Hamiltonian Mechanics Cheat Sheet

Lagrangian Mechanics

The lagrangian of a system is a function of the coordinates $\vec{q}(t)$, the velocities $\dot{\vec{q}}(t)$, and time.

$$L = T - V \tag{1}$$

The action of a system is the time integral of the lagrangian:

$$S = \int L(\vec{q}(t), \dot{\vec{q}}(t), t)dt \tag{2}$$

By varying the action, and finding a stable point, we get the Euler-Lagrange equations:

$$\delta S = 0 = \int \left(\frac{\partial L}{\partial \vec{q}} \delta \vec{q} + \frac{\partial L}{\partial \dot{\vec{q}}} \delta \dot{\vec{q}} \right) dt = \int \left(\frac{\partial L}{\partial \vec{q}} - \frac{d}{dt} \frac{\partial L}{\partial \dot{\vec{q}}} \right) \delta \vec{q} dt \tag{3}$$

And we get the Euler-Lagrange equations:

$$\frac{\partial L}{\partial \vec{q}} = \frac{d}{dt} \frac{\partial L}{\partial \dot{\vec{q}}} \tag{4}$$

The Euler-Lagrange equations are invariant under a change of the Lagrangian by a total time derivative:

$$L \to L' = L + \frac{dF}{dt} \tag{5}$$

By considering a point transformation to new coordinates $Q, \dot{Q}$: $Q(q,t), \dot{Q}(q, \dot{q}, t)$:

$$\frac{\partial \dot{q}}{\partial \dot{Q}} = \frac{\partial q}{\partial Q} \tag{6}$$

Cancellation of dots.

Generalised Momenta

The Legendre transform conjugate to the velocities $\dot{\vec{q}}$ are the generalised momenta $\vec{p}$:

$$\frac{\partial L}{\partial \dot{\vec{q}}} = \vec{p} \tag{7}$$

If the Lagrangian is independent of a coordinate q, then this is called a cyclic coordinate, and the generalised momentum is conserved:

$$\frac{\partial L}{\partial q} = \frac{d}{dt} \frac{\partial L}{\partial \dot{q}} = \dot{p} = 0 \tag{8}$$

Noether's Theorem

If the Euler-Lagrange equations are invariant under some infinitesimal coordinate transform $\vec{q} \to \vec{q} + \epsilon \vec{w}(s_1, \ldots, s_n, \vec{q}, t)$, then this transform changes the Lagrangian by a total time derivative. We can taylor expand in the infinitesimal ϵ to give:

$$L - \epsilon \frac{dG}{dt} = L + \epsilon \left(\frac{\partial L}{\partial \vec{q}} \cdot \vec{w} + \frac{\partial L}{\partial \dot{\vec{q}}} \cdot \dot{\vec{w}} \right) = L + \epsilon \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{\vec{q}}} \cdot \vec{w} \right) \tag{9}$$

Which implies:

$$I = G + \frac{\partial L}{\partial \dot{\vec{q}}} \cdot \vec{w} = \text{constant} \tag{10}$$

The Hamiltonian

The Hamiltonian is the Legendre transform of the Lagrangian with respect to the velocity.

$$H(\vec{q}, \vec{p}, t) = \frac{\partial L}{\partial \dot{\vec{q}}} \cdot \dot{\vec{q}} - L \tag{11}$$

We can get Hamilton's equations by considering the differential of H:

$$dH = \frac{\partial H}{\partial \vec{q}} \cdot d\vec{q} + \frac{\partial H}{\partial \vec{p}} \cdot d\vec{p} + \frac{\partial H}{\partial t} \tag{12}$$

$$dH = \dot{\vec{q}} \cdot d\vec{p} - \dot{\vec{p}} \cdot d\vec{q} - \frac{\partial L}{\partial t} \tag{13}$$

Equating the coefficients of the differentials, we get Hamilton's equations of motion:

$$\vec{v} = \frac{d}{dt} \begin{pmatrix} \vec{q} \\ \vec{p} \end{pmatrix} = \begin{pmatrix} \nabla_{\vec{p}} H \\ -\nabla_{\vec{q}} H \end{pmatrix} \quad \text{and} \quad \frac{\partial H}{\partial t} = -\frac{\partial L}{\partial t} \tag{14}$$

