $\angle OAB = \angle ABO = 60°$. Hence $\triangle OAB$ is equilateral and $AB = 1$, and $CB = 2$.

Since OA is the terminal arm of $60°$, the coordinates of A are $\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$. The coordinates of C are $(-1, 0)$. Then, use the Pythagorean Theorem in the right triangle with hypotenuse AC.

$$
\begin{aligned}
AC^2 &= \left(\frac{\sqrt{3}}{2}\right)^2 + \left(\frac{3}{2}\right)^2 \\
AC^2 &= \frac{3}{4} + \frac{9}{4} \\
AC^2 &= 3 \\
AC &= \sqrt{3}
\end{aligned}
$$

Then, in $\angle ABC$, $BC^2 = 4$, and

$$AC^2 + AB^2 = 3 + 1 = 4$$

So the sides of $\triangle ABC$ satisfy the Pythagorean Theorem, with $\angle CAB = 90°$ because BC is the hypotenuse.

Section 3 Solutions

3.1. No. The two shorter sides combined must be longer than the longest side.

3.2. Side c is the shortest and a is the longest side. The shortest side is opposite the smallest angle ($\angle C = 53°$), and the longest side is opposite the largest angle ($\angle A = 65°$).

3.3.

a) The largest angle ($\angle C = 73°$) must have the largest side opposite it. This is not the case.

b) The three angles do not add up to 180°.

c) The two smaller sides added are less than the largest side.

d) Two angles of different degrees, $\angle A$ and $\angle C$, cannot have sides of the same value.

3.4.

a) 170°   b) 150°   c) 138°   d) 109°   e) 59°   f) 43°

3.5.

a) This is an ambiguous case. First find the acute measure of $\angle C$:

$$\begin{aligned}
\frac{\sin C}{c} &= \frac{\sin B}{b} \\
\frac{\sin C}{13} &= \frac{\sin 67^\circ}{12} \\
\sin C &= \frac{13 \sin 67^\circ}{12} \\
\angle C &= \sin^{-1}\left(\frac{13 \sin 67^\circ}{12}\right) \\
\angle C &= 85.721\ldots
\end{aligned}$$

The acute measure of $\angle C$ is 86°, to the nearest degree. Then, the obtuse measure of $\angle C$ is $180^\circ - 85.721^\circ = 94.279\ldots\circ$. The obtuse measure of $\angle C$ is 94°, to the nearest degree.
Then, find the measure of $\angle A$.

$$\begin{aligned}
\angle A &= 180^\circ - (85.721\ldots^\circ + 67^\circ) \\
\angle A &= 27.279\ldots^\circ
\end{aligned}$$

or

$$\begin{aligned}
\angle A &= 180^\circ - (94.279\ldots^\circ + 67^\circ) \\
\angle A &= 18.721\ldots^\circ
\end{aligned}$$

The measure of $\angle A$ is 27° or 19°, to the nearest degree. Now find the measure of side a.

$$\begin{aligned}
\frac{a}{\sin A} &= \frac{b}{\sin B} \\
\frac{a}{\sin 27.279...0^\circ} &= \frac{12}{\sin 67^\circ} \\
a &= \frac{12 \sin 27.279...^\circ}{\sin 67^\circ} \\
a &= 5.974\ldots
\end{aligned}$$

or

$$\begin{aligned}
\frac{a}{\sin A} &= \frac{b}{\sin B} \\
\frac{a}{\sin 18.721...^\circ} &= \frac{12}{\sin 67^\circ} \\
a &= \frac{12 \sin 18.721...^\circ}{\sin 67^\circ} \\
a &= 4.184\ldots
\end{aligned}$$

Summary:
Acute case: $\angle C = 86^\circ, \angle A = 27^\circ$, side a is 6.0 m, to the nearest tenth of a metre.
Obtuse case: $\angle C = 94^\circ, \angle A = 19^\circ$, side a is 4.2 m, to the nearest tenth of a metre.
$<$ Note: If you use the rounded degree values, then a is 5.9 m for the acute case. For the obtuse case, using rounded degree values results in the same answer.

b) $\angle C = 54^\circ$. The length of side a is 33.6 m, to the nearest tenth of a metre. The length of side c is 40.7 m, to the nearest tenth of a metre.

c) $\angle B = 119^\circ$. The length of side a is 12.4 mm, to the nearest tenth of a millimetre. The length of side c is 20.9 mm, to the nearest millimetre.

d) $\angle B = 71^\circ$. The length of side a is 16.5 cm, to the nearest tenth of a centimetre. The length of side c is 19.4 cm, to the nearest centimetre.

