University of Colorado Boulder

Department of Mathematics

MATH 3430

A Comprehensive Analysis of series RLC circuits using Laplace Transforms

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1. Subject Motivation

The U.S. built the first fully transistorized computer in 1953. Now, a mere 70 years later, we have self-driving cars, artificial intelligence ingrained into daily life, and quite literally the culmination of all mankind's achievements, history, and knowledge packed into a computer roughly the same size as a Cliff Bar. How about we look at the conjugate of this, that is the 70 years before the first computer. Back then, daily battles took place between the punctuality of your neighborhood ice man (an actual job that required delivering ice to people), and your entire supply of food that could potentially spoil. We had to develop what I imagine to be a cumbersome tolerance to the heat since air conditioning was science fiction. If you fell in love at a downtown bar without exchanging personal addresses, statistically you would never see that person again. What were you going to do, google their Facebook or shoot them a text? I suppose you could learned to sketch and hang some fliers. Computers propelled the world into the modern age, and nowadays they are so cheap to manufacture we stick them in things like refrigerators, coffee machines, toasters, and my bedroom blinds so they gradually let sunlight in everyday at 7:00AM over the course of a half hour while my google nest plays a soft hymn to wake me up; I mean why not?

The magnificence of energy transfer, energy generation, altering billions of bits, and holding all this power in equilibrium so everything does not catch ablaze, is all thanks to the engineering of versatile circuits. The humble electrical circuit is the bedrock for computers, energy transfer, energy regulation, and everything in between. Understanding these things called circuits, comprised of batteries, wires, maybe some bulbs and switches, and how the continuous flow of electric charge moves through them, has made all of mankind's aspirations a possibility. This paper will address the fundamentals of circuits in engineering, how to mathematically understand them, and apply these ideas through guided examples. The motivation for this comes from the fascination I hold for computers, and the desire to understand the laws previous electrical engineers and mathematicians have left, that have allowed us to advance as far, and fast as we have.

2. Project Introduction

Here we will introduce the basic definitions, terminology, and theorems required to understand circuit analysis. Since the focus of this paper is solving these systems with differential equations, we will not provide too comprehensive a look into the physics half after introducing basic circuit theory, we do this to allow more focus on the analytical side.

2.1. **Basic Circuit Theory.** All material is comprised of atoms, and all atoms consist of protons, neutrons, and electrons. Protons have a positive electrical charge, neutrons have no charge, and electrons negative. A stable atom is bound together by the attraction shared by the nucleus of the atom, and the electrons in the atoms outer cloud. Now if we separate these attracting forces from each other, they exert something that is called potential difference, which drives circuit theory.

2.1.1 Electrical Charge. This is the physical property of matter when placed in an electromagnetic field. The potential difference mentioned is the direct result of the two possible electric charges; positive (protons), negative (electrons). This is a conservative property and cannot change in an isolated system. The **Coulomb (Q)** is the unit of measure for electrical charge, it is defined as the quantity of electricity transported in one second by a current of one ampere.

2.1.2 Circuit A closed loop that carries electricity.

2.1.3 Electrical Current. This is the rate at which electrons or electrical charge is flowing, denoted (I). In a circuit the electrons bleeds out of the negative terminal of the power source to continue around the circuit. The current is measured in **Amperes (i)**, denoted i for intensity. [8]We relate the current to the charge by equating the the current in amperes to the derivative of the charge with respect to time.

$$i = \frac{dQ}{dt}$$

In this system, electrons are getting ripped away from their atoms, leaving a positive charge left over for a split moment in time. Though, in reality, electrons are flowing around the circuit with a negative charge, in analysis we choose to use the equal amount of positive charge jumping in the opposite direction. We do this because as we get into complex circuit analysis it becomes burdensome to deal with the negative values of the electrons. As a mathematical convenience we analyze something known as the hole current.

2.1.4 Hole Current. An electric charge carrier with a positive charge, equal in magnitude to the charge of an electron, but opposite in parity. This is used as a mathematical compromise in studying circuits so that we are able to look at the conventional flow, that is from positive to negative. We still use the ampere as the unit of measure with this adopted methodology.

