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```
In [1]: import numpy as np
        import sklearn as skl
        from matplotlib import pyplot as plt
        %pylab inline
```

```
Populating the interactive namespace from numpy and matplotlib
```

1 1.1 Inverse Transform sampling

For Y = 0:

$$
\begin{aligned}
F_{Y=0} &= \int_{-\inf}^{x} p_{Y=0}(x')\mathrm{d}x' = \int_{0}^{x} 2 - 2x'\mathrm{d}x' = 2x' - x'^2\big|_0^x && (1)\\
&= 2x - x^2 \overset{!}{=} u, \quad u \text{ sampled from Unif(0, 1)} && (2)\\
&\Rightarrow x^2 - 2x + u = 0 && (3)\\
&\Rightarrow x = 1 \pm \sqrt{1-u} && (4)\\
& && (5)\\
X \in [0,1] &\Rightarrow x_{Y=0} = 1 - \sqrt{1-u} && (6)
\end{aligned}
$$

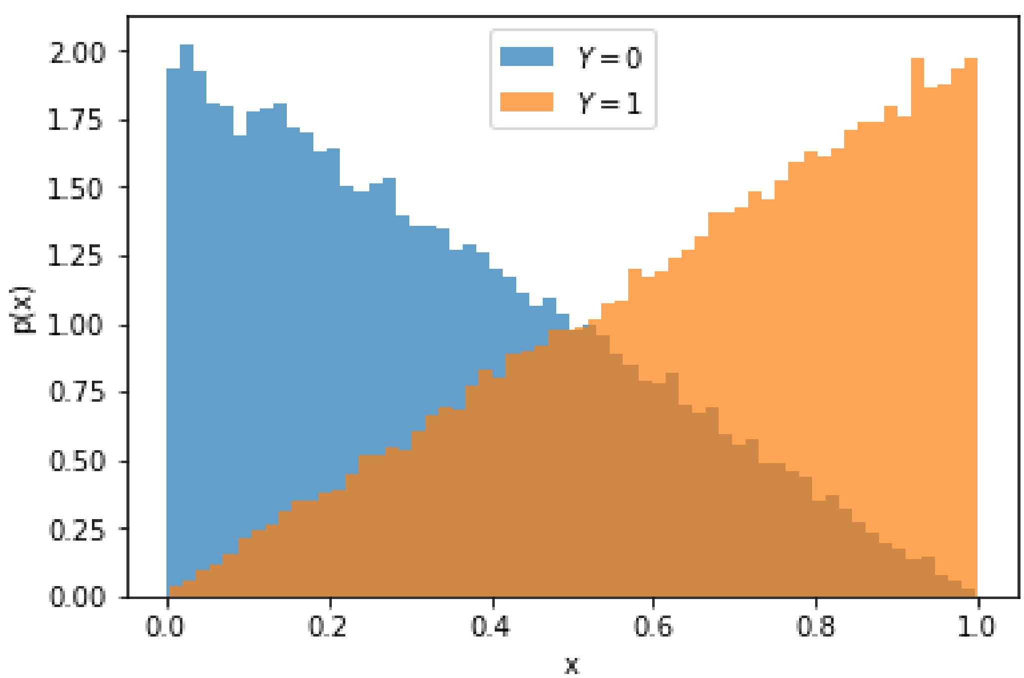
For Y = 1:

$$
\begin{aligned}
F_{Y=1} &= \int_{-\inf}^{x} p_{Y=1}(x')\mathrm{d}x' = \int_{0}^{x} 2x'\mathrm{d}x' = x'^2\big|_0^x && (7)\\
&= x^2 \overset{!}{=} u, \quad u \text{ sampled from Unif(0, 1)} && (8)\\
&\Rightarrow x = \pm\sqrt{u} && (9)\\
& && (10)\\
X \in [0,1] &\Rightarrow x_{Y=1} = \sqrt{u} && (11)
\end{aligned}
$$

```
In [2]: def sample_of_x(y):
            """
            This function applies the inverse transform sampling. The parameter Y can be 0 or 
            """
            u = np.random.rand(*np.empty_like(y).shape)
            x = np.empty_like(y, dtype=float)
```

