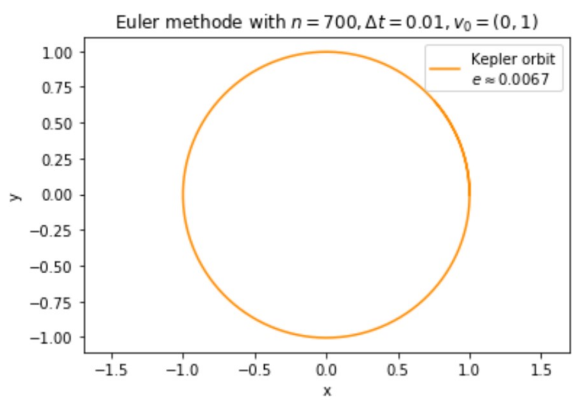
In [87]:
```
%matplotlib inline
import numpy as np
import matplotlib.pyplot as plt

#Create global arrays for error-plot
dE = np.zeros(3)
time = np.zeros(3)
```

In [88]:
```
#Defining function to do step-by-step euler integration
#n = steps
#x = initial x coord.
#y = initial y coord.
#w_x = initial x velocity
#w_y = initial y velocity
#h = stepsize/timeinterval
#title = name for saving the figure
def euler(n,x,y,w_x,w_y,h,title):

    #Create empty arrays in order to save the trajectory coordinates
    X = np.zeros(n)
    Y = np.zeros(n)

    #For-loop for integrating
    for i in range(n):

        r = np.sqrt(x**2+y**2) #Distance between the two bodies

        X[i] = x #save value to array
        w_x = w_x - x/(r)**3 * h #calculate new velocity
        x = x + w_x * h #calculate new coordinate

        Y[i] = y #save value to array
        w_y = w_y - y/(r)**3 * h #calculate new velocity
        y = y + w_y * h #calculate new coordinate

        #Calculate eccentricity
        s = np.array([x,y,0]) #3D position vector to do cross-product
        w = np.array([w_x,w_y,0]) #3D velocity vector
        e = np.cross(w,np.cross(s,w))-s #Runge-Lenz-vector
        if i==n-1: print("Eccentrity ~",np.round(np.linalg.norm(e),6)) #Print value

        #Calculate energy
        E = (w_x**2+w_y**2)/2+1/np.sqrt(x**2+y**2) #Energy
        e_i=np.abs(E-E_0)/np.abs(E_0) #Relative error
        if i==n-1: print("Energy =",E,", e_i =",e_i) #Print out

    #Plot
    plt.plot(X,Y, color="darkorange", label="Kepler orbit\n$e\\approx${}".format(np.round(
np.linalg.norm(e),4)))
    plt.title("Euler methode with $n={}, \\Delta t={}, v_0=(0,{})$".format(n,h,w_0))
    plt.axis("equal")
    plt.legend()
    plt.xlabel("x")
    plt.ylabel("y")
    plt.savefig(title)
```

In [89]:
```
#Initializing values
title = "euler.pdf"
n = 700
x = 1
y = 0
w_x = 0
w_y = 1
h = 0.01
e=0 #Intialize eccentricity
w_0=w_y #Initialize velocity

#Calculate initial Energy
E_0 = (w_x**2+w_y**2)/2+1/np.sqrt(x**2+y**2)
print("Initial energy E_0 =",E_0)

euler(n,x,y,w_x,w_y,h,title)
```

```
Initial energy E_0 = 1.5
Eccentricity ~ 0.006688
Energy = 1.5065798384798426 , e_i = 0.004386558986561706
```

