Complexity of switching chaotic maps

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Abstract

In the last years, digital systems such as Digital Signal Processors (DSP), Field Programmable Gate Arrays (FPGA) and Application-Specific Integrated Circuits (ASIC), became the standard in all experimental sciences.

In these systems digital implementations have a custom made numerical representations, therefore finite arithmetic needs to be investigated. Binary floating- and fixed-point are the numerical representations available. Fixed-point representation is prefered over floating-point when speed, low power and/or small circuit area is necessary.

Chaotic systems implemented in finite precision will always become periodic with period T, each of them produces specific period and statistical characteristics that need to be evaluated. It has been recently shown that it is convenient to describe the statistical characteristic using both, a non causal and a causal probability distribution function (PDF). The corresponding entropies, must be evaluated to quantify these PDF's.

In previous works an interesting study about period was carried out using as an example two well known chaotic maps: the tent map and the logistic map. After that, switching techniques between these two maps were tested. In this paper we complement that analysis by characterizing the behaviour of these maps from an statistical point of view using causal and non causal quantifiers.

VERSION: March 20, 2017

Preprint submitted to Elsevier *March 20, 2017*

1 Introduction

In the last years digital systems became the standard in all experimental sciences. By using virtual instruments, new programmable electronic devices such as Digital Signal Processors (DSP) and Reconfigurable electronics such as Field Programmable Gate Arrays (FPGA) or Application-Specific Integrated Circuits (ASIC), allow experimenters to design and modify their own signal generators, measuring systems, simulation models, etc.

The effect of finite precision in these new devices needs to be investigated. Floating-point is not always available when speed, low power and/or small circuit area are is required, a fixed-point solution is better in these cases. Fixed-point representation is critical if chaotic systems must be implemented, because due to roundoff errors digital implementations will always become periodic with period T and unstable orbits with a low periods may become stable destroying completely the chaotic behavior.

Grebogi and coworkers [1] studied this subject and they saw that the period T scales with roundoff ϵ as $T \sim \epsilon^{-d/2}$ where d is the correlation dimension of the chaotic attractor. To have a large period T is an important property of chaotic maps, in [2] Nagaraj et. als studied the effect of switching over the average period lengths of chaotic maps in finite precision. They saw that the period T of the compound map obtained by switching between two chaotic maps is higher than the period of each map. Liu et. als [3] studied different switching rules applied to linear systems to generate chaos. Switching issue was also addressed in [4], author considers some mathematical, physical and engineering aspects related to singular, mainly switching systems. Switching systems naturally arise in power electronics and many other areas in digital electronics. They have also interest in transport problems in deterministic ratchets [5] and it is known that synchronization of the switching procedure affects the output of the controlled system.

Stochasticity and mixing are also relevant, to characterize these properties several quantifiers were studied [6]. Among them the use of an entropy-complexity representation ($H-C$ plane) and causal-noncausal entropy ($H_{BP}-H_{Val}$ plane) deserves special consideration [7,8,9,6,10,11]. A fundamental issue is the criterium to select the probability distribution function (PDF) assigned to the time series. Causal and non causal options are possible. Here we consider the non-causal traditional PDF obtained by normalizing the histogram of the time series. Its statistical quantifier is the normalized entropy H_{Val} that is a measure of equiprobability among all allowed values. We also consider a causal PDF that is obtained by assigning ordering patterns to segments of trajectory of length D. This PDF was first proposed by Bandt & Pompe in a seminal paper [12]. The corresponding entropy H_{BP} was also proposed as a quantifier by Bandt & Pompe. Amigó and coworkers proposed the number of forbidden patterns as a quantifier of chaos [13]. Essentialy they reported the presence of forbidden patterns as an indicator of chaos. Recently it was shown that the name forbidden patterns is not convenient and it was replaced by missing patterns (MP) [14]. PONER ALGO DE BPW

In this paper we study the statistical characteristics of five maps, two well known maps: (1) the tent map (TENT) and (2) logistic map (LOG), and three additional maps generated from them: (3) SWITCH, generated by switching between TENT and LOG; (4) EVEN, generated by skipping all the elements in odd position in SWITCH time series and (5) ODD, generated by discarding all the elements in an even position in SWITCH time series. Binary floating- and fixed-point numbers are used, these specific numerical systems may be implemented in modern programmable logic devices.

The main contributions of this paper are:

(1) the definition of different statistical quantifiers and their relationship with the properties of the series generated by the studied chaotic maps.
(2) the study of how these quantifiers detect the evolution of stochasticity and mixing of the chaotic maps according as the numerical precision varies.
(3) the effect on the period and the statistical properties of the time series of switching between two different maps.
(4) the effect on the period and the statistical properties of the time series of skipping values in the switched maps.

Organization of the paper is as follows: section 2 describes the statistical quantifiers used in the paper and the relationship between their value and characteristics of the causal and non causal PDF considered; section 3 shows and discusses the results obtained for all the numerical representations. Finally section 4 deals with final remarks and future work.

2 Information theory quantifiers

All the statistical quantifiers studied here are functionals of the PDF associated to the time series. The first step to quantify the statistical properties of the values (amplitude statistics) of a time series $\{x_i, \forall\ i = 1, \ldots, N\}$ using information theory is to determine the concomitant PDF. This is an issue studied in detail in previous papers [**?**]. Let us summarize the procedure:

(1) a finite alphabet with M symbols $\mathcal{A} = \{a_1, \ldots, a_M\}$ is chosen.
(2) each one of these symbols is assigned to each: (a) value of the time series (or non-overlapped set of consecutive values) or (b) portion of length D

of the trajectory.

(3) the normalized histogram of the symbols is the desired PDF.

(4) a randomness quantifier is calculated over the PDF. In our case we calculate (a) normalyzed Shannon entropy H, (b) normalyzed statistical complexity C and (d) MP.

Note that if option (a) is chosen in step 2 then the PDF is non causal, because all the information about the time evolution of the system generating $\{x_i\}$ is completely lost. On the contrary if option (b) is chosen in step 2 then the PDF is causal, in the sense it includes information about the temporal evolution.

NOMBRAR LA BPW

NOMBRAR LOS PLANOS QUE TERMINEMOS ELIGIENDO

Of course there are infinite possibilities to choose the alphabet $\mathcal{A}$ as well as the length D. Bandt & Pompe made a proposal for a causal PDF that has been shown to be easy to implement and useful in a great variety of applications REFERENCIAS A APLICACIONES. The procedure is the following [12,15,16]: ESTA PARTE ESTÁ EN EL TIEMPO Y DEBERÍA ESTAR EN LAS MUESTRAS

- Given a series $\{x_t, \forall\ t = 0, \Delta t, \cdots, N\Delta t\}$, a sequence of vectors of length D is generated.

$$(s) \longmapsto \left(x_{t-(d-1)\Delta t}, x_{t-(d-2)\Delta t}, \ldots, x_{t-\Delta t}, x_t\right) \tag{1}$$

  Each vector turns out to be the “history” of the value x_t. Clearly, the longer the length of the vectors D, the more information about the history would the vectors have but a higher value of N is required to have an adequate statistics.

- The permutations $\pi = (r_0, r_1, \cdots, r_{D-1})$ of $(0, 1, \cdots, D-1)$ are called “order of patterns” of time t, defined by:

$$x_{t-r_{D-1}\Delta t} \leq x_{t-r_{D-2}\Delta t} \leq \cdots \leq x_{t-r_1\Delta t} \leq x_{t-r_0\Delta t} \tag{2}$$

  In order to obtain an unique result it is considered $r_i < r_{i-1}$ if $x_{t-r_i\Delta t} = x_{t-r_{i-1}\Delta t}$.

