Fourier Series Properties

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If a signal has certain properties, i.e. it is real or even, then we may deduce corresponding properties about its Fourier coefficients. Let's take a look.

$\boxed{\text{Property 1}}$ If $s(t)$ is real, then $C_k = C_{-k}^*$

why is that?

$$C_k = \frac{1}{T}\int_0^T s(t)\, e^{-j\frac{2\pi k t}{T}}\, dt$$

double minus sign for clarity

$$C_k = \frac{1}{T}\int_0^T s(t)\, e^{-(-j\frac{2\pi(-k)t}{T})}\, dt$$

$$= C_{-k}^*$$

- There's an intuitive way to see this, ask in office hours.

Remember...

$Ce^{j\phi}$ has complex conjugate

$Ce^{-j\phi} = C(\cos(-\phi) + j\sin(-\phi))$

$= C(\cos(\phi) - j\sin(\phi))$

which is clearly the complex conjugate of $C(\cos(\phi) + j\sin(\phi)) = Ce^{j\phi}$.

The caveat here is $C \in \mathbb{R}$, but we assume $s(t)$ is real for this property.

$\boxed{\text{Property 2}}$

If $s(t) = s(-t)$, then $C_k = C_{-k}$.

$$C_k = \frac{1}{T}\int_0^T \underbrace{s(t)}_{\text{|| even}} e^{-j\frac{2\pi k t}{T}}\, dt$$

$$= \frac{1}{T}\int_0^T s(-t)\, e^{-j\frac{2\pi k t}{T}}\, dt$$

Let $\alpha = -t$ so that $d\alpha = -dt$

$$C_k = \frac{1}{T}\int_{\alpha=0}^{\alpha=-T} s(\alpha)\, e^{j\frac{2\pi k\alpha}{T}} (-d\alpha)$$

$$= \frac{1}{T}\int_{\alpha=-T}^{\alpha=0} s(\alpha)\, e^{j\frac{2\pi k\alpha}{T}}\, d\alpha$$

$$= \frac{1}{T}\int_{\alpha=0}^{\alpha=T} s(\alpha)\, e^{-j\frac{2\pi(-k)\alpha}{T}}\, d\alpha \qquad \text{by periodicity.}$$

$= C_{-k}$ by definition. ☒

In other words, even functions have $C_k = C_{-k}$ $\forall k$.

$\boxed{\text{Property 3}}$ If $s(t) = -s(-t)$, then $C_{-k} = -C_k$. In other words, if $s(t)$ is odd, then $C_{-k} = -C_k$.

Proof:

$$C_{-k} = \frac{1}{T}\int_0^T s(t)\, e^{j\frac{2\pi k t}{T}}\, dt$$

$$= \frac{1}{T}\int_0^T -s(-t)\, e^{j\frac{2\pi k t}{T}}\, dt$$

Let $\alpha = -t$ so that $d\alpha = -dt$

$$C_{-k} = \frac{1}{T}\int^{-T} -s(\alpha)\, e^{-j\frac{2\pi k\alpha}{T}} (-d\alpha)$$

$$c_{-k} = \frac{1}{T}\int_0^{-T} -s(\alpha)\, e^{-j\frac{2\pi k\alpha}{T}} \left(-d\alpha\right)$$

$$= \frac{1}{T}\int_0^{-T} s(\alpha)\, e^{-j\frac{2\pi k\alpha}{T}}\, d\alpha$$

$$= -\frac{1}{T}\int_{-T}^{0} s(\alpha)\, e^{-j\frac{2\pi k\alpha}{T}}\, d\alpha$$

$$= -\frac{1}{T}\int_{0}^{T} s(\alpha)\, e^{-j\frac{2\pi k\alpha}{T}}\, d\alpha$$

$$= -c_k \quad \text{by periodicity.}$$

$\boxed{\text{Property 4}}$ The Fourier coefficients of $s(t-\tau)$ are $c_k e^{-j\frac{2\pi k\tau}{T}}$.

