Machine Intelligence 2

2.1 Independent Component Analysis

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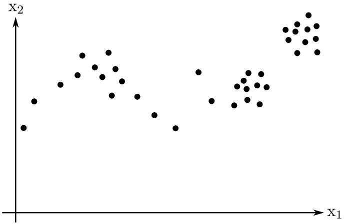
Reminder: Projection methods

observations: $\{\underline{\mathbf{x}}^{(\alpha)}\}, \alpha = 1, \ldots, p; \underline{\mathbf{x}} \in \mathbb{R}^N$

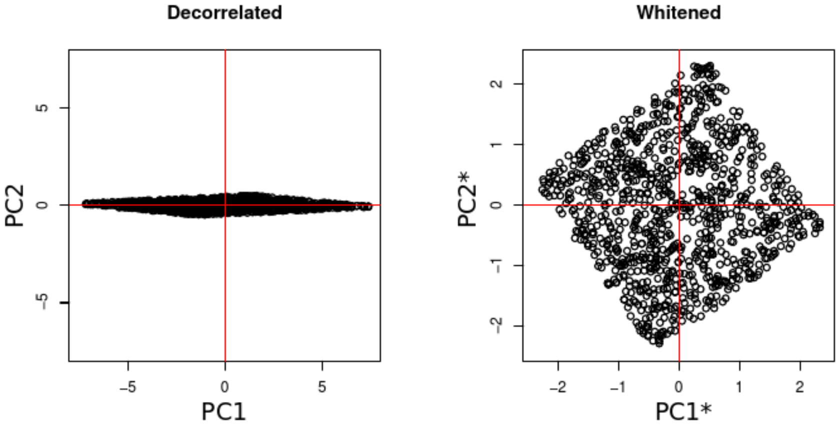


- $\rightsquigarrow$ often high-dimensional
- $\rightsquigarrow$ often grouped or clustered
- $\rightsquigarrow$ interesting directions
- $\rightsquigarrow$ different causes ($\rightsquigarrow$ unmixing)

PCA

- Directions of maximum variance
- Decorrelation

PCA may not be enough: Example 1 (uniform distribution)



Alternative methods

- Nonlinear methods (kPCA)
- Locally linear methods (mixture models)

  ...

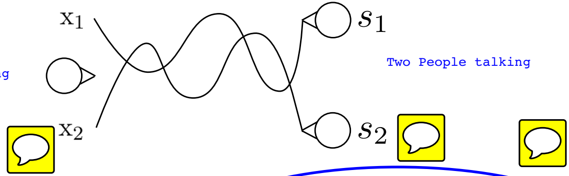
- Higher order / non-Gaussian structure (ICA)
- Finite autocorrelations (second order methods)

Blind Source Separation

The cocktail party problem

Linear superposition of acoustic signals

One listening



$$\underbrace{\underline{\mathbf{x}}}_{\text{observations}} = \underbrace{\underline{\mathbf{A}}}_{\text{mixing matrix}} \cdot \underbrace{\underline{\mathbf{s}}}_{\text{sources}} \Rightarrow \underbrace{\widehat{\underline{\mathbf{s}}}}_{\text{reconstructed sources}} = \underbrace{\underline{\mathbf{W}}}_{\text{unmixing matrix}} \cdot \underbrace{\underline{\mathbf{x}}}_{\text{observations}}$$