  In this way, all the $D!$ possible permutations π of order D, and the PDF $P = \{p(\pi)\}$ is defined as:

$$p(\pi) = \frac{\sharp\{s|s \leq N - D + 1; (s) \quad \texttt{has type}\ \pi\}}{N - D + 1} \tag{3}$$

  In the last expression the $\sharp$ symbol means “number”.

This procedure has the advantages of being *i*) simple, *ii*) fast to calculate, *iii*) robust in presence of noise, and *iv*) invariant to lineal monotonous transformations. DICE QUE ES ROBUSTO FRENTE A LA PRESENCIA DE RUIDO PERO NO, REFERENCIAS?

It is applicable to weak stationarity processes (for $k = D$, the probability that $x_t < x_{t+k}$ doesn't depend on the particulary t [12]).The causality property of the PDF allows the quantifiers (based on this PDFs) to discriminate between deterministic and stochastic systems [17].

According to this point Bandt and Pompe suggested $3 \leq D \leq 7$. $D = 6$ has been adopted in this work.

HABLAR DE BPW

The entropy $H[P]$ is the normalized version of the Entropy proposed by Shannon [18]:

$$H[P] = S[P]/S_{max}, \tag{4}$$

where $S[P] = -\sum_{j=1}^{M} \; p_j \; \ln(p_j)$ and S_{max} is the normalizing constant:

$$S_{max} = S[P_e] = \ln M, \tag{5}$$

and $P_e = \{1/M, \ldots, 1/M\}$ is the uniform distribution.

The statistical complexity $C[P]$ is given by:

$$C[P] = Q_J[P, P_e] \cdot H[P] \ , \tag{6}$$

, and Q_J is named disequilibrium and it is the distance between P and P_e in the probability space. The metric used in this paper is based on the Jensen-Shannon divergence [19]:

$$Q_J[P, P_e] = Q_0 \cdot \{S[\frac{P + P_e}{2}] - S[P]/2 - S[P_e]/2\} \ . \tag{7}$$

The normalization constant Q_0 is:

$$Q_0 = -2 \left\{ \left(\frac{N+1}{N} \right) \ln(N+1) - 2\ln(2N) + \ln N \right\}^{-1} . \tag{8}$$

From the statistical point of view the disequilibrium Q_J is an intensive magnitude, and it is 0 if and only if $P = P_e$. It has been proved that the $C[P]$ quantifies the presence of nonlinear correlations typical of chaotic systems [20,19]. The complexity $C[P]$ is independent from the entropy $H[P]$, as far as

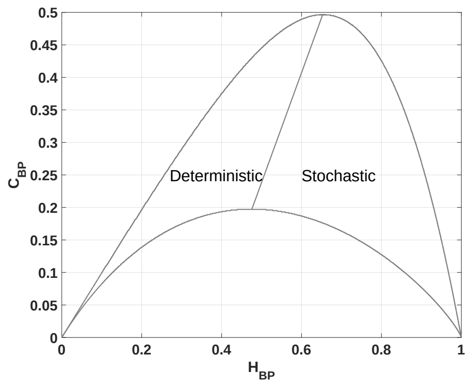

Figure 1. Entropy-Complexity plane.

different P's share the same entropy $H[P]$ but they have different complexity $C[P]$.

PONER UN PÁRRAFO QUE ENGANCHE LAS H's Y C's CON LOS HISTOGRAMAS PARA ARMAR Hval, Cval, Hbp, Cbp, Hbpw y Cbpw

Two representation planes are considered: H_{BP} vs H_{hist} [8] and H_{BP} vs C_{BP} [7]. In the first plane a higher value in any of the entropies, H_{BP} and H_{hist}, implies an increase in the uniformity of the involved PDF. The point $(1, 1)$ represents the ideal case with uniform histogram and uniform distribution of ordering patterns. In the second plane not the entire region $0 < H_{BP} < 1$, $0 < C_{BP} < 1$ is achievable. In fact for any PDF the pairs (H, C) of possible values fall between two extreme curves in the plane H-C [21]. Fig. 1 shows two regions labeled as deterministic and stochastic. In fact transition from one region to the other are smooth and the division is a bit arbitrary. A more detailed discussion can be seen in [7]. Ideal random systems having uniform Bandt & Pompe PDF, are represented by the point $(1, 0)$ [22] and a delta-like PDF corresponds with the point $(0, 0)$.

We also used the number of MP as a quantifier[14]. As shown recently by Amigó *et al.* [23,24,25,26], in the case of deterministic one-dimensional maps, not all the possible ordinal patterns can be effectively materialized into orbits, which in a sense makes these patterns forbidden. Indeed, the existence of these forbidden ordinal patterns becomes a persistent fact that can be regarded as a new dynamical property. Thus, for a fixed pattern-length (embedding dimension D) the number of forbidden patterns of a time series (unobserved patterns) is independent of the series length N. Remark that this independence does not characterize other properties of the series such as proximity and correlation, which die out with time [24,26].

A full discussion about the convenience of using these quantifiers is out of the scope of this work. Nevertheless reliable bibliographic sources do exist [27,28,29,8,10,30,14].

3 Results

Five pseudo chaotic maps were studied, two simple maps and three combination of them. For each one floating-point representation and fixed-point numbers with $1 \leq B \leq 53$ representation are considered, where B is the number of bits that represents the fractional part. For each one of these 54 representations 100 time series were generated using randomly chosen initial conditions within the interval $[0, 1]$, that is the attraction domain.

The studied maps are logistic (LOG), tent (TENT), sequential switching between TENT and LOG (SWITCH) and skipping discarding the values in the odd positions (EVEN) or the values in the even positions (ODD) respectively.DEFINIR ACRONIMOS Y EXPLICAR MEJOR.

Logistic map is interesting because is representative of the very large family of quadratic maps. Its expression is:

$$x_{n+1} = 4\,x_n(1 - x_n) \tag{9}$$

with $x_n \in \mathcal{R}$.

Note that to effectively work in a given representation it is necessary to change the expression of the map in order to make all the operations in the chosen representation numbers. For example, in the case of LOG the expression in binary fixed-point numbers is:

$$x_{n+1} = 4\epsilon \operatorname{floor}\left\{\frac{x_n(1 - x_n)}{\epsilon}\right\} \tag{10}$$

with $\epsilon = 2^B$ where B is the number of bits that represents the fractional part.

