**Statistical Inference** **Homework Sheet 1**

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**Question 1:** *Application of your knowledge of the exponential family*

(a) Let the random variable $X$ follow a Poisson distribution with paramter $\lambda > 0$. Show that the distribution of $X$ belongs to the exponential family.

(b) Let $X_1, \ldots, X_n$ be i.i.d. distributed random variables with $X_i \sim \text{Po}(\lambda)$, $\lambda > 0$, for $i = 1, \ldots, n$. Let $\boldsymbol{X} = (X_1, \ldots, X_n)^\top$. Show that the distribution of $\boldsymbol{X}$ belongs to the exponential family.

**Question 2:** *Application of your knowledge of the location- and scale family*

Let $X$ be logistically distributed with parameters $a \in \mathbb{R}$ and $b \in \mathbb{R}_+$. The density of the logistic distribution is defined as follows

$$f(x) = \frac{\exp\left(-\frac{x-a}{b}\right)}{b\left(1 + \exp\left(-\frac{x-a}{b}\right)\right)^2}$$

for $x \in \mathbb{R}$. It is given that $\mathbb{E}(X) = a$ and $Var(X) = b^2\pi^2/3$. Show that the logistic distribution is part of the location-and scale family.

**Question 3:** *Extending your knowledge of the exponential family*

Show that in general it is true that:
If the distribution of real random variable $X$ belongs to the exponential family with parameter vector $\boldsymbol{\theta} \in \mathbb{R}^p$ and $g : \mathbb{R} \to \mathbb{R}$ is a continuous, differentiable function with a continuous, differentiable inverse $g^{-1}$, then it follows that the distribution of $Z = g(X)$ also belongs to an exponential family.

*Hint:* In our situation, the rule of transformation in densities is applied to the distribution of $Z$. It is defined in the following way:

$$f_Z(z|\boldsymbol{\theta}) = \left|(g^{-1})'(z)\right| \cdot f_X(g^{-1}(z)|\boldsymbol{\theta})$$