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In class assignment 2
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1. Solution
According to the conditions of the RLC-circuit, we have the ODE

$$\frac{d^2i}{dt^2} + 2000 \cdot \frac{di}{dt} + 250 \cdot i = 110 \cdot 415 \cdot \cos(415t) = f(t) \tag{1}$$

For the general solution of equation, we have (1)

$$\frac{d^2i}{dt^2} + 2000 \cdot \frac{di}{dt} + 250 \cdot i = 0 \tag{2}$$

and

$$\lambda^2 + 2000 \cdot \lambda + 250 = 0 \tag{3}$$

So

$$\lambda = -1000 \pm 5\sqrt{39990} \tag{4}$$

The general solution is

$$y_c = C_1 e^{\lambda_1 \cdot t} + C_2 e^{\lambda_2 \cdot t}$$

where t is time, $\lambda_1 = -1000 + 5\sqrt{39990}$, $\lambda_2 = -1000 - 5\sqrt{39990}$, and C_1, C_2 are constants.

For the particular solution (steady-state), by the method of variation of parameters, we can assume $y_p = u_1 y_1 + u_2 y_2$, where $y_1 = e^{\lambda_1 \cdot t}$ and $y_2 = e^{\lambda_2 \cdot t}$.
We have

$$u_1' = -\frac{y_2 f(t)}{\det(W)} \tag{5}$$

$$u_2' = \frac{y_1 f(t)}{\det(W)} \tag{6}$$

where $\det(W)$ is Wronskian determinant

$$\det(W) = y_1 y_2' - y_1' y_2 \tag{7}$$

We have $y_1' = \lambda_1 y_1$, $y_2' = \lambda_2 y_2$, and then plug y_1' and y_2' into the equation (7)

$$\det(W) = (\lambda_2 - \lambda_1) y_1 y_2 \tag{8}$$

We have

$$u_1' = \frac{f(t)}{(\lambda_1 - \lambda_2) y_1} \tag{9}$$

$$u_2' = \frac{f(t)}{(\lambda_2 - \lambda_1) y_2} \tag{10}$$

Integration by parts, in equations (9) and (10)

$$u_1 = \frac{4565}{\sqrt{39990}} \cdot \frac{e^{-\lambda_1 \cdot t}(415\sin(415t) - \lambda_1 \cos(415t))}{\lambda_1^2 + 172225} \tag{11}$$

$$u_2 = -\frac{4565}{\sqrt{39990}} \cdot \frac{e^{-\lambda_2 \cdot t}(415\sin(415t) - \lambda_2 \cos(415t))}{\lambda_2^2 + 172225} \tag{12}$$

So, the particular solution (steady-state) is

$$y_p = 11411250 \cdot \frac{(33200\sin(415t) - 6879)}{(\lambda_1^2 + 172225)(\lambda_2^2 + 172225)}$$

where $\lambda_1 = -1000 + 5\sqrt{39990}$ and $\lambda_2 = -1000 - 5\sqrt{39990}$.

2. Solution

   The ODE

$$EIy'' - Py = -w_0 \frac{x^2}{2} \tag{1}$$

For the general solution, we have

$$EI\lambda^2 - P = 0 \tag{2}$$

So

$$\lambda = \pm\sqrt{\frac{P}{EI}} \tag{3}$$

We have the general solution

$$y_c = C_1 e^{\lambda_1 \cdot x} + C_2 e^{\lambda_2 \cdot x} \tag{4}$$

where $\lambda_1 = \sqrt{\frac{P}{EI}}$, $\lambda_2 = -\sqrt{\frac{P}{EI}}$, and C_1, C_2 are constants.

For the particular solution, we assume $y_p = Ax^2 + Bx + C$,
and plug y_p into the equation (1). We have

$$EI \cdot 2A - P \cdot (Ax^2 + Bx + C) = -w_0 \frac{x^2}{2} \tag{5}$$

After solve the equation (5), we have

$$A = -\frac{w_0}{2P}, B = 0, C = 0$$

So

$$y_p = -\frac{w_0}{2P} x^2 \tag{6}$$

Due to $y = y_c + y_p$, combine equation (4) and (6), we have

$$y = C_1 e^{\lambda_1 \cdot x} + C_2 e^{\lambda_2 \cdot x} - \frac{w_0}{2P} x^2 \tag{7}$$

$$y' = \lambda_1 \cdot C_1 e^{\lambda_1 \cdot x} + \lambda_2 \cdot C_2 e^{\lambda_2 \cdot x} - \frac{w_0}{P} x \tag{8}$$