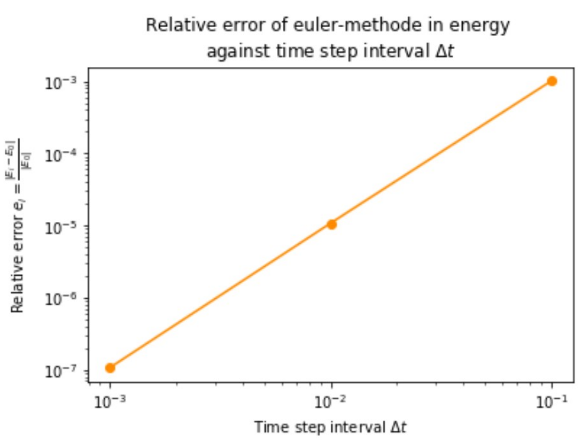


In [90]:
```
#Function to calculate relative error in energy
#t = loop-variable
def error(n,x,y,w_x,w_y,h,t):

    for i in range(n):
        #Step-by-step euler
        r = np.sqrt(x**2+y**2)

        w_x = w_x - x/(r)**3 * h
        x = x + w_x * h

        w_y = w_y - y/(r)**3 * h
        y = y + w_y * h

        #Eccentricity
        s = np.array([x,y,0])
        w = np.array([w_x,w_y,0])
        e = np.cross(w,np.cross(s,w))-s
        if i==n-1: print("Eccentrity ~",np.round(np.linalg.norm(e),6))

        #Energy
        E = np.linalg.norm(w)**2/2+1/np.linalg.norm(w)
        e_i=np.abs(E-E_0)/np.abs(E_0)
        if i==n-1: print("Energy =",E,", e_i =",e_i)
        dE[t] = e_i #Save energy in an array
        time[t] = h #Save stepsize in an array
```

In [91]:

```
#Initializing values
n=70
w_y=1
h=0.1

#Iterate over different steps and stepsizes
for t in range(3):
    E_0 = np.linalg.norm(np.array([w_x,w_y]))**2/2+1/np.linalg.norm(np.array([x,y]))
    print("Initial energy E_0 =",E_0)
    error(n*10**t,x,y,w_x,w_y,h*10**-t,t)

#Print values
print(time,dE)

#Plot
plt.plot(time,dE, "o-", color="darkorange")
plt.xscale("log")
plt.yscale("log")
plt.title("Relative error of euler-methode in energy \nagainst time step interval $\\Delta t$")
plt.xlabel("Time step interval $\\Delta t$")
plt.ylabel("Relative error $e_i=\\frac{\\|E_i-E_0\\|}{\\|E_0\\|}$")
plt.savefig("error_euler.pdf")
```

```
Initial energy E_0 = 1.5
Eccentrity ~ 0.065072
Energy = 1.501506931353755 , e_i = 0.001004620902503343
Initial energy E_0 = 1.5
Eccentrity ~ 0.006688
Energy = 1.5000161773479164 , e_i = 1.0784898610912327e-05
Initial energy E_0 = 1.5
Eccentrity ~ 0.000668
Energy = 1.5000001618634093 , e_i = 1.0790893956169612e-07
[0.1   0.01  0.001] [1.00462090e-03 1.07848986e-05 1.07908940e-07]
```