The Tent map has been extensively studied in the literature because theoretically it has nice statistical properties that can be analytically obtained. For example it is easy to proof that it has a uniform histogram and consequently an ideal $H_{val} = 1$. The Perron-Frobenius operator and its corresponding eigenvalues and eigenfunctions may be also be analytically obtained for this map [?]. PONER ALGUNA REFERNCIA A PERRON Y AGREGAR ALGO EN ITQS

This map is represented with the equation:

$$x_{n+1} \;=\; \begin{cases} 2\;x_n & ,\text{if}\;\; 0 \leq x_n \leq 1/2 \\ 2\;(1 - x_n) & ,\text{if}\;\; 1/2 < x_n \leq 1 \end{cases} \tag{11}$$

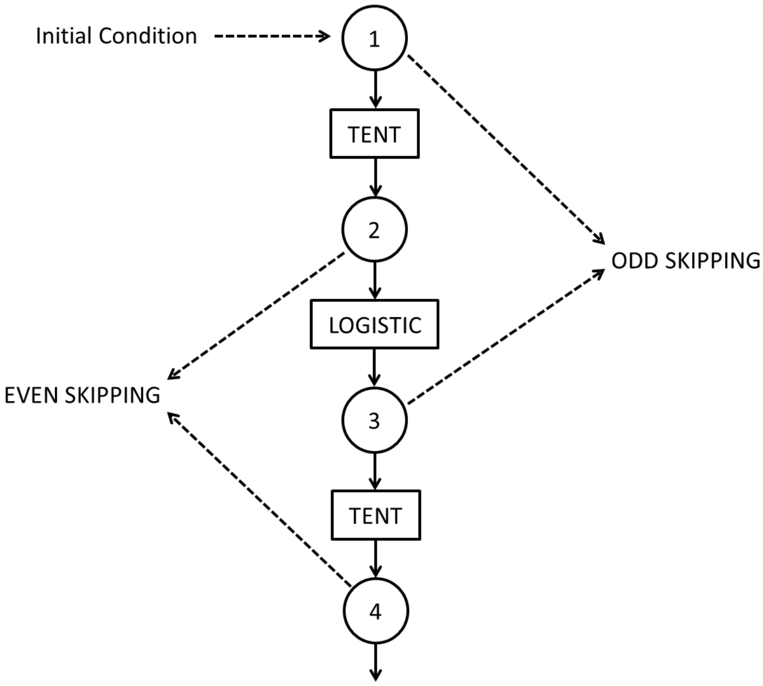

Figure 2. Sequential switching between Tent and Logistic maps. In the figure are also shown even and odd skipping strategies.

with $x_n \in \mathcal{R}$.

In base-2 fractional numbers rounding, equation 11 becomes:

$$x_{n+1} \;=\; \begin{cases} 2\ x_n & ,\text{if } 0 \leq x_n \leq 1/2 \\ \epsilon\ \text{floor}\{\frac{2\ -\ 2\ x_n}{\epsilon}\} & ,\text{if } 1/2 < x_n \leq 1 \end{cases} \tag{12}$$

with $\epsilon = 2^{-B}$.

Switching, even skipping and odd skipping procedures are shown in Fig. 2.

SWITCH map is expressed as:

$$\begin{cases} x_{n+1} = \begin{cases} 2\,x_n, & ,\text{if } 0 \leq x_n \leq 1/2 \\ 2\,(1 - x_n) & ,\text{if } 1/2 < x_n \leq 1 \end{cases} \\ x_{n+2} = 4\,x_{n+1}\,(1 - x_{n+1}) \end{cases} \tag{13}$$

with $x_n \in \mathcal{R}$ and n an even number.

Skipping is a usual randomizing technique that increases the mixing quality of a single map and correspondingly increases the value of H_{BP} and decreases C_{BP} of the time series. Skipping does not change the values of H_{val} and C_{val}

because their are evaluated using the non causal PDF (normalized histogram of values)[8].

In the case under consideration we study even and odd skipping of the sequential switching of Tent and Logistic maps:

(1) Even skipping of the sequential switching of Tent and Logistic maps (EVEN).
If $\{x_n, \forall\, n = 1, \ldots, \infty\}$ is the time series generated by 13, discard all the values in odd positions and retain the values in even positions.
(2) Odd skipping of the sequential switching of Tent and Logistic maps. If $\{x_n, \forall\, n = 1, \ldots, \infty\}$ is the time series generated by 13, discard all the values in even positions and retain all the values in odd positions.

The reason for studying even and odd skipping cases is the sequential switching map SWITCH is the composition of two different maps. Even skipping may be expressed as the composition function TENT∘LOG while odd skipping may be expressed as LOG∘TENT. The evolution of period as function of precision was reported in [2].

Let us detail our results for each of these maps.

3.1 Period T as a function of B

Grebogi and coworkers [1] have studied how the period T is related with the precision. There they saw that the period T scales with roundoff ϵ as $T \sim \epsilon^{-d/2}$ where d is the correlation dimension of the chaotic attractor.

Nagaraj et als. [2] studied the case of switching between two maps. They saw that the period T of the compound map obtained by switching between two chaotic maps is higher than the period of each map and they found that a random switching improves the results. Here we have considered sequential switching to avoid the use of another random variable, because it can include its own statistical properties in the time series.

Fig. 3 shows T vs B in semi logarithmic scale. The experimental averaged points can be fitted by a straight line expressed as $log_2 T = mB + b$ where m is the slope and b is the y-intercept. Results for all considered maps are summarized in Table 1.

Results are compatible for those obtained in [2]. Switching between maps increases de period T but skipping procedure decreases by almost half.

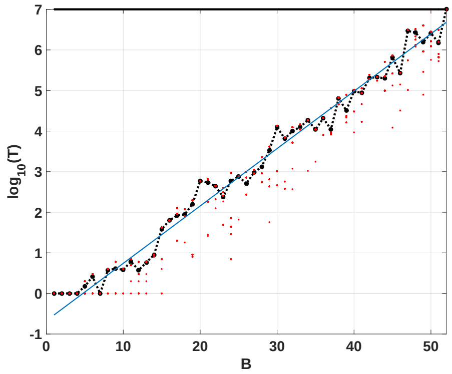

Figure 3. Period as function of precision in binary digits

Table 1
Period T as a function of B for the maps considered

map	m	b
TENT	0	0
LOG	0.139	-0.6188
SWITCH	0.1462	-0.5115
EVEN	0.1447	-0.7783
ODD	0.1444	-0.7683

3.2 Quantifiers of simple maps

Here we report our results for both simple maps, LOG and TENT

3.2.1 Logistic map (LOG)

Figs. 4 (a) to (f) show the statistical properties of LOG map in floating-point and fixed-point representation. All these figures show: 100 red points for each fixed-point precision (B) and in black their average (dashed black line connecting black dots), 100 horizontal dashed blue lines that are the results of each run in floating-point and a black solid line their average. In this case, all the lines of the floating-point are overlapped.

According as B grows, statistical properties vary until they stabilize. For $B \geq 30$ the value of H_{val} remains almost identical to the values for the floating-point representation whereas H_{BP} and C_{BP} stabilizes at $B > 21$. Their values are: $\langle H_{val} \rangle = 0.9669$; $\langle H_{BP} \rangle = 0.6269$; $\langle C_{BP} \rangle = 0.4843$. Note that the stable value of missing patterns $MP = 645$ makes the optimum $H_{BP} \leq ln(75)/ln(720) \simeq$

0.65. Then, $B = 30$ is the most convenient choice because an increase in the number of fractional digits does not improve the statistical properties.

Some conclusions can be drawn regarding BP and BPW quantifiers. For $B = 1, 2, 3$ and 4, the averaged BP quantifiers are almost 0 while the averaged BPW quantifiers can not be calculated (see in Figs. 4 c and e the missing black dashed line). For those sequences where the initial condition where 0 all iterations result to be a sequence of zeros (the fixed point of the map), this happens when using small precisions because of the roundoffs.

When B increases, $B = 7, 9$ and 12, the initial conditions are rounded to zero less frequently. So the generated sequences start from some value but many of them fall to zero with a short transitory. This can be seen in Figs.4 c and e, BPW quantifiers show a high dispersion unlike BP quantifiers. This is because BPW procedure takes into account only the transient discarding fixed points, unlike BP procedure considers all the values of the sequence.

