Complexity of switching chaotic maps

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Abstract

In the last years, digital systems such as Digital Signal Processors (DSP), Field Programmable Gate Arrays (FPGA) and Application-Specific Integrated Circuits (ASIC), became the standard in all experimental sciences. Experimenters may design and modify their own systems.

In these systems digital implementations has a custom made numerical system, therefore finite arithmetic needs to be investigated. Fixed point representation is prefered over floating point when speed, low power and/or small circuit area is necessary.

Chaotic systems implemented in finite preceision will always became periodic with period T. It has been recently shown that it is convenient to describe the statistical characteristic using both, a non causal and a causal probability distribution function (PDF). The corresponding entropies, must be evaluated to quantify these PDF's.

Binary floating and fixed point are the numerical representations available, each of them produces specific period and statistical characteristics that needs to be evaluated.

In [1] a complete study about period was carried out using as an example two well known chaotic maps: the tent map and the logistic map. After that, switching techniques between these two maps were tested. In this paper we characterize the behaviour of these maps when precision varies from an statistical point of view, using causal and non causal entropires.

VERSION: 17 February 2017

1 Introduction

In the last years digital measuring systems become the standard in all experimental sciences. By using *virtual instruments* and new programable electronic devices, such as Digital Signal Processors (DSP) and Field Programmable Gate Arrays ($FPGA$) and Application-Specific Integrated Circuits (ASIC), became the standard in all experimental sciences. This systems allow experimenters to design and modify their own signal generators, measuring systems, simulation models, etc. All these adventages gave as result the emergence of High Performance Computing (HPC) as a new research field.

The effect of finite precision in these new devices needs to be investigated. Floating point is not always available if speed, low power and/or small circuit area are in specifications. Often an fixed point solution is a better than floating point. This issue is critical if chaotic systems must be implemented, because due to roundoff errors digital implementations will always become periodic with a period T and unstable orbits with a low period may become stable destroying completely the chaotic behavior.

Stochasticity and mixing are also relevant, to characterize these properties several quantifiers were studied [2]. Among them the use of an entropy-complexity representation ($H-C$ plane) and causal-noncausal entropy ($H_{BP}-H_{Val}$ plane) deserves special consideration [3–5,2,6][Poner Paper mo]. A fundamental issue is the criterium to select the distribution function (PDF) assigned to the time series. Causal and non causal options are possible Here we consider the non-causal traditional PDF obtained by normalization of the histogram of the time series. Its statistical quantifier is the normalized entropy H_{Val} that is a measure of equiprobability among all allowed values. We also consider a causal PDF that is obtained by assigning ordering patterns to segments of trajectory of length D. This PDF were first proposed by Bandt & Pompe in a seminal paper [7]. The corresponding entropy H_{BP} was also proposed as a quantifier by Bandt & Pompe. Amigó and coworkers proposed the number of forbidden patterns as a quantifier of chaos [8]. Essentialy they reported the presence of forbidden patterns as an indicator of chaos. Recently it was shown that the name forbidden patterns is not convenient and it was replaced by *missing patterns* (MP) [9].

Grebogi and coworkers [10] studied this subject and they shaw that the period T scales with roundoff ϵ as $T \sim \epsilon^{-d/2}$ where d is the correlation dimension of the chaotic attractor.

To have a large period T is one an important property of a chaotic map, in [1] Nagaraj et. als studies the effect of switching over the average period lengths of chaotic maps in finite precision. Switching systems naturally arise

in power electronics and many other areas in digital electronics. They have also interest in transport problems in deterministic ratchets [11] and it is known that synchronization of the switching procedure affects the output of the controlled system. Nagaraj et al [1] studied the case of switching between two maps. They shaw that the period T of the compound map obtained by switching between two chaotic maps is higher than the period of each map. Liu et al [12] studied different switching rules applied to linear systems to generate chaos. Switching chaos was also addressed in [13]. Skipping values of the time series is another simple technique used to increase mixing in chaotic maps [4].

In this paper we study the statistical characteristic of two well known maps: the tent map (TENT) and logistic map (LOG). Three additional maps are generated: 1) SWITCH, generated by switching between TENT and LOG; 2) EVEN, generated by skipping all the elements in odd position in SWITCH time series and 3) ODD, generated by discarding all the elements in an even position in SWITCH time series. Binary floating and fixed point numbers are used, these specific numerical systems may be implemented in modern programmable logic boards.

The main contributions of this paper are:

(1) the definition of different statistical quantifiers and their relationship with the properties of the time series generated by the map.
(2) the study of how this quantifiers detect the evolution of stochasticity and mixing of the chaotic maps according as the numerical precision varies.
(3) the effect of switching between two different maps, on the period and the statistical properties of the time series.
(4) the effect of skipping values in any of these maps, on the period and the statistical properties of the time series.

Organization of the paper is as follows: section 2 describes the statistical quantifiers used in the paper and the relationship between their value and characteristics of the causal and non causal PDF considered; section 3 shows and discuss the results obtained for all the numerical representations. Finally section 4 deals with final remarks and future work.

2 Information theory quantifiers

The first step to quantify the statistical properties of the values (amplitude statistics) of a time series $\{x_i, \ (i = 1, ..., N)\}$, using information theory is to determine the concomitant PDF because all the quantifiers are functionals of the PDF associated to the time series. This is an issue studied in detail in

previous papers [**?**]. Let us summarize the procedure:

(1) a finite alphabet with M symbols $\mathcal{A} = \{a_1, ..., a_M\}$ is chosen.
(2) one of these symbols is assigned: (a) to each value of the time series of (b) to each portion of length D of the trajectory.
(3) the normalized histogram of the symbols is the desired PDF.
(4) an ramdomness quantifier is calculated over desired PDF. In our case we calculate (a) normalyzed Shannon entropy H, (b) normalyzed statistical complexity C and (d) missing patterns MP.

Note that if option (a) is chosen in step 2 then the PDF is *non causal*, because all the information about the time evolution of the system generating $\{x_i\}$ is completely lost. On the contrary if option (b) is chosen in step 2 then the PDF is *causal*, in the sense it has some information about the temporal evolution.

NOMBRAR DE LA BPW, SI ES QUE LA TERMINAMOS USANDO

NOMBRAR LOS PLANOS QUE TERMINEMOS ELIGIENDO

Of course there are infinite possibilities to choose the alphabet $\mathcal{A}$ as well as the length D. Bandt & Pompe made a proposal for a causal PDF that has been shown to be easy to implement and useful in a great variety of applications REFERENCIAS A APLICACIONES. The procedure is the following [7,14,15]: ESTA PARTE EST EN EL TIEMPO Y DEBERA ESTAR EN LAS MUESTRAS

- Given a series $\{x_t : t = 0, \Delta t, \cdots, M\Delta t\}$, a sequence of vectors of length d is generated.

$$(s) \mapsto \left(x_{t-(d-1)\Delta t}, x_{t-(d-2)\Delta t}, \cdots, x_{t-\Delta t}, x_t \right) \ , \tag{1}$$

  Each vector turns out to be the "history" of the value x_t. Clearly, the longer the length of the vectors D, the more information about the history would the vectors have but a higher value of N is required to have an adequate statistics.
- The permutations $\pi = (r_0, r_1, \cdots, r_{D-1})$ of $(0, 1, \cdots, D-1)$ are called "order of patterns" of time t, defined by:

$$x_{t-r_{D-1}\Delta t} \leq x_{t-r_{D-2}\Delta t} \leq \cdots \leq x_{t-r_1\Delta t} \leq x_{t-r_0\Delta t}. \tag{2}$$

  In order to obtain an unique result it is considered $r_i < r_{i-1}$ if $x_{t-r_i\Delta t} = x_{t-r_{i-1}\Delta t}$.
  In this way, all the $D!$ possible permutations π of order D, and the PDF $P = \{p(\pi)\}$ is defined as:

$$p(\pi) = \frac{\sharp\{s|s \leq M - D + 1; (s) \ \texttt{has type } \pi\}}{M - D + 1}. \tag{3}$$

In the last expression the $\sharp$ symbol means "number".