2.1.5 Voltage. The potential energy of an electrical supply stored in the form of an electrical charge. Voltage is the force that pushes electrons through a conductor in a circuit. The potential difference between two nodes in a circuit is called the voltage drop, and this is measured in **Volts (V)**. The greater the volts, the greater the pressure, or pushing force. Voltage is emitted by the energy source in a circuit, a battery being one of the possibilities. Batteries are denoted by a series of parallel lines in circuit drawings, where the positive terminal is the longer line of the two outermost lines, the negative terminal being the shorter.

We now have the knowledge to describe a basic circuit in detail. A circuit is a closed loop with an electrical current; that is the rate at which the electrical charge is moving through the confines of a circuit. Though we use an equal but opposite positive flow for convenience in analysis, known as the hole circuit. We measure the potential difference between two points in a circuit as volts, which is generated by the power source. Figure 1 shows a visual of the model we have constructed for our circuit. [2]

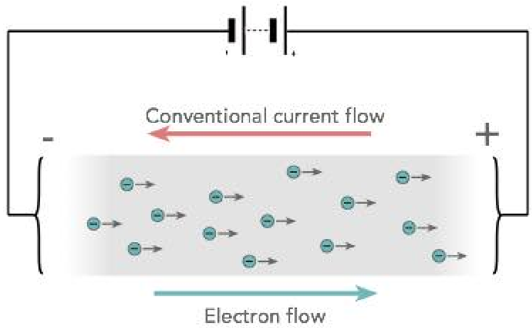

FIGURE 1. A Simple Circuit

2.2. **Building an RLC Electrical Circuit.** Now that we have a solid background on the physics of a circuit, we will introduce the components used in RLC circuits that allow us to solve these as systems. RLC circuits are widely used in signal processing and communication systems; they are main components in the upkeep of the world wide web, as well as other signals like television and radio. They are made from three two-terminal passive linear components; resistors, inductors, and capacitors. We will introduce these as the building blocks of RLC circuits, and then we will introduce some common input sources for the circuits.

2.2.1 Resistor. An electrical component that implements electrical resistance in a circuit. These are used to reduce current flow, divide voltages, terminate transmission lines, and adjust signal levels. **Resistance (R)** is a materials tendency to resist the current. This is measured in **Ohms** (Ω). We will alter define the formula for finding voltage across a resister in definition 9.

2.2.2 Inductor. An electrical component that stores energy in a magnetic field when a current flows through it; this is also called a coil and usually is displayed in the shape of a helix. This is characterized by **Inductance (L)**, or the ratio of voltage to the change of current. A **Henry (H)** is the unit of measure. In a time varying magnetic field we find the voltage drop across a conductor by taking the product of the derivative of the current with respect to time, and the inductance value L. [3] [8]

$$V_L = \frac{di}{dt}L$$

2.2.3 Capacitor. A device that stores electrical energy within a circuit in an electrical field. These typically contain two electrical conductors in the form of metallic plates. The capacitor causes a net positive charge to collect on one plate and a net negative charge to collect on the other plate. If a time varying voltage is applied across the capacitor, the circuit maintains an ongoing current as a result of the charging and discharging cycles of the capacitor. We denote the unit of capacitance as the **Farad (F)**; **Capacitance (C)** is defined as being the ratio of the positive and negative charge (Q) on each conductor to the voltage (V_C) between them, summarized by the equation below.

$$C = \frac{Q}{V_C} \Leftrightarrow V_C = \frac{Q}{C}$$

If the capacitance is varying due to a charge build up or the time variable, then the voltage across the capacitor is defined as follows. [8]

$$V_C(t) = \frac{1}{C}\int_0^t I(s)ds$$

Figure 2 (left) shows how these components are symbolized in academia and application, and then Figure 2 (right) shows the simplest RLC circuit; another way to represent a voltage emitter in a circuit is a small circle with a V in the middle, or a S like figure shown below.