The same results are shown in double entropy planes with the precision as parameter (Fig. 5). These figures show: 100 red points for each fixed-point precision (B) and in black their average (dashed black line connecting black dots), 100 blue dots that are the results of each run in floating-point and the black star is their average. In Fig. 5 the 100 blue points and their average are overlapped.

As expected, the fixed-point architecture implementation converges to the floating-point value as B increases. For both, Hbp-Hval and Hbpw-Hval, from $B = 20$, H_{val} improves but H_{BP} remains constant. It can be seen that the distribution of values reaches high values ($\langle H_{val} \rangle = 0.9669$) but their mixing is poor ($\langle H_{BP} \rangle = 0.6269$).

In Fig. 6 we show the entropy-complexity planes. Dotted gray lines are the upper and lower margins, it is expected that a chaotic system remains near the upper margin. This results characterize a chaotic behaviour, in $H_{BP} - C_{BP}$ plane we can see a low entropy and high complexity.

3.2.2 Tent map (TENT)

When this map is implemented in a computer using any numerical representation system (even floating-point!) truncation errors rapidly increase and make unstable fixed point in $x^* = 0$ to become stable. The sequences within the attractor domain of this fixed point will have a short transitory of length between 0 and B followed by an infinite number of 0's [31,32]. This issue is easily explained in [33], the problem appears because all iterations have a left-shift operation that carries the 0's from the right side of the number to the most significant positions. Some authors [**?**] have proposed to add random pertur-

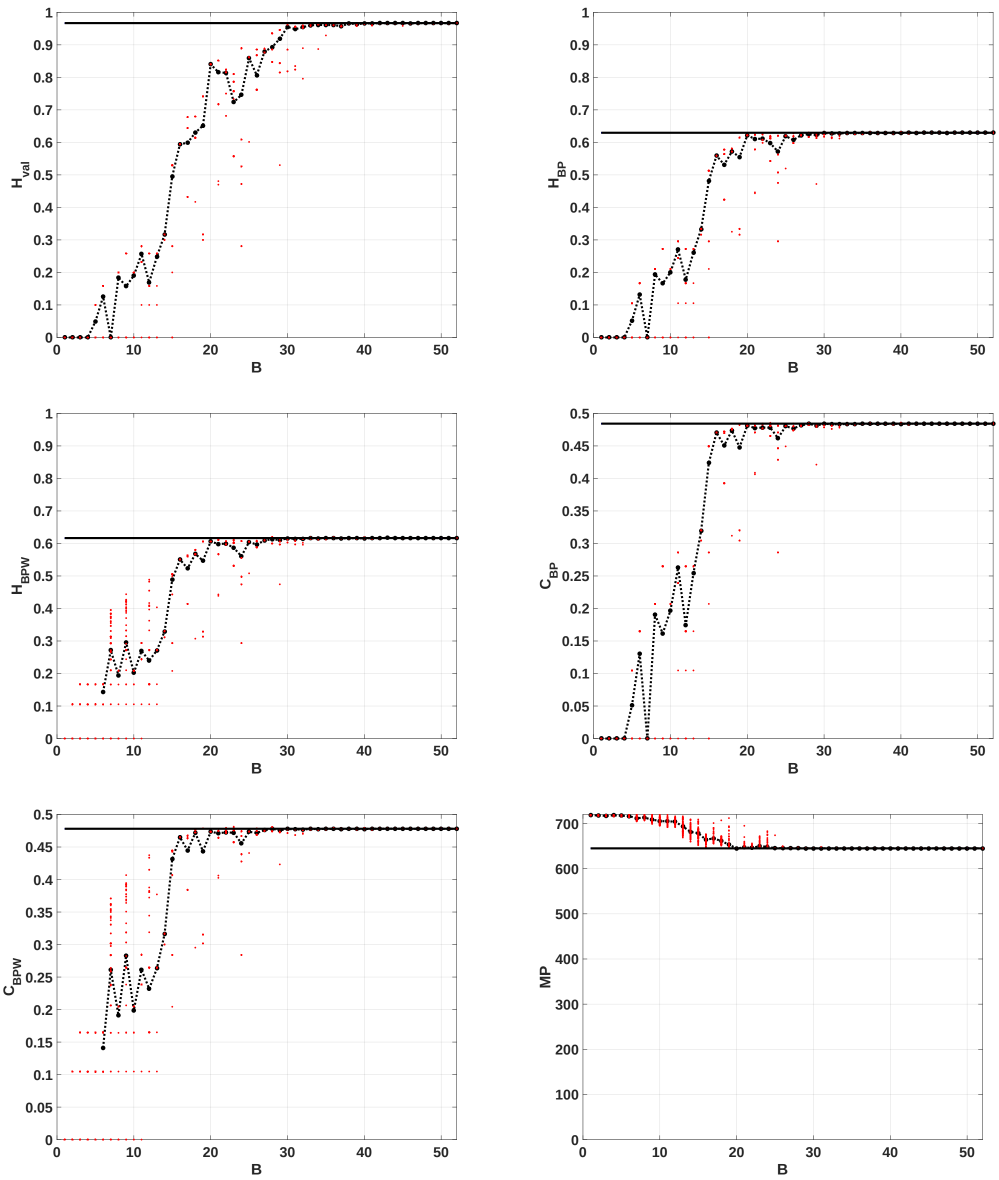

Figure 4. Statistical properties of the LOG map: (a) H_{val} vs B (b) H_{BP} vs B (c) C_{BP} vs B (d) MP vs B.

bations to avoid this drawback of the Tent map. This procedure improves the statistical properties of the time series, but what really happens is that the statistical properties of the random perturbations are mixed with those of the Tent map. Here we study the Tent map "as it is" without any artifact to evaluate its real behavior, instead of theoretical statistical properties.

Figs. 7 (a) to (e) show the quantifiers for floating- and fixed-point numerical representations. Quantifiers H_{val}, H_{BP} and C_{BP} are equal to zero for all precisions, this reflects that the series quickly converge toward a fixed point for almost all sequences. In the case of H_{BPW} and C_{BPW} quantifiers they are

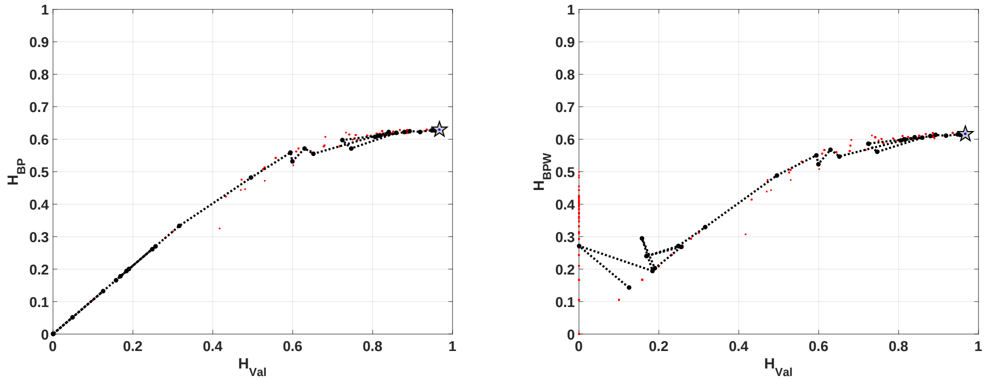

Figure 5. Evolution of statistical properties in double entropy plane of LOG map: (a) H_{val} vs H_{BP} (b) H_{val} vs H_{BPW}.

