Using Spectral Methods and the Quasilinearization Method (QLM) to Solve Non-Linear ODEs

By Maximus Smith

Spectral Methods Overview

Assume a solution of an Ordinary Differential Equation (ODE):

$y_N = \sum_{k=0}^{N} a_k \phi_k(x)$ [1]

Goal: Find a_k by **forcing** our solution to satisfy the ODE at certain points.

(Collocation points)

We can make **better solutions** by using different basis functions ($\phi_k(x)$)

(e.g. Chebyshev Polynomials)

The Problem with Spectral Methods

Linear ODEs result in Linear Equations

- Easy to solve

Non-Linear ODEs result in Non-Linear Equations

- Very hard to solve

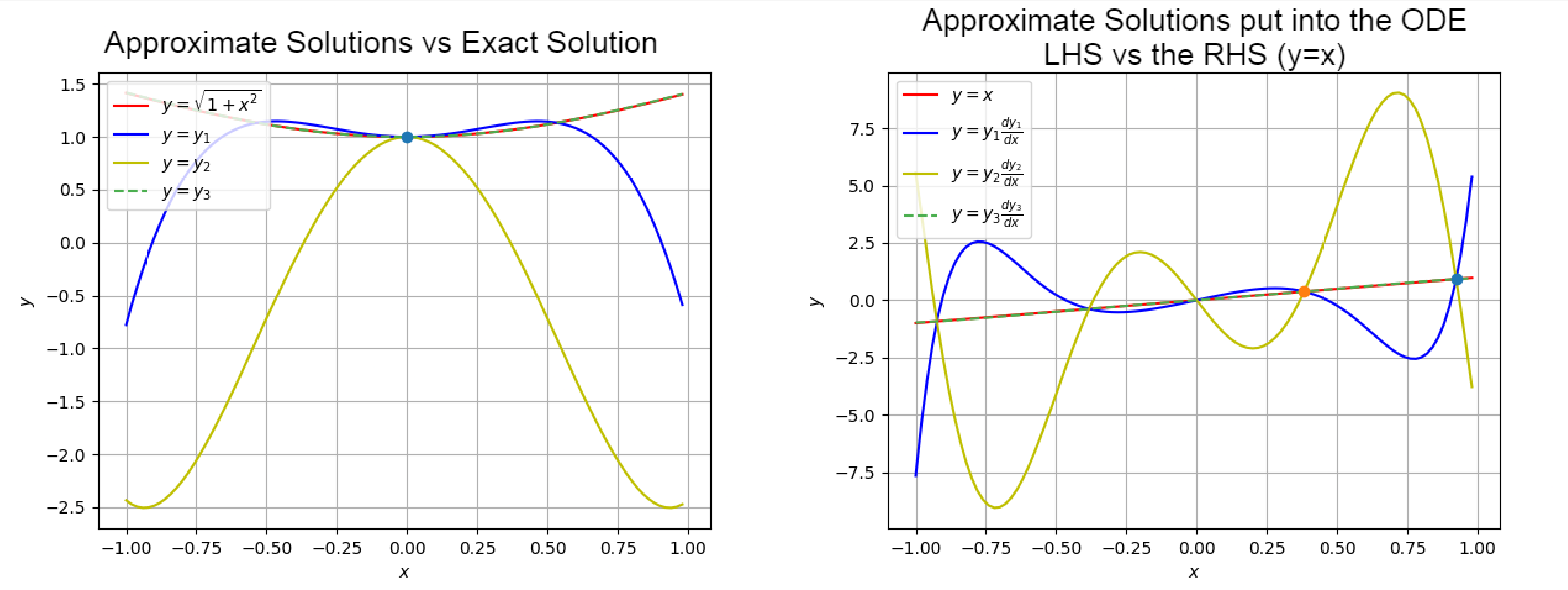

Figure 1: (Left) Shows 3 approximate solutions. (Right) Shows those solutions put into the ODE. Both cases the red line is the exact solution. The points marked on are the boundary conditions (left) and collocation points (right) respectively.

Example of *Aliasing* when solving: $y\frac{dy}{dx} = x$ and $y(0) = 1$

How can we do this on a Non-Linear ODE?