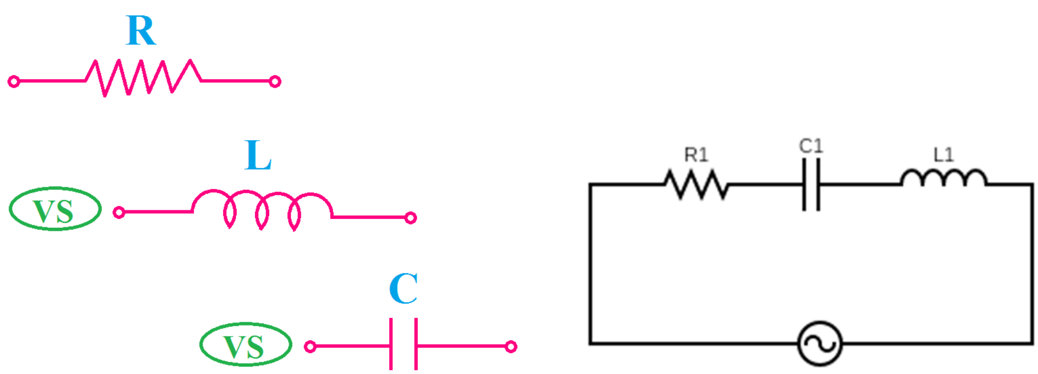

FIGURE 2. LRC Elements, Basic LRC Circuit

2.2.4 Single Coil AC Generator. A common energy source in circuits where the output is a function of time. This Generator is a displaced coil within a magnetic field that rotates anticlockwise around an axis perpendicular to the corresponding magnetic field. This is known as an Electro-Motive Force, or EMF. We will have instantaneous voltage values given by the sinusoidal waveform produced, the equation

$$V_i = V_{max}\sin(\theta)$$

describes this. Where V_{max} is the maximum voltage induced in the coil, and $\theta = \omega t$ is the rotational angle of the coil with respect to time t. If we know the peak value of the waveform, we can construct the sinusoidal shape. Figure 3 (left) shows our generator, and Figure 3 (right) shows the process where we plot the various positions of rotation in the unit circle. [3]

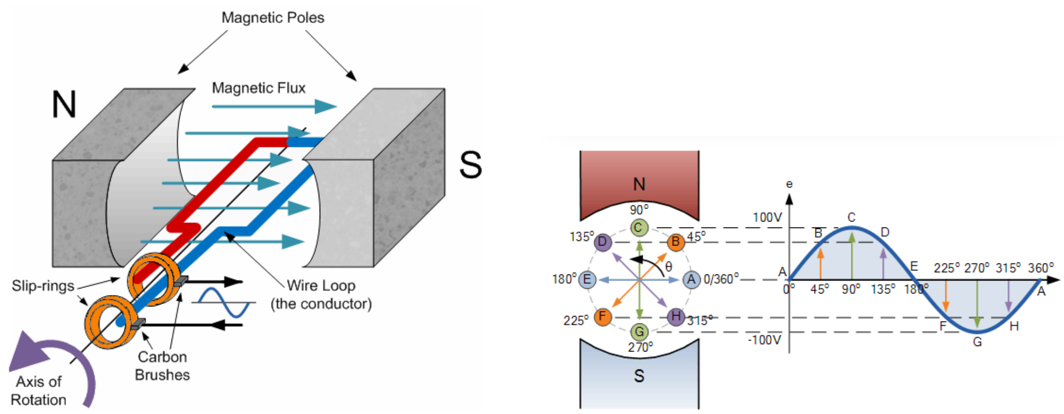

FIGURE 3. Basic Single Coil AC Generator, Sinusoidal Waveform Construction [9]

2.2.5 Switch Input. A time shifted, and time dependant input for an electrical circuit. This is also another common input for RLC circuits because this mimics an on and off switch. We define the instantaneous voltage at time t as the maximum voltage times the Heaviside function defined in definition 13. This is denoted as $V_{max}u_c(t)$ since at $t < 0$ we will simply have no input since the switch is off, and at $t \geq c$, we will have the maximum amount of input because the switch will be on, and this function has only two defined values in its codomain. Figure 4 shows the concept of a switch input. [6]