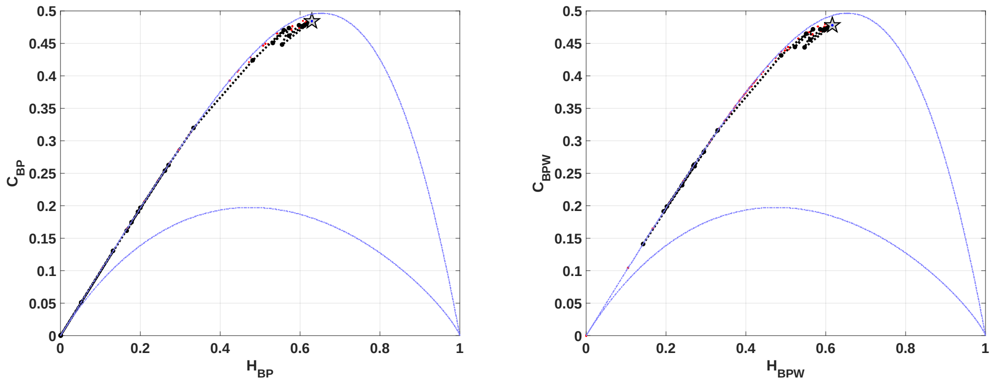

Figure 6. Evolution of statistical properties in entropy-complexity plane of LOG map: (a) C_{BP} vs H_{BP} (b) C_{BPW} vs H_{BPW}.

different from zero because BPW procedure discards the elements once they reach the fixed point. The high dispersions in H_{BPW}, C_{BPW} and MP are related to the short length of transient. These transient that converges to a fixed point has a maximum length of B iterations for fixed-point arithmetic and 54 for floating-point (double precision).

In summary in spite of using a high number of bits (with any 2-based numerical representation) to represent the digitalized TENT map it always loses the chaotic behaviour.

3.3 Quantifiers of combined Maps

Here we report our results for the three combinations of the simple maps, SWITCH, EVEN and ODD.

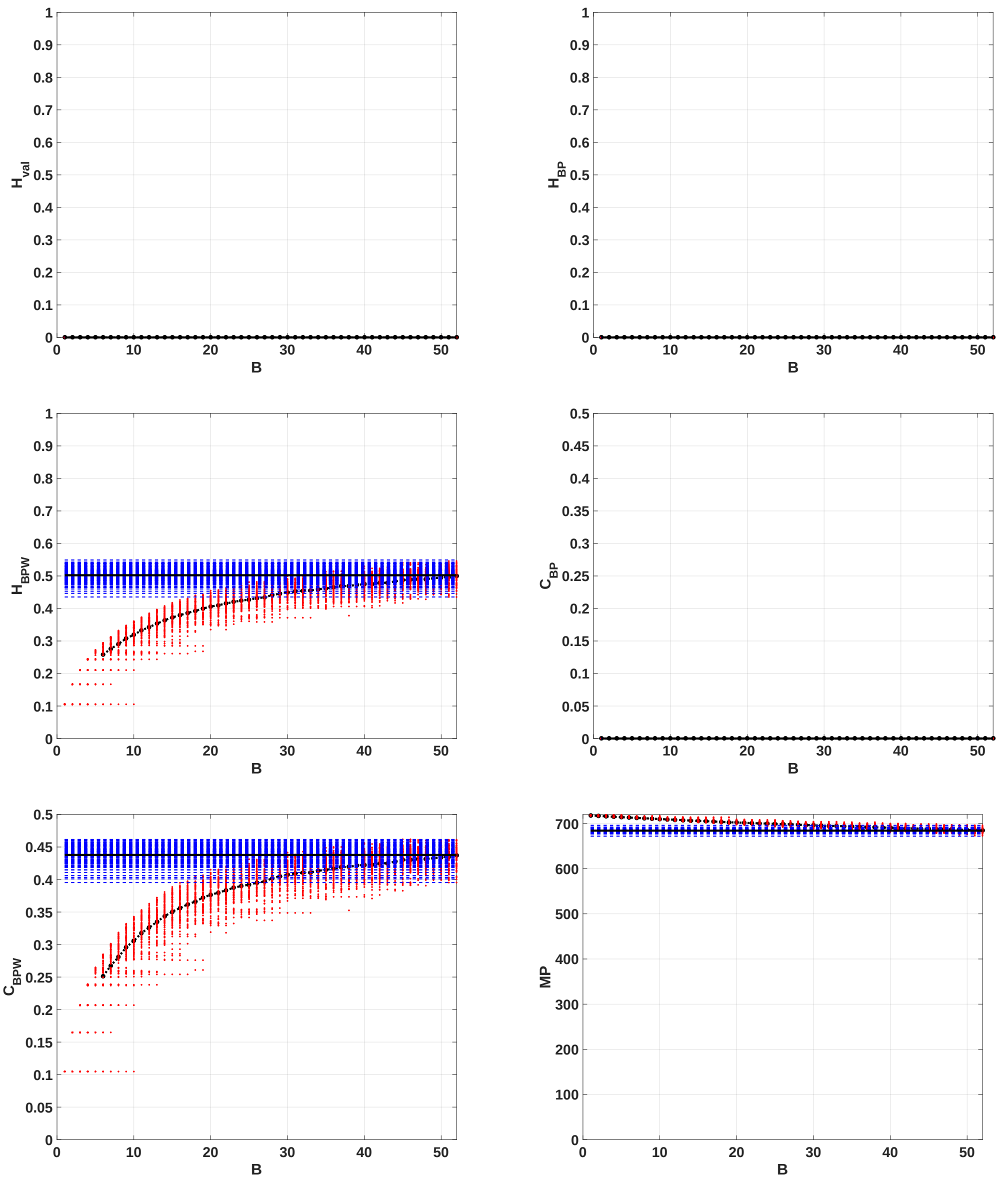

Figure 7. Statistical properties of the TENT map: (a) H_{val} vs B (b) H_{BP} vs B (c) C_{BP} vs B (d) MP vs B.

3.3.1 Sequential switching between Tent and Logistic maps (SWITCH)

Results with sequential switching are shown in Figs. 8 (a) to (f). The entropy value calculated in floating-point is $\langle H_{val} \rangle = 0.9722$, this value is slightly higher than the one obtained for the LOG map. For fixed-point arithmetic this value is reached in $B = 24$, but it stabilizes from $B = 28$. Regarding the ordering patterns the number of MP decreases to 586, this value is lower than the one obtained for LOG map. It means the entropy H_{BP} may increase up to $ln(134)/ln(720) \simeq 0.74$. BP and BPW quantifiers reach their maximum of $\langle H_{BP} \rangle = 0.6546$ and $\langle H_{BPW} \rangle = 0.6313$ at $B = 16$, but they stabilize from $B = 24$. Complexities are lower than for LOG, $\langle C_{BP} \rangle = 0.4580$ and $\langle C_{BPW} \rangle = 0.4578$, these values are reached for $B \geq 15$ but they are stable from $B \geq 23$. Compared with LOG, statistical properties are better with less amount of bits, for $B \geq 24$ this map reaches optimal characteristics in the sense of random source.

Furthermore, we encountred one initial condition in floating-point long double precision with an anomalous behavior. The quantifiers based on BPW procedure cant detect an anomaly, 8 (a), (b) and (d) an horizontal blue dashed line is far from the average value but this can not be seen in the other figures 8 (c) and (e). We detected a falling to a fixed point after a long transitory, the BPW procedure discards the values corresponding with a fixed point and calculates only the transitory.

