Information Theory HW2

1

3.1.1. 由符号的概率分布可得，符号熵为

$$H(X) = -\frac{1}{2}\log_2\frac{1}{2} - \frac{1}{4}\log_2\frac{1}{4} - 2\times\frac{1}{8}\log_2\frac{1}{8} = 1.75bit/\text{符号}.$$

3.1.2. 平均代码长度

$$\mathbb{E}L = \frac{1}{2}\times 1 + \frac{1}{4}\times 2 + 2\times\frac{1}{8}\times 3 = 1.75\text{符号}^{-1},$$

于是一个二进制码的熵为

$$H = H(X)/\mathbb{E}L = 1bit.$$

3.1.3. 由于各个符号之间互相独立,

$$p_0 = \left(\frac{1}{2}+\frac{1}{4}+\frac{1}{8}\right), \times\frac{1}{EL} = 0.5 \qquad p_1 = 1 - p_0 = 0.5.$$

又由于

$$p_{00} = \left(\frac{1}{2}+\frac{1}{4}+\frac{1}{8}\right)\times\frac{1}{2}\times\frac{1}{EL} = 0.25, \quad p_{01} = \left(\frac{1}{2}+\frac{1}{4}+\frac{1}{8}\right)\times\left(\frac{1}{4}+\frac{1}{8}+\frac{1}{8}\right)\times\frac{1}{EL} = 0.25$$
$$p_{10} = \left(\frac{1}{4}+\frac{1}{8}+\frac{1}{8}\times\frac{1}{2}\right)\times\frac{1}{EL} = 0.25, \quad p_{11} = 1 - p_{00} - p_{01} - p_{10} = 0.25.$$

于是

$$p_{0|0} = \frac{p_{00}}{p_0} = 0.5 \quad p_{0|1} = \frac{p_{10}}{p_1} = 0.5, \quad p_{1|0} = \frac{p_{01}}{p_0} = 0.5 \quad p_{1|1} = \frac{p_{11}}{p_1} = 0.5.$$

2

3.2.1 由题意，不论以前发出什么消息，概率分布都是固定的。因此，信源所发符号序列的概率分布与时间起点无关，信源是平稳的。

3.2.2

先计算信源熵

$$H(X) = -(p_0\log p_0 + p_1\log p_1) = 0.97bit.$$

对于离散无记忆信源:

$$H(X^2) = 2H(X) = 1.94bit$$
$$H(X_3|X_1, X_2) = H(X_3) = 0.97bit$$
$$\lim_{N\to\infty} H_N(X) = H(X_N) = 0.97bit$$

3.2.3

$$H(X^4) = 4H(X) = 3.88bit$$

X^4 所有可能的符号为:
0000，0001，0010，0011，
0100，0101，0110，0111，
1000，1001，1010，1011，
1100，1101，1110，1111.

3

3.3.

由方程:

$$\begin{cases} p_1 = 2/3p_1 + p_2 \\ p_2 = 1/3p_1 \\ p_1 + p_2 = 1 \end{cases} \quad \rightarrow \quad \begin{cases} p_1 = 3/4 \\ p_2 = 1/4 \end{cases}$$

于是

$$H_\infty = H_2 = -(\frac{3}{4} \times \frac{2}{3} \log \frac{2}{3} + \frac{3}{4} \times \frac{1}{3} \log \frac{1}{3}) = 0.69bit.$$

4

3.4. 信源的不确定性，即信源熵. 对于一阶Markov信源,

$H(X|X) = H_\infty(X) = H_2 = -0.3 \times (0.2 \times \log 0.2 + 0.8 \times \log 0.8) - 0.7 \times (0.9143 \log 0.9143 + 0.0857 \log 0.0857) = 0.51bit.$

状态转移图如下。

5

3.5. 由方程

$$\begin{cases} p_A = 5/8p_A + 1/4p_B + 1/8p_C \\ p_B = 1/4p_A + 1/2p_B + 1/4p_C \\ p_C = 1/8p_A + 1/4p_B + 1/8p_C \end{cases} \quad \rightarrow \quad p_A = p_B = p_C = 1/3$$

于是

$$\begin{aligned} H_0 &= \sum_{i \in \{A,B,C\}} -p_i \log p_i = 1.585bit, \\ H_\infty &= -1/3(1/2 \log 1/2 + 1/4 \log 1/4 + 1/4 \log 1/4) \\ &\quad - 1/3(5/8 \log 5/8 + 1/4 \log 1/4 + 1/8 \log 1/8) \\ &\quad - 1/3(5/8 \log 5/8 + 1/4 \log 1/4 + 1/8 \log 1/8) \\ &= 1.366bit, \end{aligned}$$

于是

$$R = 1 - \frac{H_\infty}{H_0} = 13.82\%.$$

6

3.6.1 在稳态下，由方程

$$\begin{cases} p_1 = 3/4p_1 + 1/4p_3 \\ p_2 = 2/3p_2 + 1/4p_1 \\ p_3 = 3/4p_3 + 1/3p_2 \end{cases} \quad \rightarrow \quad \begin{cases} p_1 = 4/11 \\ p_2 = 3/11 \\ p_3 = 4/11 \end{cases}$$