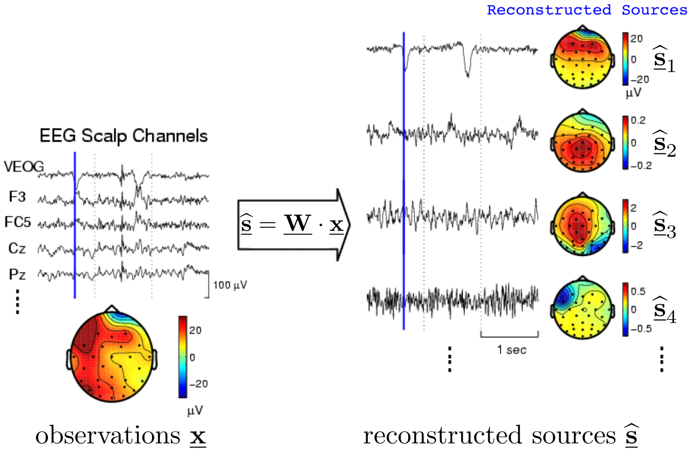
Problem: only Observations

Aim in ica: reconstruct sources

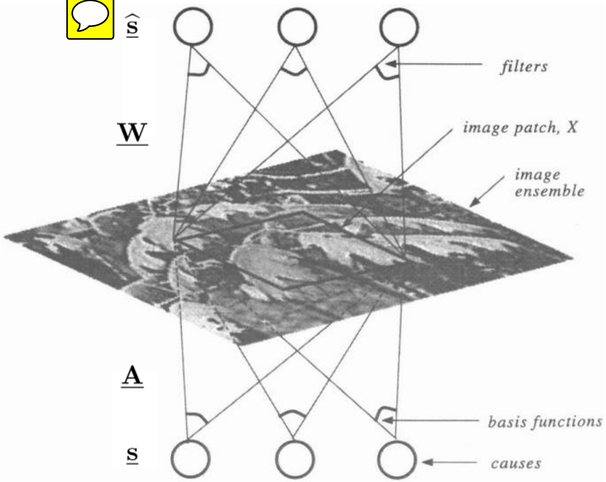
- $p * N$ observations
- $p * N + N^2$ unknowns

Source separation methods differ in what prior knowledge they exploit.

Application of BSS: EEG signals



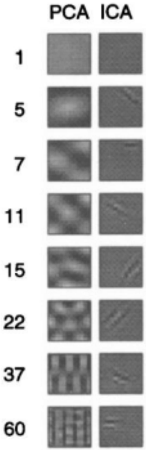
Application of BSS: Images

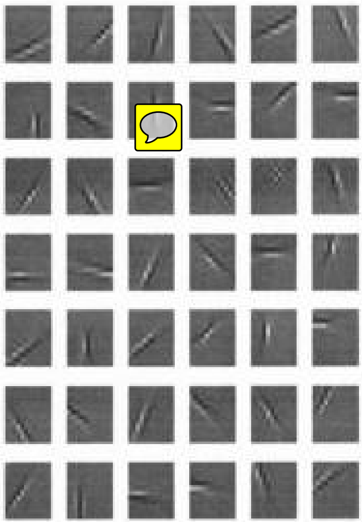


Source: Bell, Sejnowski 1997

Application of BSS: Image-derived filters

ICA on whitened data





Source: Bell, Sejnowski 1997 (modified)

Prior knowledge & cost functions

Very expensive

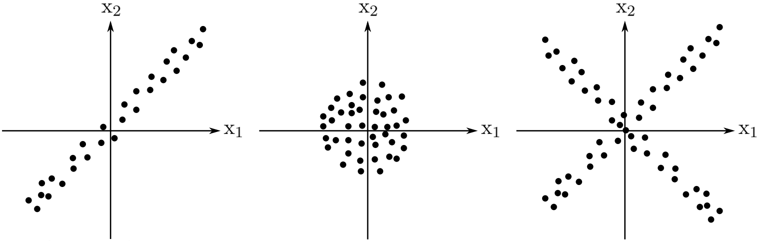
Statistical independence & infomax ("ICA")

Statistical independence		Infomax
$\underline{\mathbf{x}}(\alpha) \overset{iid}{\sim} P_{\underline{\mathbf{x}}}(\underline{\mathbf{x}})$	data	$\underline{\mathbf{x}}(\alpha) \overset{iid}{\sim} P_{\underline{\mathbf{x}}}(\underline{\mathbf{x}})$
	$\Downarrow$	
$\widehat{\underline{\mathbf{s}}} := \underline{\mathbf{W}} \cdot \underline{\mathbf{x}}$	model class	$\hat{u}_i := \hat{f}_i \overbrace{(e_i^T \cdot \underline{\mathbf{W}} \cdot \underline{\mathbf{x}})}^{=:\hat{s}_i}$
	$\Downarrow$	
$\mathrm{D_{KL}}(P_{\underline{\mathbf{s}}}(\widehat{\underline{\mathbf{s}}}))$	performance measure	$H(\widehat{\underline{\mathbf{u}}})$
	$\Downarrow$	
$\min_{\underline{\mathbf{W}}} \mathrm{D_{KL}}$	optimization	$\max_{\underline{\mathbf{W}}} H(\widehat{\underline{\mathbf{u}}})$

Other cost functions

- Vanishing cross-correlation functions (QDIAG, FFDIAG)
- Measures of non-Gaussianity (fastICA)