This procedure has the advantages of being *i)* simple, *ii)* fast to calculate, *iii)* robust in presence of noise, and *iv)* invariant to lineal monotonous transformations. DICE QUE ES ROBUSTO FRENTE A LA PRESENCIA DE RUIDO PERO NO, REFERENCIAS?

It is applicable to weak stationarity processes (for $k = D$, the probability that $x_t < x_{t+k}$ doesn't depend on the particulary t [7]).The causality property of the PDF allows the quantifiers (based on this PDFs) to discriminate between deterministic and stochastic systems [16].

According to this point Bandt and Pompe suggested $3 \leq D \leq 7$. $D = 6$ has been adopted in this work.

HABLAR DE BPW

The entropies H_{hist} and H_{BP} are the normalized version of the Of course there are infinite possibilities to choose the alphabet as well as the length d. The entropy $H[P]$ is the normalized version of the Entropy proposed by Shannon [17]:

$$H[P] = S[P]/S_{max}, \tag{4}$$

where $S[P] = -\sum_{j=1}^{M} p_j \ln(p_j)$
and S_{max} is the normalizing constant:

$$S_{max} = S[P_e] = \ln M, \tag{5}$$

and $P_e = \{1/M, \cdots, 1/M\}$ is the uniform distribution. The number of symbols M is equal to N for H_{hist} and it is equal to $D!$ for H_{BP}.

The statistical complexity $C[P]$ is given by:

$$C[P] = Q_J[P, P_e] \cdot H[P] , \tag{6}$$

, and Q_J is named "disequilibrium" and it is the distance between P and P_e in the probability space. The metric used in this paper is based on the Jensen-Shannon divergence [18]:

$$Q_J[P, P_e] = Q_0 \cdot \{S[\frac{P + P_e}{2}] - S[P]/2 - S[P_e]/2\} . \tag{7}$$

The normalization constant Q_0 is:

$$Q_0 = -2 \left\{ \left(\frac{N+1}{N} \right) \ln(N+1) - 2\ln(2N) + \ln N \right\}^{-1}. \tag{8}$$

From the statistical point of view the disequilibrium Q_J is an intensive magnitude, and it is 0 if and only if $P = P_e$. It has been proved that the $C[P]$ quantifies the presence of nonlinear correlations typical of chaotic systems [19,18]. The complexity $C[P]$ is independent from the entropy $H[P]$, as far as different P's share the same entropy $H[P]$ but they have different complexity $C[P]$.

PONER UN PRRAFO QUE ENGANCHE LAS H's Y C's CON Hbp Y Cbp, POR EJEMPLO

Two representation planes are considered: H_{BP} vs H_{hist} [4] and H_{BP} vs C_{BP} [3]. In the first plane a higher value in any of the entropies, H_{BP} and H_{hist}, implies an increase in the uniformity of the involved PDF. The point $(1,1)$ represents the ideal case with uniform histogram and uniform distribution of ordering patterns. In the second plane not the entire region $0 < H_{BP} < 1$, $0 < C_{BP} < 1$ is achievable. In fact for any PDF the pairs (H, C) of possible values fall between two extreme curves in the plane H-C [20]. Fig. **??** shows two regions labeled as *deterministic* and *stochastic*. In fact transition from one region to the other are smooth and the division is a bit arbitrary. A more detailed discussion can be seen in [3]. Ideal random systems having uniform Bandt & Pompe PDF, are represented by the point $(1,0)$ [21] and a delta-like PDF corresponds with the point $(0,0)$.

We also used the number of missing patterns MP as a quantifier[22]. As shown recently by Amigó *et al.* [23–26], in the case of deterministic one-dimensional maps, not all the possible ordinal patterns can be effectively materialized into orbits, which in a sense makes these patterns "forbidden". Indeed, the existence of these *forbidden ordinal patterns* becomes a persistent fact that can be regarded as a "new" dynamical property. Thus, for a fixed pattern-length (embedding dimension D) the number of forbidden patterns of a time series (unobserved patterns) is independent of the series length N. Remark that this independence does not characterize other properties of the series such as proximity and correlation, which die out with time [24,26].

A full discussion about the convenience of using these quantifiers is out of the scope of this work. Nevertheless reliable bibliographic sources do exist [27–29,4,6,30,22].

3 Results

Five pseudo chaotic maps were studied. For each one a floating point representation, a decimal numbers representation with $1 \leq P \leq 27$ and a binary numbers representation with $1 \leq B \leq 27$ are considered. For each representation 1000 time series were generated using randomly chosen initial conditions within the interval $[0, 1]$. The studied maps are tent (TENT), logistic (LOG) a sequential switching between TENT and LOG (SWITCH). Furthermore a skipping randomization procedure is applied to SWITCH [4], discarding the values in the odd positions (EVEN) or the values in the even positions (ODD) respectively. Let us detail our results for each of these maps.

3.1 Simple maps.

Here we report our results for both maps:

3.1.1 Logistic map (LOG)

Logistic map is representative of the very large family of quadratic maps.

$$x_{[n+1]} = 4x_{[n]}(1 - x_{[n]}) \tag{9}$$

with $x_n \in \mathcal{R}$.

Note that to effectively work in a given representation it is necessary to change the expression of the map in order to make all the operations in the chosen representation numbers. For example, in the case of LOG the expression in binary fixed point numbers is:

$$x_{n+1} = 4\epsilon floor\{\frac{x_n(1 - x_n)}{\epsilon}\} \tag{10}$$

with $\epsilon = 2^B$ where B is the length of fractional part.

Figs. 1 (a) to (f) show the statistical properties of LOG map in floating point and fixed point representation. All these figures shows: in red dots the results of each run (100 point for each presicion), in black dots the mean of these 100 points (dashed black line connects black dots), in dashed blue lines the results of each run in floating point and (100 horizontal blue dashed lines) and in black the mean for floating point.

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For $B \geq 30$ the value of H_{val} remains almost identical to the values for the floating point representation whereas H_{BP} and C_{BP} stabilizes at $B > 21$. Their values are: $< H_{val} >= 0.9669$; $< H_{BP} >= 0.6269$; $< C_{BP} >= 0.4843$. Note that the stable value of missing patterns 645 makes the optimum $H_{BP} \leq ln(75)/ln(720) \simeq 0.65$, this value of MP is reached for $B > 21$, the same of $< H_{BP} >= 0.6269$ and $< C_{BP} >= 0.4843$. Then $B = 30$ is the most convenient choice because an increase in the number of fractional figures does not improve the statistical properties.

