Sheet2

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Problem 1: Step Size Dilemma

a) Truncation error: $f'(x) - \frac{f(x+h) - f(x)}{h} = -\frac{f''(\xi)}{2} h$

$$\Rightarrow \left| f'(x) - \frac{f(x+h) - f(x)}{h} \right| \leq \frac{h}{2} \max_{\xi \in [x, x+h]} |f''(x)|$$

Round-off errors: $\varepsilon^* \hat{=}$ maximal relative error when substracting

If f is replaced by $\tilde{f}$ after substraction due to finite precision:

$$\tilde{f}(x) = f(x)(1 + \varepsilon_1) \quad \text{with } |\varepsilon_1| \leq \varepsilon^*$$
$$\tilde{f}(x+h) = f(x+h)(1 + \varepsilon_2) \quad \text{with } |\varepsilon_2| \leq \varepsilon^*$$

$$\rightarrow f'(x) - \frac{\tilde{f}(x+h) - \tilde{f}(x)}{h} = f'(x) - \frac{f(x+h) - f(x)}{h} - \frac{f(x+h)\varepsilon_2 - f(x)\varepsilon_1}{h}$$
$$= -\frac{h}{2} f''(\xi) - \frac{f(x+h)\varepsilon_2 - f(x)\varepsilon_1}{h}$$

$$\Rightarrow \left| f'(x) - \frac{\tilde{f}(x+h) - \tilde{f}(x)}{h} \right| \approx \left| -\frac{h}{2} f''(\xi) - \frac{\varepsilon_2 - \varepsilon_1}{h} f(\xi) \right|$$

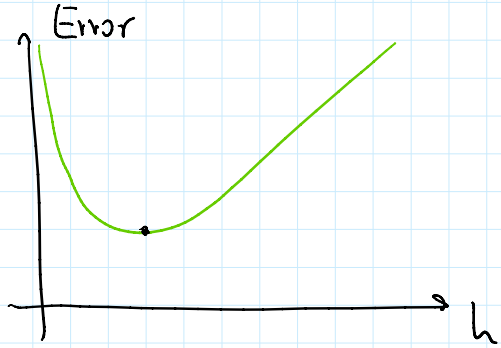
use $f(x+h) \approx f(x)$

$$\leq \frac{h}{2} |f''(\xi)| + \frac{|\varepsilon_2 - \varepsilon_1|}{2} |f(\xi)|$$
$$\leq \frac{h}{2} |f''(\xi)| + \frac{|\varepsilon_2| + |\varepsilon_1|}{h} |f(\xi)|$$
$$\leq \frac{h}{2} |f''(\xi)| + \frac{2\varepsilon^*}{h} |f(\xi)|$$

$\Rightarrow$ total error:

$$\left| f'(x) - \frac{\tilde{f}(x+h) - \tilde{f}(x)}{h} \right| \leq \underbrace{\frac{h}{2} \max_{\xi \in [x, x+h]} |f''(\xi)|}_{\text{Truncation}} + \underbrace{\frac{2\varepsilon^*}{h} \max_{\xi \in [x, x+h]} |f(\xi)|}_{\text{roundoff}}$$

Sketch:



b) $|f'_{true}(d) - TD| \leq \frac{\varepsilon |F_\varepsilon| + \delta |F_\delta|}{h^d} + |C_n| h^n$

$\frac{d}{dh}$ $\overset{!}{=}$ $0 = -d \frac{\varepsilon |F_\varepsilon| + \delta |F_\delta|}{h^{d+1}} + n |C_n| h^{n-1}$

$\Leftrightarrow \quad \frac{\varepsilon |F_\varepsilon| + \delta |F_\delta|}{h^{d+1}} = \frac{n}{d} |C_n| h^{n-1} \quad | \cdot h^{d+1}$

$\Leftrightarrow \quad \varepsilon |F_\varepsilon| + \delta |F_\delta| = \frac{n}{d} |C_n| h^{n+d}$

$\Rightarrow \quad h_{opt} = \left(\frac{\varepsilon |F_\varepsilon| + \delta |F_\delta|}{n/d \cdot |C_n|} \right)^{1/(n+d)}$

c) Forward: $|F_\delta| = \max_{\xi \in [x, x+h]} |f(\xi)| \cdot |F_\varepsilon|$

$|C_n| = \max_{\xi \in [x\ x+h]} |f''(\xi)|$

$\Rightarrow \quad h_{opt} = 2 \cdot \sqrt{\frac{(\varepsilon + \delta) \max_{\xi \in [x, x+h]} |f(\xi)|}{\max_{\xi \in [x, x+h]} |f''(\xi)|}}$

