Transform then pool or pool and then transform?

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Abstract

In this note I claim to give a proof that the order of operations between transforming and pooling a set of distributions does not matter **if and only if** the transform in question is invertible. I also give an example where the transform-then-pool and pool-then-transform with a non-invertible transform lead to distributions in the same family which are nonetheless distinct.

Key-words: logarithmic pooling; invertible; log-normal.

Background

Logarithmic pooling is a popular method for combining opinions on an agreed quantity, specially when these opinions can be framed as probability distributions. Let $\mathbf{F}_\theta := \{f_1(\theta), f_2(\theta), \ldots, f_K(\theta)\}$ be a set of distributions representing the opinions of K experts and let $\boldsymbol{\alpha} := \{\alpha_1, \alpha_2, \ldots, \alpha_K\} \in \mathcal{S}^K$ be the vector of weights, such that $\alpha_i > 0 \, \forall i$ and $\sum_{i=0}^K \alpha_i = 1$, i.e., $\mathcal{S}^K$ is the space of all open simplices of dimension K. The **logarithmic pooling operator** $\mathcal{LP}(\mathbf{F}_\theta, \boldsymbol{\alpha})$ is defined as

$$\mathcal{LP}(\mathbf{F}_\theta, \boldsymbol{\alpha}) := \pi(\theta|\boldsymbol{\alpha}) = t(\boldsymbol{\alpha}) \prod_{i=0}^K f_i(\theta)^{\alpha_i}, \tag{1}$$

where $t(\boldsymbol{\alpha}) = \int_{\boldsymbol{\Theta}} \prod_{i=0}^K f_i(\theta)^{\alpha_i} d\theta$. This pooling method enjoys several desirable properties and yields tractable distributions for a large class of distribution families (Genest et al., 1984).

Pool then transform or transform and then pool?

Definition 1. *Let $A, B \subseteq \mathbb{R}^p$. A function $h : A \to B$ is* ***invertible*** *iff $\exists \, h^{-1} : B \to A$ with $h^{-1}(h(a)) = a \, \forall \, a \in A$. Let π_A be an arbitrary probability measure in A. If h is monotonic and differentiable we can write $\pi_B(B) = \pi(h^{-1}(A))|J|$, where $|J|$ is the absolute determinant of the Jacobian matrix with entries $J_{ik} := \partial h_k^{-1}/\partial a_i$, $i, k = 1, 2, \ldots, p$.*

Suppose the we are interested in the distribution of a random variable $Y \in \mathcal{Y} \subseteq \mathbb{R}^q$ when one has a random variable $X \in \mathcal{X} \subseteq \mathbb{R}^p$ with $\phi : \mathcal{X} \to \mathcal{Y}$. Let $|J_\phi|$ be the Jacobian determinant w.r.t. ϕ. Suppose further that each expert i produces a distribution $f_i(X)$ such that we can construct the object $\mathbf{F}_X = \{f_1(X), f_2(X), \ldots, f_K(X)\}$. Then one can either:

(a) **Pool-then-transform:** construct $\pi_X(X|\boldsymbol{\alpha}) = \mathcal{LP}(\mathbf{F}_X, \boldsymbol{\alpha})$ and then apply ϕ to obtain $\pi_Y(Y|\boldsymbol{\alpha}) := \pi_X(\phi^{-1}(Y)|\boldsymbol{\alpha})|J_\phi|$;

(b) **Transform-then-pool:** apply the transform to each component i of $\mathbf{F}_X$ to build

$$\mathbf{G}_Y := \{g_i(Y), g_2(Y), \ldots, g_K(Y)\}$$

and obtain $\pi'_Y(Y|\boldsymbol{\alpha}) = \mathcal{LP}(\mathbf{G}_Y, \boldsymbol{\alpha})$.

Remark 1. *If ϕ is invertible, then $\pi_Y(Y|\boldsymbol{\alpha}) \equiv \pi'_Y(Y|\boldsymbol{\alpha})$.*