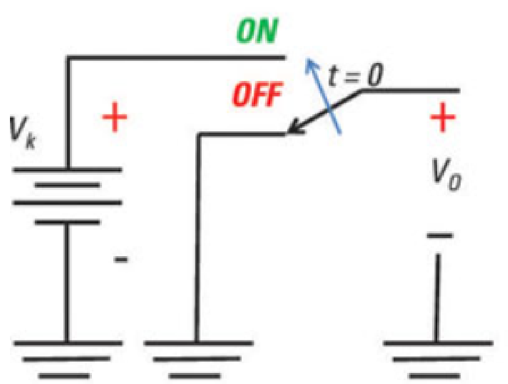

FIGURE 4. A Switch Input Circuit

2.3. **Laws and Mathematical Formula in Circuit Analysis.**

2.3.1 *Def*ⁿ Ohm's Law. [6]The current through a conductor between two nodes in a circuit is directly proportional to the voltage across the two points. This is perhaps the most important law in all of circuit analysis. We describe this relationship with the below formula. We have the current through the conductor (I), which is measured in in amperes. The voltage (V) across the conductor, measured in volts; and The resistance (R) of the conductor in units of ohms.

$$I = \frac{V}{R}$$

2.3.2 *Def^n* Kirchhoffs Current Law (KCL). [7]The total current or change entering a node is exactly equal to the charge leaving the node as it has no other place to go except to leave; as no charge is lost within the node. This says that the algebraic sum of all the currents entering and leaving a node must be zero. Another name for this is the the Law of Conservation of Charge.

$$I_{entering} + I_{exiting} = 0$$

2.3.3 *Def^n* Kirchhoffs Voltage Law (KVL). [7]In any closed circuit network, the total voltage around the loop is equal to the sum of all the voltage drops within the same loop, zero. This simply shows that the algebraic sum of all voltages in a loop is zero. Or rather for all $i \leq n$, where i is a voltage drop in loop L, and n is the total amount of voltage drops, we have that for $V_i \in L$

$$V_1 + V_2 + \cdots + V_n = 0$$

2.3.4 *Def^n* Laplace Transform. [5]Let $f(t)$ be defined for $0 \leq t < \infty$. The Laplace transform of $f(t)$, which is denoted by $F(s)$, or $\mathscr{L}\{f(t)\}$, is given by the formula

$$F(s) = \mathscr{L}\{f(t)\} = \int_0^\infty e^{-st} f(t) dt$$

where

$$\int_0^\infty e^{-st} f(t) dt = \lim_{A \to \infty} \int_0^A e^{-st} f(t) dt$$

Appendix A provides a table of Laplace transforms used in common application.

2.3.5 *Def^n* Heavyside Function. [5]A function that has major applications in Laplace transform problems, defined to be a function of time $u_c(t)$ as follows

$$u_c(t) = \begin{cases} 0 & x < 0 \\ 1 & t \geq c \end{cases}$$

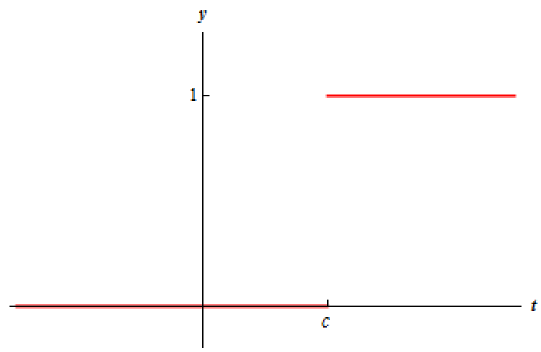

FIGURE 5. Heavyside Function

2.3.6 *Prop* Linearity of the Laplace Transform. [5]The Laplace transform is a linear operator since:

$$\begin{aligned}\mathscr{L}\{c_1 f_1(t) + c_2 f_2(t)\} = &\int_0^{\infty} e^{-st}[c_1 f_1(t) + c_2 f_2(t)]dt\\ =&c_1 \int_0^{\infty} e^{-st} f_1(t)dt + c_2 \int_0^{\infty} e^{-st} f_2(t)dt\\ =&c_1 \mathscr{L}\{f_1(t)\} + c_2 \mathscr{L}\{f_2(t)\}\end{aligned}$$