Quasilinearization Method (QLM)

1. Rewrite ODE as $L(y(x)) = f\big(y(x)\big)$ with:

   L – **Linear**, f – **Non-Linear** [2]

2. Decide a reasonably good solution $y_i(x)$
3. Find a 1^{st} *order Functional Taylor Series* of $f(y(x))$ about the point $y_i(x)$
4. Solving the resulting ODE leads to $y_{i+1}(x)$

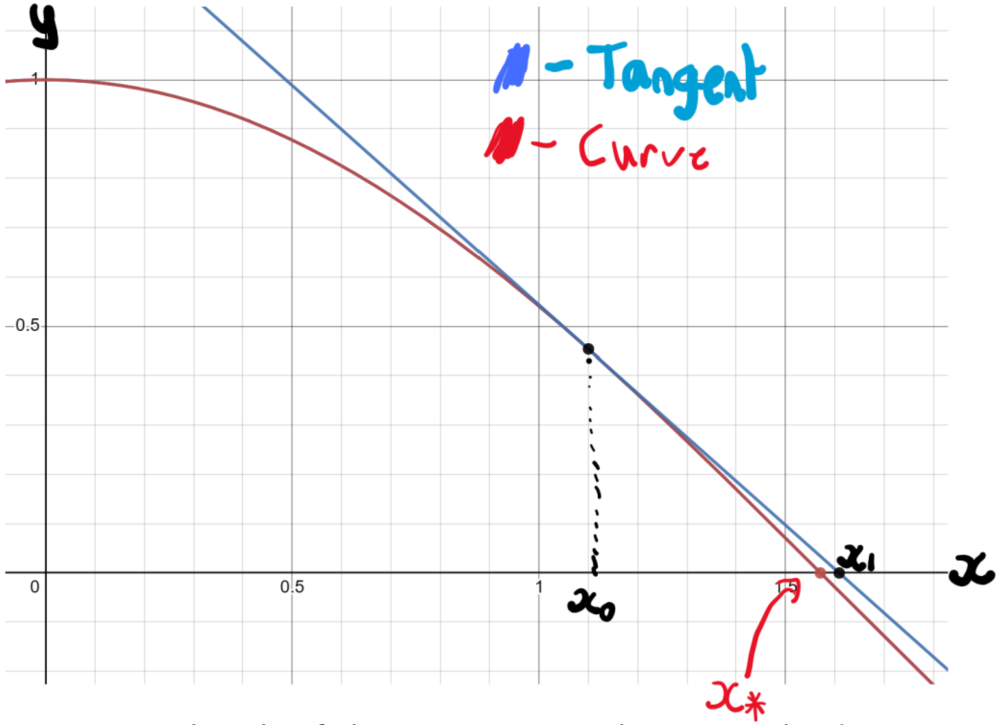

Figure 2: Sketch of the Newton-Raphson Method

Functional Taylor Series

Same as multivariate Taylor series except our “variables” are: $y(x), y'(x), y''\ (x)$ ... ext.

Example – Expand to first order about $y_i(x)$: $\boldsymbol{\cos(y(x))\, y'(x)}$

$$\Rightarrow \cos(y_i(x))y_i{}'(x) + (-\sin(y_i(x))y_i{}'(x))(y(x) - y_i(x)) + [\cos(y_i(x))](y'(x) - y_i{}'(x))$$

Remark: Multivariate Taylor Series -

$$f(x, y) = f(x_i, y_i) + \frac{\partial f(x_i, y_i)}{\partial x}(x - x_i) + \frac{\partial f(x_i, y_i)}{\partial y}(y - y_i)$$

Practical Example: Lane-Emden Equation

$$y'' + \frac{2}{x} y' + y^n = 0 \qquad y(0) = 1 \;\; and \;\; y'(0) = 0$$ [2]

1) $y'' + \frac{2}{x} y' = -y^n$

2) $y'' + \frac{2}{x} y' = (-{y_i}^n) + (-n{y_i}^{n-1})(y - y_i)$

$\Rightarrow ({y_{i+1}}'') + \frac{2}{x}({y_{i+1}}') + n{y_i}^{n-1}(y_{i+1}) = (n-1)\,{y_i}^n$

How to choose the starting point (y_0)

General case

For convergence: y_0 must satisfy the boundary conditions.

For optimal convergence: y_0 must match as many functional characteristics as the exact solution as possible.