Proof:

$$\frac{1}{T}\int_0^T s(t-\tau)\, e^{-j\frac{2\pi k t}{T}}\, dt = c_k^*$$

Let $\alpha = t-\tau$ so that $d\alpha = dt$.

$$c_k^* = \frac{1}{T}\int_{\alpha=-\tau}^{\alpha=T-\tau} s(\alpha)\, e^{-j\frac{2\pi k(\alpha+\tau)}{T}}\, d\alpha$$

$$= e^{-j\frac{2\pi k\tau}{T}}\ \underbrace{\frac{1}{T}\int_{-\tau}^{T-\tau} s(\alpha)\, e^{-j\frac{2\pi k\alpha}{T}}\, d\alpha}$$

$$= e^{-j\frac{2\pi k\tau}{T}}\, c_k$$

This integral is the same as long as the bounds are one period since $e^{-j\frac{2\pi k\alpha}{T}}$ has period of T just like $s(\alpha)$.

Property 5 Also called Parseval's theorem, we have that avg. pwr of a signal in time is the same as power calc. in frequency domain.

$$\frac{1}{T}\int_0^T s^2(t)\,dt = \sum_{k=-\infty}^{k=\infty} |c_k|^2$$

Proof:

$$\frac{1}{T}\int_0^T s^2(t)\,dt = \frac{1}{T}\int_0^T \left(\sum_{k=-\infty}^{k=+\infty} c_k e^{j\frac{2\pi k t}{T}}\right)\left(\sum_{l=-\infty}^{l=+\infty} c_l e^{j\frac{2\pi l t}{T}}\right) dt$$

$$= \frac{1}{T}\int_0^T \left(\sum_{k=-\infty}^{+\infty}\sum_{l=-\infty}^{+\infty} c_k c_l\, e^{j\frac{2\pi k t}{T}} e^{j\frac{2\pi l t}{T}}\right) dt$$

$$= \sum_{k=-\infty}^{k=+\infty}\sum_{l=-\infty}^{l=+\infty} c_k c_l \underbrace{\frac{1}{T}\int_0^T e^{j\frac{2\pi k t}{T}} e^{j\frac{2\pi l t}{T}}\,dt}_{T\delta(k+l)}$$

$$= \sum_{k=-\infty}^{k=+\infty} c_k c_{-k}$$

$$= \sum_{k=-\infty}^{k=+\infty} c_k c_k^* = \sum_{k=-\infty}^{k=+\infty} |c_k|^2 \; *$$

if $k=-l$, then this integral is 1 (see orthogonality in FS notes)

* This assumes that $s(t)$ is a real function

Bonus

c_0 is the average of a signal.

$$c_0 = \frac{1}{T}\int_0^T s(t)\,dt = \text{AVG of } s(t)$$

relation between classical Fourier

Series and complex exponentials:

$$s(t) = A_0 + \underbrace{\sum_{k=1}^{\infty} A_k \cos\left(\frac{2\pi k t}{T}\right)}_{\underbrace{\sum_{k=1}^{\infty} A_k \left(\frac{e^{j\frac{2\pi k t}{T}} + e^{-j\frac{2\pi k t}{T}}}{2}\right)}_{\frac{1}{2}\sum_{k=-\infty}^{k=+\infty} A_k e^{j\frac{2\pi k t}{T}}}} + \underbrace{\sum_{k=1}^{\infty} B_k \sin\left(\frac{2\pi k t}{T}\right)}_{\sum_{k\geq 1} B_k \left(\frac{e^{j\frac{2\pi k t}{T}} - e^{-j\frac{2\pi k t}{T}}}{2j}\right)}$$

$$= \frac{1}{2j} \sum_{k \neq 0} B_k e^{j\frac{2\pi k t}{T}}$$

$$\boxed{C_k = \begin{cases} A_0 & \text{if } k = 0 \\ \dfrac{A_k - jB_k}{2} & \text{otherwise} \end{cases}}$$