Sheet4

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Problem 1: Godunov vs. Strang Splitting

$$u_t = (\mathcal{D}_1 + \mathcal{D}_2)\, u$$

a) Godunov: $u_t = \mathcal{D}_1 u$

$u_t = \mathcal{D}_2 u$

Real solution: $u(t+h) = e^{(\mathcal{D}_1 + \mathcal{D}_2)h} u(t)$

$$e^{(\mathcal{D}_1 + \mathcal{D}_2)} = \sum_{n=0}^{\infty} \frac{(\mathcal{D}_1 + \mathcal{D}_2)^n}{n!} = \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{k=0}^{n} \binom{n}{k} \mathcal{D}_1^k \mathcal{D}_2^{n-k}$$

If commutative $[\mathcal{D}_1, \mathcal{D}_2] = 0$

$$= \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{k=0}^{n} \frac{n!}{k!(n-k)!} \mathcal{D}_1^k \mathcal{D}_2^{n-k}$$

$$= \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{1}{k!} \frac{1}{(n-k)!} \mathcal{D}_1^k \mathcal{D}_2^{n-k}$$

$$= \sum_{n=0}^{\infty} \frac{\mathcal{D}_1^n}{n!} \sum_{k=0}^{\infty} \frac{\mathcal{D}_2^k}{k!} = e^{\mathcal{D}_1} e^{\mathcal{D}_2}$$

$$\Rightarrow \hat{u}(t+h) = e^{\mathcal{D}_1 h} e^{\mathcal{D}_2 h} u$$

$$\hat{u} - u = \left(e^{h\mathcal{D}_1} e^{h\mathcal{D}_2} - e^{h(\mathcal{D}_1 + \mathcal{D}_2)}\right) u_0$$

$$= \left(1 + h\mathcal{D}_1 + \frac{h^2}{2}\mathcal{D}_1^2 + \dots\right)\left(1 + h\mathcal{D}_2 + \frac{h^2}{2}\mathcal{D}_2^2\right)$$

$$- \left(1 + h(\mathcal{D}_1 + \mathcal{D}_2) + \frac{h^2}{2}(\mathcal{D}_1 + \mathcal{D}_2)^2 + \dots\right) u_0$$

$$= \frac{h^2}{2}[\mathcal{D}_1, \mathcal{D}_2]\, u_0 + \mathcal{O}(h^3)$$

Strang: $\hat{u}(t+h) = e^{\frac{1}{2}h\mathcal{D}_1} e^{h\mathcal{D}_2} e^{\frac{1}{2}h\mathcal{D}_1} u(t)$

$$\Rightarrow \quad \hat{u} - u(h) = h^3\left(\frac{1}{12}[D_2[D_2, D_1]] - \frac{1}{24}[D_1,[D_1, D_2]]\right)u_0 + \mathcal{O}(h^4)$$

Problem 2: Butcher Tableux for Runge-Kutta Schemes

a) Euler:
$$\begin{array}{c|c} 0 & \\ \hline & 1 \end{array}$$

midpoint:
$$\begin{array}{c|cc} 0 & & \\ 1/2 & 1/2 & \\ \hline & 0 & 1 \end{array}$$

RK4:
$$\begin{array}{c|cccc} 0 & & & & \\ 1/2 & 1/2 & & & \\ 1/2 & 0 & 1/2 & & \\ 1 & 0 & 0 & 1 & \\ \hline & 1/6 & 1/3 & 1/3 & 1/6 \end{array}$$

b) RK3:
$$\begin{array}{c|ccc} 0 & & & \\ 1/2 & 1/2 & & \\ 1 & -1 & 2 & \\ \hline & 1/6 & 2/3 & 1/6 \end{array}$$

$$\rightarrow \quad y_{n+1} = y_n + h\cdot\left(\frac{1}{6}k_1 + \frac{4}{6}k_2 + \frac{1}{6}k_3\right)$$

$$k_1 = f(t_n, y_n)$$
$$k_2 = f\left(t_n + \frac{h}{2}, y_n + \frac{h}{2}k_1\right)$$
$$k_2 = f(t_n + h, y_n - hk_1 + 2hk_2)$$

The idea is to not only include the last time step, but multiple former time steps, to get a much more accurate result.

Problem 2: See Jupyter Notebook