Central: $|F_\varepsilon| = \max_{\xi \in [x-h, x+h]} |f(\xi)| = |F_\delta|$

$|C_n| = \max_{\xi \in [x-h, x+h]} |f'''(\xi)|$

$\Rightarrow \quad h_{opt} = 3 \sqrt{\frac{3(\varepsilon + \delta) \max_{\xi \in [x-h, x+h]} |f(\xi)|}{\ }}$

$$\Rightarrow h_{opt} = \sqrt[3]{\frac{3(\Sigma+\delta)\max_{\xi\in[x-h,x+h]}|f(\xi)|}{\max_{\xi\in[x-h,x+h]}|f'''(\xi)|}}$$

d) $f(x) = \exp(x)$ at $x = 0 \Rightarrow f'(x=0) = 1$

Forward: $h_{opt} = 2\cdot\sqrt{\frac{2\Sigma\exp(x)}{\exp(x)}} \approx 3\cdot 10^{-8}$

Backward: $h_{opt} \approx 3\cdot 10^{-8}$

Central: $h_{opt} = \sqrt[3]{\frac{6\Sigma\exp(x)}{\exp(x)}} \approx 9\cdot 10^{-6}$

Comparing this to the plot last time, this was expected. Forward and backward have the same order of accuracy whilst the central scheme is of higher order and hence achieve the optimal stepsize faster.

Problem 2: Lax-Friedrichs Method

Rewrite scheme as: $\frac{u_j^{n+1} - \frac{1}{2}(u_{j-1}^n + u_{j+1}^n) + a\frac{(u_{j+1}^n - u_{j-1}^n)}{2\Delta x}}{\Delta t} = 0$

local truncation error at (x_j, t_n):

$$\tau_j^n = \left\{u(x_j, t_{n+1}) - \frac{(u(x_{j-1},t_n) + u(x_{j+1},t_n))}{2}\right\}\cdot\frac{1}{\Delta t}$$

$$+ a\frac{\{u(x_{j+1},t_n) - u(x_{j-1},t_n)\}}{2\Delta x} \quad (*)$$

Taylor series expansion around (x_j, t_n)

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$$u(x_j, t_{n+1}) = u(x_j, t_n) + \Delta t \frac{\partial u}{\partial t}(x_j, t_n) + \frac{(\Delta t)^2}{2} \frac{\partial^2 u}{\partial t^2}(x_j, t_n) + \mathcal{O}(\Delta t^3)$$

$$u(x_{j+1}, t_n) = u(x_j, t_n) + \Delta x \frac{\partial u}{\partial x}(x_j, t_n) + \frac{(\Delta x)^2}{2} \frac{\partial^2 u}{\partial x^2}(x_j, t_n) + \mathcal{O}(\Delta x^3)$$

$$u(x_{j-1}, t_n) = u(x_j, t_n) - \Delta x \frac{\partial u}{\partial x}(x_j, t_n) + \frac{(\Delta x)^2}{2} \frac{\partial u}{\partial x^2}(x_j, t_n) + \mathcal{O}(\Delta x^3)$$

Insert in $(*)$:

$$\tau_j^n = \frac{\partial u}{\partial t}(x_j, t_n) + \frac{\Delta t}{2} \frac{\partial^2 u}{\partial t^2}(x_j, t_n) + \mathcal{O}(\Delta t^2)$$

$$- \frac{(\Delta x)^2}{2 \Delta t} \frac{\partial^2 u}{\partial x^2}(x_j, t_n) + \frac{\mathcal{O}(\Delta x^3)}{\Delta t}$$

$$+ a \frac{\partial u}{\partial x}(x_j, t_n) + \mathcal{O}(\Delta x^2)$$

u is exact, since $u_t + a u_x = 0$. Let $\alpha = \frac{a \Delta t}{\Delta x} = \text{const}$

The local truncation error then is

$$\tau_j^n = \frac{\Delta t}{2} \frac{\partial^2 u}{\partial t^2}(x_j, t_n) - \frac{a \Delta x}{2 \alpha} \frac{\partial^2 u}{\partial x^2}(x_j, t_n) + \mathcal{O}(\Delta t^2) + \mathcal{O}(\Delta x^2)$$

$$= \mathcal{O}(\Delta x) + \mathcal{O}(\Delta t)$$

b) Retaining error terms of the next spatial order would solve the wave equation $u_{tt} - c^2 u_{xx} = 0$

c) See Jupyter notebook

For $\Delta x = \Delta t \ll 1$ the Lax-Friedrichs method is stable for $|a| \leq 1$. In general $|\alpha| \leq 1$ for it to be stable, otherwise the solution shows strong diverging

stable, otherwise the solution shows strong diverging oscillations.