Double entropy plane H_{val} vs H_{BP} is showed in Fig. 9. The point reached in this plane for SWITCH map is similar to that reached for LOG map. The mixing is slight better in this case.

Entropy-complexity plane C_{BP} vs H_{BP} is showed in Fig. 10. If we compare with the same plane in the case of LOG (Fig. 6. a.), C_{BP} is lower for SWITCH, this fact shows a more random behaviour.

3.3.2 Skipping on sequential switching between Tent and Logistic maps (EVEN and ODD)

In figures 11.a and 12.a, quantifiers related to the normalized histogram of values slightly degrades with the skipping procedure due to finite data length. For example $\langle H_{val} \rangle$ reduces from 0.9722 without skipping to 0.9459 for EVEN and 0.9706 for ODD. This difference between EVEN and ODD in floating point is because a high dispersion was obtained for H_{val}, H_{BP} and C_{BP} but not for H_{BPW} or C_{BPW}.

Figs. 11.b to 11.f. and Figs. 12.b to 12.f show the results of BP and BPW quantifiers for EVEN and ODD respectively. Higher accuracy is required to achieve lower complexity than without using skipping. From the point of view of MP a great improvement is obtained using any of the skipping strategies but ODD is slightly better than EVEN. Missing patterns are reduced to $MP = 118$ for EVEN and ODD, increasing the maximum allowed Bandt & Pompe entropy that reaches the mean value $\langle H_{BP} \rangle = 0.8381$ for EVEN, and $\langle H_{BP} \rangle = 0.9094$. The complexity is reduced to $\langle C_{BP} \rangle = 0.224$ for EVEN and $\langle C_{BP} \rangle = 0.282$ for ODD. The amount of bits necessary to converge to this value is $B > 40$ for both EVEN and ODD maps.

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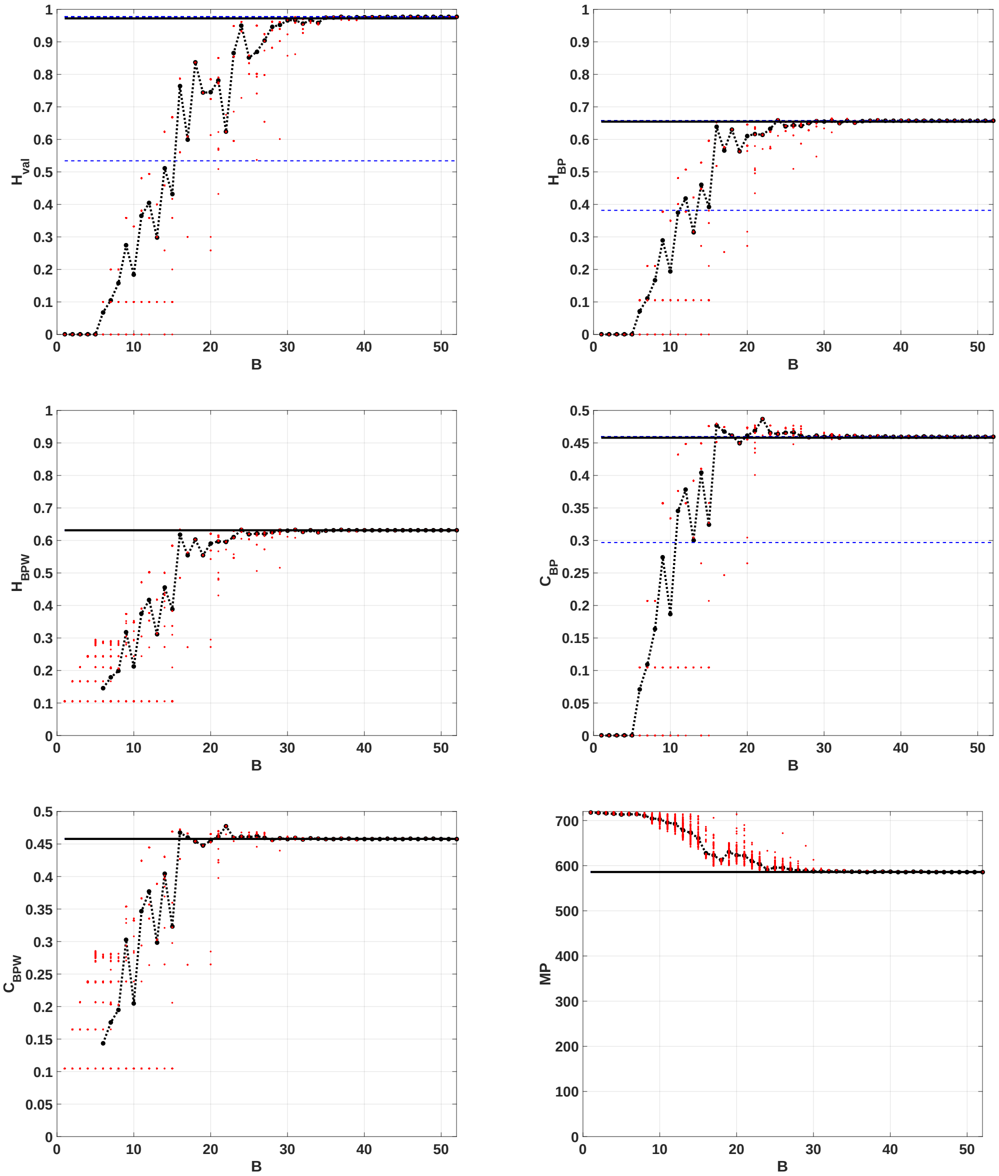

Figure 8. Statistical properties of the SWITCH map: (a) H_{val} vs B (b) H_{BP} vs B (c) C_{BP} vs B (d) MP vs B.

FALTA VER PLANOS ENTROPÍA-COMPLEJIDAD

4 Conclusions

In summary:

- Not every number base can be represented by a device with a certain base. For example, a base ten number can not be exactly represented in a con-

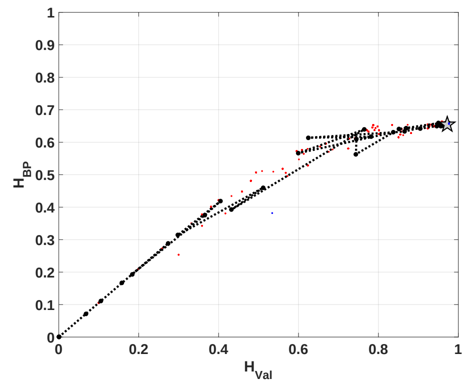

Figure 9. Evolution of statistical properties in double entropy plane of SWITCH map H_{val} vs H_{BP}.

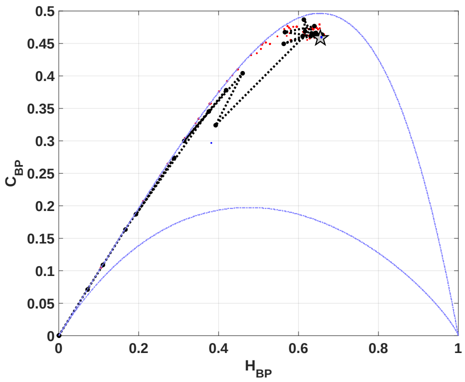

Figure 10. Evolution of statistical properties in entropy-complexity plane of SWITCH map C_{BP}.

ventional computer, it will always have an error inherent of the system..No todas las bases numéricas son representables con una máquina de base distinta. Por ejemplo, no se puede representar la base 10 con base 2.