2.3.7 *Prop* Time Differentiation of a Laplace Transform. [4]The time differentiation property states that:

$$\mathscr{L}\left\{\frac{d}{dt} f(t)\right\} = sF(s) - f(0^-)$$

2.3.8 *Prop* Time Integration of a Laplace Transform. The time integration property states that:

$$\mathscr{L}\left\{\int_0^t f(t)dt\right\} = \frac{1}{s}F(s)$$

3. Example Problems

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We will present two walked through examples, the first will look at finding the Laplace expression for V_C in a RLC circuit presented in a general solution, that is arbitrary variables are used so this solution may be applied to any and all circuits of this same skeleton. The second example will also look at finding the Laplace expression, though we will use discrete values, and invert the final Laplace expression to find the solution to express the current as a function of time.

3.4. **Example 1.** Observe the RLC circuit in Figure 6 below. Find the Laplace expression for V_C, also called the frequency domain. The input is given by a sinusoidal source, described by the function $V_i = V_A \cos(\omega t)$. [3]

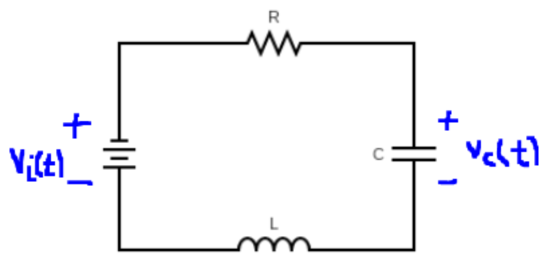

FIGURE 6. Series Circuit Example 1 Diagram

3.5. ***Solution*** **Example 1.** First, we will need to consider the resistors, inductors, and the capacitors in the form of the current-voltage relationship. For the resistor, the frequency domain is the same as the time domain, since Ohm's Law is not time-dependant. We now apply the Laplace transform directly, and we start with the resistor:

$$\begin{aligned}\mathscr{L}\{v_R(t)\} &= \mathscr{L}\{Ri_R(t)\}\\ V_R(s) &= RI_R(s)\\ \frac{V_R}{I_R} &= R\end{aligned}$$

capacitor:

$$\begin{aligned}\mathscr{L}\{i_C(t)\} &= \mathscr{L}\left\{C\frac{dv_C(t)}{dt}\right\}\\ I_C(s) &= CsV_C(s)\\ \frac{V_C}{I_C} &= \frac{1}{sC}\end{aligned}$$

inductor:

$$\begin{aligned}\mathscr{L}\{v_L\} &= L\mathscr{L}\left\{\frac{di_L(t)}{dt}\right\}\\ V_L &= LsI_L(s)\\ \frac{V_L}{I_L} &= sL\end{aligned}$$

Since we have all of these in terms of $\frac{V}{I}$, we can see per Ohm's Law that they are all in units of ohms. We now apply KVL with Ohm's Law to find the voltage drop across each resistor divided in terms of the common current:

$$
\begin{aligned}
V_C(s) &= \left(\frac{\frac{V_C}{I_C}}{\frac{V_L}{I_L} + \frac{V_L}{I_L} + \frac{V_R}{I_R}} \right) V_i(s) \\
&= \left(\frac{\frac{1}{sC}}{R + \frac{1}{sC} + sL} \right) V_i(s) \\
V_C(s) &= \left(\frac{1}{1 + sRC + s^2LC} \right) \left(\frac{V_A}{s^2 + \omega^2} \right) \\
&\boxed{= \frac{\frac{V_A}{LC}}{\left(s^2 + \omega^2\right)\left(s^2 + s\frac{R}{L} + \frac{1}{LC}\right)}}
\end{aligned}
$$

Thus we have found the general solution to this specific type of RLC circuit expressing the frequency domain output as a function of s.