We can extract some conclussions when compare BP with BPW quantifiers. For $B = 1, 2, 3 and 4$, BP quantifiers is almost 0 and BPW quantifiers not exists in the mean, the initial condition of some atractors was rounded to 0 and BPW histogram not exist. For $B = 7, 9 and 12$, an high dispersion in BPW are showed, but not in BP graphs, atractor falls to a fixed point with an short transitory in these cases. Finally, for $B = 45, 49, 51 and 52$ some BP quantifiers has a low value, but BPW quantifiers has a more predictible behavior, we can see that these atractors falls to a fixed point with a long transitory. In this case the problem has to do with the plataform that we use, the miltiplication needs the double of bits to be represented but the machine has only 64.

In Fig. 2 double entropy planes are showed. Red points are each individual measurement and black points their mean. Black dashed line shows the evolution of the mean in this plane when B growths. Blue points shows each individual measurement in floating point and their mean with an star.

Although the distribution of values reaches high values, their mixing is poor. It can be seen in the evolution of the mean values according B growths. Accross $B = 20$, H_{val} improves but H_{BP} stay around its maximum.

In Fig. 3 we shows the entropy-complexity planes. Dotted gray lines are the superior and inferior margins, is spected that an chaotic system remain near superior margin. There results characterize an chaotic behaviour, in $H_{BP} - C_{BP}$ plane we can see a low entropy and high complexity.

3.1.2 Tent map (TENT)

The Tent map has been extensively studied in the literature because theoretically it has nice statistical properties that can be analytically obtained. For example it is easy to proof that it has a uniform histogram and consequently an ideal $H_{val} = 1$. The Perron-Frobenius operator and its corresponding eigenvalues and eigenfunctions may be also be analytically obtained for this map [31].
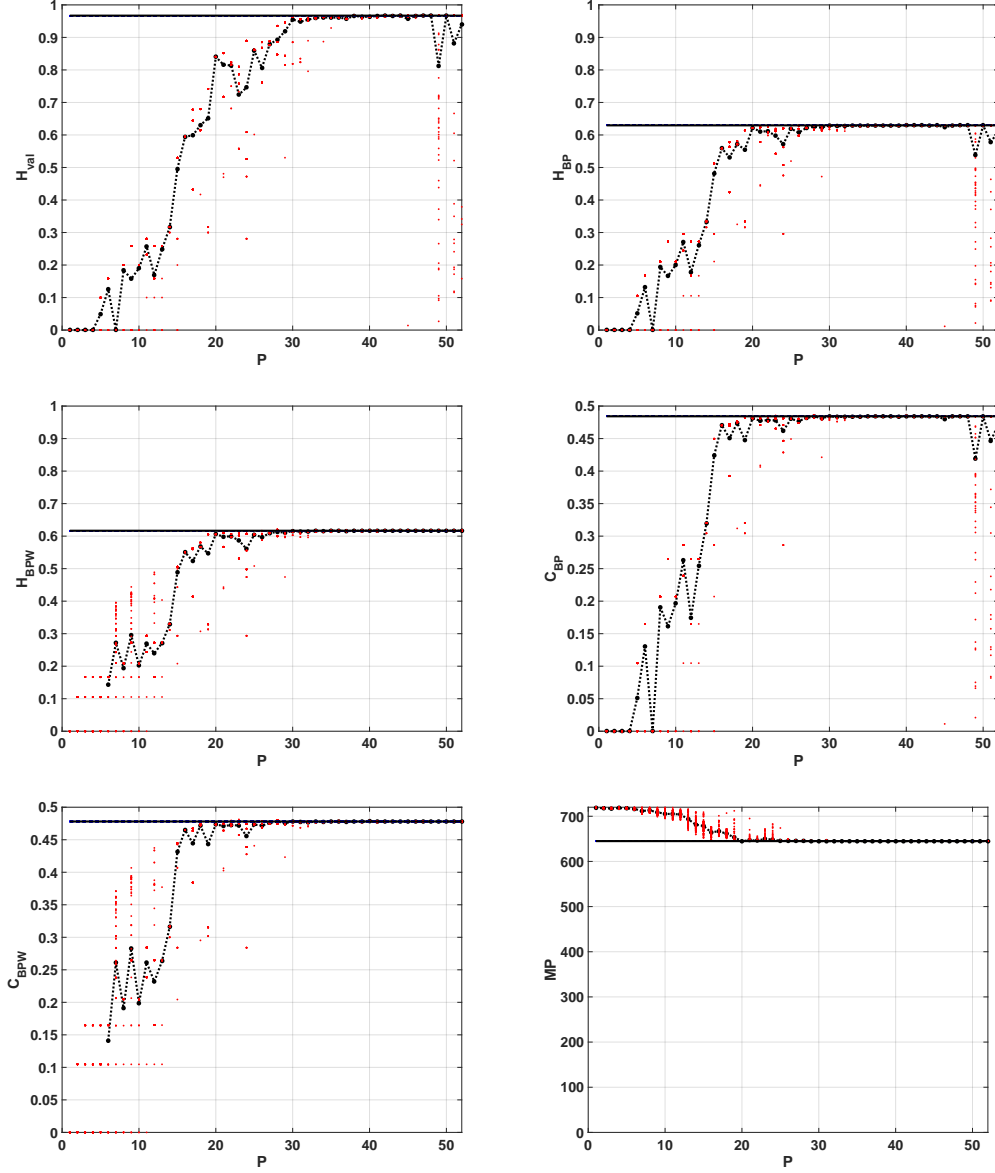
Fig. 1. Statistical properties of the LOG map using binary representation: (a) H_{val} vs B (b) H_{BP} vs B (c) C_{BP} vs B (d) Number of missing ordering patterns MP vs B.

This map is represented with the equation:

$$x_{n+1} = \begin{cases} 2\,x_n & \text{if } 0 \leq x_n \leq 1/2 \\ 2\,(1-x_n) & \text{if } 1/2 < x_n \leq 1 \end{cases}, \tag{11}$$
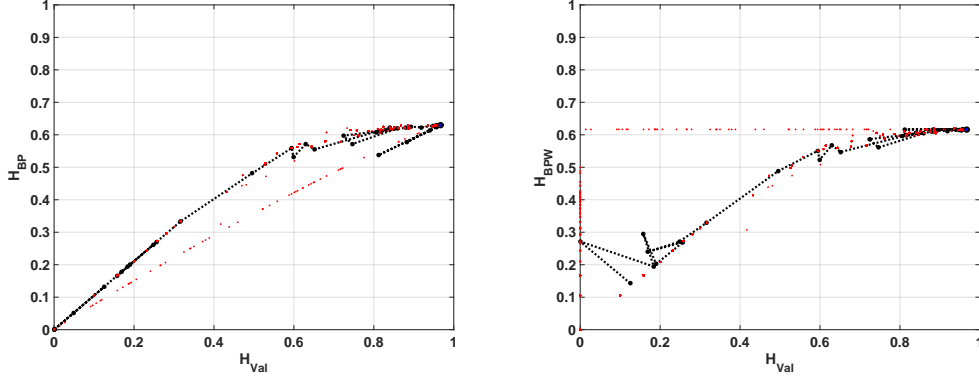
with $x_n \in \mathcal{R}$.