- En una máquina de cálculo "a medida", como la que puede implementarse en ASICs o FPGAs existen limitaciones en el bus de datos y en la electrónica de cálculo. Si la electrónica de cálculo debe ser reducida se recomienda usar mapas que puedan ser calculados sólo con sumas y restas de la variable pseudoaleatoria.
- Los mapas que sólo tienen operaciones de shifteo en la base de la máquina de cálculo inevitablemente caerán a cero en tantas iteraciones como el largo de la mantisa de representación. Por ejemplo el tent en base 2.
- La comparación entre BP y BPW permite detectar el comportamiento del sistema. Puede detectarse si el atractor cae a un punto fijo y diferenciar si el transitorio es corto o largo, respecto de la cantidad de itaraciones del mapa.
- Como se menciona en el paper de referencia, el período del mapa SWITCH aumenta respecto del simple. También se nota una mejora marginal en la mezcla de la secuencia. La distribución de valores es buena en todos los casos.

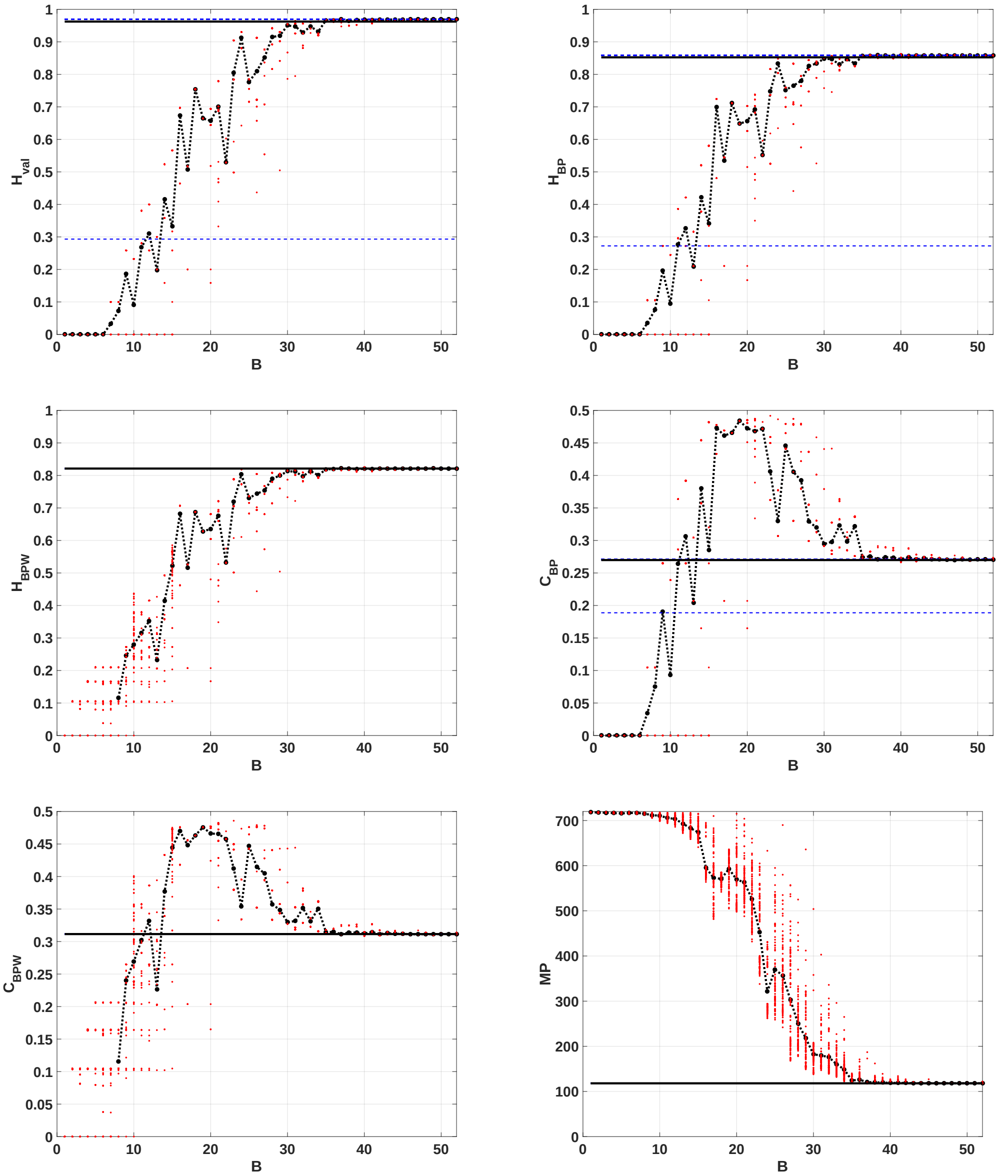

Figure 11. Statistical properties of the EVEN map: (a) H_{val} vs B (b) H_{BP} vs B (c) C_{BP} vs B (d) MP vs B.

- el skipping empeora el período pero mejora sustancialmente la mezcla de los valores. Esto puede verse en BP, BPW y MP.

produces a non-monotonous evolution toward the floating point result. This result is relevant because it shows that increasing the precision is not always recommended. It is specially interesting to note that some systems (TENT) with very nice statistical properties in the world of the real numbers, become "pathological" when binary numerical representations are used.

