Exercises sheet 6

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Exercise 2

i) For the given formula of the time derivative of a population

$$\frac{dN}{dt} = rN(1 - N/K) - \frac{BN^2}{A^2 + N^2} \tag{1}$$

to begin analyzing the postive Parameters r, K, A and B we started by requiring the dimension of N being the number of inhabitants ("$[N] \equiv \#$") of the population. Thus we find that:

$$[\frac{dN}{dt}] = \#t^{-1} = [r]\#(1 - \frac{\#}{[K]}) - [B]\frac{\#^2}{[A]^2 + \#^2} \tag{2}$$

We assume that K gives us the maximum capacity of the habitat $\implies [K] = \#$

$$\Leftrightarrow \#t^{-1} = [r]\# - [B]\frac{\#^2}{[A]^2 + \#^2} \implies [r] = t^{-1} \wedge [A] = \# \tag{3}$$

Which gives us a second (predatory/harmful) population A inside the habitat.

$$\implies [B] = \#t^{-1}$$

With B being the rate of the external reduction of the observed poplation N through A scaled by the factor $\frac{N^2}{A^2+N^2}$
To convert the formula into a non-dimensional form we'll use $n = \frac{N}{A}$ as non-dimensional form of N. Since we are required to define the non-dimensional t without r we have to use B. We decide to define τ as non-dim. t as:

$$\tau \equiv \frac{tB}{A} \implies \frac{dn}{d\tau} = \frac{dN}{dt}\frac{1}{B} = \frac{r}{B}N(1 - N/K) - \frac{N^2}{A^2 + N^2} \tag{4}$$

Wit $N = nA$ this can be written as:

$$\frac{dn}{d\tau} = \frac{r}{B}nA(1 - \frac{nA}{K}) - \frac{(nA)^2}{A^2 + (nA)^2} \tag{5}$$

$$\Leftrightarrow \frac{dn}{d\tau} = \frac{rA}{B}n(1 - n\frac{A}{K}) - \frac{n}{1 + n^2} \tag{6}$$

We then define $\alpha = \frac{K}{A}$ and $\beta = \frac{Ar}{B}$ as our two system defining parameters. This simplifies the equation to:

$$\frac{dn}{d\tau} = \beta n(1 - \frac{n}{\alpha}) - \frac{n}{1 + n^2} \tag{7}$$

ii) To determine the stationary points n^* of the population for $\alpha = 7.5$ we require the derivative of (7) with respect to n to be equal 0:

$$\beta - \frac{2n}{7.5} + \frac{2n^2}{(1+n^2)^2} - \frac{1}{1+n^2} = 0 \tag{8}$$

Solving this numerically to determine the stationary points of this differential equation results in following pictures and values for it

β	Zeropoints n^*	stability
0.95	-0.09	not stable
	0.19	stable
	3.61	not stable
1	0	not stable
	0.09	stable
	3.77	not stable
1.05	3.94	not stable

So we can conclude, that there are three or one stationary points of this differential equation. If $\beta < 1$ we get three real zero- and thus stationary points, one of them is a stable stationary point, the other two are unstable. This is also true for $\beta = 1$ (at least numerically, analytically it probably could also be two). For $\beta > 1$ we only get one unstable stationary point.

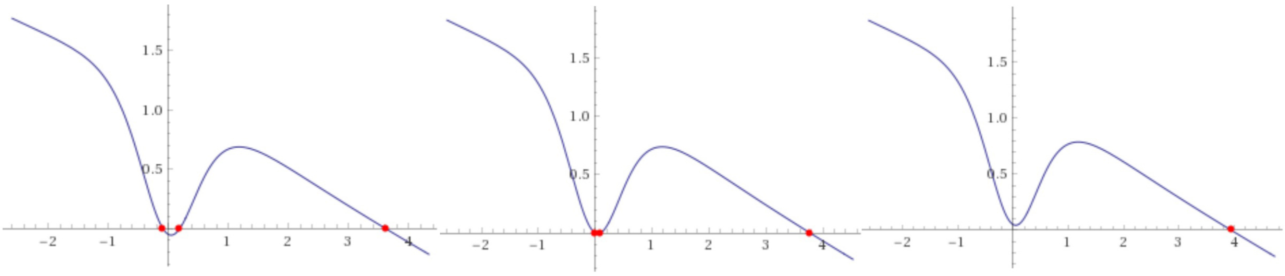

Figure 1: Left: $\beta = 0.95$, middle: $\beta = 1$, right: $\beta = 1.05$