Fourier Series

Wednesday, June 15, 2022 8:16 AM

Question: How can we represent periodic signals as the sum of sinusoids?

Let $s(t)$ be periodic w/ period T.

Its fundamental frequency is $f_0 = 1/T$.

Let's try expressing $s(t)$ as the superposition of harmonics of f_0.

$$s(t) = \sum_{k=-\infty}^{k=\infty} c_k e^{j\frac{2\pi k t}{T}} = \sum_{k=-\infty}^{k=\infty} c_k e^{j 2\pi k f_0 t}$$

$$= \cdots + c_{-1} e^{j2\pi(-1)f_0 t} + c_0 e^{j2\pi(0)f_0 t} + c_1 e^{j2\pi(1)f_0 t} + \cdots$$

But how to find c_k?

Orthogonality

Recall from linear algebra that two vectors are orthogonal when their dot product is 0.

$\langle \vec{a}, \vec{b} \rangle = 0$

This is useful for extending our 3D intuition of what orthogonality means to n dimensions and more abstract vector spaces.

$\{ e^{j\frac{2\pi k}{T}t} \mid k \in \mathbb{Z} \}$ are basis functions for Fourier series, and can be thought of as basis vectors.

Let's say $\phi_k(t)$ corresponds to $e^{j2\pi f_0 k t}$

$$\langle \phi_m, \phi_n \rangle \equiv \int_a^b \phi_m(x)\,\overline{\phi_n(x)}\,dx$$

where $[a,b]$ is some chosen interval.
This is an extension of the original dot product definition to continuous functions from discrete ones.

But note something important:

$$! \quad \int_0^T e^{j\frac{2\pi k t}{T}} e^{-j\frac{2\pi \ell t}{T}}\,dt = \ldots$$

(1) suppose $k \neq \ell$

$$\int_0^T e^{j\frac{2\pi t}{T}(k-\ell)}\,dt =$$

$$\left[e^{j\frac{2\pi t}{T}(k-\ell)} \cdot \frac{T}{2\pi j(k-\ell)} \right]_{t=0}^{t=T} =$$

$$\frac{T}{2\pi j(k-\ell)} \left[e^{j2\pi(k-\ell)} - 1 \right] = \boxed{0}$$

because

$$e^{j2\pi(k-\ell)} = \left(e^{j2\pi}\right)^{k-\ell} = 1^{k-\ell} = 1$$

(2) Suppose $k = \ell$

$$\int_0^T 1\,dt = \boxed{T}$$

This is the Dirac delta. more on this in ELEC 242.

So we have shown $\langle \phi_k, \phi_\ell \rangle = \begin{cases} T & \text{if } k=\ell \\ 0 & \text{otherwise} \end{cases} = T\delta(k-\ell)$

In other words, harmonics (pairs of) of a complex exponential with a fundamental frequency are orthogonal.

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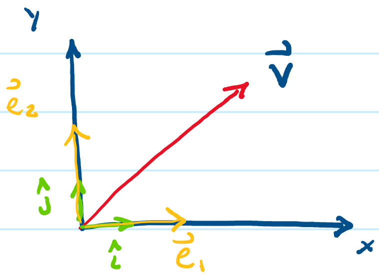
Back to Fourier Series. Why cover orthogonality?



$$\vec{v} = \alpha\,\vec{e}_1 + \beta\,\vec{e}_2$$

$$\underbrace{\frac{\langle \vec{v}, \vec{e}_1 \rangle}{\|\vec{e}_1\|}}_{\substack{\text{Projection of } \vec{v} \text{ onto} \\ \text{the normalized} \\ \text{basis vector}}} \cdot \underbrace{\frac{1}{\|\vec{e}_1\|}}_{\substack{\text{How many} \\ \vec{e}_1 \text{ vectors} \\ \text{do we need} \\ \text{to stack?}}} = \alpha = \frac{\langle \vec{v}, \vec{e}_1 \rangle}{\langle \vec{e}_1, \vec{e}_1 \rangle}$$

$\|\vec{v}\|$ is the L2 length of $\vec{v}$, and is defined as $\sqrt{\langle \vec{v}, \vec{v} \rangle}$

You can see the above formula will give use coefficients to express a vector $\vec{v}$ in terms of an orthogonal, not necessarily normal basis set. We can apply this directly to finding C_k for basis function $\phi_k(t)$.

$$C_k = \frac{\langle s(t), \phi_k(t) \rangle}{\;}$$

sort of like projecting $s(t)$ onto $\phi_k(t)$

$$C_k = \frac{\langle s(t), \phi_k(t) \rangle}{\langle \phi_k(t), \phi_k(t) \rangle}$$ s(t) onto $\phi_k(t)$

"length" of $\phi_k(t)$

Back to integral form, we get

complex conjugate!

$$\langle s(t), \phi_k(t) \rangle = \int_0^T s(t)\, e^{-j\frac{2\pi k}{T}t}\, dt$$

$\langle \phi_k(t), \phi_k(t) \rangle = T$ (shown previously)

$$\boxed{C_k = \frac{1}{T}\int_0^T s(t)\, e^{-j\frac{2\pi k t}{T}}\, dt}$$

So we can now express periodic functions as sinusoids similar to how we express geometric vectors as the sum of perpendicular vectors. I like to think about Fourier Series like this since it is more intuitive than just memorizing the formula. Later we'll see how to generalize this to aperiodic signals with the Fourier Transform.