Welcome to this presentation on Using Spectral Methods and the Quasilinearization method aka the QLM to solve Non-linear ODEs.

What are Spectral Methods and how do they help us with solving Ordinary Differential equations. To begin we assume there is some approximate solution yN of the ODE, where phi k are basis functions that we choose and ak are coefficients that we aim to find.

To find ak we force our assumed solution to satisfy the ODE at some particular points, which we will call our collocation points.

To improve the accuracy of yN we can increase the number of collocation points ie. Increase the number of places where yN satisfies the ODE but this will require more computing power. Although we won’t look at the specifics of this area today another way to improve yN would be to change our choice of basis, this is such a large area of study and we will just look at results from particular choices today.

By forcing our solution to satisfy the ODE at a collocation point we create an equation in ak , using all of the collocation points creates a set of equations. To be more specific Linear ODEs lead to linear equations and Non-Linear ODEs lead to Non-Linear equations. This leads us to our first problem, Non-Linear equations are very challenging to solve, they need to be solved numerically and these numerical algorithms require start points which are hard to choose. On top of this Non-Linear equations have multiple solutions, this creates a phenomenon known as aliasing.

Here’s an example of Aliasing. So we are solving the ODE ydy/dx=x, it has exact solution plotted in red. By using spectral methods we get 3 solutions drawn in blue, yellow and green respectively. On the left where we plot the 3 lines we get the green line looks like the best fit whereas the other two are pretty bad. Nevertheless they all satisfy the boundary conditions marked in blue and on the right is a plot of all of the functions put into the ODE and we see all 3 pass through the collocation points which are also marked on. So we have 3 solutions but only one makes sense.

To summarise Non-Linear ODEs are hard to solve. So we would like to convert them to the closest Linear ODE, as they result in Linear equations which can be solved very easily. This is analogous to how we find a tangent to a curve to approximate roots in the Newton-Raphson method as finding where a straight line hits the x axis is super easy.

So how do we do the QLM. First, we split the ODE into a Linear and Non-linear part. Then we choose a reasonably good solution yi(x) which we will talk about how to do later. Then to linearize this we use what is called a first order functional taylor series about yi(x), again we’ll discuss how to find this later.

Solving this approximation of our ODE then leads to a solution closer to our exact solution than y\_i(x), we will call this y\_i+1(x). The diagram below shows how the Newton-Raphson method converges, the QLM is the same procedure as the Newton-Raphson method except in Functional Space.

The process for finding functional taylor series is exactly the same as a multivariate taylor series except our variables are y(x), y’(x) ext. Let’s find the functional taylor series of cos(y(x))y’(x), the first term is evaluating the non-linear part at yi(x), the second part is first partially differentiating with respect to y(x) then evaluating this at yi(x) and multiplying by (y(x)-yi(x)) like you would with say a variable x in a multivariate taylor and finally doing the same for y’(x) by partially differentiating with respect to y’(x) and evaluating at yi(x). If you pay close attention the result is in fact linear in y(x).

Here's a practical example the Lane-Emden equation. First we split this ode into a linear and non-linear part.

Then we use our functional taylor series to approximate the non-linear part. Finally we do some clearing up and replace y with yi+1 to arrive at this Linear ODE in yi+1(x).

So, we have created an iterative sequence, meaning for this to work we just need a start point ie. y0(x) but this will depend on the basis functions used. As a minimum for convergence, we require y0 to satisfy the boundary conditions. For better convergence we require the start point to match other functional characteristics such as featuring singularities in matching areas.

In the case of the Lane-Emden equation it can be shown that the exact solution decays ie. As x goes to infinity y goes to 0. For better convergence we would aim y0 to match this.

Here’s the first result, in this case our basis is the Chebyshev Polynomials. The graph shows a plot of the iterate in green and the converged solution in red, each movement is the next iterate. We get fast convergence in just 6 iterations. We have set y0=1 as it satisfies the boundary conditions, we couldn’t enforce y0 to go to 0 in this basis as it’s a polynomial basis and this has resulted in a maximum domain width of [0,10] anything bigger leads to blow up.

On the other hand, here is a really good approximation using a Rational Chebyshev Basis. In this example we picked a starting point that did have the property of decomposing to 0 as x goes to infinity. For this method we could find a solution on a domain of around [0,25] but anything bigger leads to blow-up.

Here’s an example using domain patching, this involves splitting our domain into subdomains. Finding the solution on the first subdomain using the first boundary conditions, then finding the solution by forcing the boundary to be continuously differentiable. Using this method I was able to produce a domain of over [0,100].

Overall the QLM offers quadratic convergence, a method for picking a start point and it’s easy to implement. But it can be unstable, large domains lead to blowup and conditions for convergence aren’t fully known.

For future research, I’d like to find to a find more complete set of conditions for convergence for this method, so we know the problems we can apply it to. On top of this I’d like to research more domain extending algorithms. I’ve already conducted some research producing satisfactory results by using a more complicated basis known as a fractional rational Chebyshev basis.

Thank you for listening, if you would like to reproduce the graphs seen here today or access my Linear ODE solver then you can access my GitHub here. Does anyone have any questions?

**Feedback:**

Make “The Problem” more clear

Also mention QLM = Quasilinearization Method

Emphasise the point on how to choose starting point

Make sure to mention Newton-Raphson

Explain the red solution is the converged solution

Be specific on the pros and cons slide about the METHOD

Emphasise use QLM and Spectral Methods together but focus is on QLM

Clear up the future research part

Working on the concluding section

**Add xlabel and ylabel to graphs**