Fourier Transform

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How can we generalize the Fourier series to aperiodic signals? Let $s_T(t)$ be a signal with period T.

Let us define $S_T(f) \equiv T c_k(T) = \int_{-T/2}^{T/2} s_T(t) e^{\frac{-j2\pi k t}{T}} dt$

$$s_T(t) = \sum_{k=-\infty}^{k=+\infty} S_T\left(\frac{2\pi k}{T}\right) e^{\frac{j2\pi k t}{T}} \frac{1}{T}$$

An aperiodic signal has $T \to \infty$, so spectral lines become infinitely close together.

$$\lim_{T\to\infty} s_T(t) = \lim_{T\to\infty} \sum_{k=-\infty}^{k=+\infty} S_T\left(\frac{2\pi k}{T}\right) e^{j2\pi f t} \frac{1}{T}$$

Riemann Sum　　becomes continuous　　kf　　df

$$s(t) = \int_{-\infty}^{+\infty} S(f) e^{j2\pi f t} df$$ Inv. Fourier Transform

$$S(f) = \lim_{T\to\infty} S_T(f) = \int_{-\infty}^{+\infty} s(t) e^{-j2\pi f t} dt$$ Fourier Transform

Properties of FT

Since the FT is derived from the FS, many of its properties mirror the properties of FS coefficients. Take as an example of FS coefficients. Take as an example the effect of a time delay.

$$\mathcal{F}\{s(t)\} = \int_{-\infty}^{+\infty} s(t) e^{-j2\pi f t}\, dt$$

$$\mathcal{F}\{s(t-\tau)\} = \int_{-\infty}^{+\infty} s(t-\tau)\, e^{-j2\pi f t}\, dt$$

Let $\alpha = t - \tau$ such that $d\alpha = dt$

$$\mathcal{F}\{s(t-\tau)\} = \int_{-\infty}^{+\infty} s(\alpha)\, e^{-j2\pi f(\alpha+\tau)}\, d\alpha$$

$$= e^{-j2\pi f\tau} \int_{-\infty}^{+\infty} s(\alpha)\, e^{-j2\pi f\alpha}\, d\alpha$$

$$= e^{-j2\pi f\tau}\, S(f).$$

For a complete list of properties, please refer to the textbook.

$\mathcal{F}^4\{s(t)\}$

An interesting property of FT is that $\mathcal{F}^4\{s(t)\} = (\mathcal{F}\circ\mathcal{F}\circ\mathcal{F}\circ\mathcal{F})\{s(t)\} = s(t)$.

Let's see how to show this.

$$\mathcal{F}\{s(t)\}(f) = \int_{-\infty}^{+\infty} s(t) e^{-j2\pi f t}\, dt = S(f)$$

$$\mathcal{F}\{\mathcal{F}\{s(t)\}\}(\tau) = \int_{-\infty}^{+\infty} df\, e^{-j2\pi\tau f} \int_{-\infty}^{+\infty} dt\, e^{-j2\pi f t}\, s(t)$$

$$= \int_{-\infty}^{+\infty} dt\, s(t) \underbrace{\int_{-\infty}^{+\infty} df\, \underbrace{e^{-j2\pi\tau f}}\, e^{-j2\pi t f}}_{\mathcal{F}\{e^{-j2\pi\tau f}\}(f)}$$

by changing the order of integration.

A side note about FT of complex exponentials...

$e^{j\alpha t} = f(t)$

Note that $f(t) \notin L^1(\mathbb{R})$, in other words

$\int_{-\infty}^{+\infty} |f(t)|\, dt = \infty$. But we can use the

Dirac delta, which is kinda like a function but not really, to "define" an FT for complex exponentials b/c it's mathematically useful.

Using IFT, note that

$$\int_{-\infty}^{+\infty} \delta(f-\alpha)\, e^{j2\pi f t}\, df = e^{j2\pi\alpha t}$$

by def. of the Dirac delta.

So we understand

$$e^{jf_0 t} \longleftrightarrow \delta(f-f_0).$$

Side note over.

$$\mathcal{F}\{\mathcal{F}\{s(t)\}\}(\tau) = \int_{-\infty}^{+\infty} dt\, s(t)\, \delta(t+\tau)$$

$= s(-\tau)$.

So $\mathcal{F} \circ \mathcal{F}$ reverses a function in the time domain. From here, it is sensible to reason that $\mathcal{F} \circ \mathcal{F} \circ \mathcal{F} \circ \mathcal{F}$ maps each function $s(t)$ to itself.

Relation Between FT and FS

Suppose $s(t)$ is periodic with period T. This means we can write $s(t)$ as a Fourier series of harmonics of the fundamental frequency $1/T$.

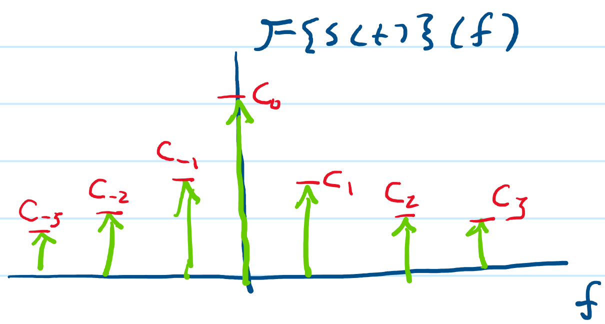
$$s(t) = \sum_{k=-\infty}^{k=+\infty} C_k e^{j\frac{2\pi k}{T}t}$$

What is the Fourier Transform of $s(t)$?

$$\mathcal{F}\{s(t)\} = \sum_{k=-\infty}^{k=+\infty} C_n \mathcal{F}\{e^{j\frac{2\pi k}{T}t}\}$$ by linearity of FT.

$$\mathcal{F}\{e^{j\frac{2\pi k}{T}t}\} = \delta\left(f - \frac{k}{T}\right)$$

$$\mathcal{F}\{s(t)\} = \sum_{k=-\infty}^{k=+\infty} C_n \delta\left(f - \frac{k}{T}\right)$$



In the Freq. domain, periodic signals are a bunch of scaled dirac deltas.