Exercise Sheet 2

Max Simon

1) Global error:

$$\varepsilon_n = y_{n+1} - y(t_n + \Delta t)$$
$$= y_{n+1} - \left(y_n + \int_{t_n}^{t_n+\Delta t} f(y(t),t)\, dt\right)$$
$$\underset{\text{trapezoidal rule}}{=} y_{n+1} - y_n - \frac{f(y_n,t_n) + f(y_{n+1},t_{n+1})}{2}\Delta t + \mathcal{O}(\Delta t^2)$$

$$y_{n+1} = y_n + f(y_n)\Delta t + \mathcal{O}(\Delta t^2) \quad \text{(Euler)}$$

$$\Rightarrow \varepsilon = f(y_n,t_n)\Delta t - \frac{1}{2} f(y_n,t_n)\Delta t - \frac{1}{2} f(y_n + f(y_n)\Delta t, t_{n+1}) + \mathcal{O}(\Delta t^2)$$
$$= \frac{1}{2}\Delta t\left(f(y_n(t_n), t_n) - f(y_n(t_n) - f(y_n(t_n)) + f(y_n(t_n))\cdot \Delta t, t_{n+1}\right) + \mathcal{O}(\Delta t^2)$$

So the RK2 is second order accurate for the global integration error.

2)

$$\frac{dT}{dt} = -\frac{2}{3k_B} n_H \Lambda_0 \begin{cases} \left(\frac{T}{T_0}\right)^{10} & T \le T_0 \\ \left(\frac{T}{T_0}\right)^{-0,5} & T > T_0 \end{cases}$$

$$\vartheta := \frac{T}{T_0} \Rightarrow \frac{d\vartheta}{dt} = \frac{1}{T_0}\frac{dT}{dt} = -k' \begin{cases} \vartheta^{10} \\ \vartheta^{-0,5} \end{cases}$$

$$k' = \frac{2}{3k_B} n_H \Lambda_0 \frac{1}{T_0} = 2{,}415 \cdot 10^{-11} \frac{1}{\text{sec}}.$$

$$\tau := \frac{t}{t_0},\ t_0 = 10^{10}\,\text{sec} \quad \Rightarrow dt = t_0 \cdot d\tau$$

$$\Rightarrow \frac{d\vartheta}{d\tau} = -k \begin{cases} \vartheta^{10} \\ \vartheta^{-0,5} \end{cases}, \quad k = k' \cdot t_0 = 0{,}2415.$$

a, b) The ODE was integrated with a RK2 - Algorithm with different timesteps $\Delta\tau$.

You find the results in p2_1.png.

As you can see there one gets large deviations for $\Delta\tau = 6$, but $\Delta\tau = 7$ is okay again. $\Delta\tau = 8$ has also some deviations, but converges to the right value in the end. Above this timestep the algorithm does not converge.

For $\Delta\tau = 7$ one needs 7739 integration steps.

c)

I implemented the adaptive stepsize as described in the lecture notes.

In plot p2_2.png you see the integration using an adaptive stepsize compared to the integration with a fixed stepsize.

The adaptive stepsize algorithm gives a very good approximation and needs only 81 integration steps (compared to more than 2000 for the best of fixed stepsize). It varies the stepsize between $\tau = 0.2$ and $\tau = 20000$, as you can see in p2_2.png.

The adaptive stepsize algorithm is very robust. For an initial stepsize of $1 \cdot 10^{-3}$ it needs only 115 integration steps, but a initial stepsizes of $\tau = 10000$ are also possible. (stable over 7 magnitudes, always less than 100 integration steps).

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a) $\frac{\partial L}{\partial \dot\phi_1} = q_1 = m_1 l_1^2 \dot\phi_1 + m_2 l_1^2 \dot\phi_1 + m_2 l_1 l_2 \dot\phi_2 \cos(\phi_1 - \phi_2)$

$\frac{\partial L}{\partial \dot\phi_2}$ $q_2 = m_2 l_2^2 \dot\phi_2 + m_2 l_1 l_2 \dot\phi_1 \cos(\phi_1 - \phi_2)$

$$\Rightarrow \dot\phi_1 = \frac{d\phi_1}{dt} = \frac{l_2 q_1 - l_1 q_2 \cos(\phi_1 - \phi_2)}{l_1^2 l_2 (m_1 + m_2 \sin^2(\phi_1 - \phi_2))} =: f_1$$

$$\dot\phi_2 = \frac{d\phi_2}{dt} = \frac{l_1 (m_1 + m_2) q_2 - l_2 m_2 q_1 \cos(\phi_1 - \phi_2)}{l_1 l_2^2 m_2 (m_1 + m_2 \sin^2(\phi_1 - \phi_2))} = f_2$$

$$\frac{dq_1}{dt} \underset{\text{Lagrange}}{=} \frac{\partial L}{\partial \phi_1} = -m_2 l_1 l_2 \dot\phi_1 \dot\phi_2 \sin(\phi_1 - \phi_2) - (m_1 + m_2) g l_1 \sin(\phi_1) =: f_3$$

$$\frac{dq_2}{dt} = \frac{\partial L}{\partial \phi_2} = + m_2 l_1 l_2 \dot\phi_1 \dot\phi_2 \sin(\phi_1 - \phi_2) - m_2 g l_2 \sin(\phi_2) =: f_4$$

One can insert f_1 and f_2 for $\dot\phi_1$ and $\dot\phi_2$.

Now we have the ODE

$$\frac{dy}{dt} = f(y)$$

where $y := \begin{pmatrix} \phi_1 \\ \phi_2 \\ q_1 \\ q_2 \end{pmatrix}$.

