# Sampling

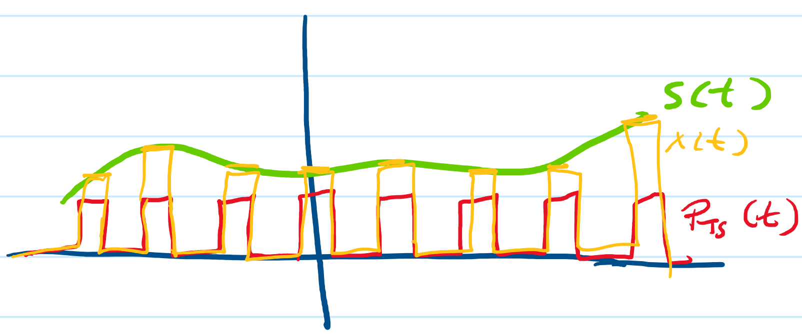
Sunday, July 10, 2022 3:39 PM

To store a signal in a computer, we need to quantize it in both time and value. Quantizing in time allows (sometimes) perfect reconstruction of the original signal due to the sampling theorem, but quantizing value leads to unavoidable sampling error.

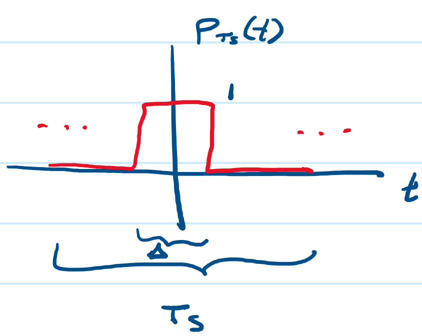
## $\boxed{\text{Sampling Theorem}}$

We can model sampling a signal with a sampling interval $T_s$ as multiplying by a periodic pulse.

$$x(t) = s(t)\, P_{T_s}(t)$$




Let's express $P_{T_s}(t)$ as a Fourier Series.



$$C_K = \frac{1}{T_s}\int_{-T_s/2}^{T_s/2} P_{T_s}(t)\, e^{-j\frac{2\pi k}{T_s}t}\, dt$$

$$= \frac{1}{T_s}\int_{-\Delta/2}^{\Delta/2} e^{-j\frac{2\pi k}{T_s}t}\, dt$$

$$= -\frac{1}{2\pi j k}\left[ e^{-j\frac{2\pi k t}{T_s}} \right]_{t=-\Delta/2}^{t=\Delta/2}$$

$$= -\frac{1}{j2\pi k}\left[ e^{-j\frac{\pi k \Delta}{T_s}} - e^{+j\frac{\pi k\Delta}{T_s}} \right]$$

$$= \frac{1}{\pi k}\left[ \frac{e^{j\frac{\pi k\Delta}{T_s}} - e^{-j\frac{\pi k\Delta}{T_s}}}{2j} \right]$$

$$= \frac{1}{\pi k}\sin\left(\frac{\pi k\Delta}{T_s}\right)$$

Nice. Now we can write $x(t)$ like this:

$$x(t) = \sum_{k=-\infty}^{k=+\infty} C_k e^{j\frac{2\pi k t}{T_s}} s(t)$$

What is $X(f)$? Suppose that $s(t) \xleftrightarrow{F} S(f)$.

$$X(f) = \int_{-\infty}^{+\infty} \sum_{k=-\infty}^{k=+\infty} C_k e^{j\frac{2\pi k t}{T_s}} e^{-j2\pi f t} s(t)\, dt$$

$$= \sum_{k=-\infty}^{k=+\infty} C_k \int_{-\infty}^{+\infty} e^{-j2\pi\left(f - \frac{k}{T_s}\right)t} s(t)\, dt$$

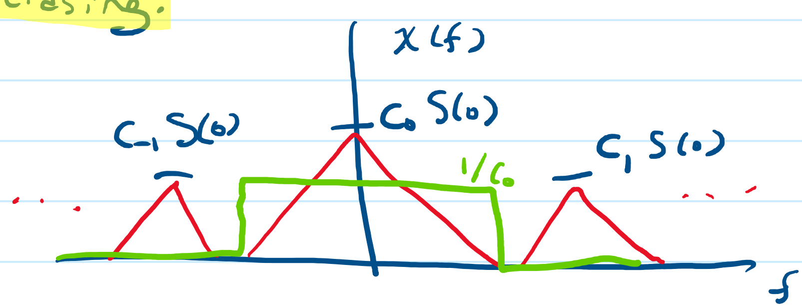
$$= \sum_{k=-\infty}^{k=+\infty} C_k\, S\left(f - \frac{k}{T_s}\right)$$

$$= \sum_{k=-\infty}^{\infty} C_k \, S\left(f - \frac{k}{T_s}\right)$$

Suppose $S(f)$ is bandlimited to $W$.