Lane-Emden $n \geq 2$

For convergence: $y_0(0) = 1 \; and \; {y_0}'(0) = 0$

For optimal convergence: Require $\lim_{x \to \infty} y_0(x) = 0$

Using Chebyshev Polynomials *(Lane-Emden n=3)*

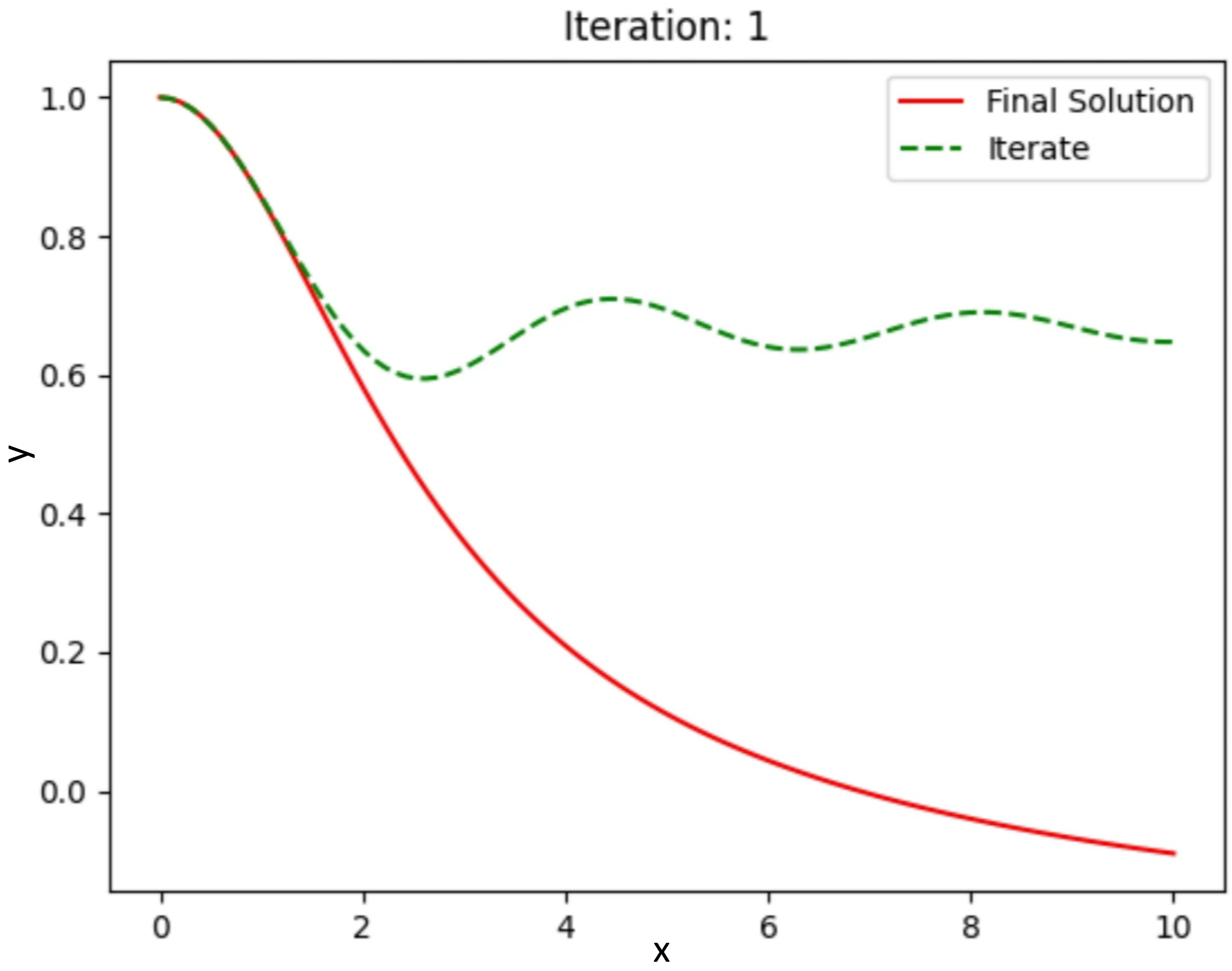

Figure 3: Shows iterates of the QLM (green), converging to the converged solution (red), with $y_0 = 1$.