```
    x[y == 0] = 1 - np.sqrt(1 - u[y == 0])
    x[y == 1] = np.sqrt(u[y == 1])

    return x

def create_data(N):
    """
    This function creates a set of N labels and the corresponding features.
    """
    # draw a uniform variable, set y to 0, if below 0.5, 1 if larger than 0.5.
    # => p(y = 0) = p(y = 1) = 0.5
    labels = np.array([0 if np.random.rand() < 0.5 else 1 for _ in range(N)])
    features = sample_of_x(labels)
    return features, labels
```

In [3]:
```
# Checking the distributions

features, labels = create_data(100000)

fig, ax = plt.subplots(1, 1)
ax.hist(features[labels == 0], normed=1, bins=60, alpha=0.7, label="$Y = 0$")
ax.hist(features[labels == 1], normed=1, bins=60, alpha=0.7, label="$Y = 1$")
ax.set_xlabel('x')
ax.set_ylabel('p(x)')
ax.legend()
plt.show()
```



You can see in the plot, that for $Y = 0$ the distribution follows $p(x) = 2x - 2$ and for $Y = 1$ the distribution follows $p(x) = 2x$.

2 1.2 Classification by threshold

In [4]:
```python
def thresh_rule_a(x, t):
    """
    This function returns the result (label) of rule A for a given t and x.
    """
    f = np.empty_like(x, dtype=int)
    f[x <= t] = 0
    f[x > t] = 1
    return f

def thresh_rule_b(x, t):
    """
    This function returns the result (label) of rule B for a given t and x.
    """
    f = np.empty_like(x, dtype=int)
    f[x <= t] = 1
    f[x > t] = 0
    return f

def analytic_error_rule_a(t):
    """
    This function returns the analytic error of rule A for a given t
    """
    return 1/4 + (t - 1/2)**2

def analytic_error_rule_b(t):
    """
    This function returns the analytic error of rule B for a given t
    """
    return 3/4 - (t - 1/2)**2


def calculate_error(decisions, labels):
    """
    This function calculates the error rate of the decisions with the real labels.
    """
    diff = decisions - labels
    return np.count_nonzero(diff)/labels.shape[0]
```

2.1 Get error for different thresholds

```
In [5]: for t in [0.2, 0.5, 0.6]:
            features, labels = create_data(10000)

            decision_a = thresh_rule_a(features, t)
            decision_b = thresh_rule_b(features, t)