According the prompt, we have conditions $y(0) = 0$ and $y'(L) = 0$.
Plug $y(0) = 0$ into the equation (7)

$$y(0) = C_1 + C_2 = 0 \tag{9}$$

Plug $y'(L) = 0$ and the equation (9) into the equation (8)

$$y'(L) = \lambda_1 \cdot C_1 e^{\lambda_1 \cdot L} + \lambda_1 \cdot C_1 e^{-\lambda_1 \cdot L} - \frac{w_0}{P} L = 0 \tag{10}$$

$$C_1 = \frac{w_0 L}{P\lambda_1 (e^{\lambda_1 L} + e^{-\lambda_1 L})} \tag{11}$$

then

$$C_2 = -\frac{w_0 L}{P\lambda_1 (e^{\lambda_1 L} + e^{-\lambda_1 L})} \tag{12}$$

So the solution is

$$y = \frac{w_0 L \cdot (e^{\lambda_1 x} - e^{-\lambda_1 x})}{P\lambda_1 (e^{\lambda_1 L} + e^{-\lambda_1 L})} - \frac{w_0}{2P} x^2$$

where $\lambda_1 = \sqrt{\frac{P}{EI}}$

3. Solution
   Let $y'' = m$, we have the new ODE

$$EIm'' + Pm = 0 \tag{1}$$

We have

$$EI\lambda^2 + P = 0 \tag{2}$$

$$\lambda = \pm \sqrt{\frac{P}{EI}} i \tag{3}$$

The general solution of the equation (1) is

$$y_c = C_1 \cos \lambda_1 x + C_2 \sin \lambda_2 x \tag{4}$$

where $\lambda_1 = \sqrt{\frac{P}{EI}}$, $\lambda_2 = -\sqrt{\frac{P}{EI}}$, and C_1, C_2 are constants.
According to the prompt, we have four conditions $y(0) = 0$, $y''(0) = 0$, $y(L) = 0$, and $y''(L) = 0$.
Plug $y(0) = 0$ into the equation (4), we have

$$y(0) = C_1 = 0$$

So the equation (4) is

$$y_c = C_2 \sin \lambda_2 x \tag{5}$$

Plug $y(L) = 0$ into the equation (5)

$$y(L) = C_2 \sin \lambda_2 L = 0 \tag{6}$$

So, $\lambda_2 \cdot L = n \cdot \pi$, where n is integers.
We also have

$$y'' = -\lambda_2^2 \cdot C_2 \sin \lambda_2 x \tag{7}$$

Plug $y''(0)$ and $y''(L)$ into the equation (7), we have $y''(0) = 0$ and $y''(L) = 0$, which are the same result given by the prompt.
So the solution is

$$y_c = C_2 \sin \lambda_2 x$$

where C_2 is constant.

4. Solution
Let $\frac{du}{dr} = k$, the new ODE is

$$r \cdot \frac{dk}{dr} + 2 \cdot k = 0 \tag{1}$$

By separating dk and dr, and integrating both sides

$$\frac{1}{k} dk = -\frac{2}{r} dr \tag{2}$$

$$\int \frac{1}{k} dk = -2 \int \frac{1}{r} dr \tag{3}$$

We have

$$ln|k| = -2ln|r| + C_1 \tag{4}$$

$$k = C_2 e^{ln(r^{-2})} = \frac{C_2}{r^2} = \frac{du}{dr} \tag{5}$$

By integrate $\frac{du}{dr}$ in the equation (5)

$$u = \int du = \int \frac{C_2}{r^2} dr = -\frac{C_2}{r} + C_3 \tag{6}$$

where C_2 and C_3 are constants.
Plug $u(a) = u_0$ and $u(b) = u_1$ into the equation (6), we have

$$u_0 = -\frac{C_2}{a} + C_3 \tag{7}$$

$$u_1 = -\frac{C_2}{b} + C_3 \tag{8}$$

Solve equations (7) and (8)

$$C_2 = \frac{ab(u_0 - u_1)}{a - b}$$

$$C_3 = \frac{au_0 - bu_1}{a - b}$$

So the solution is

$$u(r) = -\frac{ab(u_0 - u_1)}{r(a - b)} + \frac{au_0 - bu_1}{a - b}$$