Using Rational Chebyshev Functions *(Lane-Emden n=3)*

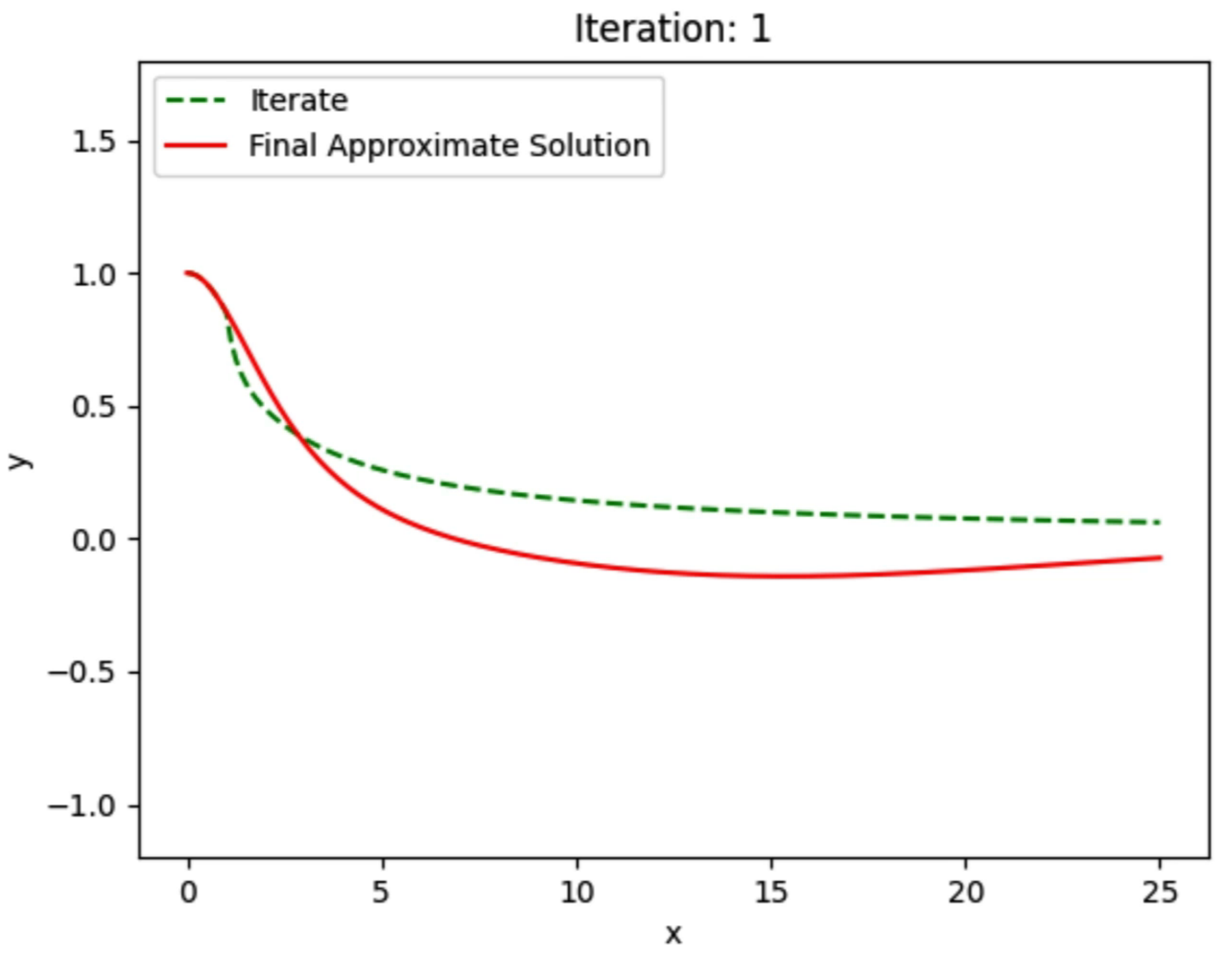

Figure 4: Shows iterates of the QLM (green), converging to the converged solution (red). In this case: $\lim_{x\to\infty} y_0 = 0$ *and* y_0 *satisfies the boundary conditions.*

Using Domain Patching and Rational Chebyshev Basis (Lane-Emden n=3)

