(1) а) $x^2+z^3+f(x-y^2)=0$ – неявн.

$\frac{\partial z}{\partial x}=-\frac{F'_x}{F'_z}$; $\frac{\partial z}{\partial y}=-\frac{F'_y}{F'_z}$

$F'_x=2x+f'$; $F'_y=f'\cdot(-2y)=-2yf'$; $F'_z=3z^2$

$\underbrace{\frac{\partial z}{\partial x}=-\frac{2x+f'}{3z^2}}$; $\underbrace{\frac{\partial z}{\partial y}=-\frac{-2yf'}{3z^2}=\frac{2yf'}{3z^2}}$.

б) $F\left(\frac{y}{x},\frac{z}{y}\right)=0$ Пусть $F(u,v)=0$, $u=\frac{y}{x}$, $v=\frac{z}{y}$

$\frac{\partial z}{\partial x}=-\frac{F'_x}{F'_z}$; $\frac{\partial z}{\partial y}=-\frac{F'_y}{F'_z}$

$\frac{\partial u}{\partial x}=-\frac{y}{x^2}$ $\frac{\partial u}{\partial y}=\frac{1}{x}$

$\frac{\partial v}{\partial x}=0$ $\frac{\partial v}{\partial y}=-\frac{z}{y^2}$

$\frac{\partial u}{\partial z}=0$ $\frac{\partial v}{\partial z}=\frac{1}{y}$

$F'_x=\frac{\partial F}{\partial u}\frac{\partial u}{\partial x}+\frac{\partial F}{\partial v}\frac{\partial v}{\partial x}=F'_u\cdot\left(-\frac{y}{x^2}\right)=-F'_u\cdot\frac{y}{x^2}$

$F'_y=\frac{\partial F}{\partial u}\frac{\partial u}{\partial y}+\frac{\partial F}{\partial v}\frac{\partial v}{\partial y}=F'_u\cdot\frac{1}{x}+F'_v\cdot\left(-\frac{z}{y^2}\right)=F'_u\frac{1}{x}-\frac{F'_v z}{y^2}$

$F'_z=\frac{\partial F}{\partial u}\frac{\partial u}{\partial z}+\frac{\partial F}{\partial v}\frac{\partial v}{\partial z}=F'_v\cdot\frac{1}{y}$

$\frac{\partial z}{\partial x}=-\frac{-F'_u\, y/x^2}{F'_v\cdot 1/y}=\frac{F'_u y^2}{F'_v x^2}$

$\frac{\partial z}{\partial y}=-\frac{1/x\, F'_u-F'_v\, z/y^2}{F'_v\, 1/y}$

(2) $z=f\left(\frac{2xy}{x+y},x^3-2y\right)$ найти $\frac{\partial z}{\partial x}$, $\frac{\partial z}{\partial y}$. ← явн.

$\frac{\partial u}{\partial x}=\frac{2y(x+y)-2xy}{(x+y)^2}=\frac{2y^2}{(x+y)^2}$

$\frac{\partial u}{\partial y}=\frac{2x^2}{(x+y)^2}$

$\frac{\partial v}{\partial x}=3x^2$ $\frac{\partial v}{\partial y}=-2$

$z=f(u,v)$, $u=\frac{2xy}{x+y}$, $v=x^3-2y$

$\frac{\partial z}{\partial x}=\underbrace{\frac{\partial f}{\partial u}}_{f'_u}\frac{\partial u}{\partial x}+\underbrace{\frac{\partial f}{\partial v}}_{f'_v}\frac{\partial v}{\partial x}=f'_u\cdot\frac{2y^2}{(x+y)^2}+f'_v\cdot 3x^2$

$\frac{\partial z}{\partial y}=\frac{\partial f}{\partial u}\frac{\partial u}{\partial y}+\frac{\partial f}{\partial v}\cdot\frac{\partial v}{\partial y}=f'_u\cdot\frac{2x^2}{(x+y)^2}-2f'_v$