3.6. Let C be the position of the coast guard ship and H the foot of the perpendicular from C to the shoreline.

$$\begin{aligned}\frac{\sin \mathrm{D}}{500} &= \frac{\sin 20^\circ}{250}\\ \sin \mathrm{D} &= \frac{500 \sin 20^\circ}{250}\\ \angle \mathrm{D} &= \sin^{-1}\left(\frac{500 \sin 20^\circ}{250}\right)\\ \angle \mathrm{D} &= 43.160\ldots\end{aligned}$$

Then, in $\triangle$CDH :

$$\begin{aligned}\cos 43.160\ldots^\circ &= \frac{\mathrm{DH}}{250}\\ \mathrm{DH} &= 250 \cos 43.160\ldots^\circ\\ \mathrm{DH} &= 182.361\ldots\end{aligned}$$

Since $\mathrm{CA} = \mathrm{CD}, \Delta\mathrm{ACH} \approx \triangle\mathrm{DCH}$ and AH + HD. So $\mathrm{AD} = 2(182.361\ldots)$ or $364.722\ldots$ The length of shoreline that is illuminated by the spotlight is 364.7 m, to the nearest tenth of a metre.

3.7.

a) Law of Cosines

b) Law of Sines

c) none

d) Law of Sines

e) Law of Cosines

f) Law of Sines

3.8.

a) $\angle B = 41.8^\circ, \angle C = 88.2^\circ, a = 11.5$

b) $\angle A = 19.2^\circ, \angle C = 124.8^\circ, a = 7.2$

c) $\angle A = 90^\circ, \angle B = 30^\circ, c = 6.9$

d) $\angle A = 29.0^\circ, \angle B = 46.5^\circ, \angle C = 104.5^\circ$

e) $\angle A = 16.3^\circ, \angle B = 73.7^\circ, \angle C = 90^\circ$

f) $\angle A = 40^\circ, \angle B = 88.3^\circ, \angle C = 51.7^\circ$

g) $\angle Y = 49.1^\circ, \angle Z = 10.9^\circ, x = 4.6$

h) Impossible triangle.

i) $\angle D = 33.2^\circ, \angle K = 19.0^\circ, s = 3829.8$

j) $\angle Q = 85.3^\circ, \angle R = 55.4^\circ, \angle S = 39.3^\circ$

3.9.

a) $\angle B = 15°, b = 8.1, c = 13.1$

b) $\angle A = 27.5°, \angle C = 114.5°, c = 11.8$

c) $\angle A = 107°, b = 90.3, c = 57.0$

d) Impossible triangle.

e) $\angle A = 90°, \angle C = 30°, c = 4$

f) $\angle A = 41°, \angle B = 98°, b = 13.6$

g) Impossible triangle.

h) $\angle C = 74°, a = 34.8, c = 39.5$

i) $\angle C = 99°, a = 11.9, b = 10.7$

j) $\angle B = 14.4°, \angle C = 132.6°, c = 36.7$

k) $\angle B = 38.7°, 141.3°, \angle C = 111.3°, 8.7°, c = 14.9.2.4$

l) $\angle B = 70.4°, 109.6°, \angle C = 51.6°, 12.4°, c = 8.3, 2.3$

m) $\angle C = 110°, b = 22.4, c = 24.4$

n) $\angle A = 35°, a = 48.7, b = 14.7$

o) Impossible triangle.

p) Impossible triangle.

3.10.

a) $\angle B = 22.6°, \angle C = 31.4°, a = 18.9$

b) $\angle C = 110°, a = 9.1, b = 13.0$

c) Impossible triangle.

d) $\angle A = 107.7°, \angle C = 31.3°, b = 7.6$

e) $\angle A = 83.3°, \angle B = 50.8°, \angle C = 45.9°$

f) $\angle A = 89.7°, 14.3°, \angle B = 52.3°, 127.7°, a = 11.4, 2.8$

g) Impossible triangle.

h) $\angle B = 30°, \angle C = 90°, b = 2$

i) $\angle A = 118.5°, \angle B = 35.9°, \angle C = 25.6°$

j) $\angle B = 30°, \angle C = 75°, c = \sqrt{6} - \sqrt{2}$

3.11. 13.48 km

3.12. 1.42 km

3.13. 921 km

3.14. 426.4 km

3.15. $AB' = 133.6$ million miles $AB = 26.4$ million miles

3.16. 12.65 cm

3.17. 82.8°

Section 4 Solutions

4.1.

a) $150\mathrm{ft}^2, 125\mathrm{ft}^3$

b) $220\ \mathrm{cm}^2, 200\ \mathrm{cm}^3$

c) $120\ \mathrm{mm}^2, 72\ \mathrm{mm}^3$

d) $1468\ \mathrm{m}^2, 3240\ \mathrm{m}^3$

e) $614.1\mathrm{yd}^2, 700\mathrm{yd}^3$

f) $2219.7\mathrm{ft}^2, 6600\mathrm{ft}^3$

g) $396\ \mathrm{cm}^2, 432\ \mathrm{cm}^3$

h) $448\ \mathrm{m}^2, 512\ \mathrm{m}^3$

i) $400\ \mathrm{m}^2, 528\ \mathrm{m}^3$

j) $564\ \mathrm{m}^2, 756\ \mathrm{m}^3$

k) $36\ \mathrm{mm}^2, 12\ \mathrm{mm}^3$

l) $216\mathrm{ft}^2, 144\mathrm{ft}^3$

4.2. 2100000 L

4.3.

