Discrete Fourier Transform (DFT)

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Definition

- There are some issues with the DTFT
  - ↳ It requires the whole signal
  - ↳ The produced spectrum has uncountably infinite frequencies in the range $[-1/2, 1/2]$

We define the DFT as follows:

$$S(k) = \sum_{n=0}^{N-1} s(n) e^{-j\frac{2\pi nk}{K}} \quad k \in [0, 1, \ldots, K-1]$$

Assuming $s[n]$ is an N-length signal.
Here, we only compute $S(k)$ at K frequencies, all equally spaced from $[0, \frac{K-1}{K}]$.

$$S(k) = S(e^{j2\pi f})\Big|_{f = k/K}$$

How can we construct $s[n]$ from $S(k)$? $s[n] \stackrel{?}{=}$

$$\sum_{k=0}^{K-1} \left(\sum_{m=0}^{N-1} s[m] e^{-j\frac{2\pi mk}{K}} \right) e^{j\frac{2\pi nk}{K}} =$$

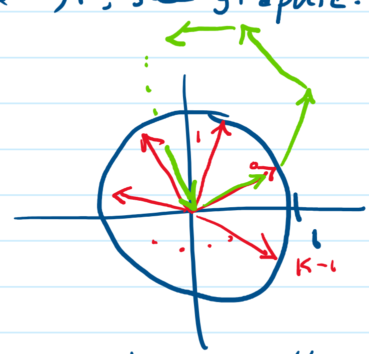
$$\sum_{m=0}^{N-1} s[m] \sum_{k=0}^{K-1} e^{-j\frac{2\pi mk}{K}} e^{-j\frac{2\pi nk}{K}} = \sum_{m=0}^{N-1} s[m] K \sum_{l=-\infty}^{l=+\infty} \delta[m-n-Kl] = K s[n]\text{, pretty close!}$$

side note:

There is also an orthogonality notion for sampled complex exponentials:

$$\sum_{k=0}^{K-1} e^{-j\frac{2\pi mk}{K}} e^{j\frac{2\pi nk}{K}} = \sum_{k=0}^{K-1} e^{j\frac{2\pi(m-n)k}{K}}$$

$= 0$ if $(m-n) \nmid K$, see graphic:



Here, we just illustrate the addition of the complex exponentials as tip-to-tail vector

Here, we just illustrate the addition of the complex exponentials as tip-to-tail vector addition (in green) which wraps back to the origin.

If $(m-n)|K$ the sum is K.

$$\langle e^{j\frac{2\pi mk}{K}}, e^{j\frac{2\pi nk}{K}} \rangle = \begin{cases} K & \text{if } (m-n)\,|\,K \\ 0 & \text{if } (m-n) \nmid K \end{cases}$$

$$= \sum_{l=-\infty}^{l=+\infty} \delta(m-n-lK)$$

——— end sidenote

To be able to recover $s[n]$ from K samples of the signal in the frequency domain, $K \geq N$ is required.

$$S(0) = s[0] + s[1] + \cdots + s[N-1]$$

$$S(1) = s[0] + s[1]e^{-j\frac{2\pi}{K}} + \cdots + s[N-1]e^{-j\frac{2\pi (N-1)}{K}}$$

$$\vdots$$

$$S(K-1) = s[0] + s[1]e^{-j\frac{2\pi K}{K}} + \cdots + s[N-1]e^{-j\frac{2\pi (K-1)(N-1)}{K}}$$

So solve for $s[0], s[1], \ldots, s[N-1]$, we need at least N equations.

So we arrive at the formal definition of a DFT:

$$S(k) = \sum_{n=0}^{N-1} s[n] e^{-j\frac{2\pi nk}{N}}$$

$$s[n] = \frac{1}{N} \sum_{k=0}^{N-1} S(k) e^{j\frac{2\pi kn}{N}}$$

from "length" of complex exponentials

Properties

Note the IDFT generates an $s[n]$ periodic with period N, while the original $s[n]$ is only defined for $n = 0, 1, \ldots, N-1$.

Let's resolve this by just considering $s[n]$

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Let's resolve this by just considering $s[n]$ periodic with period N. This means that
$s[-n] = s[N-n]$

$$S(k) = \sum_{n=0}^{N-1} s[n] e^{-j\frac{2\pi k n}{N}}$$

If $s[n]$ is "even", then
$s[n] = s[N-n]$ and
$S(k) = S(N-k)$

If $s[n]$ is "odd", then
$s[n] = -s[N-n]$ and
$S(k) = -S(N-k)$

Proof

$$S(N-k) = \sum_{n=0}^{N-1} s[n] e^{\frac{-j2\pi(N-k)n}{N}}$$

$$= \sum_{n=0}^{N-1} s[n] e^{-j2\pi n} e^{+j\frac{2\pi k n}{N}}$$

$$= \sum_{n=0}^{N-1} s[n] e^{j\frac{2\pi k n}{N}}$$

Let $\alpha = -n$ $s[\alpha] = s[n]$ since $s[n]$ is even.

$$S(N-k) = \sum_{n=0}^{N-1} s[\alpha] e^{-j\frac{2\pi k \alpha}{N}}$$

$$= S(k). \quad \blacksquare$$

An important property of the DFT is that
if $s[n] : \mathbb{Z} \leftrightarrow \mathbb{R}$, then $S(k) = S^*(-k) = S^*(N-k)$.
Proof:

$$S(k) = \sum_{n=0}^{N-1} s[n] e^{-j\frac{2\pi k n}{N}}$$

$$= \left(\sum_{n=0}^{N-1} s[n] e^{j\frac{2\pi k n}{N}}\right)^*$$

$$= \left(\sum_{n=0}^{N-1} s[n] e^{j\frac{2\pi k n}{N}} e^{-j\frac{2\pi N n}{N}}\right)^*$$

$$= \left(\sum_{n=0}^{N-1} s[n] e^{-j\frac{2\pi n}{N}(N-k)}\right)^*$$

$$= S^*(N-k). \quad \blacksquare$$

This fact will come in handy when we discuss FFT.