Final Project Part 2

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Previously, we proved the Parallel Axis Theorem. It says that for any random variable X and any constant c:

$$E[(X-c)^2] = Var[X] + (E[X]-c)^2$$

Now, lets use this theorem to prove a general property of numbers.

Problem: Let $a_1, a_2, ..., a_n$ be any numbers. Express $\frac{1}{n}\sum_{i=1}^{n}(a_i-\bar{a})^2$ in terms of $\sum_{i=1}^{n}(a_i-c)^2$ and $(\bar{a}-c)^2$ for any constant c.

Recall we previously defined a random variable X where X takes on the values $a_1, a_2, ..., a_n$ each with a probability of $1/n$. We will use this random variable here. In the previous example we showed that $E[X] = \sum_{i=1}^{n} a_i \cdot \frac{1}{n} = \bar{a}$ where $\bar{a}$ is defined to be the sample mean of $a_1, a_2, ..., a_n$.

Notice that we can rearrange the above Parallel Axis Theorem as follows:

$$Var[X] = E[(X-c)^2] - (E[X]-c)^2$$

We have the expression $\frac{1}{n}\sum_{i=1}^{n}(a_i-\bar{a})^2$ which does not look like like any of the terms above.

What if we apply LOTUS?

Recall that for discrete random variables LOTUS gives:

$$E[g(X)] = \sum_x g(x)f(x)$$

Consider the following random variable: $(X-\bar{a})^2$.

For this random variable $g(x) = (x-\bar{a})^2$ and $f(x) = 1/n$. Where $g(x)$ is the transformation applied to the random variable X and f(x) is the p.d.f. of X.

If we want to calculate the expected value of $(X-\bar{a})^2$ we apply LOTUS:

$$E[(X-\bar{a})^2] = \sum_x (x-\bar{a})^2 \frac{1}{n} = \frac{1}{n}\sum_x (x-\bar{a})^2$$

But note, this sum is over all possible values x of the random variable X. But X is defined to have $a_1, a_2, ..., a_n$ as its possible values. We thus get:

$$E[(X-\bar{a})^2] = \frac{1}{n}\sum_{i=1}^{n}(a_i-\bar{a})^2$$

But notice $E[(X-\bar{a})^2]$ is the random variable X minus its mean (as mentioned before). So by definition of variance we get:

$$Var[X] = E[(X-\bar{a})^2] = \frac{1}{n}\sum_{i=1}^{n}(a_i-\bar{a})^2$$

Similarly, we have the term: $E[(X-c)^2]$.

Let $g(x) = (x-c)^2$ and $f(x) = 1/n$.

By LOTUS we get:

$$E[(X-c)^2] = \sum_x (x-c)^2\frac{1}{n} = \frac{1}{n}\sum_x (x-c)^2$$

By the definition of the random variable X:

$$E[(X-c)^2] = \frac{1}{n}\sum_{i=1}^{n}(a_i-c)^2$$

Finally, we have the term $(E[X]-c)^2$. Since $E[X] = \bar{a}$ we get:

$$(E[X]-c)^2 = (\bar{a}-c)^2$$

No need to use LOTUS.

Putting all the pieces together and substituting we get the non-trivial fact of numbers:

$$\frac{1}{n}\sum_{i=1}^{n}(a_i-\bar{a})^2 = \frac{1}{n}\sum_{i=1}^{n}(a_i-c)^2 - (\bar{a}-c)^2$$

Numerical Example:

```
a <- c(39,32,-399,49,319,-12)
a.bar <- mean(a)
c <- 389.2398
```

Left Side:

```
1/length(a) * sum((a - a.bar)^2)
```

```
## [1] 44320.22
```

Right Side:

```r
1/length(a) * sum((a-c)^2) - (a.bar-c)^2
```

```
## [1] 44320.22
```

Using properties of probability we were able to prove a lemma which will be incredibly useful in showing that $\frac{1}{n}\sum_{i=1}^{n}(X_i - \bar{X})^2$ is a biased estimator of σ.