Where $\vec{v}$ is the phase space velocity. If the Hamiltonian is time-independent, then the motion in phase space lies along paths of constant energy:

$$\vec{v} \cdot \nabla H = \nabla_{\vec{p}} H \cdot \nabla_{\vec{q}} H - \nabla_{\vec{q}} H \cdot \nabla_{\vec{p}} H = 0 \tag{15}$$

Liouville's Theorem

The "fluid" system is incompressible in phase space, and if the Hamiltonian is independent of time then the phase space fluid is constant.
Incompressibility is shown by:

$$\nabla \cdot \vec{v} = \nabla_{\vec{q}} \cdot \nabla_{\vec{p}} H - \nabla_{\vec{p}} \cdot \nabla_{\vec{q}} H = 0 \tag{16}$$

The density ρ obeys the equation:

$$\frac{d\rho}{dt} = \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{v}) = 0 \tag{17}$$

The virial theorem tells us that when averaged over time:

$$\overline{\vec{p} \cdot \frac{\partial H}{\partial \vec{p}}} = \overline{\vec{q} \cdot \frac{\partial H}{\partial \vec{q}}} \tag{18}$$

Qualitative Dynamics

Have systems described by a velocity $\dot{\vec{r}}$. These systems have fixed points at $\dot{\vec{r}} = \vec{0}$, which we call $\vec{r^*}$.
We can expand linearly around these fixed points to find the behaviour of the system:

$$\dot{r}^i(\vec{r^*} + \delta \vec{r}) \approx \dot{r}^i(\vec{r^*}) + \left. \frac{\partial \dot{r}^i}{\partial x^j} \right|_{\vec{r^*}} \delta r^j = A^i_j \delta r^j \tag{19}$$

Where x^i are the coordinates, and the matrix A^i_j is the jacobian of the velocity vector.
We can then look for solutions of $\delta \dot{\vec{r}}(t) = \mathbf{A} \delta \vec{r}$:

$$\lambda w e^{\lambda t} = \mathbf{A} \vec{w} e^{\vec{A} t} \tag{20}$$

And we have an eigenvalue equation:

$$\det(\mathbf{A} - \lambda \mathbf{I}) = 0 \tag{21}$$

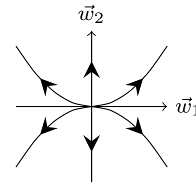
And can work out the eigenvectors:

$$(\mathbf{A} - \lambda_i \mathbf{I}) \vec{w}_i = 0 \tag{22}$$

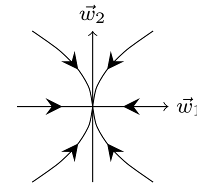
Eigenvalue Behaviours

The behaviour of 2D systems for a range of eigenvalues is illstrated next.

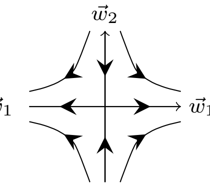
For real and distinct eigenvalues, $\lambda_1 \neq \lambda_2 \neq 0$, the system has the behaviours:



$\boxed{\lambda_1 > \lambda_2 > 0}$



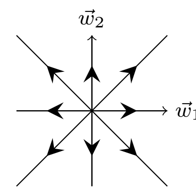
$\boxed{0 > \lambda_1 > \lambda_2}$



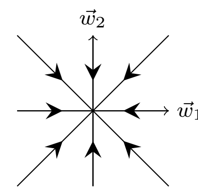
$\boxed{\lambda_1 > 0 > \lambda_2}$

An unstable repeller, a stable attractor and an unstable saddle point.

For real degenerate eigenvalues, $\lambda_1 = \lambda_2 \neq 0$:



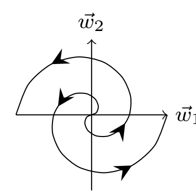
$\boxed{\lambda_1 = \lambda_2 > 0}$



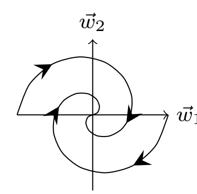
$\boxed{0 > \lambda_1 = \lambda_2}$

An unstable repeller and a fixed attractor.