a) $48\sqrt{3} + 120 \approx 203.14\ \mathrm{cm}^2$

b) $120\sqrt{3} \approx 207.84\ \mathrm{m}^3$

4.4. $x^3 + x^2$

4.5.

a) $96\ \mathrm{cm}^2, 48\ \mathrm{cm}^3$

b) $360\ \mathrm{cm}^2, 400\ \mathrm{cm}^3$

c) $800\ \mathrm{m}^2, 1280\ \mathrm{m}^3$

d) $173.2\ \mathrm{m}^2, 101.0\ \mathrm{m}^3$

4.6. $1 : 6$

4.7.

a) $V = \frac{1}{3}\left(A_{\text{base}}\right)(h) = \frac{1}{3}\left(\frac{3\sqrt{3}}{2}\right)(\sqrt{3}) = \frac{3}{2}\ \mathrm{cm}^2$

b) $S.A. =$ area of base $+6$ sides $= \frac{3\sqrt{3}}{2} + 6\left(\frac{1}{2} \cdot 1 \cdot \frac{\sqrt{15}}{2}\right) = \left(\frac{3\sqrt{3}}{2} + \frac{3\sqrt{15}}{2}\right) \approx 8.41\ \mathrm{cm}^2$

4.8. Surface Area is 4 times as large. Volume is 8 times as large.

4.9. $V = \frac{1}{3}\left(A_{\text{base}}\right)(h) \rightarrow 256 = \frac{1}{3}(x \cdot 2x)(6) \rightarrow x^2 = 64 \rightarrow x = 8$; Therefore 8 cm $\times$ 16 cm

4.10.

a) $301.6\ \mathrm{m}^2, 402.1\ \mathrm{m}^3$

b) $392.1\ \mathrm{mm}^2, 289.3\ \mathrm{mm}^3$

c) $270.2\ \mathrm{m}^2, 273.3\ \mathrm{m}^3$

d) $486.4\ \mathrm{m}^2, 556.3\ \mathrm{m}^3$

e) $108.8\ \mathrm{cm}^2, 75.4\ \mathrm{cm}^3$

f) $659.73\ \mathrm{cm}^2, 980.18\ \mathrm{cm}^3$

g) $105.11\ \mathrm{cm}^2, 56.55\ \mathrm{cm}^3$

h) $722.57\ \mathrm{m}^2, 1413.72\ \mathrm{m}^3$

i) $314.16\ \mathrm{cm}^2, 523.60\ \mathrm{cm}^3$

j) $314.16\mathrm{ft}^2, 523.60\mathrm{ft}^3$

4.11. Can B holds twice the volume of can A, so they are of equal value.

4.13. $V = \frac{1}{6}\pi r^2 h \rightarrow 18\pi = \frac{1}{6}\pi \cdot 6^2 \cdot h \rightarrow h = 3$ cm

4.14. $A = \pi r^2 = 4 \to r^2 = \frac{4}{\pi}; S.A. = 2\pi r^2 = 2\pi \cdot \frac{4}{\pi} = 8$ cans

4.15. $\frac{4}{3}\pi r^3 = \frac{1}{3}\pi r^2 h \quad \to \quad h = 4r$

4.16. S.A. $= \frac{7}{8}\left(4\pi r^2\right) + \pi r^2 = \frac{7}{8}\left(4\pi \cdot 4^2\right) + \pi \cdot 4^2 = 72\pi \approx 226.19 \text{ cm}^2$

4.17. $4\pi r^2 : 6(2r)^2 \to 4\pi r^2 = 24r^2 \to \pi : 6$

4.18. $\frac{4}{3}\pi r^3 : \frac{1}{3}\pi R^2 h \to 4r^3 : (\sqrt{3}r)^2(3r) \to 4r^3 : 9r^3 \to 4 : 9$

4.19.

a) $3\left(\frac{4}{3}\pi r^3\right) : (2r)(2r)(6r) \to 4\pi r^3 : 24r^3 \to \pi : 6$

b) $3\left(\frac{4}{3}\pi r^3\right) : \pi \cdot r^2 \cdot 6r \to 4\pi r^3 : 6\pi r^3 \to 2 : 3$

c) $3\left(\frac{4}{3}\pi r^3\right) : \frac{1}{2}(2\sqrt{3}r)(3r)(6r) \to 4\pi r^3 : 18\sqrt{3}r^3 \to 2\pi : 9\sqrt{3}$