(3) dz – ? $M(-1;1;1)$ где $y^3=ze^{x+z}$

$d(y^3=ze^{x+z})\Rightarrow 3y^2dy=(e^{x+z}+ze^{x+z})dz+ze^{x+z}dx$

$dz|_M=\frac{3y^2dy-ze^{x+z}dx}{e^{x+z}+ze^{x+z}}\Big|_M=\frac{3dy-dx}{1+1}=\frac{3}{2}dy-\frac{1}{2}dx$

(4) dz $M(2,1,2)$ $3x^2y^2+2xyz^2-2x^3z+4y^3z-4=0$

(I) $6xy^2dx+6x^2ydy+2yz^2dx+2xz^2dy+4xyzdz-$
$-6x^2zdx-2x^3dz+12y^2zdy+4y^3dz=0$

$(6xy^2+2yz^2-6x^2z)dx+(6x^2y+2xz^2+12y^2z)dy+(4xyz-2x^3+4y^3)dz=0$

$(6\cdot2\cdot1+2\cdot1\cdot4-6\cdot4\cdot2)dx+(6\cdot4\cdot1+2\cdot2\cdot4+12\cdot1\cdot2)dy+(4\cdot2\cdot1\cdot2-2\cdot8+4)dz=0\,\big|_M$

$-28dx+64dy+4dz=0\Rightarrow \underbrace{dz=7dx-16dy}$

(II) $dz=\frac{\partial z}{\partial x}dx+\frac{\partial z}{\partial y}dy$

$\frac{\partial z}{\partial x}=-\frac{F'_x}{F'_z}$; $\frac{\partial z}{\partial y}=-\frac{F'_y}{F'_z}\Big|_M$.

$F'_x=6xy^2+2yz^2-6x^2z|_M=-28$

$F'_y=6x^2y+2xz^2+12y^2z|_M=64$

$F'_z=4xyz-2x^3+4y^3|_M=4$

$\frac{\partial z}{\partial x}\Big|_M=-\frac{-28}{4}=7$

$\frac{\partial z}{\partial y}\Big|_M=-\frac{64}{4}=-16$

(5) $z=f(y/x)\rightarrow x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}=0$.

$z=f(u)$, $u=y/x$ – явн.

$\left(\cos\left(\frac{y}{x}\right)\right)'_y=\cos'(\ldots)\cdot\left(\frac{y}{x}\right)'_y=f'\cdot\frac{1}{x}$

$\left(\cos\left(\frac{y}{x}\right)\right)'_y=\underbrace{-\sin\left(\frac{y}{x}\right)}_{f'}\cdot\frac{1}{x}$

$\frac{\partial z}{\partial x}=f'\cdot\frac{\partial u}{\partial x}=f'\cdot\left(-\frac{y}{x^2}\right)=-\frac{f'y}{x^2}$

$\frac{\partial z}{\partial y}=f'\cdot\frac{\partial u}{\partial y}=f'\cdot\left(\frac{1}{x}\right)=f'\cdot\frac{1}{x}$

$-x\cdot\frac{f'\cdot y}{x^2}+y\cdot\frac{f'}{x}=0$. верн.

