# Computer Graphics - Exercise 06

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## 1 2D Transformations

a) **Transformation description**

- Reflecting about the $x$-axis and scaling the $y$-value by 0.5.
- Rotation by $\pm 180$ degrees or equivalently, mirroring about $x$- and $y$-axis
- Small subscript $h$ indicates homogeneous coordinates. Therefore the matrix performs a translation by the vector $(0,2)^\top$.

b) **Rotation matrices**

- The exercise contains a mistake because the given points do not form a rectangle. By changing $V_4 = (3,4)$ instead of $V_4 = (3,3)$ would lead to the form of a rectangle.
  The desired transformation can be performed with three transformation matrices

$$M_1 = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix} \quad M_2 = \begin{pmatrix} \cos\frac{\pi}{4} & \sin\frac{\pi}{4} & 0 \\ -\sin\frac{\pi}{4} & \cos\frac{\pi}{4} & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad M_3 = \begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{pmatrix} \tag{1}$$

  The points after the transformation can be found by matrix multiplication

$$V_1 = \begin{pmatrix} 2 \\ 2 \end{pmatrix}, W_1 = \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix}_h \quad \Rightarrow \quad W_1' = M_3 \cdot M_2 \cdot M_1 \cdot W_1 = \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix}_h \quad \Rightarrow \quad V_1' = \begin{pmatrix} 2 \\ 2 \end{pmatrix} \tag{2}$$

$$V_2 = \begin{pmatrix} 3 \\ 1 \end{pmatrix}, W_2 = \begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix}_h \quad \Rightarrow \quad W_2' = M_3 \cdot M_2 \cdot M_1 \cdot W_2 = \begin{pmatrix} 2+\sqrt{2} \\ 2 \\ 1 \end{pmatrix}_h \quad \Rightarrow \quad V_2' = \begin{pmatrix} 2+\sqrt{2} \\ 2 \end{pmatrix} \tag{3}$$

$$V_3 = \begin{pmatrix} 4 \\ 2 \end{pmatrix}, W_3 = \begin{pmatrix} 4 \\ 2 \\ 1 \end{pmatrix}_h \quad \Rightarrow \quad W_3' = M_3 \cdot M_2 \cdot M_1 \cdot W_3 = \begin{pmatrix} 2+\sqrt{2} \\ 2+\sqrt{2} \\ 1 \end{pmatrix}_h \quad \Rightarrow \quad V_3' = \begin{pmatrix} 2+\sqrt{2} \\ 2+\sqrt{2} \end{pmatrix} \tag{4}$$

$$V_4 = \begin{pmatrix} 3 \\ 3 \end{pmatrix}, W_4 = \begin{pmatrix} 3 \\ 3 \\ 1 \end{pmatrix}_h \quad \Rightarrow \quad W_4' = M_3 \cdot M_2 \cdot M_1 \cdot W_4 = \begin{pmatrix} 2 \\ 2+\sqrt{2} \\ 1 \end{pmatrix}_h \quad \Rightarrow \quad V_4' = \begin{pmatrix} 2 \\ 2+\sqrt{2} \end{pmatrix} \tag{5}$$

$$\tag{6}$$

- Q1: Rotating without translating would in this case lead to the rotation with respect to the origin of the coordinate system instead of $V_1$.
- Q2: Translating the center of the quadrangle to (0, 0) before the rotation would rotate it around the vertex its center.
- Q3: The rectangle would be rotated around the vertex $V_1$
- Q4: It is possible by multiplying the matrices

$$M = M_3 \cdot M_2 \cdot M_1 = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 2 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 2-2\sqrt{2} \\ 0 & 0 & 1 \end{pmatrix} \tag{7}$$

# 2 Euler Angles and Transformations

The Euler rotations are given by the three matrices for the three coordinate axes

$$R_x(\phi) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & -\sin\phi \\ 0 & \sin\phi & \cos\phi \end{pmatrix} \quad R_y(\phi) = \begin{pmatrix} \cos\phi & 0 & \sin\phi \\ 0 & 1 & 0 \\ -\sin\phi & 0 & \cos\phi \end{pmatrix} \quad R_z(\phi) = \begin{pmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{8}$$

a) The general rotation matrix for the three axes $zyz$ can be found by matrix multiplication of the Euler rotation matrices

$$R_{zyz} = R_z(\psi)R_y(\theta)R_z(\phi) = \begin{pmatrix} \cos\psi\cos\theta\cos\phi - \sin\phi\sin\psi & -\cos\theta\cos\psi\sin\phi - \sin\psi\cos\phi & \sin\theta\cos\psi \\ \cos\theta\sin\phi\cos\psi + \sin\phi\cos\psi & -\cos\theta\sin\phi\sin\psi + \cos\phi\cos\psi & \sin\theta\sin\psi \\ -\sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \end{pmatrix} \tag{9}$$

$$= \begin{pmatrix} 0.7071 & 0 & -0.7071 \\ 0 & -1 & 0 \\ -0.7071 & 0 & -0.7071 \end{pmatrix} \tag{10}$$

The angles can be calculated by exploiting the entries with only one or two trigonometric functions

$$\cos\theta = -0.7071 \quad \Rightarrow \quad \theta = \cos^{-1}(-0.7071) \approx 135^\circ \tag{11}$$

$$-\sin\theta\cos\phi = -0.7071 \quad \Rightarrow \quad \cos\phi = \frac{0.7071}{\sin(135^\circ)} = 0.99998... \quad \Rightarrow \quad \phi = \cos^{-1}(0.99998...) \approx 0.35^\circ \tag{12}$$

$$\sin\theta\cos\psi = -0.7071 \quad \Rightarrow \quad \cos\psi = \frac{-0.7071}{\sin(135^\circ)} = -0.99998... \quad \Rightarrow \quad \psi = \cos^{-1}(-0.99998...) \approx 177^\circ \tag{13}$$

The second rotation matrix is calculated by

$$R_{zyx} = R_z(\psi)R_y(\theta)R_x(\phi) \tag{14}$$

where the angles are computed analogously

$$\theta = -30^\circ \quad \psi \approx 30^\circ \quad \phi \approx 30^\circ \tag{15}$$

(i) The algorithm describes our approach of the first exercise

**Algorithm 1** 45° Rotation around $V_1$

**pushMatrix()**<br>
    Translate $V_i$ by $-V_1$, such that $V_1$ is at the coordinate center<br>
    Rotate $V_i$ by 45°<br>
**popMatrix()**<br>
Translate $V_i$ by $+V_1$

(ii) The sun is at the coordinate center, rotates around itself ($y$-axis) with arbitrary angle $\phi_{\text{sun}}$. The earth rotates around the sun around the $y$-axis around coordinate center with angle $\phi_{\text{earth}}$. The earth spins around its own, by 23.5 rotated, axis with angle $\psi_{\text{earth}}$. The moon rotates around the center of the earth but in the $x$-$z$-layer, not the rotated axes.

**Algorithm 2** Sun-Earth-Moon System

Set sun, earth and moon to coordinate center.<br>
**pushMatrix()**<br>
    Rotate sun about $\phi_{\text{sun}}$ around $y$-axis<br>
**popMatrix()**<br>
**pushMatrix()**<br>
    Translate earth and moon about distance $d_{\text{sun-earth}}$<br>
    **pushMatrix()**<br>
        Rotate earth about 23.5° around $z$-axis<br>
        Rotate earth about $\phi_{\text{earth}}$ around new $y$-axis<br>
    **popMatrix()**<br>
    **pushMatrix()**<br>
        Translate moon about distance $d_{\text{earth-moon}}$<br>
    **popMatrix()**<br>
**popMatrix()**

b) Similarity transforms preserve angles but also contain affine (non-linear) transformations. Linear transforms can also preserve angles, but also stretch and shear. They do however not contain translations, because they do not satisfy the linearity condition. Similarity transforms and linear transforms therefore have a non-empty intersection.

c) The geometric interpretation of the translation depends on the kind of transformation:

- Identity and mirroring: No effect because there are no off-diagonal elements.
- Shear: The axes of shear are switched.
- Rotation: The rotation gets inverted.

d) A way to efficiently compute the inverse of a rotation matrix is by using the orthogonality of the rotation group, i.e. $R^\top R = \mathbb{1}$, so in case of rotation matrices the transpose is the inverse $R^\top = R^{-1}$. If the rotation is given by a function $R(\phi)$, the inverse can be obtained by plugging in the negative angle, i.e. $R^{-1}(\phi) = R(-\phi)$.

e) An orthogonal matrix contains mirroring if and only if its determinant is $-1$.