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Fig. 2. Evolution of statistical properties in double entropy plane of LOG map using binary representation: (a) H_{val} vs H_{BP} (b) H_{val} vs H_{BPW}.
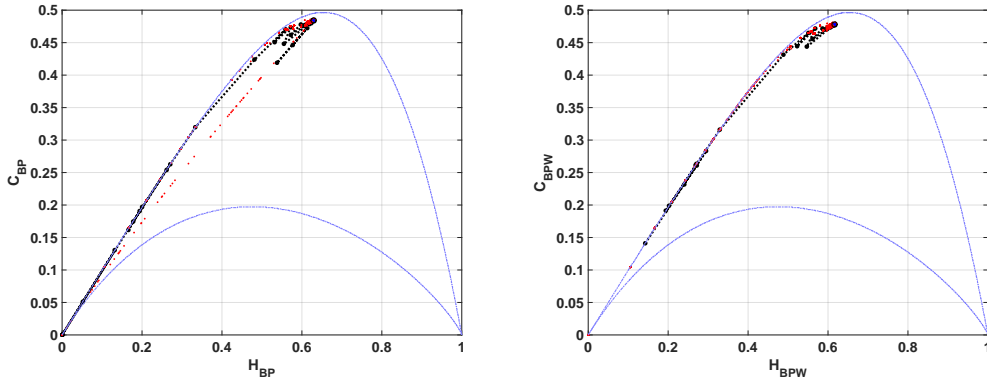
Fig. 3. Evolution of statistical properties in entropy-complexity plane of LOG map using binary representation: (a) C_{BP} vs H_{BP} (b) C_{BPW} vs H_{BPW}.

In Base-2 fractional numbers rounding, equation 11 became

$$x_{n+1} = \begin{cases} 2\ x_n & \text{if } 0 \leq x_n \leq 1/2 \\ \epsilon \times floor\{\frac{2\ -\ 2\ x_n}{\epsilon}\} & \text{if } 1/2 < x_n \leq 1 \end{cases}, \tag{12}$$

with $\epsilon = 2^{-B}$ for binary numbers.

When this map is implemented in a computer using any numerical representation system (even floating point!) truncation errors rapidly increases and makes the unstable fixed point in $x^* = 0$ becomes stable producing a short transitory of length between 0 and B followed by an infinite number of 0's[32,33]. This issue is easily explained in [with chaos meet computers], problem appears because all itarations has an shift-to-left operation that carry the $0's$ from right side. Some authors [34] have proposed to add a random perturbation to avoid this drawback of the Tent map. But this procedure introduces statistical properties of the random perturbation that are mixed with those of

the Tent map itself. Skew Tent is an option that allows reach nice statistical properties even in base 2 fractional numbers. Here we study the Tent map "as it is" without any artifact to evaluate its real instead of theoretical statistical properties.

Figs. 4 (a) to (e) show the different quantifiers for floating point and fixed point numerical representation. Quantifiers H_{val}, H_{BP} and C_{BP} are equal to zero for all presitions, this reflects that the series remains in a fixed point for almost all samples. When we compares with H_{BPW} and C_{BPW} quantifiers they are different of 0 because BPW procedure discards the elements once they reach the fixed point. High dispersions in H_{BPW}, C_{BPW} and MP are related with the short length of data taken into account, transient to fixed point has a maximum length of B iterations for fixed point arithmetic and 54 for floating point.

When fixed point is discarded, complexity remains near maximum for all presicions. It can be seen in Fig. 5, where all points are close to superior margin.

In summary, a comparison between LOG and TENT maps shows that only LOG can be used because TENT is highly anomalous.

3.2 Sequential switching

3.2.1 Sequential switching between Tent and Logistic maps (SWITCH)

SWITCH may be expressed as:

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$$\begin{cases} x_{n+1} & = \begin{cases} 2\ x_n & \text{if } 0 \leq x_n \leq 1/2 \\ 2\ (1-x_n) & \text{if } 1/2 < x_n \leq 1 \end{cases} \\ x_{n+2} & = \quad 4\ x_{n+1}\ (1-x_{n+1}) \end{cases}$$

with $x_n \in \mathcal{R}$ and n an even number.

Results with sequential switching are shown in Figs. 6 (a) to (f). The floating point entropy value is $H_{val} = 0.9722$, a marginal increment value to that obtained for LOG. For fractial numbers this value is reached for $B = 24$, but it is stable from $B = 28$. Regarding ordering patterns the number of MP decreases to 586, a value lower than the one obtained for LOG. It means the entropy H_{BP} may increase up to $ln(134)/ln(720) \simeq 0.74$. BP and BPW
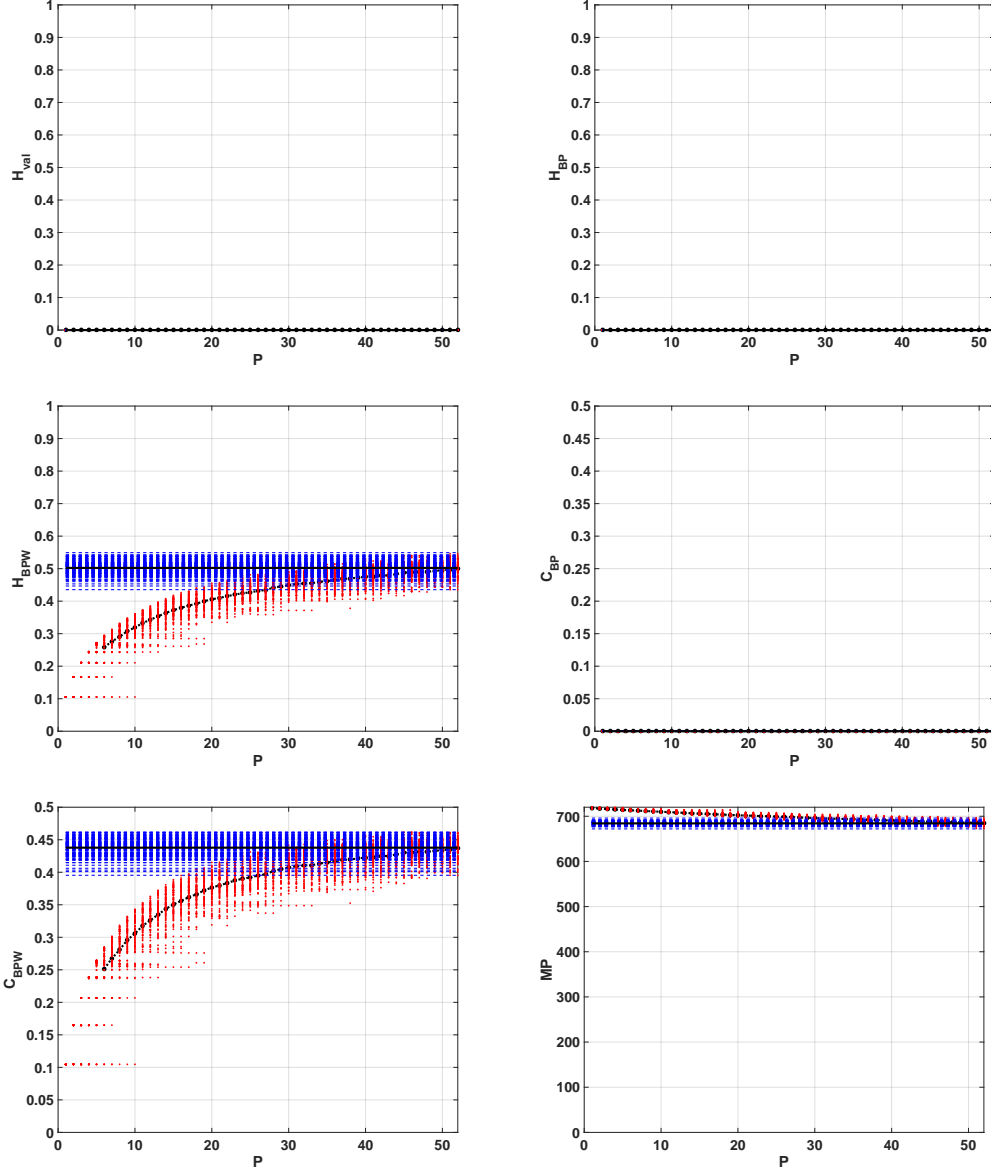
Fig. 4. Statistical properties of the TENT map using binary representation: (a) H_{val} vs B (b) H_{BP} vs B (c) C_{BP} vs B (d) Number of missing ordering patterns MP vs B.