3.6. **Example 2.** Observe the series RLC circuit below. The switch moves from point α to point β at time $t = 0$. We are given the initial conditions $i_L(0^-) = 2A$ and $i_C(0^-) = 2V$. Find the instantaneous current, measured in amperes, of the circuit as a function of time.

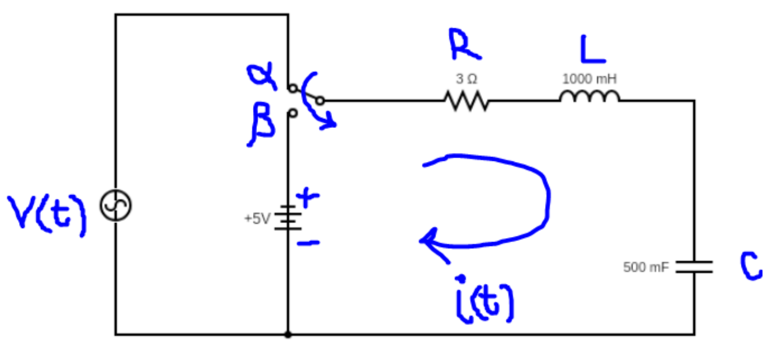

FIGURE 7. Series Circuit Example 2 Diagram

3.7. ***Solution* Example 2.** Since we are analyzing this circuit for an arbitrary t, we will solve for $t > 0$, per the definition of Laplace transforms which are only defined on intervals of time greater than 0. Notice at $t > 0$, the switch will move from α to β, omitting the voltage source $v(t)$ on the far left of the circuit. Lets now summarize the relationships we have thus far in the circuit:

$$\begin{aligned} R =& 3\Omega \\ L =& 1000mH = 1H \\ C =& 500mF = \frac{1}{2}F \end{aligned}$$

By 2.3.3 (KVL) we have:

$$\begin{aligned} 5 =& V_R + V_L + V_C \\ =& Ri + L\frac{di}{dt} + \frac{1}{C}\int_{-\infty}^{t} I(s)ds \end{aligned}$$

Now taking the Laplace transforms of each of these, our goal is to express the entire system in terms of s, and then take the inverse Laplace transform for out final result. We reference to the table in appendix A and prop 2.3.8 to get:

$$\begin{aligned} \mathscr{L}\{5\} =& \frac{5}{s} \\ \mathscr{L}\{Ri(t)\} =& RI(s) \\ \mathscr{L}\left\{L\frac{di(t)}{dt}\right\} =& L(sI(s) - i(0^-)) \\ \mathscr{L}\left\{\frac{1}{C}\int_{-\infty}^{t} idt\right\} =& \frac{1}{C}\left(\mathscr{L}\left\{\int_{-\infty}^{0^-} idt\right\} + \mathscr{L}\left\{\int_{0^-}^{t} idt\right\}\right) \\ =& \frac{2}{s} + \frac{1}{Cs}I(s) \end{aligned}$$

Therefore we can substitute for the values given by the circuit into our new new Laplace transform equation to get:

$$\begin{aligned} \frac{5}{s} =& RI(s) + L(sI(s) - i(0^-)) + \frac{2}{s} + \frac{1}{Cs}I(s) \\ \frac{5}{s} =& RI(s) + L(sI(s) - 2) + \frac{2}{s} + \frac{1}{Cs}I(s) \\ \frac{5}{s} =& 3I(s) + sI(s) - 2 + \frac{2}{s} + \frac{2}{s}I(s) \\ \frac{5}{s} + 2 - \frac{2}{s} =& \left(3 + s + \frac{2}{s}\right) I(s) \\ \frac{3}{s} + 2 =& \frac{(3s + s^2 + 2)}{s} I(s) \\ 3 + 2s =& \left(3s + s^2 + 2\right) I(s) \\ I(s) =& \frac{3 + 2s}{s^2 3s + 2} \\ I(s) =& \frac{3 + 2s}{(s+2)(s+1)} \end{aligned}$$