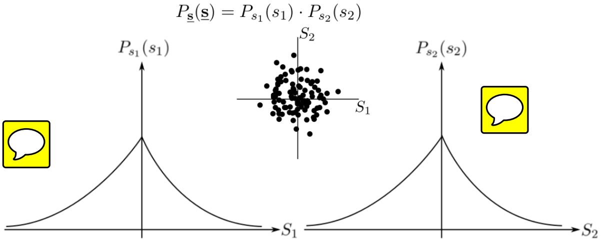
Decorrelation vs. independence



$C_{12} = C_{21} > 0$ $\qquad$ $C_{12} = C_{21} \approx 0$ $\qquad$ $C_{12} = C_{21} \approx 0$

Uncorrelatedness does
Not imply independence

Decorrelation vs. independence



$$\underline{\mathbf{x}} = \underline{\mathbf{A}} \cdot \underline{\mathbf{s}} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \cdot \begin{pmatrix} s_1 \\ s_2 \end{pmatrix} = \begin{pmatrix} a_{11}s_1 + a_{12}s_2 \\ a_{21}s_1 + a_{22}s_2 \end{pmatrix}$$

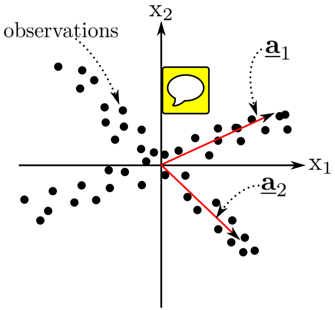
for $\underline{\mathbf{A}} = (\underline{\mathbf{a}}_1, \underline{\mathbf{a}}_2)$:

direction induced by 1st source $\quad \underline{\mathbf{s}} := \begin{pmatrix} 1 \\ 0 \end{pmatrix} \rightsquigarrow \underline{\mathbf{x}} = \begin{pmatrix} a_{11} \\ a_{21} \end{pmatrix} = \underline{\mathbf{a}}_1$

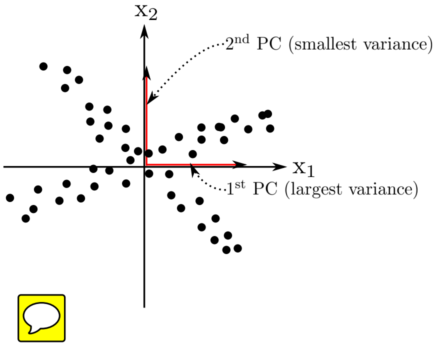
direction induced by 2nd source $\quad \underline{\mathbf{s}} := \begin{pmatrix} 0 \\ 1 \end{pmatrix} \rightsquigarrow \underline{\mathbf{x}} = \begin{pmatrix} a_{12} \\ a_{22} \end{pmatrix} = \underline{\mathbf{a}}_2$

Decorrelation vs. independence

generative model



PCA solution



Limits to recovery

1. Permutations of sources
2. Source amplitudes
3. Gaussian distributed sources

Limits to recovery: Permutations of sources

$$\begin{pmatrix} \widehat{s}_1 \\ \widehat{s}_2 \end{pmatrix} = \begin{pmatrix} \mathrm{w}_{11} & \mathrm{w}_{12} \\ \mathrm{w}_{21} & \mathrm{w}_{22} \end{pmatrix} \begin{pmatrix} \mathrm{x}_1 \\ \mathrm{x}_2 \end{pmatrix} \quad \widehat{=} \quad \begin{pmatrix} \widehat{s}_2 \\ \widehat{s}_1 \end{pmatrix} = \begin{pmatrix} \mathrm{w}_{21} & \mathrm{w}_{22} \\ \mathrm{w}_{11} & \mathrm{w}_{12} \end{pmatrix} \begin{pmatrix} \mathrm{x}_1 \\ \mathrm{x}_2 \end{pmatrix}$$

$$P_{s_1}(\widehat{s}_1) \cdot P_{s_2}(\widehat{s}_2) \qquad\qquad P_{s_2}(\widehat{s}_2) \cdot P_{s_1}(\widehat{s}_1)$$