quantifiers reachs their maximum of $H_{BP} = 0.6546$ and $H_{BPW} = 0.6313$ at $B = 16$, but they stabilishes at $B = 24$. Complexities are lower than LOG, $C_{BP} = 0.4580$ and $C_{BPW} = 0.4578$, these values are reached for $B \geq 15$ but is stable from $B \geq 23$. Compared with LOG, statistical proerties are better and they are reached with a less amount of bits, for $B \geq 24$ this map reachs optimal characteristics.

We can see some points with anomalies from $B \geq 49$ due to same problem of LOG map, an multiplication needs double of precision to be done. Furthermore, we encountred one initial condition in fixed point with an anomalous

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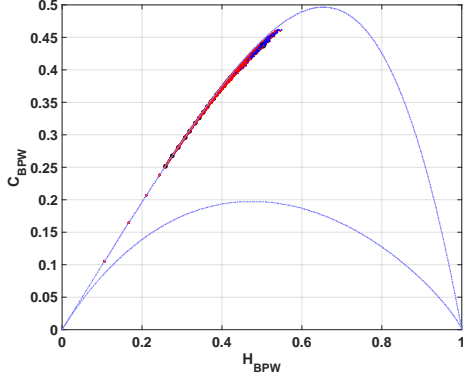
Fig. 5. Evolution of statistical properties in entropy-complexity plane of LOG map using binary representation: C_{BPW} vs H_{BPW}.

behaviour.

Double entropy plane H_{val} vs H_{BP} is showed in Fig. 7. The point reached in this plane for SWITCH map is similar than reached for LOG map, there is a small difference on vertical axis. It means that mixing is beter for SWITCH.

TENDRA QUE SACAR MAS CONCLUSIONES DE LOS PLANOS TENDRA QUE VERLAS EN LAS SERIES DE DATOS, ESPECIALMENTE EL PUNTO ESE QUE DA CORRIDO

Entropy complexity plane C_{BP} vs H_{BP} is showed in Fig. 8. When we compare with the convergent point of LOG in this plane we can see that C_{BP} is lower for SWITCH, this behaviour is more similar to an random generator.

3.2.2 Skipping on sequential switching between Tent and Logistic maps (EVEN and ODD)

Skipping is a usual randomizing technique that increases the mixing quality of a single map and correspondingly increases the value of H_{BP} and decreases C_{BP} of the time series. Skipping does not change the values of H_{val} and C_{val} evaluated using the non causal PDF (normalized histogram)[4].

In the case under consideration we study Even and Odd skipping of the sequential switching of Tent and Logistic maps.

(1) Even skipping of the sequential switching of Tent and Logistic maps (EVEN).
If $\{x_n, \ (n = 1, ...\infty)\}$ is the time series generated by 3.2.1 discard all the values in odd positions and retain the values in even positions.
(2) Odd skipping of the sequential switching of Tent and Logistica maps. If $\{x_n, \ (n = 1, ...\infty)\}$ is the time series generated by 3.2.1 discard all the
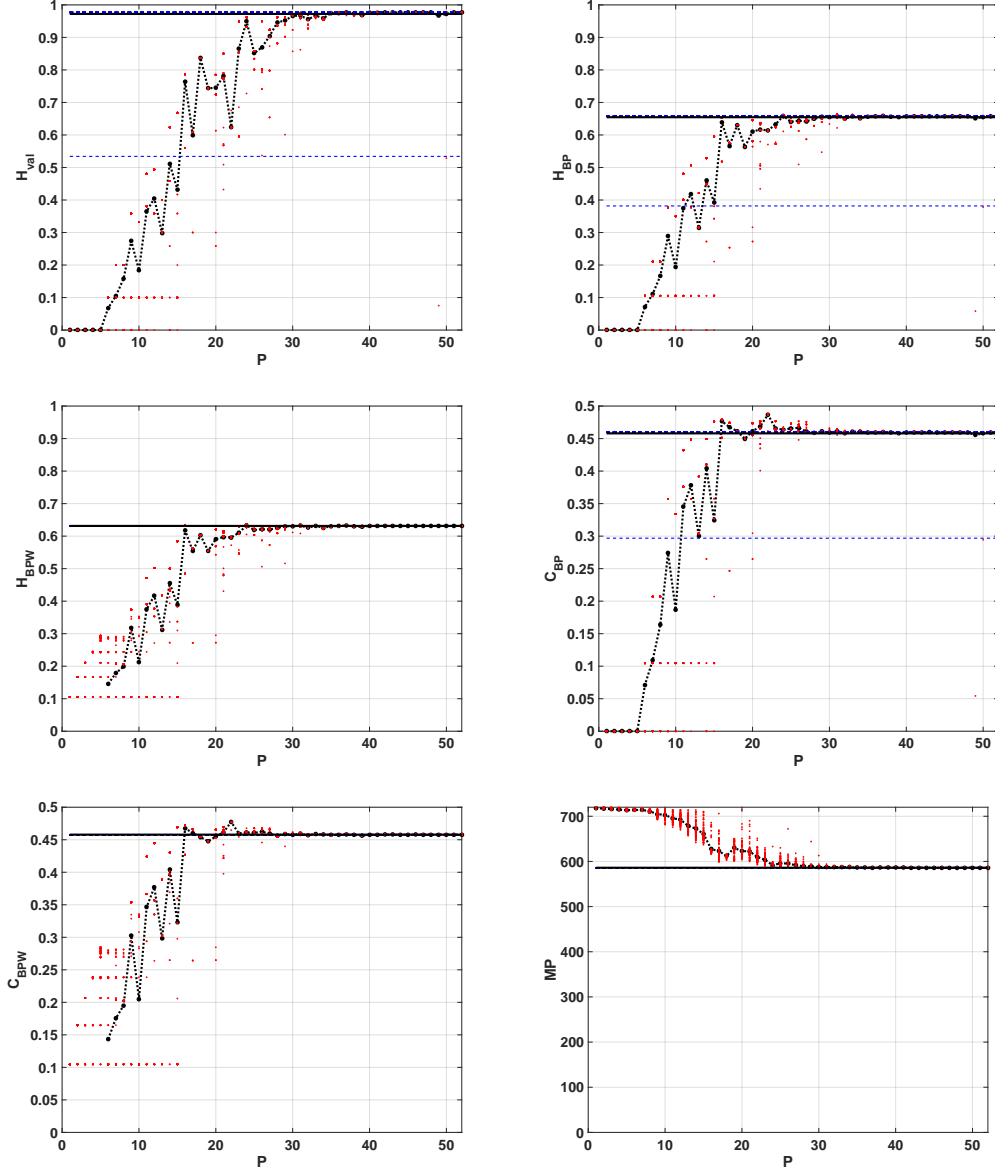
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Fig. 6. Statistical properties of the SWITCH map using binary representation: (a) H_{val} vs B (b) H_{BP} vs B (c) C_{BP} vs B (d) Number of missing ordering patterns MP vs B.