Proof. First,

$$\pi_Y(y|\boldsymbol{\alpha}) \propto \pi_X(\phi^{-1}(y))|J_\phi|, \tag{2}$$

$$= \prod_{i=0}^K \left[f_i(\phi^{-1}(y))\right]^{\alpha_i} |J_\phi|. \tag{3}$$

For situation (b) we have:

$$g_i(y) = f_i(\phi^{-1}(y))|J_\phi|. \tag{4}$$

And,

$$
\begin{aligned}
\pi'_Y(y|\boldsymbol{\alpha}) &\propto \prod_{i=0}^{K} g_i(y)^{\alpha_i} && (5)\\
&= \prod_{i=0}^{K} \left[f_i(\phi_x^{-1}(y))|J_\phi|\right]^{\alpha_i} && (6)\\
&= \prod_{i=0}^{K} \left[f_i(\phi^{-1}(y))\right]^{\alpha_i} |J_\phi|, && (7)
\end{aligned}
$$

as claimed. $\square$

An interesting idea is whether Remark 1 is an iff result. Let $\eta : \mathcal{X} \to \mathcal{Y}$ be a surjective non-injective differentiable function, which is not invertible on the whole of $\mathcal{Y}$, but instead is **piece-wise invertible**. Let $\mathcal{Y}_1, \mathcal{Y}_2, \ldots, \mathcal{Y}_T$ be a partition of $\mathcal{Y}$, i.e., $\mathcal{Y}_i \cap \mathcal{Y}_j = \emptyset$, $\forall i \neq j \in \{1, 2, \ldots, T\}$ and $\bigcup_{t=1}^{T} \mathcal{Y}_t = \mathcal{Y}$. Then define the inverse functions $\eta_t^{-1}(y) : \mathcal{Y}_t \to \mathcal{X}$, $t \in \{1, 2, \ldots, T\}$. Lastly, let $|J_t|$ be the Jacobian of $\eta_t^{-1}(\cdot)$. Then we are prepared to write:

$$
\begin{aligned}
\pi_Y(y|\boldsymbol{\alpha}) &\propto \sum_{t=1}^{T}\left(\prod_{i=0}^{K} f_i(\eta_t^{-1}(y))^{\alpha_i}\right)|J_t| \quad \text{and} && (8)\\
\pi'_Y(y|\boldsymbol{\alpha}) &\propto \prod_{i=0}^{K}\left[\sum_{t=1}^{T} f_i(\eta_t^{-1}(y))|J_t|\right]^{\alpha_i} && (9)
\end{aligned}
$$

which, I claim, will only be equal if $T = 1$, i.e. if $\eta(\cdot)$ is invertible in the usual sense. Below I try to establish a result in general for any surjective non-injective mapping, not just piece-wise invertible ones.

Remark 2. *$\pi_Y(Y|\boldsymbol{\alpha}) \equiv \pi'_Y(Y|\boldsymbol{\alpha})$ if and only if ϕ is invertible.*

Proof. In general, we can define $\Omega(y) := \{x : \eta(x) = y\}$ and thus[1]

$$
g_i(y) = \sum_{x\in\Omega(y)} f_i(x). \tag{10}
$$

Notice that for any $x \in \mathcal{X}$, there exists y and $\Omega(y)$ such that $x \in \Omega(y)$. Assume $|\Omega(y)| > 1$ for some $y \in \mathcal{Y}$ and $f_i \not\equiv f_j$, for some i, j. We have

$$
\begin{aligned}
\pi_Y(y|\boldsymbol{\alpha}) &\propto \sum_{x\in\Omega(y)}\left(\prod_{i=0}^{K} f_i(x)^{\alpha_i}\right) \quad \text{and} && (11)\\
\pi'_Y(y|\boldsymbol{\alpha}) &\propto \prod_{i=0}^{K}\left[\sum_{x\in\Omega(y)} f_i(x)\right]^{\alpha_i}. && (12)
\end{aligned}
$$

Define

$$
T = \int_{\mathcal{Y}} \sum_{x\in\Omega(y)}\left(\prod_{i=0}^{K} f_i(x)^{\alpha_i}\right) dy \quad \text{and} \quad T' = \int_{\mathcal{Y}} \prod_{i=0}^{K}\left[\sum_{x\in\Omega(y)} f_i(x)\right]^{\alpha_i} dy.
$$

It is not hard to show[2] that $T = \left(\int_{\mathcal{X}} \prod_{i=0}^{K} f_i(x)^{\alpha_i} dx\right)^{-1}$. Since for any given y and $x \in \Omega(y)$ we have $\sum_{\omega\in\Omega(y)} f_i(\omega) > f_i(x)$ *a.e.*, it follows that $T' < T$ and hence the densities would have different normalising constants, which is impossible if $\pi_Y(y|\boldsymbol{\alpha}) = \pi'_Y(y|\boldsymbol{\alpha}) \,\forall\, y \in \mathcal{Y}$. $\square$

There probably exists a measure-theoretic proof that is way more elegant, but this should suffice.

[1] Notice there is no guarantee that $|\Omega(y)| < \infty$.