→ Inverting A

$$A \dot{\vec\phi} = \vec q \,, \quad A = \begin{pmatrix} m_1 l_1^2 + m_2 l_1^2 & m_2 l_1 l_2 \cos(\phi_1 - \phi_2) \\ m_2 l_1 l_2 \cos(\phi_1 - \phi_2) & m_2 l_2^2 \end{pmatrix}$$

$$\left(\begin{array}{cc|cc} m_1 l_1^2 + m_2 l_1^2 & m_2 l_1 l_2 \cos & 1 & 0 \\ m_2 l_1 l_2 \cos & m_2 l_2^2 & 0 & 1 \end{array}\right)$$

$$\mathrm{I} - \frac{l_1 \cos}{l_2}: \left(\begin{array}{cc|cc} m_1 l_1^2 + m_2 l_1^2 \sin^2 & 0 & 1 & -\frac{l_1 \cos}{l_2} \\ m_2 l_1 l_2 \cos & m_2 l_2^2 & 0 & 1 \end{array}\right)$$

$$\mathrm{II} - \frac{m_2 l_1 l_2 \cos}{m_1 l_1^2 + m_2 l_1^2 \sin^2}: \left(\begin{array}{cc|cc} m_1 l_1^2 + m_2 l_1^2 \sin^2 & 0 & 1 & -\frac{l_1 \cos}{l_2} \\ 0 & m_2 l_2^2 & -\frac{m_2 l_1 l_2 \cos}{m_1 l_1^2 + m_2 l_1^2 \sin^2} & 1 + \frac{m_2 l_1^2 l_2 \cos^2}{l_2 l_1^2 (m_1 + m_2 \sin^2)} \end{array}\right)$$

$$\begin{array}{l} \mathrm{I} / (m_1 l_1^2 + m_2 l_1^2 \sin^2) \\ \mathrm{II} / m_2 l_2^2 \end{array} \left(\begin{array}{cc|cc} 1 & 0 & \frac{1}{l_1^2 (m_1 + m_2 \sin^2)} & -\frac{l_1 \cos}{l_2 (m_1 l_1^2 + m_2 l_1^2 \sin^2)} \\ 0 & 1 & -\frac{l_1 \cos}{l_2 m_1 l_1^2 + m_2 l_1^2 \sin^2 l_2} & \underbrace{\frac{1}{m_2 l_2^2} + \frac{m_2 l_1^2 \cos^2 l_2}{l_1^2 l_2^3 m_2 (m_1 + m_2 \sin^2)}} \end{array}\right)$$

$$= \frac{l_1^2 (m_1 + m_2 \sin^2) + m_2 l_1^2 \cos^2}{l_1^2 l_2^2 m_2 (m_1 + m_2 \sin^2)}$$

$$\rightarrow \dot{\phi}_1 = \frac{q_1}{m_1 l_1^2 + m_2 l_1^2 \sin^2} - \frac{q_2 l_1 \cos}{l_2 (m_1 l_1^2 + m_2 l_1^2 \sin^2)}$$

$$= \frac{l_2 q_1 - q_2 l_1 \cos(\phi_1 - \phi_2)}{l_1^2 l_2 (m_1 + m_2 \sin^2(\phi_1 - \phi_2))}$$

$$\dot{\phi}_2 = \cancel{-\frac{m_2 l_1 l_2 \cos(\phi_1-\phi_2)}{l_1 l_2 m_2 (m_1 + m_2 \sin^2(\phi_1 - \phi_2))}}$$

$$= -\frac{l_2 \cancel{l_1} m_2 \cos(\phi_1 - \phi_2)}{l_2^2 l_1^2 m_2 (m_1 + m_2 \sin^2(\phi_1 - \phi_2))} q_1$$

$$+ \frac{l_1 (m_1 + m_2 \sin^2(\phi_1 - \phi_2) + m_2 \cos^2(\phi_1 - \phi_2))}{l_1 l_2^2 m_2 (m_1 + m_2 \sin^2(\phi_1 - \phi_2))} q_2$$

$$= \frac{l_1 (m_1 + m_2) q_2 - l_2 m_2 \cos(\phi_1 - \phi_2) q_1}{l_1 l_2^2 m_2 (m_1 + m_2 \sin^2(\phi_1 - \phi_2))}$$

c)

Potential energy:

$$V = m_1 g y_1 + m_2 g y_2$$

$$= -m_1 g \underbrace{l_1 \cos(\phi_1)}_{y_1} - m_2 g \underbrace{\left(l_1 \cos(\phi_1) + \cancel{m_2 g}\, l_2 \cos(\phi_2)\right)}_{y_2}$$

$$= -(m_1 + m_2) g l_1 \cos(\phi_1) - m_2 g l_2 \cos(\phi_2).$$

$$T = \frac{1}{2} m_1 v_1^2 + \frac{1}{2} m_2 v_2^2$$

$$= \frac{1}{2} m_1 l_1^2 \dot{\phi}_1^2 + \frac{1}{2} m_2 l_1^2 \dot{\phi}_1^2 + \frac{1}{2} m_2 l_2^2 \dot{\phi}_2^2$$

$$+ \cancel{2}\, m_2 l_1 l_2 \dot{\phi}_1 \dot{\phi}_2 \cos(\phi_1 - \phi_2).$$

$$E_{tot} = V + T$$

$$\underline{\underline{\varepsilon = \frac{E_{tot}}{E_{tot}(0)} - 1.}}$$

RK 2: We see a relatively large increase of ε of about 0,27 over the time. (Plot p3_1.png)

RK 4: With the more precise RK 4 I got an ε increase of only $7 \cdot 10^{-3}$. (Plot P3_2.png).

You can find the motion of the pendulum in the movie p3_mov.mp4.