3.6.2 信源熵为

$$\begin{aligned} H_\infty &= -4/11(3/4 \log 3/4 + 1/4 \log 1/4) \\ &\quad - 3/11(2/3 \log 2/3 + 1/3 \log 1/3) \\ &\quad - 4/11(3/4 \log 3/4 + 1/4 \log 1/4) \\ &= 0.84bit \end{aligned}$$

7

4.1 概率分布

$$P(Y=0) = 1/3 \times 2/3 + 2/3 \times 1/6 = 1/3, \quad P(Y=1) = 1/3 \times 1/6 + 2/3 \times 2/3 = 1/2,$$
$$P(Y=?) = 1/3 \times 1/6 + 2/3 \times 1/6 = 1/6.$$

于是

$$\begin{gathered} H(X) = -(1/3 \log 1/3 + 2/3 \log 2/3) = 0.92bit \\ H(Y) = -(1/3 \log 1/3 + 1/2 \log 1/2 + 1/6 \log 1/6) = 1.46bit \\ H(Y|X) = -(2/9 \log 2/3 + 1/18 \log 1/6 + 1/18 \log 1/6 + 1/9 \log 1/6 + 1/9 \log 1/6 + 4/9 \log 2/3) = 1.25bit \\ H(XY) = H(Y|X) + H(X) = 2.17bit \rightarrow H(X|Y) = H(XY) - H(Y) = 0.71bit \\ I(X;Y) = H(X) - H(X|Y) = 0.21bit. \end{gathered}$$

8

4.2 依题意可得:

$$P(R=2)=0.7,\quad P(R=5)=0.3$$
$$P(W=1/8)=0.64,\quad P(W=1/4)=0.36$$
$$P(W=1/8|R=2)=0.8,\quad P(W=1/4)=0.2$$

于是:

$$P(W=1/8|R=5)=\frac{P(W=1/8)-P(R=2)P(W=1/8|R=2)}{P(R=5)}=4/15, P(W=1/4|R=5)=1-P(W=1/8|R=5)=11/15.$$
$$H(W|R)=-0.7\times(0.8\log 0.8+0.2\log 0.2)-0.3\times(11/15\log 11/15+4/15\log 4/15)=0.76bit$$
$$H(W)=-(0.64\log 0.64+0.36\log 0.36)=0.94bit$$
$$I(W;R)=H(W)-H(W|R)=0.18bit$$

9

4.3.1 概率分布

$$P(Y=0)=2/3\times 3/4+1/3\times 1/4=7/12,\quad P(Y=1)=1-7/12=5/12.$$

于是

$$H(X)=-(3/4\log 3/4+1/4\log 1/4)=0.81bit$$
$$H(Y)=-(7/12\log 7/12+5/12\log 5/12)=0.98bit$$
$$H(Y|X)=-(3/4+1/4)\times(2/3\log 2/3+1/3\log 1/3)=0.92bit$$
$$H(XY)=H(Y|X)+H(X)=1.73bit\rightarrow H(X|Y)=H(XY)-H(Y)=0.75bit$$
$$I(X;Y)=H(X)-H(X|Y)=0.06bit.$$

4.3.2 容量

$$C=\log 2-(2/3\log 2/3+1/3\log 1/3)=0.08bit/\text{符号}$$

输入概率分布为

$$P_x=[1/2,1/2].$$

10

4.4.a 首先等到信道矩阵为

$$\begin{bmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{6} & \frac{1}{6} \\ \frac{1}{6} & \frac{1}{6} & \frac{1}{3} & \frac{1}{3} \end{bmatrix},$$

对于对称信道，可以得到信道容量:

$$C=\log 4-H(1/3,1/3,1/6,1/6)=2+(2\times 1/3\log 1/3+2\times 1/6\log 1/6)=0.08bit$$

此时最佳输入概率分布为

$$P_x=[1/4,1/4,1/4,1/4].$$

4.4.b 首先等到信道矩阵为

$$\begin{bmatrix} \frac{1}{2} & \frac{1}{3} & \frac{1}{6} \\ \frac{1}{6} & \frac{1}{2} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{6} & \frac{1}{2} \end{bmatrix},$$

对于对称信道，可以得到信道容量:

$$C=\log 3-H(1/2,1/3,1/6)=2+(1/2\log 1/2+1/3\log 1/3+1/6\log 1/6)=0.13bit$$

此时最佳输入概率分布为

$$P_x=[1/3,1/3,1/3].$$

11

4.5 14000个二元符号，对应的信息量

$$H(X)=-(2\times 1/2\log 1/2)=1bit/\text{符号}$$
$$I=14000\times H(X)=14000bit$$

信道容量为

$$C = \log 2 - H(0.02, 0.98) = 0.86bit/\text{符号}$$
$$V = 1500 \times 0.86 = 1290bit/s$$

于是时间

$$t = \frac{I}{V} = 14000/1290s = 10.85s > 10s$$

因此不能无失真地传送完。

12

4.6 依题意，可以得到

$$C = \log 2 - H(0.01, 0.99) = 0.92bit/\text{符号}.$$