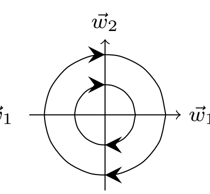
For $\mathbf{A} = \begin{pmatrix} \alpha & -\omega \\ \omega & \alpha \end{pmatrix}$, we have complex eigenvalues $\lambda_{\pm} = \alpha \pm i\omega$, which cause rotations. It is easier to think polar coordinates in this case, with $\delta \dot{r} = \alpha \delta r$ and $\dot{\theta} = \omega$. There are 3 types of fixed point in this case:



$\boxed{\alpha > 0, \omega \neq 0}$



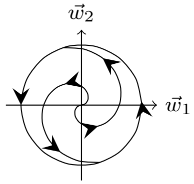
$\boxed{\alpha < 0, \omega \neq 0}$



$\boxed{\alpha = 0, \omega \neq 0}$

Limit Cycles

Can also have limit cycles, where instead of a fixed point, there might be a fixed trajectory that the system is attracted to, eg the circle with constant $r = R$. We would expand around this radius and evaluate the system there:



Hamiltonian Behaviours

For Hamiltonian systems, we have $\lambda_\pm = \pm\sqrt{-|\mathbf{A}|}$, and so there are only **elliptic** fixed points for $|\mathbf{A}| > 0$ and **hyperbolic** fixed points for $|\mathbf{A}| < 0$.

Poisson Brackets

The poisson bracket of two functions f and g is defined in a coordinate system with positions and momenta $\vec{q}, \vec{p}$ as:

$$\{f, g\}_{p,q} = \left(\frac{\partial f}{\partial q^i}\frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q^i}\right) \tag{23}$$

We have the equation of motion of an observable $A(\vec{q}, \vec{p}, t)$:

$$\frac{dA}{dt} = \{A, H\} + \frac{\partial A}{\partial t} \tag{24}$$

Hamilton's equations of motion can be written:

$$\dot{\vec{q}} = \{\vec{q}, H\} \qquad \dot{\vec{p}} = \{\vec{p}, H\} \tag{25}$$

If A is a constant of the motion:

$$\frac{dA}{dt} = 0, \qquad \text{then,} \qquad \{A, H\} = \frac{\partial A}{\partial t} \tag{26}$$

If u and v are constants of motion, then $\{u, v\}$ is a constant of motion too.

Canonical Transformations

Canonical transformations are a transformation of the coordinates $\vec{q}, \vec{p}, H \to \vec{Q}(\vec{q}, \vec{p}, t), \vec{P}(\vec{q}, \vec{p}, t), K$ that satisfies:

$$\dot{\vec{Q}} = \frac{\partial K}{\partial \dot{\vec{P}}} \qquad \dot{\vec{P}} = -\frac{\partial K}{\partial \dot{\vec{Q}}} \tag{27}$$

Using the fact that the Euler-Lagrange equations are invariant under a total time derivative added to the Lagrangian, we find a generating function:

$$\vec{p} \cdot \dot{\vec{q}} - H - (\vec{P} \cdot \dot{\vec{Q}} - K) = \frac{dF_1(\vec{q}, \vec{Q}, t)}{dt} \tag{28}$$

We can multiply equation 28 by dt and look at the differential form of F_1 to see how to get $\vec{p}, \vec{P}$ from it:

$$dF_1 = \vec{p} \cdot d\vec{q} - \vec{P} \cdot d\vec{Q} - (H - K)dt \tag{29}$$

$$dF_1 = \frac{\partial F_1}{\partial \vec{q}} \cdot d\vec{q} + \frac{\partial F_1}{\partial \vec{Q}} \cdot d\vec{Q} + \frac{\partial F_1}{\partial t} dt \tag{30}$$

Equating the coefficients:

$$\vec{p} = \frac{\partial F_1}{\partial \vec{q}} \qquad \vec{P} = -\frac{\partial F_1}{\partial \vec{Q}} \qquad K = H + \frac{\partial F_1}{\partial t} \tag{31}$$

