Fundamental of Simulation Methods

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Problem Set 2: Integration of ordinary differential equations

Problem 2.1: Order of an ODE integration scheme

Not done because i had a busy week... :)

Problem 2.2: Double Pendulum

In [ ]:
```
import numpy as np
import sympy as sp
import matplotlib.pyplot as plt
from IPython.display import Math, display

plt.style.use("light")

def printf(expr, prefix=""):
    """Function to print sympy nice

    Args:
        expr (sympy.expr): Any sympy expression
        prefix (str, optional): Prefix to expression in LaTeX. Defaults to "".
    """
    display(Math(prefix + sp.latex(expr)))


# Define Symbols
t, g, m_1, m_2, l_1, l_2 = sp.symbols("t g m_1 m_2 l_1 l_2")

# Define angle functions
phi_1 = sp.symbols("phi_1", cls=sp.Function)(t)
phi_2 = sp.symbols("phi_2", cls=sp.Function)(t)

# First derivative / angular velocity
dphi_1 = sp.diff(phi_1, t)
dphi_2 = sp.diff(phi_2, t)

# Second derivative / angular accelaration
ddphi_1 = sp.diff(dphi_1, t)
ddphi_2 = sp.diff(dphi_2, t)

# Change from phase space coordinates to cartesian coordinates
x1 = l_1 * sp.sin(phi_1)
y1 = -l_1 * sp.cos(phi_1)

x2 = l_1 * sp.sin(phi_1) + l_2 * sp.sin(phi_2)
y2 = -l_1 * sp.cos(phi_1) - l_2 * sp.cos(phi_2)

# Kinetic Energy
T1 = 1 / 2 * m_1 * (x1.diff(t) ** 2 + y1.diff(t) ** 2)
T2 = 1 / 2 * m_2 * (x2.diff(t) ** 2 + y2.diff(t) ** 2)
T = T1 + T2

# Potential Energy
V1 = m_1 * g * y1
V2 = m_2 * g * y2
V = V1 + V2

# Lagrange Function
L = T - V

printf(L.simplify(), "L=")
```

$$L = gl_1m_1\cos(\phi_1(t)) + gm_2(l_1\cos(\phi_1(t)) + l_2\cos(\phi_2(t))) + 0.5l_1^2m_1\left(\frac{d}{dt}\phi_1(t)\right)^2 + 0.5m_2$$

(a) Lagrangian Equations of Motion

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial\dot{\phi}_i} - \frac{\partial L}{\partial\phi_i} = 0 \quad \text{with} \quad i = 1, 2$$

In [ ]:
```
EL_1 = sp.diff(L, phi_1) - sp.diff(sp.diff(L, dphi_1), t).simplify()
EL_2 = sp.diff(L, phi_2) - sp.diff(sp.diff(L, dphi_2), t).simplify()

printf(EL_1.simplify(), "0=")
printf(EL_2.simplify(), "0=")
```

$$0 = -l_1\left(gm_1\sin(\phi_1(t)) + gm_2\sin(\phi_1(t)) + l_1m_1\frac{d^2}{dt^2}\phi_1(t) + l_1m_2\frac{d^2}{dt^2}\phi_1(t) + l_2m_2\sin(\phi_1\right.$$

$$0 = 1.0l_2m_2\left(-g\sin(\phi_2(t)) + l_1\sin(\phi_1(t) - \phi_2(t))\left(\frac{d}{dt}\phi_1(t)\right)^2 - l_1\cos(\phi_1(t) - \phi_2(t))\frac{d^2}{dt^2}\right.$$

(b) Cast System of Equation into 1st-order form

Solve for $\ddot{\phi}_i$ and define $\dot{\phi} = \omega$ such that $\ddot{\phi} = \frac{\mathrm{d}}{\mathrm{d}t}\omega$