[2] This hinges on the fact that $\int_{\mathcal{Y}} g_i(y)dy = 1 \,\forall\, i$.

An example

Suppose $Z = U/V$ and each expert i elicits $U \sim \text{log-normal}(\mu_{iU}, \sigma^2_{iU})$, $V \sim \text{log-normal}(\mu_{iV}, \sigma^2_{iV})$, i.e.

$$
\begin{aligned}
f_{iU}(u|\mu_{iU}, \sigma^2_{iU}) &= \frac{1}{u\sqrt{2\pi\sigma^2_{iU}}} \exp\left(-\frac{(\ln u - \mu_{iU})^2}{2\sigma^2_{iU}}\right), \\
g_{iV}(v|\mu_{iV}, \sigma^2_{iV}) &= \frac{1}{v\sqrt{2\pi\sigma^2_{iV}}} \exp\left(-\frac{(\ln v - \mu_{iV})^2}{2\sigma^2_{iV}}\right).
\end{aligned}
$$

Again, let $\mathbf{F}_U = \{f_{1U}(U), f_{2U}(U), \ldots, f_{KU}(U)\}$ and $\mathbf{G}_V = \{g_{1V}(V), g_{2V}(V), \ldots, g_{KV}(V)\}$. First, let us derive $\pi_Z(Z)$ under scheme (a). It is not hard to show that $\pi_U(U|\boldsymbol{\alpha}) := \mathcal{LP}(\mathbf{F}_U, \boldsymbol{\alpha}) = \text{log-normal}(\mu^*_U, v^*_U)$ with

$$
\begin{aligned}
\mu^*_U &:= \frac{\sum_{i=0}^K w_{iU}\mu_{iU}}{\sum_{i=0}^K w_{iU}}, && (13) \\
v^*_U &:= \frac{1}{\sum_{i=0}^K w_{iU}}, && (14) \\
w_{iU} &:= \frac{\alpha_i}{\sigma^2_{iU}}. && (15)
\end{aligned}
$$

See our paper for a proof. Analagously, $\pi_V(V|\boldsymbol{\alpha}) := \mathcal{LP}(\mathbf{G}_V, \boldsymbol{\alpha}) = \text{log-normal}(\mu^*_V, v^*_V)$. Then $\pi_Z(Z|\boldsymbol{\alpha}) = \text{log-normal}(\mu^*_Z, v^*_Z)$, with

$$
\begin{aligned}
\mu^*_Z &= \mu^*_U - \mu^*_V, \\
&= \frac{\sum_{i=0}^K w_{iU}\mu_{iU}}{\sum_{i=0}^K w_{iU}} - \frac{\sum_{i=0}^K w_{iV}\mu_{iV}}{\sum_{i=0}^K w_{iV}} \quad \text{and} && (16) \\
v^*_Z &= v^*_U + v^*_V, \\
&= \frac{1}{\sum_{i=0}^K w_{iU}} + \frac{1}{\sum_{i=0}^K w_{iV}}. && (17)
\end{aligned}
$$

Now let us consider case (b). Since $r_{iZ} = \text{log-normal}(\mu_{iU} - \mu_{iV}, \sigma^2_{iU} + \sigma^2_{iV})$, we arrive at $\pi'_Z(Z|\boldsymbol{\alpha}) = \text{log-normal}(\mu^{**}_Z, v^{**}_Z)$,

$$
\begin{aligned}
\mu^{**}_Z &:= \frac{\sum_{i=0}^K w_{iZ}\mu_{iU}}{\sum_{i=0}^K w_{iZ}} - \frac{\sum_{i=0}^K w_{iZ}\mu_{iV}}{\sum_{i=0}^K w_{iZ}}, && (18) \\
v^{**}_Z &:= \frac{1}{\sum_{i=0}^K w_{iZ}}, && (19) \\
w_{iZ} &:= \frac{\alpha_i}{\sigma^2_{iU} + \sigma^2_{iV}}. && (20)
\end{aligned}
$$

Clearly, $v^*_Z \leq v^{**}_Z$ and hence $\mu^{**}_Z \leq \mu^*_Z \; \forall \, \boldsymbol{\alpha}$.

Minimising Kullback-Leibler divergence in transformed space

One might argue that procedure (b) makes little sense, given that the set of opinions $\mathbf{F}_X$ concerns only X, i.e, it was not necessarily constructed taking the transformation $\phi(\cdot)$ into account. An example is a situation where experts are asked to provide distributions on the probability p of a particular event. In general, elicitation for $f_i(p)$ will not take into account the induced distribution on the log-odds, $\phi(p) = \log p/(1-p)$. Nevertheless, the decision-maker may wish to assign the weights $\boldsymbol{\alpha}$ in a way that takes $\phi(\cdot)$ into account, e.g., by giving lower weights to experts whose distributions on the log-odds scale are unreasonable.