If we have two sets of coordinates: $\vec{\eta} = (\vec{q}, \vec{p})$ and $\vec{\zeta} = (\vec{Q}, \vec{P})$, then the transformation between them is canonical if and only if:

$$\left\{\zeta^i, \zeta^j\right\}_{\vec{\eta}} = \begin{pmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0} & -\mathbf{I} \end{pmatrix} = J^{ij} \tag{32}$$

Where the $\mathbf{I}$ are block identity matrices.
We can use generating functions of different combinations of q, p, Q, P. These are all related by a Legendre transform:

$$F_2(\vec{q}, \vec{P}) = F_1(\vec{q}, \vec{Q}) + \vec{Q} \cdot \vec{P} \tag{33}$$

$$F_3(\vec{p}, \vec{Q}) = F_1(\vec{q}, \vec{Q}) - \vec{q} \cdot \vec{p} \tag{34}$$

$$F_4(\vec{p}, \vec{P}) = F_1(\vec{q}, \vec{Q}) - \vec{q} \cdot \vec{p} + \vec{Q} \cdot \vec{P} \tag{35}$$

Under these generating functions, the equations for the other coordinates are:

$$F_1(\vec{q}, \vec{Q}): \qquad \vec{p} = \frac{\partial F_1}{\partial \vec{q}} \qquad \vec{P} = -\frac{\partial F_1}{\partial \vec{Q}} \tag{36}$$

$$F_2(\vec{q}, \vec{P}): \qquad \vec{p} = \frac{\partial F_2}{\partial \vec{q}} \qquad \vec{Q} = \frac{\partial F_2}{\partial \vec{P}} \tag{37}$$

$$F_3(\vec{p}, \vec{Q}): \qquad \vec{q} = -\frac{\partial F_3}{\partial \vec{p}} \qquad \vec{P} = -\frac{\partial F_3}{\partial \vec{Q}} \tag{38}$$

$$F_4(\vec{p}, \vec{P}): \qquad \vec{q} = -\frac{\partial F_4}{\partial \vec{p}} \qquad \vec{Q} = \frac{\partial F_4}{\partial \vec{P}} \tag{39}$$

Under transformations with generating functions of the form $F_1 = \vec{q} \cdot \vec{Q}$ and $F_4 = \vec{p} \cdot \vec{P}$, we can perform an exchange transformation, where the old coordinates become the new moment and vice-versa. With generating functions of the form $F_2 = \vec{q} \cdot \vec{P}$ and $F_3 = -\vec{p} \cdot \vec{Q}$, we get an identity transformation.

Infinitesimal Canonical Transformations

Consider canonical transformations close to the indentity:

$$F_2(\vec{q}, \vec{P}, t) = \vec{q} \cdot \vec{P} + \epsilon G(\vec{q}, \vec{P}, t) \tag{40}$$

$G(\vec{q}, \vec{P}, t)$ is the generator of the infinitesimal canonical transform. Considering the changes in coordinates and momenta under this transformation, we have:

$$\delta\vec{q} = \vec{Q} - \vec{q} \quad = \epsilon\frac{\partial G(\vec{q}, \vec{P}, t)}{\partial \vec{P}} = \epsilon\frac{\partial G(\vec{q}, \vec{p}, t)}{\partial \vec{p}} + O(\epsilon^2) \tag{41}$$

$$\delta\vec{p} = \vec{P} - \vec{p} \quad = -\epsilon\frac{\partial G(\vec{q}, \vec{P}, t)}{\partial \vec{q}} = -\epsilon\frac{\partial G(\vec{q}, \vec{p}, t)}{\partial \vec{q}} + O(\epsilon^2) \tag{42}$$