In [ ]:
```
deq = sp.solve([EL_1, EL_2], (ddphi_1, ddphi_2), simplify=False, rational=False)

deq1 = deq[ddphi_1]
deq2 = deq[ddphi_2]

printf(deq1.simplify(), "\ddot{\phi}_1=")
printf(deq2.simplify(), "\ddot{\phi}_2=")
```

$$\ddot{\phi}_1 = \frac{-1.0gm_1\sin(\phi_1(t)) - 0.5gm_2\sin(\phi_1(t) - 2\phi_2(t)) - 0.5gm_2\sin(\phi_1(t)) - 0.5l_1m_2\sin}{l_1(m_1 - m_2\cos^2(\phi_1(t) - \phi}$$

$$\ddot{\phi}_2 = \frac{0.5gm_1\sin(2\phi_1(t) - \phi_2(t)) - 0.5gm_1\sin(\phi_2(t)) + 0.5gm_2\sin(2\phi_1(t) - \phi_2(t)) - 0.5g}{}$$

(c) + (d) Integration of Equations of Motion with Runge Kutta 2nd and 4th Order

In [ ]:

```
# Lambdify sympy functions to actually evaluate EoMs with values
accel1 = sp.lambdify((t, g, m_1, m_2, l_1, l_2, phi_1, phi_2, dphi_1, dphi_2), deq1
accel2 = sp.lambdify((t, g, m_1, m_2, l_1, l_2, phi_1, phi_2, dphi_1, dphi_2), deq2
vel1 = sp.lambdify(dphi_1, dphi_1)
vel2 = sp.lambdify(dphi_2, dphi_2)
```

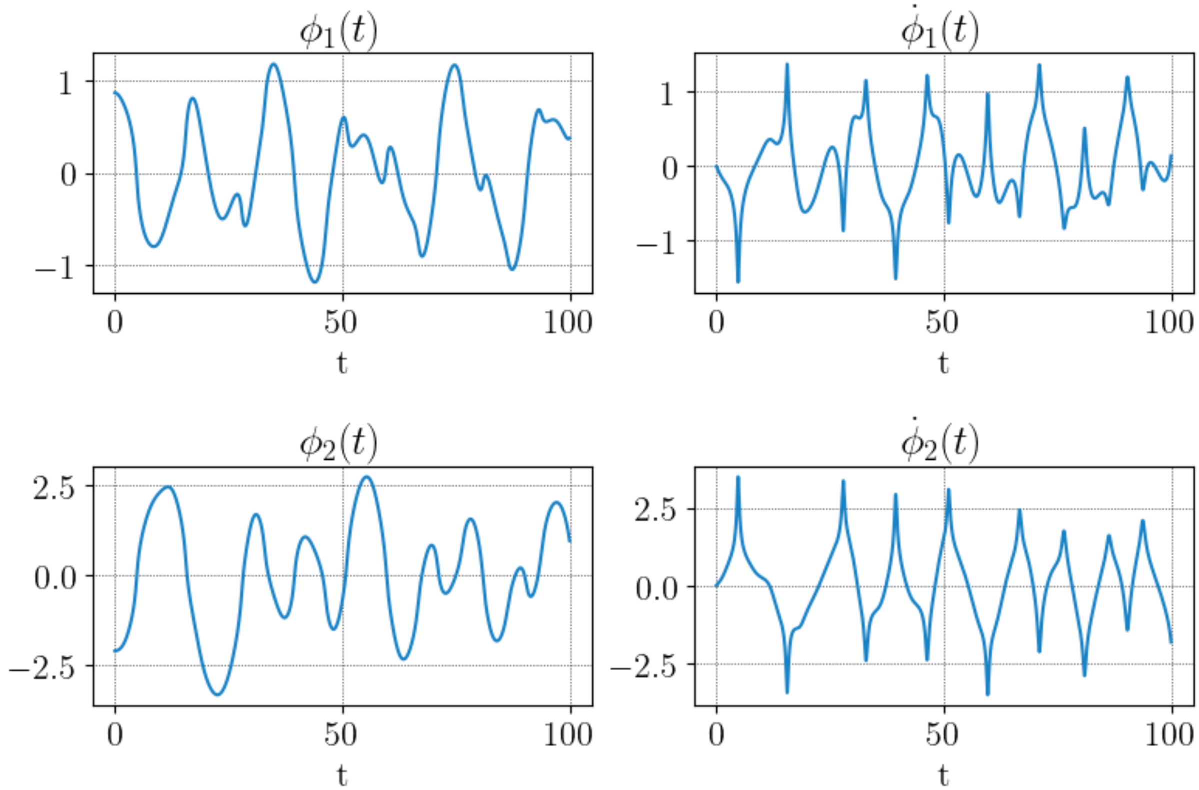
In [ ]:

```
def dydt(S, t, g, m_1, m_2, l_1, l_2):
    phi_1, z_1, phi_2, z_2 = S
    return np.array(
        [
            vel1(z_1),
            accel1(t, g, m_1, m_2, l_1, l_2, phi_1, phi_2, z_1, z_2),
            vel2(z_2),
            accel2(t, g, m_1, m_2, l_1, l_2, phi_1, phi_2, z_1, z_2),
        ]
    )


def rungeKutta2nd(t0, y0, t, h, *args):
    n = round((t - t0) / h)
    y = y0
    solution = []
    for _ in range(n):
        k1 = h * dydt(y, t0, *args)
        k2 = h * dydt(y + 0.5 * k1, t0 + 0.5 * h, *args)

        y = y + (1.0 / 6.0) * (k1 + 2 * k2)
        t0 = t0 + h

        solution.append(y)
    return np.array(solution)


def rungeKutta4th(t0, y0, t, h, *args):
    n = round((t - t0) / h)
    y = y0
    solution = []
    for i in range(1, n + 1):
        k1 = h * dydt(y, t0, *args)
        k2 = h * dydt(y + 0.5 * k1, t0 + 0.5 * h, *args)
        k3 = h * dydt(y + 0.5 * k2, t0 + 0.5 * h, *args)
        k4 = h * dydt(y + k3, t0 + h, *args)

        y = y + (1.0 / 6.0) * (k1 + 2 * k2 + 2 * k3 + k4)
        t0 = t0 + h

        solution.append(y)
    return np.array(solution)
```