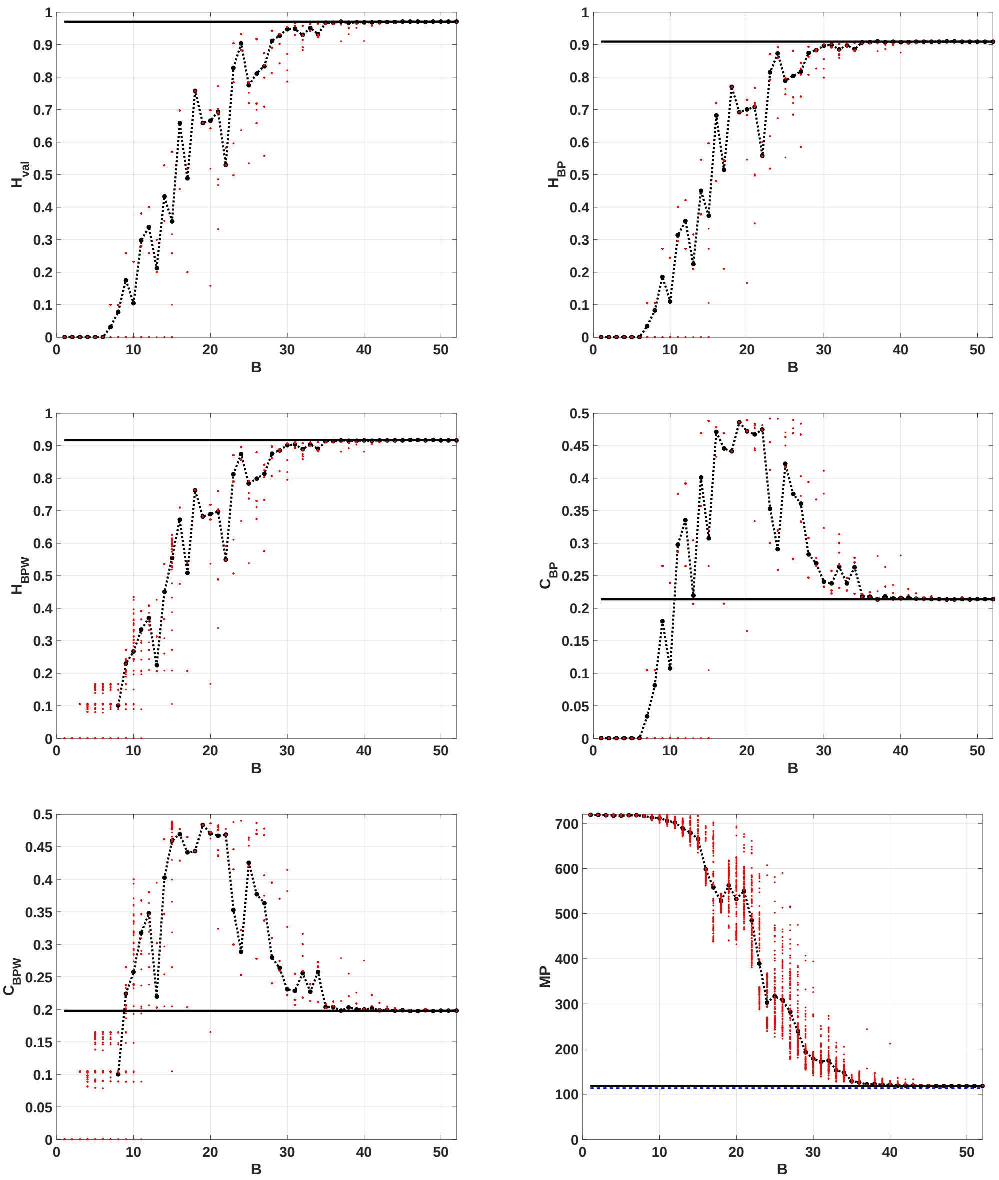

Figure 12. Statistical properties of the ODD map: (a) H_{val} vs B (b) H_{BP} vs B (c) C_{BP} vs B (d) MP vs B.

Acknowledgment

This work was partially financed by CONICET (PIP2008), and UNMDP.

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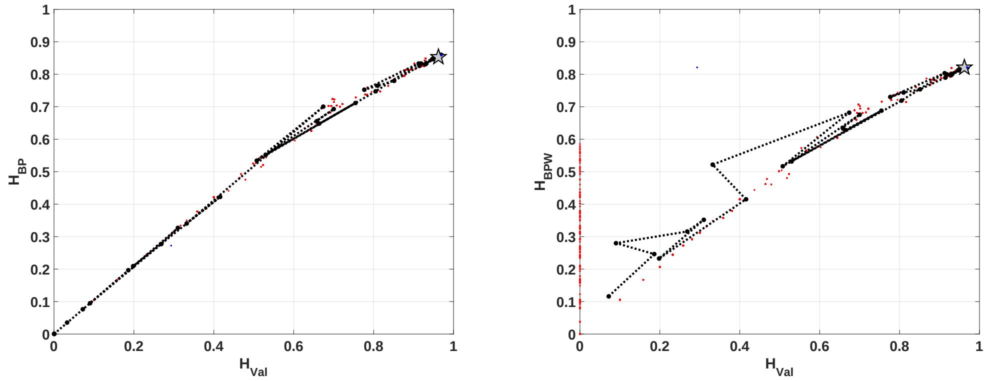

Figure 13. Evolution of statistical properties in double entropy plane of EVEN map: (a) H_{val} vs H_{BP} (b) H_{val} vs H_{BPW}.

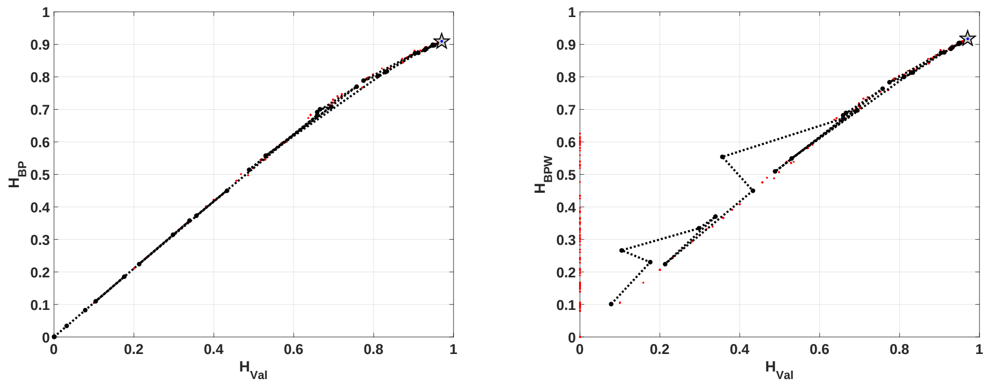

Figure 14. Evolution of statistical properties in double entropy plane of ODD map: (a) H_{val} vs H_{BP} (b) H_{val} vs H_{BPW}.

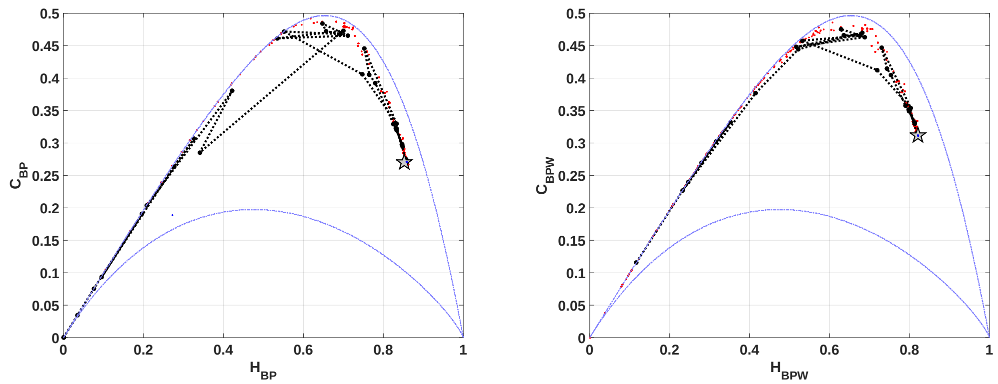

Figure 15. Evolution of statistical properties in entropy-complexity plane of EVEN map: (a) C_{BP} vs H_{BP} (b) C_{BPW} vs H_{BPW}.

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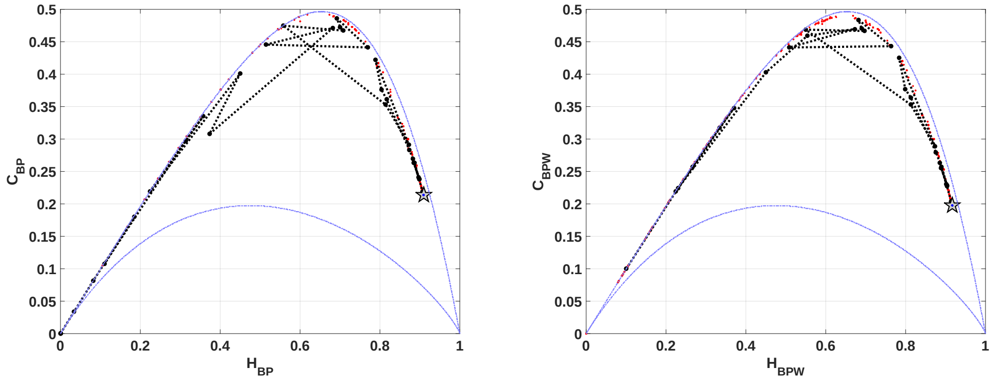

Figure 16. Evolution of statistical properties in entropy-complexity plane of ODD map: (a) C_{BP} vs H_{BP} (b) C_{BPW} vs H_{BPW}.

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