If some infinitesimal generator leaves the functional form of the Hamiltonian the same, then $H(\vec{Q}, \vec{P}, t) - H(\vec{q}, \vec{p}, t) = \epsilon\frac{\partial G}{\partial t} = -\epsilon\{G, H\}$, and the infinitesimal generator of the transformation is a constant of motion:

$$\frac{dG}{dt} = \{G, H\} + \frac{\partial G}{\partial t} = 0 \tag{43}$$

We can iterate the poisson bracket of a variable with an infinitesimal canonical transformation to build up finite motions. Let $\hat{G}u = \{G, u\}$, then we can take derivatives of u with:

$$\frac{du}{d\alpha} = -\hat{G}u \tag{44}$$

And can build up the taylor expansion of $u(\alpha)$:

$$u(\alpha) = \sum_{n=0}^{\infty} \frac{(-\alpha)^n}{n!}\hat{G}u(\alpha)|_{\alpha=0} = e^{-\alpha\hat{G}}u(\alpha)|_{\alpha=0} \tag{45}$$

Hamilton-Jacobi Theory

Find a canonical transformation, $S(\vec{q}, \vec{\alpha})$ that takes the coordinates $(\vec{q}, \vec{p})$ to constant coordinates $(\vec{\beta}, \vec{\alpha})$. We take the transformed Hamiltonian, $K(\vec{\beta}, \vec{\alpha}) = 0$ to give the Hamilton-Jacobi equation:

$$H(\vec{q}, \frac{\partial S}{\partial \vec{q}}, t) + \frac{\partial S}{\partial t} = 0 \tag{46}$$

For systems of constant energy with separable coordinates, we can look for solutions of the form:

$$S(\vec{q}, \vec{\alpha}, t) = \sum_i W_i(q_i, \vec{\alpha}) - Et \tag{47}$$

Where each $W_i(q_i, \vec{\alpha})$ is a function of a coordinate and the transformed momenta.

Constants of Motion

Some function of coordinates $I(q, p)$ is a constant of motion if and only if $\{I, H\} = 0$
If a set of constants of motion, $\mathcal{C} = \{I_i(q, p)\}$, all mutually commute ($\{I_i, I_j\} = 0, \forall I_i, I_j \in \mathcal{C}$), then they are said to be in **involution**

Liouville's Theorem on Integrability

If the phase space of some Hamiltonian with f degrees of freedom is bounded, and there are f constants of motion α_i in involution, then the motion in the new coordinates β^i has constant velocity $\dot{\beta}^i = \frac{\partial H(\vec{\alpha})}{\partial \alpha_i} = \omega^i$, and the system evolves on f-dimensional tori in the new coordinates.

Action-Angle Coordinates

Can define an action and angle variable J_i, θ_i for each of the coordinates q_i of a time-independent Hamiltonian:

$$J_i = \frac{1}{2\pi}\oint_{\text{cycle}} p_i dq_i \tag{48}$$

Since the original Hamiltonian was time-independent, the new Hamiltonian has the same numerical value, with the new coordinates $H(\vec{\theta}, \vec{J})$. Since $\vec{J}$ is constant then the new Hamiltonian is a function of $\vec{J}$ only:

$$\frac{dJ_i}{dt} = 0 = \frac{\partial H}{\partial \theta_i} \tag{49}$$

And the frequencies of the angles are:

$$\frac{d\theta^i}{dt} = \omega^i = \frac{\partial H}{\partial J_i} \tag{50}$$

Over one cycle, the angle changes by 2π:

$$\Delta\theta^i = \oint \frac{\partial\theta^i}{\partial q^j}dq^j = \oint \frac{\partial W}{\partial q^j \partial J_i}dq^j = \frac{\partial}{\partial J_i}\oint p^j dq^j = 2\pi \tag{51}$$

Canonical Perturbation Theory

Look at perturbations to the Hamiltonian of the form $H = H_0 + \epsilon H_1$. We first find solutions to the unperturbed Hamiltonian H_0, and the transformed variables $\vec{\alpha}_0, \vec{\beta}_0$. We can then find the progressively higher order corrections with the recursive equations:

$$\dot{\vec{\beta}}_n = \frac{\partial H_1(\vec{\beta}_{n-1}, \vec{\alpha}_{n-1}, t)}{\vec{\alpha}_{n-1}} \tag{52}$$

$$\dot{\vec{\alpha}}_n = -\frac{\partial H_1(\vec{\beta}_{n-1}, \vec{\alpha}_{n-1}, t)}{\vec{\beta}_{n-1}} \tag{53}$$

Where the $(n - 1^{th})$ term contains all terms of lower order.