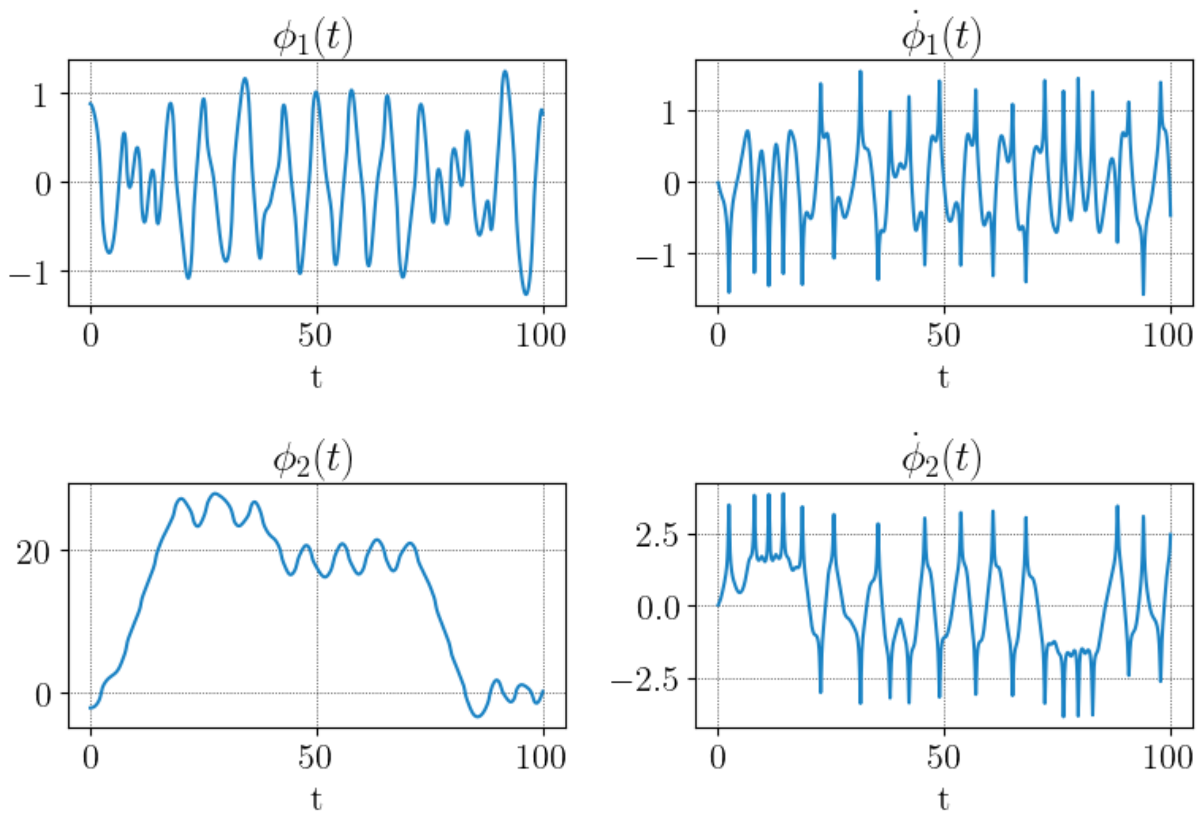
In [ ]:
```
# Constants
g   = 1.0 # Gravitational acceleration
l_1 = 2.0 # Length of first pendulum
l_2 = 1.0 # Length of second pendulum
m_1 = 0.5 # Mass of first pendulum
m_2 = 1.0 # Mass of second pendulum

# Initial angles
phi1_0 = 50 / 360 * np.pi * 2
phi2_0 = -120 / 360 * np.pi * 2

# Initial angular velocities
dphi1_0 = 0
dphi2_0 = 0

# Initial state
y0 = [phi1_0, dphi1_0, phi2_0, dphi2_0]

t0 = 0  # Start time
t = 100  # Total time steps
h = 0.05  # Step size

# Perform integration
sol_rk2 = rungeKutta2nd(t0, y0, t, h, g, m_1, m_2, l_1, l_2)
sol_rk4 = rungeKutta4th(t0, y0, t, h, g, m_1, m_2, l_1, l_2)
```