This decision process can be made more precise. In a similar spirit to the paper, one can construct $\boldsymbol{\alpha}$ so as to minimise the Kullback-Leibler divergence between each distribution in $\mathbf{F}_{\mathbf{y}}^{-\mathbf{1}}$ and a transformation of the distribution obtained by procedure (a), $\pi_Y(y|\boldsymbol{\alpha}) = \pi_\theta(\phi^{-1}(y)|\boldsymbol{\alpha})|J_\phi|$. Let $d_i = \text{KL}(h_i(y)||\pi_Y(y|\boldsymbol{\alpha}))$.

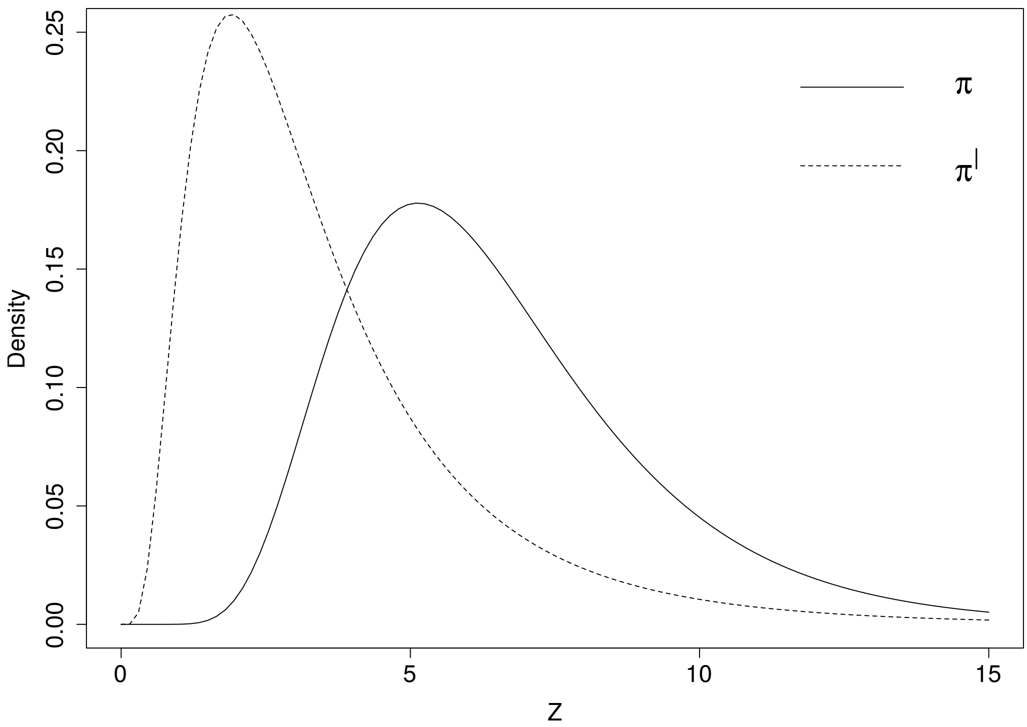

Figure 1: **Log-normal example**. Solid line displays $\pi_Z(Z|\boldsymbol{\alpha})$, obtained by first pooling the distributions on U and V and then computing the induced distribution on Z. Dashed line displays the logarithmic pooling of individual distributions r_{iZ}, $\pi'_Z(Z|\boldsymbol{\alpha})$. For this example $K = 2$, $\alpha_0 = 0.70$, $\mu_{0U} = 0.80$, $\sigma^2_{0U} = 0.40$, $\mu_{1U} = 0.5$, $\sigma^2_{1U} = 0.05$, $\mu_{0V} = -1.60$, $\sigma^2_{0V} = 0.024$, $\mu_{1V} = -1.25$ and $\sigma^2_{1V} = 0.4$.

We then aim at solving the problem

$$L(\boldsymbol{\alpha}) = \sum_{i=0}^{K} d_i \tag{21}$$
$$\hat{\boldsymbol{\alpha}} := \arg\min L(\boldsymbol{\alpha})$$

This procedure therefore choses weights for each expert by how coherent the prior provided by each expert is with the pool-then-Transform – procedure (a) – prior in the transformed space induced by $\phi(\cdot)$.

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References

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