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In class assignment 1
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1. Solution
   The Gompertz Model
   $$y' = -Aylny \tag{1}$$
   where $y(t)$ is the mass of tumor cells at time t, A is a constant greater than 0 .

   In the equation (1), by separating $dy(t)$ and dt, and integrating both sides
   $$\int (\frac{1}{y(t)})(\frac{1}{lny(t)})dy = -A\int dt \tag{2}$$
   $$ln(lny(t)) = -At + C_1 \tag{3}$$
   $$lny(t) = B_1 e^{-At} \tag{4}$$
   $$y(t) = e^{B_1 e^{-At}} \tag{5}$$
   If the initial mass of tumor cells is m_0, in the equation (5) we have
   $$y(0) = e^{B_1} = m_0 \tag{6}$$
   So
   $$B_1 = ln(m_0) \tag{7}$$
   Find the constant solution in the equation (1), let $y'(t) = 0$
   $$\begin{aligned} y' &= -Aylny \\ &= Ay(ln1 - lny) \\ &= 0 \end{aligned}$$
   Due to there is no 0^- in the domain of lny, the limit in
   $$\lim_{y \to 0} Ay(ln1 - lny)$$
   is not exist. $y = 0$ is an unstable point.
   We have a constant solution $y = 1$.
   When $0 < y \leq 1$, in the equation (1), we have
   $$y' \geq 0$$
   it means in the range $0 < y \leq 1$, the mass of tumor cells is growing.
   When $y \geq 1$, in the equation in the equation (1), we have
   $$y' \leq 0$$
   it means in the range $y' \leq 0$, the mass of tumor cells is declining.
   We can conclude $y = 1$ is a stable point (attractor).

2. Solution
   Assume the function y(t) represents the volume of salt in the tank at time t
   The rate salt into the tank is 0, and the rate salt out off the tank is $2 \times \frac{y(t)}{400}$
   So we can have the separable first order differential equation below:

$$\frac{dy(t)}{dt} = \text{Rate In - Rate Out} = -\frac{y(t)}{200} \tag{1}$$

   Solve equation (1) by separating $dy(t)$ and dt

$$\frac{1}{y(t)} dy(t) = -\frac{1}{200} dt \tag{2}$$

   By integrating both sides

$$\int \frac{1}{y(t)} dy(t) = -\int \frac{1}{200} dt \tag{3}$$

$$\ln |y(t)| = -\frac{1}{200} t + C_1 \tag{4}$$

$$y(t) = Ae^{-\frac{1}{200} t} \tag{5}$$

   where $y(t) \geq 0$, A is a constant coefficient

   Use the initial condition $y(0) = 100(lb)$ in equation (5), so that

$$y(0) = A = 100 \tag{6}$$

   We solve the ODE

$$y(t) = 100e^{-\frac{1}{200} t} \tag{7}$$

   Plug $t = 60$ into equation (7), we conclude the volume of salt in the tank at the end of 1 hour (60 minutes)

$$y(60) = 100e^{-\frac{60}{200}} \approx 74.08$$

3. Solution
   Assume the contact rate is constant k, and the proportion of infected person is function $i(t)$ and of non-infected person is function $n(t)$, we have the equation

$$i(t) + n(t) = 1 \tag{1}$$

   So, the ODE model (Logistic Model) for the spread of contagious disease is

$$\frac{di(t)}{dt} = k \times i(t)(1 - i(t)) \tag{2}$$

   Solve equation (2) by separating $di(t)$ and dt

$$\frac{1}{i(t)(1 - i(t))} di(t) = kdt \tag{3}$$

By integrating both sides

$$\int \frac{1}{i(t)(1-i(t))} di(t) = \int kdt \tag{4}$$

$$\int [\frac{1}{i(t)} + \frac{1}{1-i(t)}] di(t) = kt + C_1 \tag{5}$$

So

$$ln|i(t)| + ln|1-i(t)| = kt + C_1 \tag{6}$$

$$ln|\frac{i(t)}{1-i(t)}| = kt + C_1 \tag{7}$$

$$\frac{i(t)}{1-i(t)} = Ae^{kt} \tag{8}$$

$$i(t)(1 + Ae^{kt}) = Ae^{kt} \tag{9}$$

We solve the ODE

$$i(t) = \frac{Ae^{kt}}{(1 + Ae^{kt})} = \frac{A}{(A + e^{-kt})} \tag{10}$$

By solving the equation $(2) = 0$, we have two critical points. The first one $i(0) = i_0$ is an unstable point where i_0 is the initial proportion of infected person. And the second one $i(t_n) = 1$ is a stable point when time $t_n \to \infty$, it means, in this model, the whole population will be infected in a long term.

4. Solution

According to the prompt, we have the model

$$\frac{dC(t)}{dt} = -k \times C(t) \tag{1}$$

where k is the constant rate and $C(t)$ is the amount of drug present at time t .

Solve the ODE by separating $C(t)$ and dt

$$\frac{1}{C(t)} dC(t) = -kdt \tag{2}$$

By integrating both sides

$$\int \frac{1}{C(t)} dC(t) = \int -kdt \tag{3}$$

So

$$ln|C(t)| = -kt + C_1 \tag{4}$$

$$C(t) = Be^{-kt} \tag{5}$$

Plug the initial condition $C(0) = A$ into the equation (5), and we solve the ODE

$$C(t) = Ae^{-kt}$$

5. Solution

According to the prompt, we have

$$a = \frac{1}{v} \tag{1}$$

where a is acceleration and v is velocity.

We know v is the first derivative of position and a is the second derivative of position. Assume the $y(t)$ is the distance(position) at time t, so the equation (1) can be

$$y''(t) = \frac{1}{y'(t)} \tag{2}$$

By substitution $y'(t) = u(t)$, the equation (2) can be

$$\frac{du(t)}{dt} = \frac{1}{u(t)} \tag{3}$$

Solve the ODE by separating $u(t)$ and dt

$$u(t)du(t) = dt \tag{4}$$

By integrating both sides

$$\int u(t)du(t) = \int dt \tag{5}$$

$$\frac{1}{2}u^2(t) = t + C_1 \tag{6}$$

We have

$$u(t) = \sqrt{2t + C_2} = (2t + C_2)^{\frac{1}{2}} \tag{7}$$

So

$$\frac{dy(t)}{dt} = (2t + C_2)^{\frac{1}{2}} \tag{8}$$

By separating $y(t)$ and t, and integrating both sides

$$\int dy = \int (2t + C_2)^{\frac{1}{2}} \tag{9}$$

By chain rule

$$y = \frac{2}{3}(2t + C_2)^{\frac{3}{2}} \times \frac{1}{2} \tag{10}$$

We solve the ODE

$$y(t) = \frac{1}{3}(2t + C_2)^{\frac{3}{2}}$$

where the constant C_2 dependents on a given initial position $y(0)$.