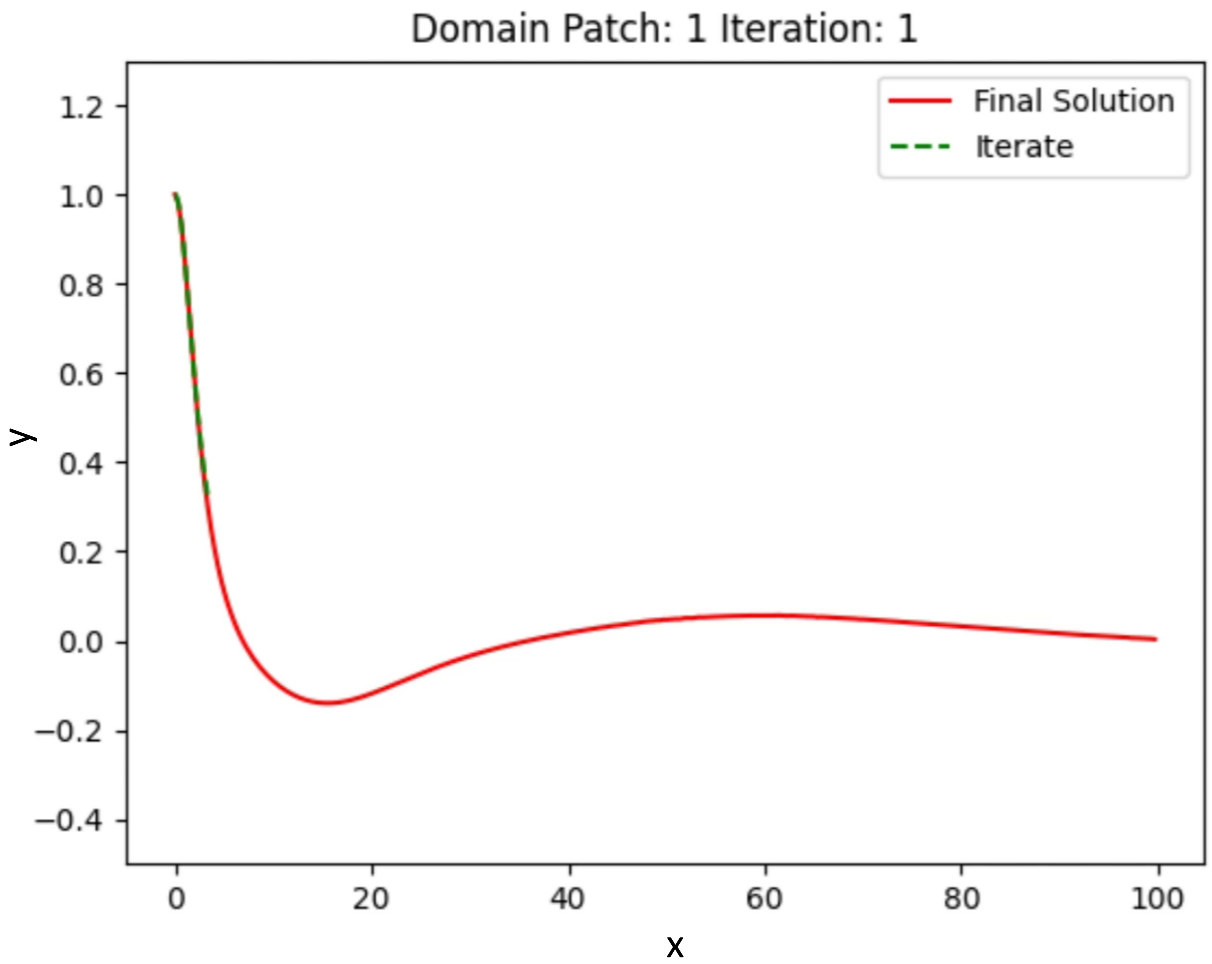

Figure 5: Shows iterates of the QLM (green), converging to the converged solution (red). In this case we split the domain into subdomains and solve on each subdomain. The boundary conditions come from forcing the overall solution to be continuously differentiable across each boundary.

Pros and Cons of QLM

Pros:

- *Quadratic Convergence for most cases [2][4]*
- *We have a method to pick our starting point*
- *Easy to implement*

Cons:

- *Resulting Linear ODEs can be hard to solve to high precision*
- *Can be very unstable*
- *Currently unknown specifically when the QLM converges*

Future Research

Concluding Remarks

Collect a full set of conditions for convergence of the QLM

Find an efficient method for solving Non-Linear ODEs with the QLM on large domains

References:

[1]J.P Boyd Chebyshev and Fourier Spectral Methods Heidelberg 1989 ISBN 0-387-51487-2 S

[2]Mandelzweig VB, Tabakin F (2001) Quasilinearization approach to nonlinear problems in physics with application to nonlinear ODEs. Comput Phys Commun 141:268–281

[3] Zwillinger, D. Handbook of Differential Equations, 3rd ed. Boston, MA: Academic Press, pp. 124 and 126, 1997.

[4] R. Bellman, R. Kalaba, Quasilinearization and Nonlinear Boundary-Value Problems, Elsevier, New York, 1965.

Any Questions?

Try my code: *(QR Code for Github)*