Limits to recovery: Source amplitudes

$$\begin{pmatrix} \widehat{s}_1 \\ \widehat{s}_2 \end{pmatrix} = \begin{pmatrix} \mathrm{w}_{11} & \mathrm{w}_{12} \\ \mathrm{w}_{21} & \mathrm{w}_{22} \end{pmatrix} \begin{pmatrix} \mathrm{x}_1 \\ \mathrm{x}_2 \end{pmatrix} \quad \widehat{=} \quad \begin{pmatrix} a\widehat{s}_1 \\ b\widehat{s}_2 \end{pmatrix} = \begin{pmatrix} a\mathrm{w}_{11} & a\mathrm{w}_{12} \\ b\mathrm{w}_{21} & b\mathrm{w}_{22} \end{pmatrix} \begin{pmatrix} \mathrm{x}_1 \\ \mathrm{x}_2 \end{pmatrix}$$

$$P_{s_1}(\widehat{s}_1) \cdot P_{s_2}(\widehat{s}_2) \qquad\qquad aP_{s_1}(a\widehat{s}_1) \cdot bP_{s_2}(b\widehat{s}_2)$$

Limits to recovery: Gaussian distributed sources

$$\begin{pmatrix} \widehat{s}_1 \\ \widehat{s}_2 \end{pmatrix} = \begin{pmatrix} \mathrm{w}_{11} & \mathrm{w}_{12} \\ \mathrm{w}_{21} & \mathrm{w}_{22} \end{pmatrix} \begin{pmatrix} \mathrm{x}_1 \\ \mathrm{x}_2 \end{pmatrix}$$

$$\begin{aligned} P_{\underline{\mathbf{s}}}(\underline{\widehat{\mathbf{s}}}) &= \frac{1}{2\pi} \exp\left\{ - \frac{\|\underline{\widehat{\mathbf{s}}}\|^2}{2} \right\} \\ &= \underbrace{\left[\frac{1}{\sqrt{2\pi}} \exp\left\{ - \frac{\widehat{s}_1{}^2}{2} \right\} \right] \left[\frac{1}{\sqrt{2\pi}} \exp\left\{ - \frac{\widehat{s}_2{}^2}{2} \right\} \right]}_{\text{solution to the unmixing problem}} \end{aligned}$$

Limits to recovery: Gaussian distributed sources

ICA solution:

$$\widehat{\underline{\mathbf{s}}} = \underline{\mathbf{W}} \cdot \underline{\mathbf{x}}$$

Let $\underline{\mathbf{B}}$ be an orthogonal (rotation) matrix: $\underline{\mathbf{B}}^T \underline{\mathbf{B}} = \underline{\mathbf{1}}$

$$\widetilde{\underline{\mathbf{s}}} = \underline{\mathbf{B}} \cdot \widehat{\underline{\mathbf{s}}} \quad = \quad \underline{\mathbf{B}} \cdot \underline{\mathbf{W}} \cdot \underline{\mathbf{x}} \quad = \quad \underline{\mathbf{W}}' \cdot \underline{\mathbf{x}}$$

For the (density) relevant term $\widetilde{\underline{\mathbf{s}}}^2$:

$$\begin{aligned} \|\widetilde{\underline{\mathbf{s}}}\|^2 &= \left(\underline{\mathbf{B}} \cdot \widehat{\underline{\mathbf{s}}}\right)^T \left(\underline{\mathbf{B}} \cdot \widehat{\underline{\mathbf{s}}}\right) \\ &= \widehat{\underline{\mathbf{s}}}^T \underbrace{\left(\underline{\mathbf{B}}^T \underline{\mathbf{B}}\right)}_{\stackrel{!}{=} \underline{\mathbf{1}}} \widehat{\underline{\mathbf{s}}} \qquad = \|\widehat{\underline{\mathbf{s}}}\|^2 \end{aligned}$$

Limits to recovery: Gaussian distributed sources

Therefore:

$$\begin{pmatrix} \tilde{s}_1 \\ \tilde{s}_2 \end{pmatrix} = \begin{pmatrix} \mathrm{w'}_{11} & \mathrm{w'}_{12} \\ \mathrm{w'}_{21} & \mathrm{w'}_{22} \end{pmatrix} \begin{pmatrix} \mathrm{x}_1 \\ \mathrm{x}_2 \end{pmatrix}$$

With:

$$\begin{aligned} P_{\underline{\mathbf{s}}}(\widetilde{\underline{\mathbf{s}}}) &= \frac{1}{2\pi} \exp\left\{ -\frac{\|\widetilde{\underline{\mathbf{s}}}\|^2}{2} \right\} \\ &= \underbrace{\left[\frac{1}{\sqrt{2\pi}} \exp\left\{ -\frac{\widetilde{s_1}^2}{2} \right\} \right] \left[\frac{1}{\sqrt{2\pi}} \exp\left\{ -\frac{\widetilde{s_2}^2}{2} \right\} \right]}_{\text{alternative solution to the unmixing problem}} \end{aligned}$$