            print('t = {:.1f}:'.format(t))
            print('\t p_err_a = {:.3f} (analytic: {:.3f})'.format(calculate_error(decision_a, l
            print('\t p_err_b = {:.3f} (analytic: {:.3f})'.format(calculate_error(decision_b, l
```

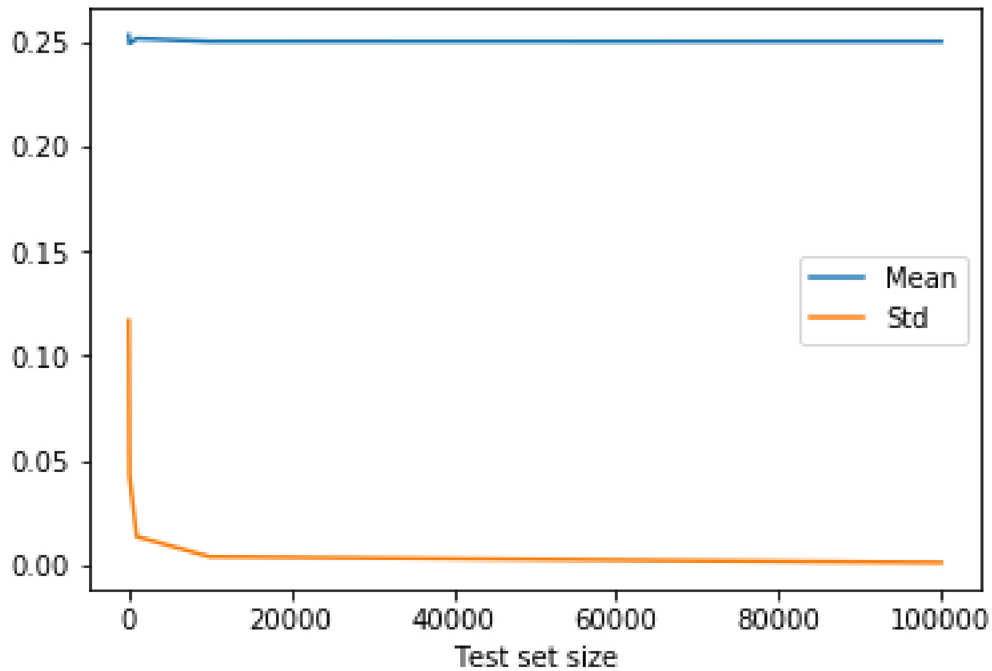
```
t = 0.2:
         p_err_a = 0.345 (analytic: 0.340)
         p_err_b = 0.655 (analytic: 0.660)
t = 0.5:
         p_err_a = 0.255 (analytic: 0.250)
         p_err_b = 0.745 (analytic: 0.750)
t = 0.6:
         p_err_a = 0.259 (analytic: 0.260)
         p_err_b = 0.741 (analytic: 0.740)
```

As you can see above, the best results are reached for $t = 0.5$ with decision rule A (≈ 0.25). The observations fit very well with the analytic results.

Now we want to see, how test set size influence the mean value and standard deviation of p_{error}. Therefore, we calculate p_{error} for rule A with $t = 0.5$ 10 times to estimate the mean and standard-deviation. This is repeated for different test set sizes.

```
In [6]: means = []
        errors = []
        ms = [10, 100, 1000, 10000, 100000]
        for m in ms:
            test_results = []
            for _ in range(100):
                features, labels = create_data(m)
                decision_a = thresh_rule_a(features, 0.5)
                test_results.append(calculate_error(decision_a, labels))
            means.append(np.mean(test_results))
            errors.append(np.std(test_results))
```

```
In [7]: fig, ax = plt.subplots(1, 1)
        ax.plot(ms, means, label = "Mean")
        ax.plot(ms, errors, label = "Std")

        ax.set_xlabel('Test set size')
        ax.legend()

        plt.show()
```



As you can see in the plot, the mean value drops to 25% already for a small test set size. The standard deviation decreases towards 0. The decay is quite rapid, for 20000 elements it is already negligable against the error for 10 elements (≈ 15%).

3 1.3 Nearest Neighbour classifier

In [8]:
```
class NearestNeighbour1D:

    def __init__(self):
        self.training = {"features": [], "labels": []}

    def forget(self):
        self.training = {"features": [], "labels": []}

    def train(self, features, labels):
        self.training = {"features": features, "labels": labels}

    def classify(self, x):
        # get index of nearest neighbour
        ndx = np.argmin(np.abs(x - self.training["features"]))
        # return label of this guy
        return self.training["labels"][ndx]

    def test(self, features, labels):
        errors = 0
```

```
            for i in range(len(features)):
                result = self.classify(features[i])
                if result != labels[i]:
                    errors += 1
            return errors/len(features)
```

```
In [9]: # create a classifier
        NN = NearestNeighbour1D()

        error_rates = []

        for _ in range(100):
            # reset this guy
            NN.forget()
            # create training set and train classifier
            training_features = [0, 0]
            training_labels = [0, 0]
            while training_labels[0] == training_labels[1]:
                training_features, training_labels = create_data(2)

            NN.train(training_features, training_labels)
            # create test set and test classifier
            test_features, test_labels = create_data(20000) # 20000 from above
            error_rates.append(NN.test(test_features, test_labels))
```

```
In [10]: print('Average error rate: {:f}, Standarddeviation: {:f}'.format(np.mean(error_rates)
```

```
Average error rate: 0.339008, Standarddeviation: 0.159542
```

In the range of the standard deviation, the average fits to the expected value of 35%.