In [ ]:
```
plt.rcParams['figure.figsize'] = [8,6]

def plot_evolution(sol, title):
    """Plot time evolution of phase space variables

    Args:
        sol (np.array): Solution of integration
        title (string): Title of the Integrator
    """
    fig, ax = plt.subplots(2, 2)
    fig.suptitle(title, fontsize=16)
    t = np.arange(0, 100, 0.05)
    ax[0, 0].plot(t, sol[:, 0])
    ax[0, 1].plot(t, sol[:, 1])
    ax[1, 0].plot(t, sol[:, 2])
    ax[1, 1].plot(t, sol[:, 3])

    ax[0, 0].set_title("$\phi_1(t)$")
    ax[0, 1].set_title("$\dot{\phi}_1(t)$")
    ax[1, 0].set_title("$\phi_2(t)$")
    ax[1, 1].set_title("$\dot{\phi}_2(t)$")

    ax[0, 0].set_xlabel("t")
    ax[0, 1].set_xlabel("t")
    ax[1, 0].set_xlabel("t")
    ax[1, 1].set_xlabel("t")

    plt.tight_layout()
    plt.show()


plot_evolution(sol_rk2, "Runge Kutta 2nd Order")
plot_evolution(sol_rk4, "Runge Kutta 4th Order")
```

