Moments (derivations)

1) Gaussian : $X \sim \mathcal{N}(\mu, \sigma^2)$

CF : $\varphi_X(\cancel{x}\, t) = \exp\left(i\mu t - \frac{\sigma^2 t^2}{2}\right)$

first Moment ~~mean~~ $(E(X))$

$$\frac{\partial}{\partial t} \varphi_X(t) = (i\mu - \sigma^2 t) \cdot \exp\left(-i\mu t - \frac{\sigma^2 t^2}{2}\right)$$

→ evaluate at $t=0$

$$iE[X] = i\mu \cdot \exp(0) = i\mu$$

$$\Leftrightarrow E[X] = \mu$$

second moment $(E(X^2))$

$$\frac{\partial^2}{\partial t^2} \varphi_X(t) = -\sigma^2 \exp\left(i\mu t - \frac{\sigma^2 t^2}{2}\right) + (i\mu - \sigma^2 t) \cdot (i\mu - \sigma^2 t) \exp\left(-i\mu t - \frac{\sigma^2 t^2}{2}\right)$$

→ evaluate at $t=0$

$$i^2 E[X^2] = -\sigma^2 \exp(0) + i^2\mu^2 \exp(0)$$
$$= -\sigma^2 - \mu^2$$

$$\Leftrightarrow E[X^2] = \sigma^2 + \mu^2$$

for centered moment set $\mu = E[X] = 0$

$$\Rightarrow E[X^2]_c = \sigma^2$$

third moment $(E(X^3))$

$$\frac{\partial^3}{\partial t^3} \varphi_X(t) = -(i\mu - \sigma^2 t)\sigma^2 \exp(\cdot\cdot) + 2(i\mu - \sigma^2 t)(-\sigma^2) \exp(\cdot\cdot) + (i\mu - \sigma^2 t)^3 \exp(\cdots)$$

→ evaluate at $t=0$: $i^3 E[X^3] = -3i\mu\sigma^2 + i^3\mu^3$

$$\Leftrightarrow E[X^3] = -\frac{1}{i^2} 3\mu\sigma^2 + \mu^3 = 3\mu\sigma^2 + \mu^3$$

For standardized moment set $\mu = E[X] = 0$ and $\sigma^2 = E[X^2]_c = 1$ $\Rightarrow E[X_3]_s = 0$

fourth moment:

$$\frac{\partial^4}{\partial t^4} \varphi_X(t) = \dots$$

→ evaluate at $t=0$ $\Rightarrow E[X^4] = \mu^4 + 6\mu^2\sigma^2 + 3\sigma^4$

⇒ standardize:

$$E[X^4]_s = 3\underbrace{\sigma^2}_{=1}\underbrace{\sigma^2}_{=1} = 3$$

$$Y \sim \text{Laplace}(\mu, b)$$

$$CF: \varphi_Y(t) = \frac{e^{it\mu}}{1+b^2t^2}$$

First moment:

$$\frac{\partial}{\partial t}\varphi_Y(t) = \frac{\partial}{\partial t}(1+b^2t^2)^{-1}\exp(it\mu)$$

$$= -(2b^2t)\exp(it\mu) + (1+b^2t^2)^{-1} i\mu \cdot \exp(it\mu)$$

$\rightarrow$ evaluate at $t=0$

$$iE[Y] = \cancel{ea+}\ i\mu$$

$$\Leftrightarrow E[Y] = \mu$$

Second moment:

$$\frac{\partial^2}{\partial t^2}\varphi_Y(t) = -2b^2\exp(it\mu) - 2b^2t^2 i\mu\exp(it\mu) + i^2\mu^2\exp(it\mu) +$$
$$+ 2b^2ti\mu\exp(it\mu) + (bti\mu)^2\exp(it\mu)$$

$\rightarrow$ evaluate at $t=0$

$$i^2E[\cancel{x}Y^2] = -2b^2 + i^2\mu^2 = -2b^2 - \mu^2$$

$$\Leftrightarrow E[Y^2] = 2b^2 + \mu^2$$

center i.e. set $\mu = E[Y] = 0$

$$\Rightarrow E[Y^2]_c = 2b^2$$

same proceedure for third and fourth moment

Uniform distr. $z \sim Unif(a,b)$

first moment:

$\cancel{E[X] = \int_{-\infty}^{\infty} x f_z(x)}$ $f_z(z) = \begin{cases} \frac{1}{b-a} & , a \leq z \leq b \\ 0 & , else \end{cases}$ density

$$E[Z] = \int_{-\infty}^{\infty} z \cdot f_z(z)\, dz$$

$$= \frac{1}{b-a} \int_a^b z\, dz$$

$$= \frac{1}{2} \frac{b^2 - a^2}{b-a} = \frac{a+b}{2}$$

second centered moment:

$$E[Z^2]_{cen} = E[Z^2] - E[Z]^2$$

$$= \frac{1}{b-a} \int_a^b z^2\, dz - \left(\frac{a+b}{2}\right)^2$$

$$= \frac{1}{3} \frac{b^3 - a^3}{b-a} - \left(\frac{a+b}{2}\right)^2$$

$$= \frac{1}{12} (b-a)^2$$

same proceedure for third and fourth moment