Exercise sheet 5

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Pertubated quantum mechanical oscillator

Consider a dimensionless Hamiltonian

$$h = \frac{H}{\hbar\omega} = \left(\frac{1}{2}\Pi^2 + \frac{1}{2}Q^2 + \lambda Q^4\right) \tag{1}$$

$$(h)_{nm} = (h_0)_{nm} + \lambda(Q^4)_{nm} \tag{2}$$

where

$$(h_0)_{nm} = \left(n + \frac{1}{2}\right)\delta_{nm} \tag{3}$$

is the unperturbated Hamiltonian

a) Determining the matrix form of Q^4

Using

$$Q_{nm} = \frac{1}{\sqrt{2}}\left(\sqrt{n+1}\delta_{n,m-1} + \sqrt{n}\delta_{n,m+1}\right) \tag{4}$$

and the properties of the creation and annihilation operator a and $a^\dagger$ we can obtain Q^4 with the following approach:

$$\begin{aligned}
(a+a^\dagger)|n\rangle =& \sqrt{n+1}|n+1\rangle + \sqrt{n}|n-1\rangle \\
(a+a^\dagger)^2|n\rangle =& \sqrt{(n+1)(n+2)}|n+2\rangle + (2n+1)|n\rangle + \sqrt{n(n-1)}|n-2\rangle \\
(a+a^\dagger)^3|n\rangle =& \sqrt{(n+1)(n+2)(n+3)}|n+3\rangle + (3n+3)\sqrt{n+1}|n+1\rangle \\
&+ 3n\sqrt{n}|n-1\rangle + \sqrt{n(n-1)(n-2)}|n-3\rangle \\
(a+a^\dagger)^4|n\rangle =& \sqrt{(n+1)(n+2)(n+3)(n+4)}|n+4\rangle + (4n+6)\sqrt{(n+1)(n+2)}|n+2\rangle \\
&+ (6n^2+6n+3)|n\rangle + (4n-2)\sqrt{n(n-1)}|n-2\rangle + \sqrt{n(n-1)(n-2)(n-3)}|n-4\rangle
\end{aligned}$$

This way we can finally derive the matrix representation of this expression by multiplying with the dual vector

$$(Q^4)_{mn} = \langle m|(a+a^\dagger)^4|n\rangle \tag{5}$$

and using the relation

$$\langle m|n\rangle = \delta_{mn} \tag{6}$$

we can find

$$\begin{aligned}
(Q^4)_{mn} =& \sqrt{(n+1)(n+2)(n+3)(n+4)}\delta_{m,n+4} && (7)\\
&+ (4n+6)\sqrt{(n+1)(n+2)}\delta_{m,n+2} && (8)\\
&+ (6n^2+6n+3)\delta_{m,n} && (9)\\
&+ (4n-2)\sqrt{n(n-1)}\delta_{m,n-2} && (10)\\
&+ \sqrt{n(n-1)(n-2)(n-3)}\delta_{m,n-4} && (11)
\end{aligned}$$

b) Eigenvalues of the pertubated oscillator

In order to find the eigenvalues of the pertubated oscillator, we started by defining the matrix Q^4, adding it to the unpertubated Hamiltonian, finding the tridiagonal form and finally finding the eigenvalues of the pertubated Hamiltonian using the off-diagonal elements of the tridiagonal matrix. λ has been set to $\lambda = 0.1$
Eigenvalues for $n = 15$:

$$\begin{aligned}
n = 0 : E_n &= 0.669 \\
n = 1 : E_n &= 2.217 \\
n = 2 : E_n &= 4.104 \\
n = 3 : E_n &= 6.218 \\
n = 4 : E_n &= 8.521 \\
n = 5 : E_n &= 11.302 \\
n = 6 : E_n &= 14.582 \\
n = 7 : E_n &= 21.075 \\
n = 8 : E_n &= 27.266 \\
n = 9 : E_n &= 42.837
\end{aligned}$$

Eigenvalues for $n = 20$:

$$\begin{aligned}
n = 0 : E_n &= 0.669 \\
n = 1 : E_n &= 2.217 \\
n = 2 : E_n &= 4.103 \\
n = 3 : E_n &= 6.216 \\
n = 4 : E_n &= 8.514 \\
n = 5 : E_n &= 10.966 \\
n = 6 : E_n &= 13.644 \\
n = 7 : E_n &= 16.654 \\
n = 8 : E_n &= 21.458 \\
n = 9 : E_n &= 26.497
\end{aligned}$$

Eigenvalues for $n = 30$:

$$\begin{aligned}
n = 0 : E_n &= 0.669 \\
n = 1 : E_n &= 2.217 \\
n = 2 : E_n &= 4.103 \\
n = 3 : E_n &= 6.216 \\
n = 4 : E_n &= 8.511 \\
n = 5 : E_n &= 10.963 \\
n = 6 : E_n &= 13.554 \\
n = 7 : E_n &= 16.268 \\
n = 8 : E_n &= 19.095 \\
n = 9 : E_n &= 22.065
\end{aligned}$$

c) Analytical solution

To derive the analytical solution we use pertubation theory

$$\langle n|(a+a^\dagger)^4|n\rangle = 6(n^2+n+1/2) \tag{12}$$

therefore we get a correction of the eigenvalues of

$$E_n' = \lambda 6(n^2+n+1/2) \tag{13}$$

So finally we get

$$E_n = E_0 + E_n' = \left(n+\frac{1}{2}\right) + \lambda 6(n^2+n+1/2) \tag{14}$$

This yields

$$\begin{aligned}
n = 0 &: E_n = 0.8\\
n = 1 &: E_n = 3\\
n = 2 &: E_n = 6.4\\
n = 3 &: E_n = 11\\
n = 4 &: E_n = 16.8\\
n = 5 &: E_n = 23.8\\
n = 6 &: E_n = 32\\
n = 7 &: E_n = 41.4\\
n = 8 &: E_n = 52.\\
n = 9 &: E_n = 63.8
\end{aligned}$$