Runge Kutta 2nd Order

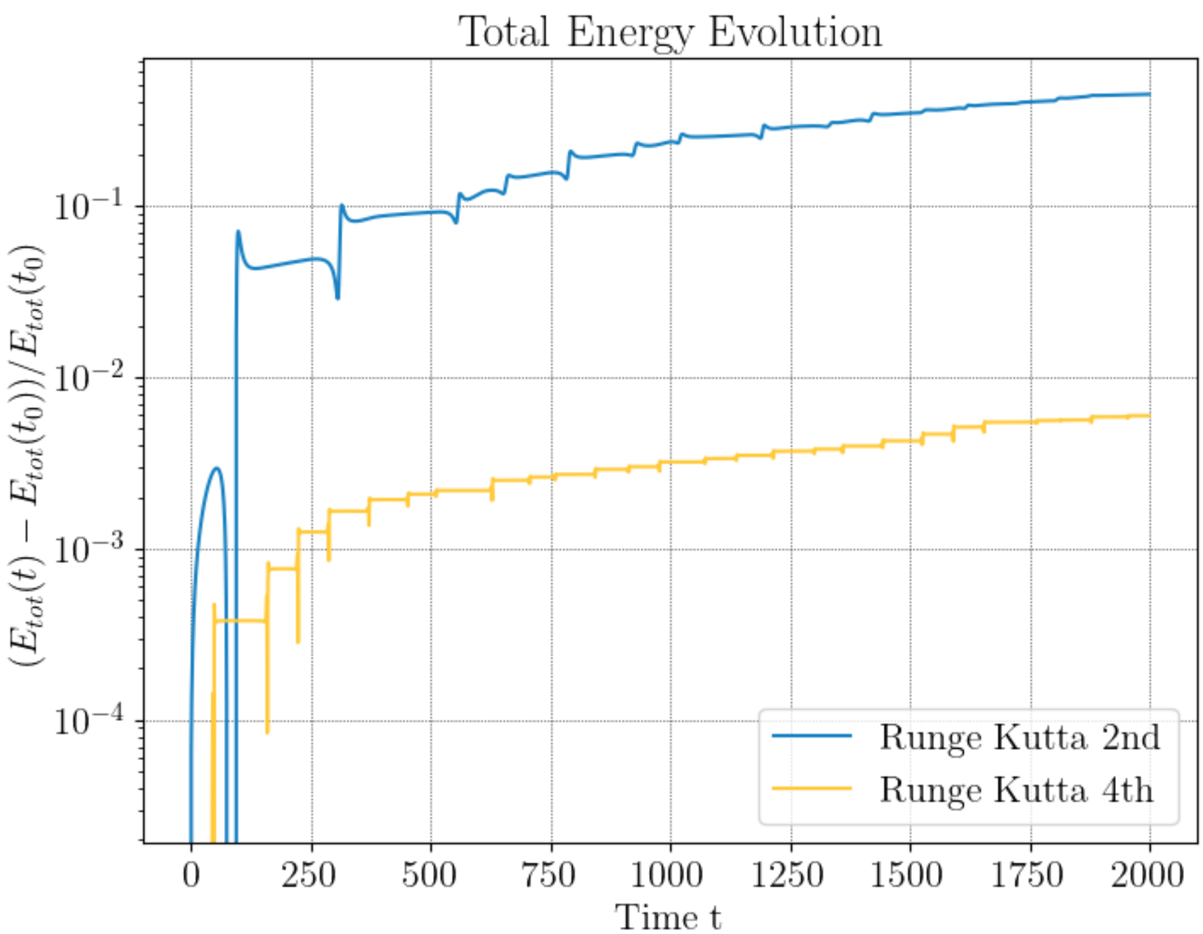


Runge Kutta 4th Order



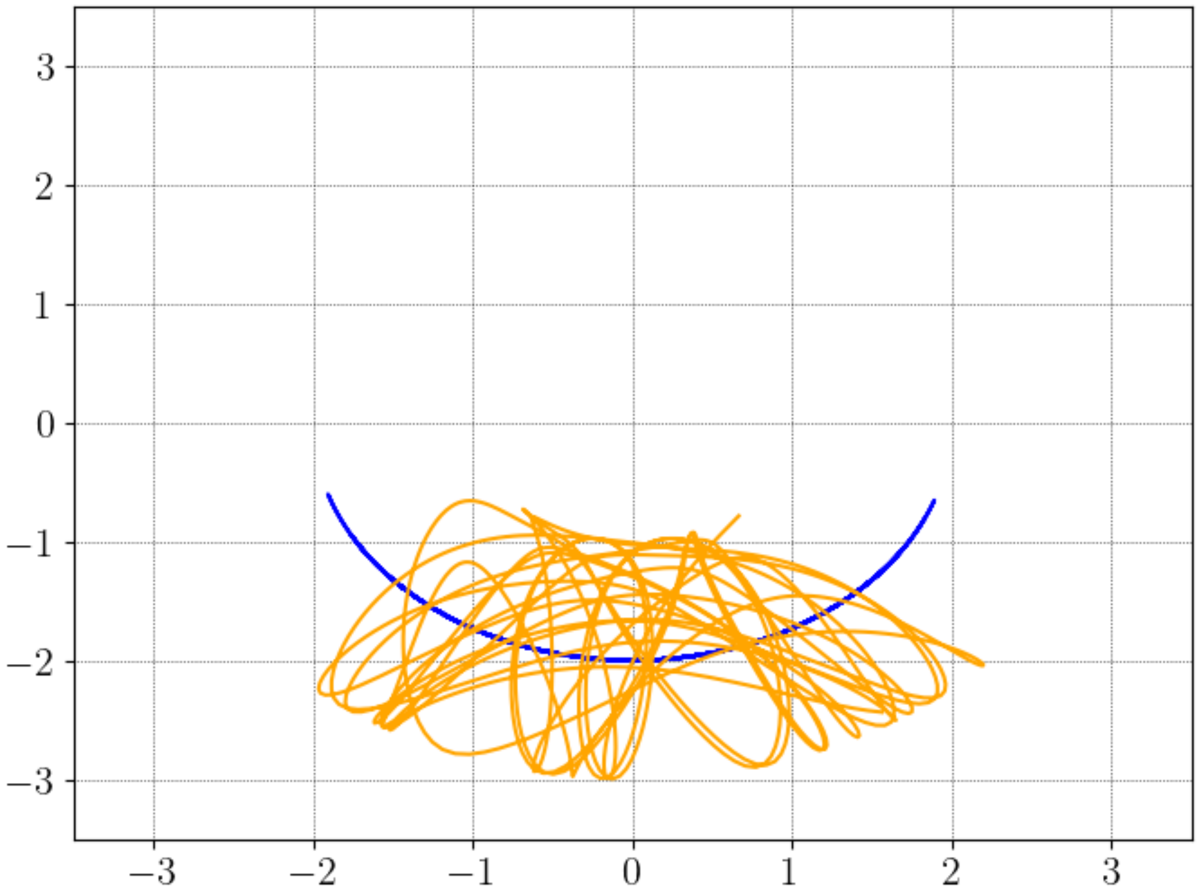
In [ ]:
```
def get_coords(phi_1, phi_2):
    """Get cartesian coordinates from angles

    Args:
        phi_1 (float): Angle of first pendulum
        phi_2 (float): Angle of second pendulum

    Returns:
        Tuple: Cartesian coordinates
    """
    x1 = l_1 * np.sin(phi_1)
    y1 = -l_1 * np.cos(phi_1)

    x2 = l_1 * np.sin(phi_1) + l_2 * np.sin(phi_2)
    y2 = -l_1 * np.cos(phi_1) - l_2 * np.cos(phi_2)
    return x1, y1, x2, y2


def get_energy(w1, w2, l1, y1, l2, y2, m1, m2, phi1, phi2, g):
    """Get tota energy from all double pendulum variables"""
    # Kinetic Energy
    T = (
        1
        / 2
        * (
            m1 * l1**2 * w1**2
            + m2 * (l1**2 * w1**2 + l2**2 * w2**2 + 2 * l1 * l2 * w1 * w2 * np.cos(
        )
    )
    # Potential Energy
    V = g * m1 * y1 + g * m2 * y2
    return T + V