Now we can use partial fraction decomposition on I(s):

$$
\begin{aligned}
I(s) =& \frac{3+2s}{(s+2)(s+1)} \\
I(s) =& \frac{A}{(s+2)} + \frac{B}{(s+1)} \\
3+2s =& A(s+1) + B(s+2) \\
3+2s =& As + A + Bs + 2B \\
3+2s =& (A+B)s + A + 2B \\
& \Rightarrow \begin{cases} A+B=2 \\ A+2B=3 \end{cases} \\
& \Rightarrow \begin{cases} A=1 \\ B=1 \end{cases} \\
I(s) =& \frac{3+2s}{s^2 3s+2} \\
I(s) =& \frac{1}{s+2} + \frac{1}{s+1}
\end{aligned}
$$

Now, it is a simple inverse Laplace transform that can be done by comparison. We reach our final answer of:

$$
\begin{aligned}
i(t) =& \mathscr{L}^{-1}\left\{\frac{1}{s+2} + \frac{1}{s+1}\right\} \\
i(t) =& \mathscr{L}^{-1}\left\{\frac{1}{s+2}\right\} + \mathscr{L}^{-1}\left\{\frac{1}{s+1}\right\} \\
& \boxed{i(t) = e^{-2t} + e^{-t}}
\end{aligned}
$$

. ...

[1]

Function Table

Function	Transform	Region of convergence
1	$1/s$	$\mathrm{Re}(s) > 0$
e^{at}	$1/(s-a)$	$\mathrm{Re}(s) > \mathrm{Re}(a)$
t	$1/s^2$	$\mathrm{Re}(s) > 0$
t^n	$n!/s^{n+1}$	$\mathrm{Re}(s) > 0$
$\cos(\omega t)$	$s/(s^2+\omega^2)$	$\mathrm{Re}(s) > 0$
$\sin(\omega t)$	$\omega/(s^2+\omega^2)$	$\mathrm{Re}(s) > 0$
$e^{at}\cos(\omega t)$	$(s-a)/((s-a)^2+\omega^2)$	$\mathrm{Re}(s) > \mathrm{Re}(a)$
$e^{at}\sin(\omega t)$	$\omega/((s-a)^2+\omega^2)$	$\mathrm{Re}(s) > \mathrm{Re}(a)$
$\delta(t)$	1	all s
$\delta(t-a)$	e^{-as}	all s
$\cosh(kt) = \dfrac{e^{kt}+e^{-kt}}{2}$	$s/(s^2-k^2)$	$\mathrm{Re}(s) > k$
$\sinh(kt) = \dfrac{e^{kt}-e^{-kt}}{2}$	$k/(s^2-k^2)$	$\mathrm{Re}(s) > k$
$\dfrac{1}{2\omega^3}(\sin(\omega t) - \omega t\cos(\omega t))$	$\dfrac{1}{(s^2+\omega^2)^2}$	$\mathrm{Re}(s) > 0$
$\dfrac{t}{2\omega}\sin(\omega t)$	$\dfrac{s}{(s^2+\omega^2)^2}$	$\mathrm{Re}(s) > 0$
$\dfrac{1}{2\omega}(\sin(\omega t) + \omega t\cos(\omega t))$	$\dfrac{s^2}{(s^2+\omega^2)^2}$	$\mathrm{Re}(s) > 0$
$u(t-a)$	e^{-as}/s	$\mathrm{Re}(s) > 0$
$t^n e^{at}$	$n!/(s-a)^{n+1}$	$\mathrm{Re}(s) > \mathrm{Re}(a)$
Interesting, but not included in this course.		
$\dfrac{1}{\sqrt{\pi t}}$	$\dfrac{1}{\sqrt{s}}$	$\mathrm{Re}(s) > 0$
t^a	$\dfrac{\Gamma(a+1)}{s^{a+1}}$	$\mathrm{Re}(s) > 0$

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