$f=\cos$

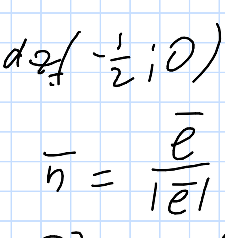
$f'=-\sin$

$f'_u\cdot\frac{\partial u}{\partial x}$

$f'_u\cdot\frac{\partial u}{\partial y}$

(6) $z=\frac{y}{x^2+y^2}$, $M(1,1)$

а) $z(M)=\frac{1}{1+1}=\frac{1}{2}$

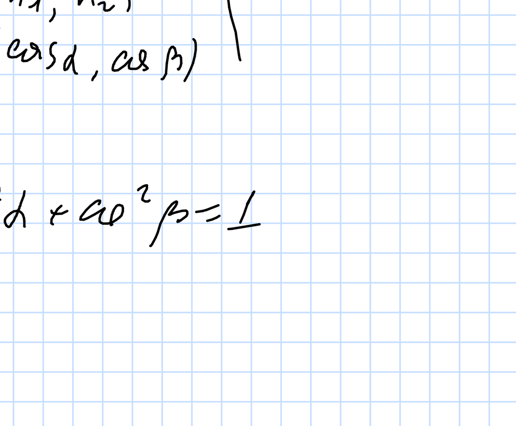


$c=\frac{1}{2}$

лин. ур: $\frac{y}{x^2+y^2}=\frac{1}{2}\Rightarrow x^2+y^2=2y\Rightarrow \underbrace{x^2+(y-1)^2=1}_{\text{лин. ур.}}$ – окр-ть (0;1) $R=1$

б) $z'_x=\frac{-2xy}{(x^2+y^2)^2}\Big|_M=\frac{-2}{4}=-\frac{1}{2}$

$z'_y=\frac{x^2+y^2-2y^2}{(x^2+y^2)^2}=\frac{x^2-y^2}{(x^2+y^2)^2}\Big|_M=0$



$\text{grad}\,z|_M=-\frac{1}{2}i$ $\text{grad}\,z\left(-\frac{1}{2};0\right)$

в) $\bar{l}=4i+3j$ $\bar{n}=\frac{\bar{e}}{|\bar{e}|}$ $\bar{n}=(n_1,n_2)$ $(\cos\alpha,\cos\beta)$

$\frac{\partial z}{\partial \bar{e}}\Big|_M=(\text{grad}\,z,\bar{n})|_M=\frac{\partial z}{\partial x}\Big|_M\cos\alpha+\frac{\partial z}{\partial y}\Big|_M\cos\beta$

$|\bar{e}|=\sqrt{4^2+3^2}=5$ $\cos^2\alpha+\cos^2\beta=1$

$\bar{n}=\left(\frac{4}{5};\frac{3}{5}\right)$ cos α, cos β

$\frac{\partial z}{\partial \bar{e}}=-\frac{1}{2}\cdot\frac{4}{5}+\frac{3}{5}\cdot 0=-\frac{2}{5}$

г) наиб. зн-е.
$|\text{grad}\,z|=\frac{1}{2}$

(7) $u=xy^3+3yz+z^2$ в $M(0,1,1)$:

а) пов-ть ур.

$C=U(M)=3\cdot1\cdot1+1=4$

пов-ть ур: $xy^3+3yz+z^2=4$

б) гр-т.

$u'_x=y^3|_M=1$

$u'_y=3xy^2+3z|_M=3$

$u'_z=3y+2z|_M=5$

$\text{grad}\,z=i+3j+5k$

в) пр-ю по нап. в-ра $\overline{MN}$, $N(2,2,3)$

$\overline{MN}(2,1,2)$ $|\overline{MN}|=\sqrt{2^2+1^2+2^2}=3$

$\bar{n}=\frac{\overline{MN}}{|\overline{MN}|}=\left(\frac{2}{3};\frac{1}{3};\frac{2}{3}\right)$ cos α, cos β, cos γ

$\frac{\partial u}{\partial \overline{MN}}\Big|_M=(\text{grad}\,u|_M,\bar{n})=1\cdot\frac{2}{3}+3\cdot\frac{1}{3}+5\cdot\frac{2}{3}=\frac{15}{3}=5$

г) $|\text{grad}\,u|=\sqrt{1+3^2+5^2}=\sqrt{35}$.

(9) $3x^4-4yz^3+xy^2+x^2z+5y-6=0$ $M(1,1,1)$

$F'_x=12x^3+y^2+2xz|_M=15$

$F'_y=-4z^3+2xy+5|_M=3$

$F'_z=-12yz^2+x^2|_M=-11$

ур-е кас. пл-ти: $F'_x(M)(x-x_0)+F'_y(M)(y-y_0)+F'_z(z-z_0)=0$

$15(x-1)+3(y-1)-11(z-1)=0\Rightarrow$

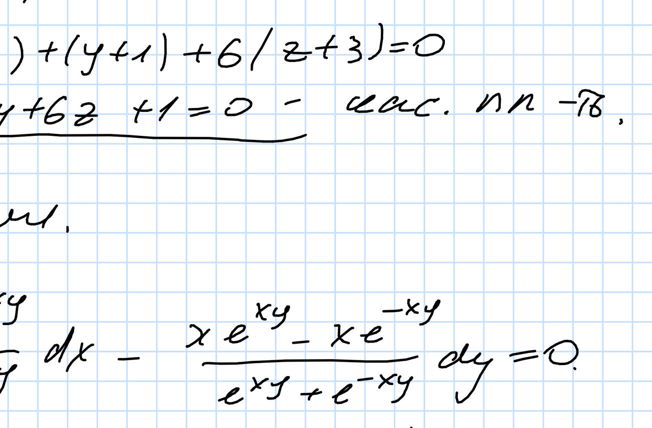
$\underline{15x+3y-11z-7=0}$ – кас. пл-ть.

ур-е нор. $\frac{x-x_0}{F'_x(M)}=\frac{y-y_0}{F'_y(M)}=\frac{z-z_0}{F'_z(M)}$

$\frac{x-1}{15}=\frac{y-1}{3}=\frac{z-1}{-11}$ – норм.

(10) $y^2+3y-6x-z^2=7$ // $y-6x+6z+5=0$ пл-ть $\bar{n}_1(-6,1,6)$

$(\bar{n}_2\parallel\bar{n}_1)$ $\frac{x_1}{x_2}=\frac{y_1}{y_2}=\frac{z_1}{z_2}=k$.



$\bar{n}_2(F'_x(M),F'_y(M),F'_z(M))$.

$F'_x=-6$

$F'_y=2y+3$

$F'_z=-2z$

$\begin{cases}-6=-6k & \Rightarrow k=1\\ 2y_0+3=k\cdot1\\ -2z_0=6k\\ y_0^2+3y_0-6x_0-z_0^2=7 & \Rightarrow x_0=-3\end{cases}$ $\begin{cases}y_0=-1\\ z_0=-3\end{cases}$

$M(x_0,y_0,z_0)$ $TM(-3;-1;-3)$

кас. пл-ти: $-6(x+3)+(y+1)+6(z+3)=0$

$\underline{-6x+y+6z+1=0}$ – кас. пл-ть.

(11) $1+xy-\ln(e^{xy}+e^{-xy})=0$ – неяв.

$\frac{dy}{dx}$ – ? $\frac{d^2y}{dx^2}$

$ydx+xdy-\frac{ye^{xy}-ye^{-xy}}{e^{xy}+e^{-xy}}dx-\frac{xe^{xy}-xe^{-xy}}{e^{xy}+e^{-xy}}dy=0$.

$\left[y-y\left(\frac{e^{xy}-e^{-xy}}{e^{xy}+e^{-xy}}\right)\right]dx+\left[x-x\left(\frac{e^{xy}-e^{-xy}}{e^{xy}+e^{-xy}}\right)\right]dy=0$

$y\underbrace{\left(1-\frac{e^{xy}-e^{-xy}}{e^{xy}+e^{-xy}}\right)}dx+x\underbrace{\left(1-\frac{e^{xy}-e^{-xy}}{e^{xy}+e^{-xy}}\right)}dy=0$

$ydx+xdy=0\Rightarrow\frac{dy}{dx}=-\frac{y}{x}$ $y'=-y/x$

$\frac{d^2y}{dx^2}=\frac{d}{dx}\left(\frac{dy}{dx}\right)=-\frac{d}{dx}\left(\frac{y}{x}\right)=-\frac{y'x-y}{x^2}=-\frac{-y/x\cdot x-y}{x^2}=\frac{2y}{x^2}$

$-\frac{d}{dx}\left(\frac{y(x)}{x}\right)$

(12) $z=x\ln\frac{y}{x}$ d^2z – ?