def plot_total_energy(sol, label):
    """Plot total energy with label

    Args:
        sol (np.array): Solution of integration
        label (str): Title of Integrator
    """
    _, y1, _, y2 = get_coords(sol[:, 0], sol[:, 2])
    phi1, w1, phi2, w2 = (sol[:, 0],sol[:, 1],sol[:, 2],sol[:, 3])

    Etot = get_energy(w1, w2, l_1, y1, l_2, y2, m_1, m_2, phi1, phi2, g)

    plt.plot((Etot - Etot[0]) / Etot[0], label=label)
    plt.yscale("log")


plot_total_energy(sol_rk2, "Runge Kutta 2nd")
plot_total_energy(sol_rk4, "Runge Kutta 4th")
plt.xlabel("Time t")
plt.ylabel("$(E_{tot}(t)-E_{tot}(t_0))/E_{tot}(t_0)$")
plt.title("Total Energy Evolution")
plt.legend()
```

Out[ ]: <matplotlib.legend.Legend at 0x7f51c623f510>



In [ ]:
```python
# Check if result seems valid
x1, y1, x2, y2 = get_coords(sol_rk4[:, 0], sol_rk4[:, 2])

extent = l_1 + l_2 + 0.5
plt.xlim(-extent, extent)
plt.ylim(-extent, extent)

plt.plot(x1, y1, "-", color="blue")
plt.plot(x2, y2, "-", color="orange")
```

