Sheet3

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Problem 1: Von Neumann stability analysis

$$u_j^{n+1} = u_j^n - \frac{\Delta t}{2\Delta x} a \left(u_{j+1}^n - u_{j-1}^n \right)$$

One Fourier component: $\hat{u}^n(\xi) = \exp(-i \xi j \Delta x)\, u_j^n$

let $\phi := \xi j \Delta x$

$$\Rightarrow \quad \hat{u}^{n+1} - \hat{u}^n + \frac{a \Delta t}{2 \Delta x} \hat{u}^n \left(e^{i\phi} - e^{-i\phi} \right) = 0$$

Amplification factor:

$$G(\phi) = \frac{\hat{u}^{n+1}}{\hat{u}^n} = 1 - i \frac{a \Delta t}{\Delta x} \sin\phi$$

The scheme is stable in the L_2-norm $\Leftrightarrow |G| \leq 1 \;\; \forall \phi$.

This is not the case, since

$$|G|^2 = 1 + \left(\frac{a \Delta t}{\Delta x} \right)^2 \sin^2\phi$$

$\Rightarrow$ The scheme is unstable for fixed $\frac{\Delta t}{\Delta x}$

Problem 2: Accuracy and stability

a) $u_t + c u_x = d u$ with $u(x,0) = u_0(x)$ and $c > 0$, $d \in \mathbb{R}$

Method $u_i^{n+1} = u_i^n - \frac{c \Delta t}{\Delta x} \left(u_i^n - u_{i-1}^n \right) + \Delta t\, d\, u_i^n \equiv \mathcal{N}(u^n)$

Exact solution: $u(x_i, t_n)$

Taylor-expansion:

$$u(x_i, t_{n+1}) = u(x_i, t_n) + \frac{\partial u}{\partial t}(x_i, t_n) \Delta t + \frac{1}{2} \frac{\partial^2 u}{\partial t^2}(x_i, t_n) \Delta t^2 + \mathcal{O}(\Delta t^3) \equiv u_i^{n+1}$$

$$u(x_{i-1}, t_n) = u(x_i, t_n) - \frac{\partial u}{\partial x}(x_i, t_n) \Delta x + \frac{1}{2} \frac{\partial^2 u}{\partial x^2}(x_i, t_n) \Delta x^2 + \mathcal{O}(\Delta x^3) \equiv u_{i-1}^n$$

Local truncation error:

$$L^n = \frac{1}{\Delta t} \left(\mathcal{N}(u^n) - u^{n+1} \right)$$

$$\mathcal{N}(u^n) - u^{n+1} = -c \frac{\partial u}{\partial x}(x_i, t_n) \Delta t + \frac{1}{2} c \frac{\partial^2 u}{\partial x^2}(x_i, t_n) \Delta t \Delta x + c \Delta t\, \mathcal{O}(\Delta x^3)$$

$$\mathcal{N}(u^n) - u^{n+1} = -c\frac{\partial u}{\partial x}(x_i,t_n)\Delta t + \frac{1}{2}c\frac{\partial^2 u}{\partial x^2}(x_i,t_n)\Delta t\Delta x + c\Delta t\,\mathcal{O}(\Delta x^2)$$

$$+ \Delta t\, d\, u(x_i,t_n)$$

$$- \frac{\partial u}{\partial t}(x_i,t_n)\Delta t - \frac{1}{2}\frac{\partial^2 u}{\partial t^2}(x_i,t_n)\Delta t^2 + \mathcal{O}(\Delta t^3)$$

Discard higher orders

$$= -\Delta t\left[\underbrace{\frac{\partial u}{\partial t}(x_i,t_n) + c\frac{\partial u}{\partial x}(x_i,t_n) - d\,u(x_i,t_n)}_{=0,\text{ from PDE}} - \mathcal{O}(\Delta t) - \mathcal{O}(\Delta x)\right]$$

$$= \Delta t\left[\mathcal{O}(\Delta t) + \mathcal{O}(\Delta x)\right]$$

$$\Rightarrow \quad L^n = \mathcal{O}(\Delta t) + \mathcal{O}(\Delta x)$$

b) $\| U^n \|_1 = \Delta x \sum_i |U_i^n|$

$$\|\mathcal{N}(U^n) - \mathcal{N}(u^n)\| = \|E^{n+1}\|$$

$$\|E^{n+1}\| \leq \|\mathcal{N}(u^n + E^n) - \mathcal{N}(u^n)\| + \Delta t\|L^n\|$$

$$\leq \|E^n\| + \Delta t\|L^n\|$$

$$\rightarrow \|E^N\| \leq \Delta t\sum_{n=1}^{N}\|L^n\| \qquad (\text{assume } t_0 = 0)$$

L^n is bounded

$$\Rightarrow \|E^N\| \leq T\max_{1\leq n\leq N}\|L^n\| \quad \text{with} \quad T = N\Delta t$$

$$\rightarrow \|E^{n+1}\| \leq (1+\alpha\Delta t)\|E^n\| + \Delta t\|L^n\|$$

$\Rightarrow$ For our scheme:

$$\left|\frac{c\Delta t}{\Delta x}\right| \leq 1$$

c) See jupyter notebook