values in even positions and retain all the values in odd positions.

The reason for studying even and odd skipping cases is the sequential switching map M_{switch} is the composition of two different maps. Even skipping may be expressed as $M_{TENT} \circ M_{LOG}$ while odd skipping may be expressed as $M_{LOG} \circ M_{TENT}$.

Quantifiers related to the normalized histogram of values slightly degrades with the skipping procedure. For example H_{val} reduces from 0.9722 without skipping to 0.9459 for EVEN and 0.9706 for ODD. This difference between
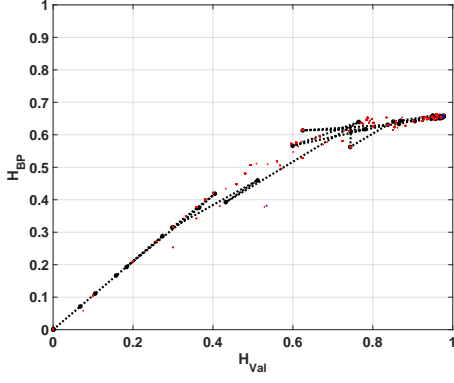
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Fig. 7. Evolution of statistical properties in double entropy plane of SWITCH map using binary representation: (a) H_{val} vs H_{BP} (b) H_{val} vs H_{BPW}.
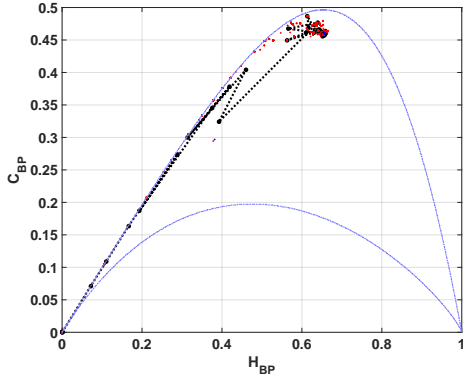

Fig. 8. Evolution of statistical properties in entropy-complexity plane of SWITCH map using binary representation: (a) C_{BP} vs H_{BP}.

EVEN and ODD is because in floating point an high dispersion was obtained for H_{val}, H_{BP} and C_{BP} but not for H_{BPW} or C_{BPW}. VER LOS ATRACTORES Y EXPLICAR

From the point of view of order patterns a great improvement is obtained using any of the skipping strategies but ODD is slightly better than EVEN. MP are reduced to $MP \simeq 118$ for EVEN and ODD, increasing the maximum allowed Bandt & Pompe entropy that reaches the mean value $< H_{BP} >= 0.8381$ for EVEN, and $< H_{BP} >= 0.9094$. The complexity is reduced to $< C_{BP} >\simeq 0.224$ for EVEN and $< C_{BP} >\simeq 0.282$ with for ODD.

As a counterpart, higher accuracy is required to achieve low complexity, characteristic of stochastic systems. Amount of bits to represent the fractional part was $B > 40$ for both EVEN and ODD generators.

In Figs. ?? and Figs. ?? are shown the results for EVEN and ODD respsctivelly.
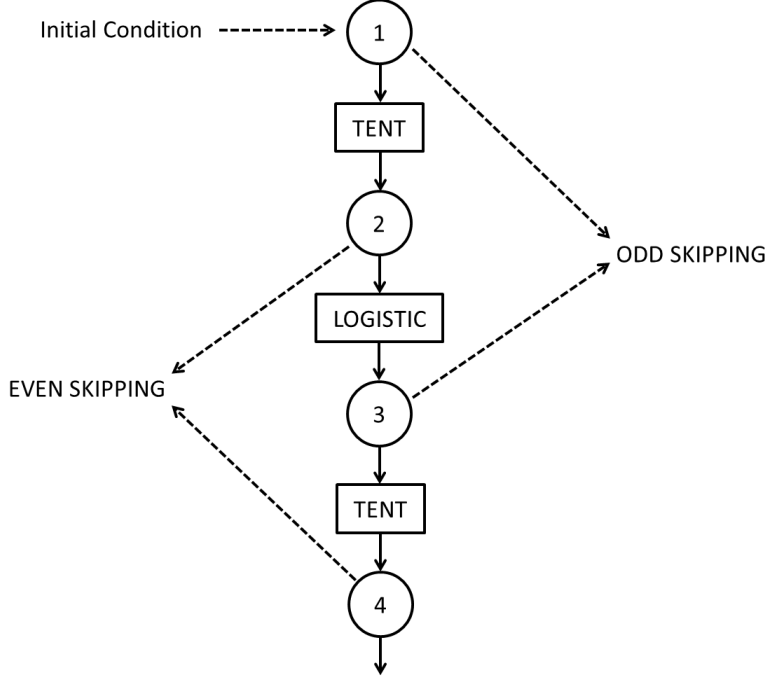
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Fig. 9. Sequential switching between Tent and Logistic maps. In the figure are also shown even and odd skipping strategies
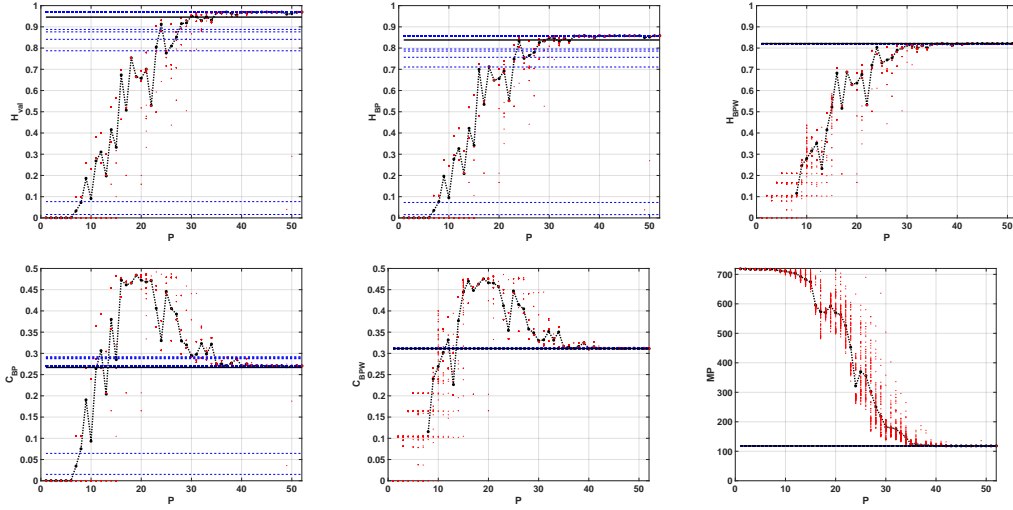

Fig. 10. Statistical properties of the EVEN map using binary representation: (a) H_{val} vs B (b) H_{BP} vs B (c) C_{BP} vs B (d) Number of missing ordering patterns MP vs B.