Out[ ]: [<matplotlib.lines.Line2D at 0x7f51c41ce350>]



In [ ]:
```
def plot_timestep(timestep):
    """Plot double pendulum snapshot in cartesian coordinates and save as picture

    Args:
        timestep (int): Index for evaluation
    """
    space = l_1 + l_2 + 0.5
    plt.xlim(-space, space)
    plt.ylim(-space, space)

    # Plot origin and mass points
    plt.plot(0, 0, "o", color="black")
    plt.plot(x1[timestep], y1[timestep], "o", color="blue")
    plt.plot(x2[timestep], y2[timestep], "o", color="orange")

    # Plot trajectories of masses
    plt.plot(x1[:timestep], y1[:timestep], "-", color="blue")
    plt.plot(x2[:timestep], y2[:timestep], "-", color="orange")

    # Plot joints
    plt.plot([0, x1[timestep]], [0, y1[timestep]], color="black")
    plt.plot([x1[timestep], x2[timestep]], [y1[timestep], y2[timestep]], color="bla

    # Save as PNG
    plt.savefig(f"./pictures/output{str(timestep).zfill(4)}.png", dpi=100)
    plt.clf()
```

In [ ]:
```
# Use multiprocessing because i am inpatient and waiting is annoying
import multiprocessing as mp

# Use commong figure
plt.figure(figsize=(5, 5))

# Good old friend the mp.Pool, using all CPUs available
with mp.Pool() as p:
    p.map(plot_timestep, np.arange(0, int(t / 0.05)))
```

```
<Figure size 500x500 with 0 Axes>
```