$X(f)$ is $S(f)$ repeated every $1/T_s$ periods and scaled by the Fourier Series coefficients of a periodic pulse centered at the origin.

Note that if $1/T_s < 2W$, the triangles overlap and we can't filter to recover the original signal. Overlapping is called aliasing.

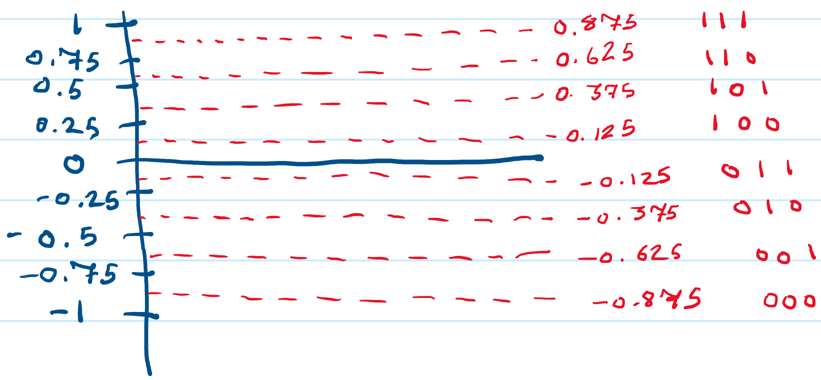



An LPF w/ $1/C_0$ gain can recover $s(t)$.
The pulse width determines the scaling coefficient the LPF requires.

The sampling theorem states for a signal bandlimited to $W$, we must sample at a frequency $\geq 2W$ to be able to fully reconstruct the signal. $2W$ is called able to fully reconstruct the signal. 2W is called the Nyquist frequency.

Anti-aliasing filters bandlimit a signal to avoid aliasing.

## Amplitude Quantization



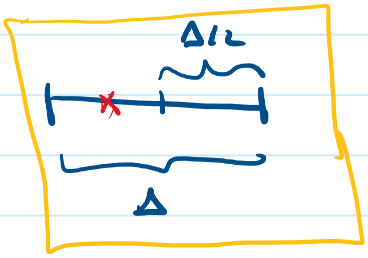

To convert a domain of continuous values to a range of discrete values, intervals are mapped to some representative value. Above shows a 3-bit quantization of the domain [-1, 1]. Note [0.75, 1] gets mapped to 0.875. In this case, for a B-bit ADC, the quantization width $\Delta = \frac{2}{2^B}$.

Let's quantify quantization error. First, we need to choose a representative signal to sample. Let's choose $s(t) = \sin(t)$, a simple sinusoid.

We'll use signal-to-noise ratio (SNR), defined as

$$SNR = \frac{\text{power}\{s(t)\}}{\text{power}\{\epsilon(t)\}}.$$

$\epsilon(t)$ is error.




The graphic above should convince you that
$|\epsilon(t)| < \Delta/2$.

$$\text{power}\{\sin(t)\} = \frac{1}{2\pi}\int_0^{2\pi} \sin^2(t)\, dt$$

$$\sin^2(t) = \frac{1-\cos(2t)}{2}$$

$$\frac{1}{2}\int_0^{2\pi} (1-\cos(2t))\, dt = \frac{1}{2}\left[t - \frac{\sin(2t)}{2}\right]_{t=0}^{t=2\pi}$$

$$= \frac{1}{2}\left[2\pi\right] = \pi = \int_0^{2\pi} \sin^2(t)\, dt$$

$$\text{Power}\{\sin(t)\} = \frac{1}{2\pi}\cdot\pi = \frac{1}{2}$$

Assuming the signal is relatively equally spaced in $[-1, 1]$, the rms of error should be close to

$$\left\{\frac{1}{\Delta}\int_{-\Delta/2}^{+\Delta/2} \varepsilon^2\, d\varepsilon\right\}^{1/2} = \left\{\frac{1}{\Delta}\cdot\frac{1}{3}\cdot\frac{\Delta^3}{4}\right\}^{1/2}$$

$$= \sqrt{\frac{\Delta^2}{12}}$$

, and the power of error should be close to $(\text{rms}\{\epsilon\})^2 = \Delta^2/12$.

$$\text{SNR} = \frac{1/2}{\Delta^2/12} = \frac{6}{\left(\frac{2}{2^B}\right)^2} = \frac{6\cdot 2^{2B}}{4}$$

$$= \frac{3}{2} \cdot 2^{2B}$$

In dB, this is $10 \log_{10}\left(\frac{3}{2} \cdot 2^{2B}\right) = 10 \log_{10}\left(\frac{3}{2}\right) + 10 \cdot 2B \log_{10}(2)$

$$\approx 6B + 1.76$$

Each additional bit in an ADC yields an additional 6 dB of SNR.