$dz=\left(\ln\frac{y}{x}+x\cdot\frac{-y}{x^2}\cdot\frac{1}{y/x}\right)dx+\left(x\cdot\frac{1}{x}\cdot\frac{1}{y/x}\right)dy=\underbrace{\left(\ln\frac{y}{x}-1\right)}_{\frac{\partial z}{\partial x}}dx+\underbrace{\frac{x}{y}}_{\frac{\partial z}{\partial y}}dy$

$dz=\frac{\partial z}{\partial x}dx+\frac{\partial z}{\partial y}dy$

$\boxed{d^2z=\frac{\partial^2z}{\partial x^2}dx^2+2\frac{\partial^2z}{\partial x\partial y}dxdy+\frac{\partial^2z}{\partial y^2}dy^2}$

$\frac{\partial^2z}{\partial x^2}=\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right)=\frac{\partial}{\partial x}\left(\ln\frac{y}{x}-1\right)=-\frac{y}{x^2}\cdot\frac{1}{y/x}=-\frac{1}{x}$

$\frac{\partial^2z}{\partial y^2}=\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right)=\frac{\partial}{\partial y}\left(\frac{x}{y}\right)=-\frac{x}{y^2}$

$\frac{\partial^2z}{\partial x\partial y}=\frac{\partial}{\partial y}\left(\overbrace{\ln\frac{y}{x}-1}^{\frac{\partial z}{\partial x}}\right)=\frac{1}{x}\cdot\frac{1}{y/x}=\frac{1}{y}=\frac{\partial^2z}{\partial y\partial x}=\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right)=$
$=\frac{\partial}{\partial x}\left(\frac{x}{y}\right)=\frac{1}{y}$

$d^2z=-\frac{1}{x}dx^2+2\cdot\frac{1}{y}dxdy-\frac{x}{y^2}dy^2$

$d(xyz^2+2x^2yz+4x^2y+5z^2=0)$ ← неяв.

$(\ldots)dx+(\ldots)dy+(\ldots)dz=0$

$dz=\ldots$

$yz^2dx+xz^2dy+2xyzdz\ \ldots\ =0$.

$F'_x=yz^2+4xyz+8xy$

$F'_y=xz^2+2x^2z+4x^2$

$F'_z=2xyz+2x^2y+10z$

$\Rightarrow\frac{\partial z}{\partial x}=-\frac{F'_x}{F'_z}$

$\frac{\partial z}{\partial y}=-\frac{F'_y}{F'_z}$

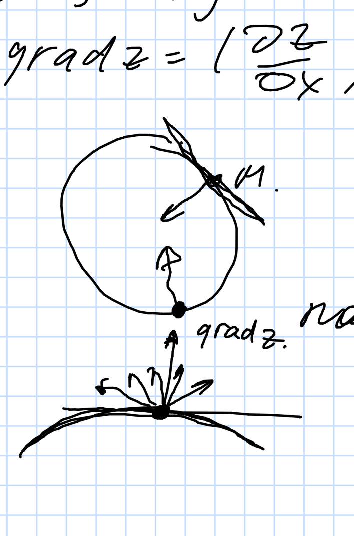
$dz=\frac{\partial z}{\partial x}dx+\frac{\partial z}{\partial y}dy$.

$z=f(x,y)$.

$\text{grad}\,z=\left(\frac{\partial z}{\partial x},\frac{\partial z}{\partial y}\right)\Big|_M$.

$\frac{\partial y}{\partial z}=-\frac{F'_z}{F'_y}$

$\frac{\partial y}{\partial x}=-\frac{F'_x}{F'_y}$.



наиб. з-е $\Rightarrow|\text{grad}\,z|$