In [ ]:
```
# Create MP4 movie with ffmpeg
!ffmpeg -y -r 30 -i ./pictures/output%04d.png movie.mp4
!rm -rf ./pictures/*
```

```
ffmpeg version 4.2.2 Copyright (c) 2000-2019 the FFmpeg developers
  built with gcc 7.3.0 (crosstool-NG 1.23.0.449-a04d0)
  configuration: --prefix=/tmp/build/80754af9/ffmpeg_1587154242452/_h_env_placehold_
placehold_placehold_placehold_placehold_placehold_placehold_placehold_placehold_plac
ehold_placehold_placehold_placehold_placehold_placehold_placehold_placehold_placehol
d_placehold_placehold_placeho --cc=/tmp/build/80754af9/ffmpeg_1587154242452/_build_e
nv/bin/x86_64-conda_cos6-linux-gnu-cc --disable-doc --enable-avresample --enable-gmp
--enable-hardcoded-tables --enable-libfreetype --enable-libvpx --enable-pthreads --e
nable-libopus --enable-postproc --enable-pic --enable-pthreads --enable-shared --ena
ble-static --enable-version3 --enable-zlib --enable-libmp3lame --disable-nonfree --e
nable-gpl --enable-gnutls --disable-openssl --enable-libopenh264 --enable-libx264
  libavutil      56. 31.100 / 56. 31.100
  libavcodec     58. 54.100 / 58. 54.100
  libavformat    58. 29.100 / 58. 29.100
  libavdevice    58.  8.100 / 58.  8.100
  libavfilter     7. 57.100 /  7. 57.100
  libavresample   4.  0.  0 /  4.  0.  0
  libswscale      5.  5.100 /  5.  5.100
  libswresample   3.  5.100 /  3.  5.100
  libpostproc    55.  5.100 / 55.  5.100
Input #0, image2, from './pictures/output%04d.png':
  Duration: 00:01:20.00, start: 0.000000, bitrate: N/A
    Stream #0:0: Video: png, rgba(pc), 500x500 [SAR 3937:3937 DAR 1:1], 25 fps, 25 t
br, 25 tbn, 25 tbc
Stream mapping:
  Stream #0:0 -> #0:0 (png (native) -> h264 (libx264))
Press [q] to stop, [?] for help
[libx264 @ 0x561f6a788dc0] using SAR=1/1
[libx264 @ 0x561f6a788dc0] using cpu capabilities: MMX2 SSE2Fast SSSE3 SSE4.2 AVX FM
A3 BMI2 AVX2
[libx264 @ 0x561f6a788dc0] profile High 4:4:4 Predictive, level 3.0, 4:4:4, 8-bit
[libx264 @ 0x561f6a788dc0] 264 - core 157 - H.264/MPEG-4 AVC codec - Copyleft 2003-2
018 - http://www.videolan.org/x264.html - options: cabac=1 ref=3 deblock=1:0:0 analy
se=0x1:0x111 me=hex subme=7 psy=1 psy_rd=1.00:0.00 mixed_ref=1 me_range=16 chroma_me
=1 trellis=1 8x8dct=0 cqm=0 deadzone=21,11 fast_pskip=1 chroma_qp_offset=4 threads=1
6 lookahead_threads=2 sliced_threads=0 nr=0 decimate=1 interlaced=0 bluray_compat=0
constrained_intra=0 bframes=3 b_pyramid=2 b_adapt=1 b_bias=0 direct=1 weightb=1 open
_gop=0 weightp=2 keyint=250 keyint_min=25 scenecut=40 intra_refresh=0 rc_lookahead=4
0 rc=crf mbtree=1 crf=23.0 qcomp=0.60 qpmin=0 qpmax=69 qpstep=4 ip_ratio=1.40 aq=1:
1.00
Output #0, mp4, to 'movie.mp4':
  Metadata:
    encoder         : Lavf58.29.100
    Stream #0:0: Video: h264 (libx264) (avc1 / 0x31637661), yuv444p, 500x500 [SAR 1:
1 DAR 1:1], q=-1--1, 30 fps, 15360 tbn, 30 tbc
    Metadata:
      encoder         : Lavc58.54.100 libx264
    Side data:
      cpb: bitrate max/min/avg: 0/0/0 buffer size: 0 vbv_delay: -1
frame= 2000 fps=0.0 q=-1.0 Lsize=    1053kB time=00:01:06.56 bitrate= 129.5kbits/s s
peed=68.2x
video:1030kB audio:0kB subtitle:0kB other streams:0kB global headers:0kB muxing over
head: 2.172636%
[libx264 @ 0x561f6a788dc0] frame I:8     Avg QP:11.75  size: 19137
[libx264 @ 0x561f6a788dc0] frame P:853   Avg QP:19.24  size:   636
[libx264 @ 0x561f6a788dc0] frame B:1139  Avg QP:22.66  size:   315
```

```
[libx264 @ 0x561f6a788dc0] consecutive B-frames: 16.2% 20.0% 10.8% 53.0%
[libx264 @ 0x561f6a788dc0] mb I  I16..4: 76.0%  0.0% 24.0%
[libx264 @ 0x561f6a788dc0] mb P  I16..4:  0.1%  0.0%  0.1%  P16..4:  0.7%  0.5%  0.
6%  0.0%  0.0%    skip:98.0%
[libx264 @ 0x561f6a788dc0] mb B  I16..4:  0.0%  0.0%  0.0%  B16..8:  0.8%  0.4%  0.
3%  direct: 0.2%  skip:98.2%  L0:38.9% L1:36.8% BI:24.3%
[libx264 @ 0x561f6a788dc0] coded y,u,v intra: 19.4% 7.4% 6.9% inter: 0.6% 0.3% 0.2%
[libx264 @ 0x561f6a788dc0] i16 v,h,dc,p: 74% 19%  6%  0%
[libx264 @ 0x561f6a788dc0] i4 v,h,dc,ddl,ddr,vr,hd,vl,hu: 36% 21% 23%  3%  4%  5%
3%  3%  2%
[libx264 @ 0x561f6a788dc0] Weighted P-Frames: Y:0.0% UV:0.0%
[libx264 @ 0x561f6a788dc0] ref P L0: 66.0%  7.3% 15.3% 11.3%
[libx264 @ 0x561f6a788dc0] ref B L0: 88.6%  9.0%  2.4%
[libx264 @ 0x561f6a788dc0] ref B L1: 97.5%  2.5%
[libx264 @ 0x561f6a788dc0] kb/s:126.52
```

In [ ]:

```python
# Plot movie in markdown
from IPython.display import Video

Video("movie.mp4")
```

Out[ ]:

Video can't be played because the file is corrupt.

In [ ]: