Discrete Time Fourier Transform

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$S(f) = \int_{-\infty}^{+\infty} s(t)\, e^{-j2\pi f t}\, dt$ for analog signals.

For discrete-time signals, we can define the Discrete-Time Fourier Transform (DTFT) as follows:

$$S(e^{j2\pi f}) = \sum_{n=-\infty}^{n=+\infty} s[n]\, e^{-j2\pi f n}$$

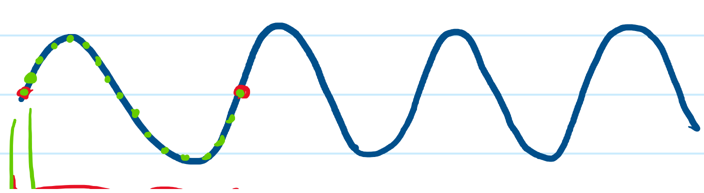
This is analogous to the definition above.
Since $e^{-j2\pi f(m+n)} = e^{-j2\pi f m} e^{-j2\pi f n}$
$= e^{-j2\pi f n}$ for $m \in \mathbb{Z}$, we only need to consider $S(e^{j2\pi f t})$ over a unit-length interval. It is common to select $[-\frac{1}{2}, \frac{1}{2}]$ as the interval.

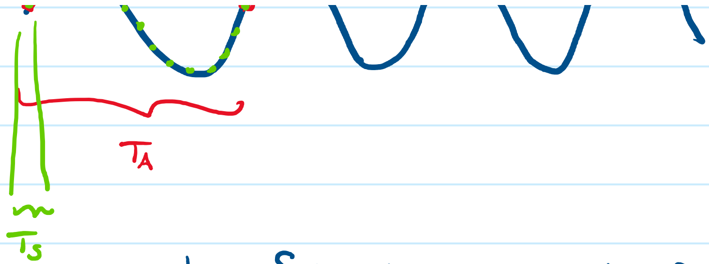
If $s[n] \in \mathbb{R}$, $S(e^{j2\pi f}) = S^*(e^{-j2\pi f})$, so we only need to plot $S(e^{j2\pi f})$ over $[0, \frac{1}{2}]$ to understand the frequency-domain representation of the signal.

f is digital frequency (let's write f_d for clarity).

$f_d = f_a T_s$ is the relation between digital and analog frequency.
why?

$$\frac{T_A}{T_D} = T_S \Rightarrow \frac{T_A}{T_S} = T_D$$





How many samples fit into a period of the analog function being sampled? This is the digital period, which is dimensionless.

Example Find the DTFT of $s[n] = a^n u[n]$.

$$S(e^{j2\pi f}) = \sum_{n=-\infty}^{n=+\infty} a^n u[n]\, e^{-j2\pi f n}$$

$$= \sum_{n=0}^{n=+\infty} \left(a e^{-j2\pi f}\right)^n$$

$$= \frac{1}{1 - a e^{-j2\pi f}}$$ if $|a| < 1$ since this is a geometric series.

$$= \frac{1}{1 - a\cos(2\pi f) + j a \sin(2\pi f)}$$

So $$|S(e^{j2\pi f})| = \frac{1}{\sqrt{(1 - a\cos(2\pi f))^2 + a^2 \sin^2(2\pi f)}}$$

and $$\angle S(e^{j2\pi f}) = -\tan^{-1}\left(\frac{a\sin(2\pi f)}{1 - a\cos(2\pi f)}\right)$$

See the textbook for graphs of these fxns.

$\boxed{\text{Inverse DTFT}}$

$$\int_{-1/2}^{1/2} e^{j2\pi f n} e^{-j2\pi f m}\, df = \left[\frac{e^{j2\pi f(n-m)}}{j2\pi(n-m)}\right]_{-1/2}^{1/2}$$ if $n \neq m$

$$= \frac{e^{j\pi(n-m)}}{} - \frac{e^{-j\pi(n-m)}}{}$$

$$= \frac{e^{j\pi(n-m)}}{j2\pi(n-m)} - \frac{e^{-j\pi(n-m)}}{j2\pi(n-m)}$$

$$= \frac{(-1)^{n-m} - (-1)^{n-m}}{j2\pi(n-m)} = 0$$

If $n=m$, the integral is 1.
Remember, complex exponentials are orthogonal.

$$\int_{-1/2}^{1/2} S(e^{j2\pi f}) e^{j2\pi fn} \, df =$$

$$\int_{-1/2}^{1/2} \sum_{m=-\infty}^{m=+\infty} s[m] e^{-j2\pi fm} e^{j2\pi fn} \, df$$

$$= \sum_{m=-\infty}^{m=+\infty} s[m] \int_{-1/2}^{+1/2} e^{-j2\pi fm} e^{j2\pi fn} \, df$$

$$= \sum_{m=-\infty}^{m=+\infty} s[m] \, \delta[m-n] = s[n]$$

So Inverse DTFT is in fact analogous to inverse FT.

$\boxed{\text{Parseval's Theorem}}$

$$\int_{-1/2}^{+1/2} \left| S(e^{j2\pi f}) \right|^2 df = \int_{-1/2}^{+1/2} \left(\sum_{n=-\infty}^{n=+\infty} s[n] e^{-j2\pi fn} \right) \left(\sum_{n=-\infty}^{n=+\infty} s^*[m] e^{j2\pi fm} \right) df$$

$$= \sum_{n=-\infty}^{n=+\infty} \sum_{m=-\infty}^{m=+\infty} s[n] s^*[m] \int_{-1/2}^{+1/2} e^{-j2\pi fn} e^{j2\pi fm} \, df$$

$$= \sum_{m,n} s[n] s^*[m] \, \delta[m-n]$$

$$= \sum_{n=-\infty}^{n=+\infty} s[n] s^*[n] = \sum_{n=-\infty}^{n=+\infty} |s[n]|^2$$

↑ definition of energy for a

$$= \sum_{n=-\infty} s[n]\, s^*[n] = \sum_{n=-\infty} |s[n]|$$

definition of energy for a discrete-time signal

Signal energy can be found in either the time or the frequency domain.

$\boxed{\text{Recap}}$

$$S(e^{j2\pi f}) = \sum_{n=-\infty}^{n=+\infty} s[n]\, e^{-j2\pi f n}$$

$$s[n] = \int_{-1/2}^{+1/2} S(e^{j2\pi f})\, e^{j2\pi f n}\, df$$