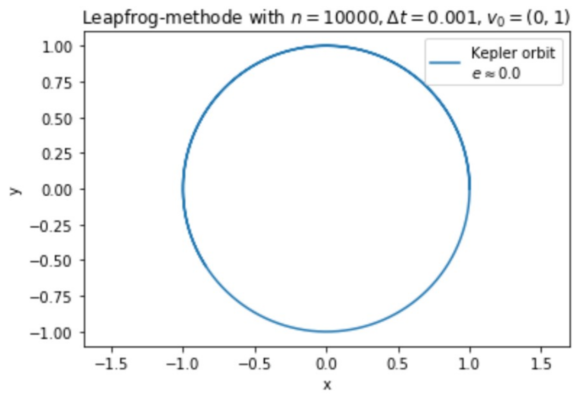


In [92]:

```
#Leapfrog methode
#n = steps
#x0 = initial x coord.
#y0 = initial y coord.
#v_x0 = initial x velocity
#v_y0 = initial y velocity
#dt = stepsize/time-interval
def leapfrog(n,x0,y0,v_x0,v_y0,dt):

    #Empty arrays
    x = np.zeros(n)
    y = np.zeros(n)
    v_x = np.zeros(n)
    v_y = np.zeros(n)
    E = np.zeros(n)
    r = np.zeros(n)

    #Giving inital values
    x[0] = x0
    y[0] = y0
    v_x[0] = v_x0
    v_y[0] = v_y0

    #Leapfrog-implementation
    for i in range(n-1):

        #Distance between the bodies
        r[i] = np.sqrt(x[i]**2+y[i]**2)

        #Calculate new coordinates
        x[i+1] = x[i]+v_x[i]*dt-0.5*x[i]/r[i]**3*dt**2
        y[i+1] = y[i]+v_y[i]*dt-0.5*y[i]/r[i]**3*dt**2

        #calculate new distance
        r[i+1] = np.sqrt(x[i+1]**2+y[i+1]**2)

        #calculate new velocity
        v_x[i+1] = v_x[i]+0.5*(-x[i]/r[i]**3-x[i+1]/r[i+1]**3)*dt
        v_y[i+1] = v_y[i]+0.5*(-y[i]/r[i]**3-y[i+1]/r[i+1]**3)*dt

        #Calculate energy
        E[i] = (v_x[i]**2+v_y[i]**2)-1/np.sqrt(x[i]**2+y[i]**2)

        #Calculate eccentricity
        s = np.array([x[i],y[i],0])
        w = np.array([v_x[i],v_y[i],0])
        e = np.cross(w,np.cross(s,w))-s
        if i==n-2: print("Eccentrity ~",np.round(np.linalg.norm(e),6))

    #Plot
    plt.plot(x,y, label="Kepler orbit\n$e\\approx${}".format(np.round(np.linalg.norm(e),4)
))
    plt.title("Leapfrog-methode with $n={}, \\Delta t={}, v_0=(0,{})$".format(n,dt,v_y0))
    plt.axis("equal")
    plt.legend()
    plt.xlabel("x")
    plt.ylabel("y")
    plt.savefig("leap.pdf")
```

In [96]:
```
#Initializing values
n = 10000
x0 = 1
y0 = 0
v_x0 = 0
v_y0 = 1
dt = 0.001

leapfrog(n,x0,y0,v_x0,v_y0,dt)
```

```
Eccentrity ~ 1e-06
```



In [94]:
```
#Function to calculate error in leapfrog-methode
def error_leap(n,x0,y0,v_x0,v_y0,dt):

    #Empty arrays
    x = np.zeros(n)
    y = np.zeros(n)
    v_x = np.zeros(n)
    v_y = np.zeros(n)
    E = np.zeros(n)
    r = np.zeros(n)

    #Setting initial values
    x[0] = x0
    y[0] = y0
    v_x[0] = v_x0
    v_y[0] = v_y0

    #Leapfrog
    for i in range(n-1):
        E[i] = (v_x[i]**2+v_y[i]**2)/2-1/np.sqrt(x[i]**2+y[i]**2)

        r[i] = np.sqrt(x[i]**2+y[i]**2)

        x[i+1] = x[i]+v_x[i]*dt-0.5*x[i]/r[i]**3*dt**2
        y[i+1] = y[i]+v_y[i]*dt-0.5*y[i]/r[i]**3*dt**2

        r[i+1] = np.sqrt(x[i+1]**2+y[i+1]**2)

        v_x[i+1] = v_x[i]+0.5*(-x[i]/r[i]**3-x[i+1]/r[i+1]**3)*dt
        v_y[i+1] = v_y[i]+0.5*(-y[i]/r[i]**3-y[i+1]/r[i+1]**3)*dt

    dE[s]=np.abs(E[n-2]-E[0])/np.abs(E[0]) #Calculate relative error and save to array
    time[s]=dt #save timestep to array
```

In [95]:
```
#Iterate for different steps and stepsize
for s in range(3):
    error_leap(100*10**s,1,0,0,1.2,0.1*10**-s)

#Print out values
print(time)
print(dE)

#Plot
plt.plot(time,dE,"o-")
plt.xscale("log")
plt.yscale("log")
plt.title("Relative error of leapfrog-methode in energy \nagainst time step interval $\\Delta t$")
plt.xlabel("Time step interval $\\Delta t$")
plt.ylabel("Relative error $e_i=\\frac{\\|E_i-E_0\\|}{\\|E_0\\|}$")
plt.savefig("error_leap.pdf")
```

```
[0.1   0.01  0.001]
[2.80696651e-03 2.80006008e-05 2.80009355e-07]
```