3.3 Period T as a function of P and B

The issue of how the period T is related with the representation with P decimal digits was studied by Grebogi and coworkers [10]. There they shaw that the period T scales with roundoff ϵ as $T \sim \epsilon^{-d/2}$ where d is the correlation
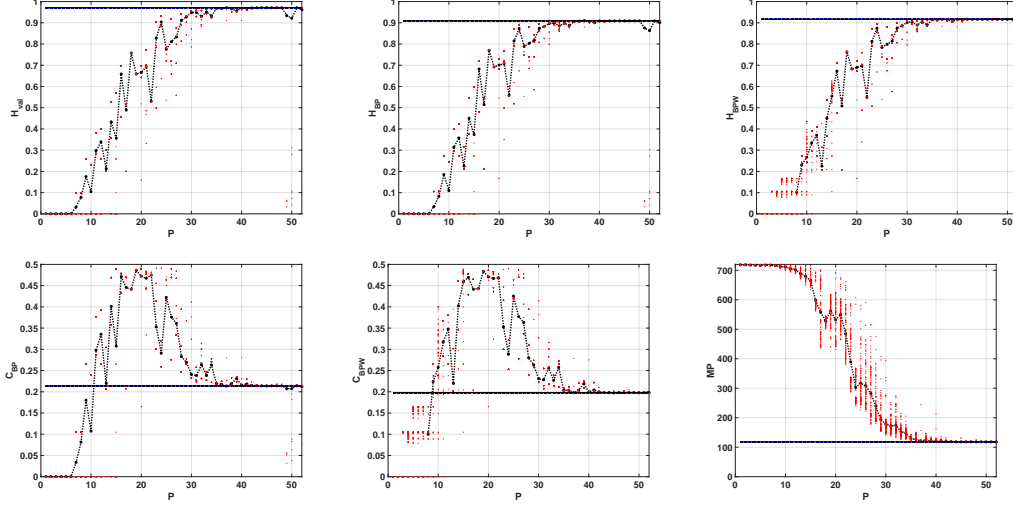
Fig. 11. Statistical properties of the LOG map using binary representation: (a) H_{val} vs B (b) H_{BP} vs B (c) C_{BP} vs B (d) Number of missing ordering patterns MP vs B.

Table 1
Period T as a function of B for the maps considered

map	m	b
TENT	0	0
LOG	0.139	-0.6188
SWITCH	0.1462	-0.5115
EVEN	0.1447	-0.7783
ODD	0.1444	-0.7683

dimension of the chaotic attractor. Nagaraj et al [1] studied the case of switching between two maps. They shaw that the period T of the compound map obtained by switching between two chaotic maps is higher than the period of each map and they found that a "random" switching improves the results. Here we considered sequential switching to avoid the use of another random variable, because it can include its own statistical properties in the time series. We studied decimal and binary numbers representations. Fig. 12 shows T vs P in semi logarithmic scale. A straight line can fit the points and has the expression $log_{10}T = m \times P + b$ for decimal numbers and $log_2 T = m \times B + b$ for binary numbers, where m is the slope and b is the y-intercept. Results for all considered maps are summarized in Table ?? and 1.

Results are compatible for those obtained in [1]. Switching between maps increase de period T but the skipping procedure decrease it esentially to one half.
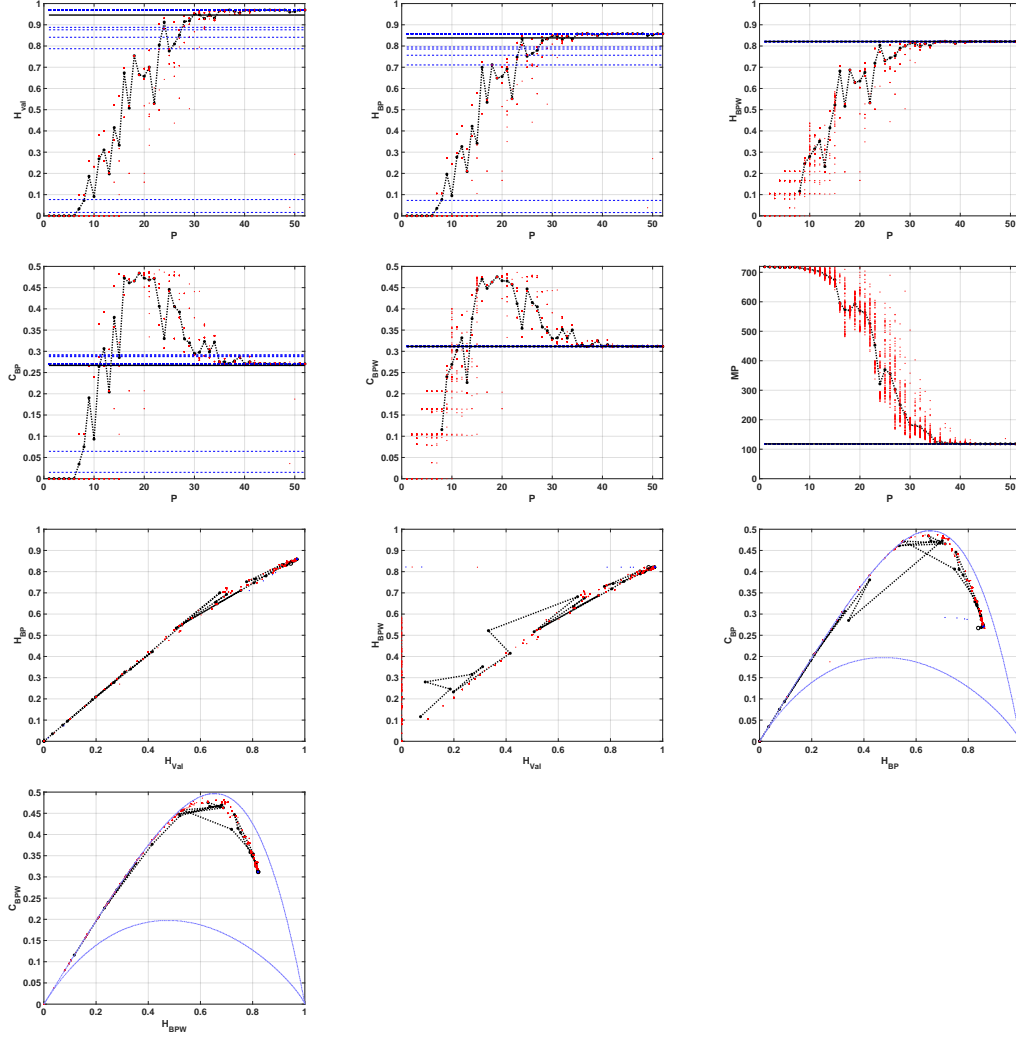
Fig. 12. Statistical properties of EVEN, obtained by skipping the values in the odd position of the time series of SWITCH, using binary representation: (a) H_{val} vs P (b) H_{BP} vs P (c) C_{BP} vs P (d) Number of missing ordering patterns MP vs P. In Figures (a) to (d) dashed line correspond to floating point numbers. (e) representation in the H_{val}, H_{BP} plane in the the binary numerical system. The star represents the state for floating points numbers. (f) representation in the H_{BP}, C_{BP} plane. The star represents the state for floating points numbers.

4   Conclusions

In summary:

- Not every number base can be represented by a device with a certain base. For example, a base ten number can not be exactly represented in a conventional computer, it will always have an error inherent of the system..No todas las bases numricas son representables con una mquina de base distinta. Por ejemplo, no se puede representar la base 10 con base 2.
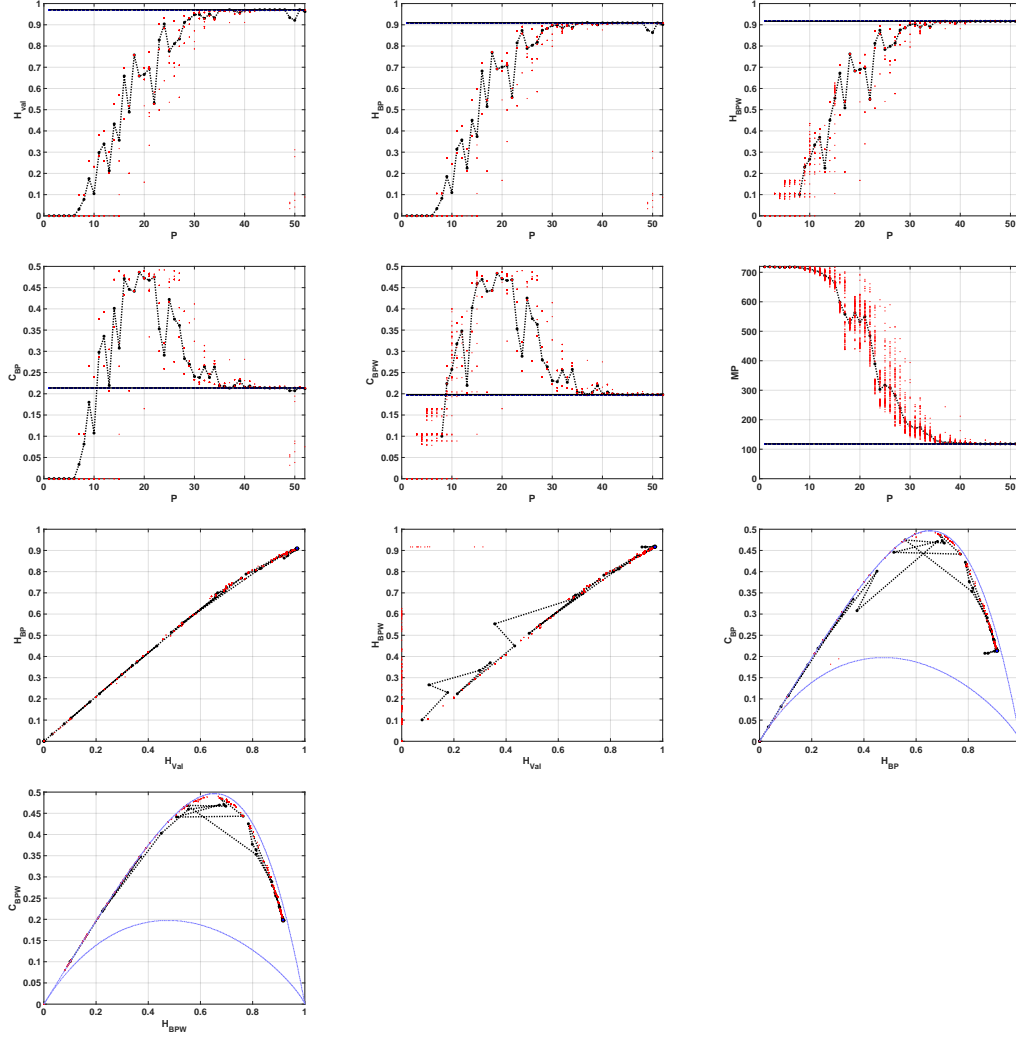
18

Fig. 13. Statistical properties of ODD, obtained by skipping the values in the odd position of the time series of SWITCH, using binary representation: (a) H_{val} vs P (b) H_{BP} vs P (c) C_{BP} vs P (d) Number of missing ordering patterns MP vs P. In Figures (a) to (d) dashed line correspond to floating point numbers. (e) representation in the H_{val}, H_{BP} plane in the the binary numerical system. The star represents the state for floating points numbers. (f) representation in the H_{BP}, C_{BP} plane. The star represents the state for floating points numbers.

- En una mquina de clculo "a medida", como la que puede implementarse en ASICs o FPGAs existen limitaciones en el bus de datos y en la electrnica de clculo. Si la electrnica de clculo debe ser reducida se recomienda usar mapas que puedan ser calculados slo con sumas y restas de la variable pseudoaleatoria.
- Los mapas que slo tienen operaciones de shifteo en la base de la mquina de clculo inevitablemente caern a cero en tantas iteraciones como el largo de la mantisa de representacin. Por ejemplo el tent en base 2.
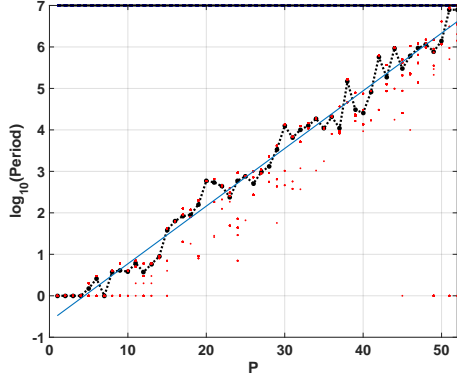- La comparacin entre BP y BPW permite detectar el comportamiento del

Fig. 14. Period as function of presition in binary digits

sistema. Puede detectarse si el atractor cae a un punto fijo y diferenciar si el transitorio es corto o largo, respecto de la cantidad de itaraciones del mapa.

- Como se menciona en el paper de referencia, el perodo del mapa iterado aumenta respecto del simple. Tambin se nota una mejora marginal en la mezcla de la secuencia. La distribucin de valores es buena en todos los casos.
- el skipping empeora el perodo pero mejora sustancialmente la mezcla de los valores. Esto puede verse en BP, BPW y MP.

produces a non-monotonous evolution toward the floating point result. This result is relevant because it shows that increasing the precision is not always recommended.

ESTA FRASE ESTABA ARRIBA, VER DONDE LA PONEMOS: It is specially interesting to note that some systems (TENT) with very nice statistical properties in the world of the real numbers, become "pathological" when binary numerical representations are used.

Acknowledgment

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[34] hay que buscar el trabajo que propone modificar en forma aleatoria.