

Logarithmic pooling and log-concavity

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Abstract

In this brief note I claim to show that logarithmic pooling is the *only* pooling operator that will *always* produce a log-concave opinion when all expert opinions are also log-concave.

Key-words: logarithmic pooling; log-concavity; uniqueness.

Background

Logarithmic pooling is a popular method for combining opinions on an agreed quantity, specially when these opinions can be framed as probability distributions. Let $\mathbf{F}_\theta := \{f_0(\theta), f_1(\theta), \dots, f_K(\theta)\}$ be a set of distributions representing the opinions of $K + 1$ experts and let $\boldsymbol{\alpha} := \{\alpha_0, \alpha_1, \dots, \alpha_K\} \in \mathcal{S}^K$ be the vector of weights, such that $\alpha_i > 0 \forall i$ and $\sum_{i=0}^K \alpha_i = 1$, i.e., \mathcal{S}^{K+1} is the space of all open simplices of dimension $K + 1$. The **logarithmic pooling operator** $\mathcal{LP}(\mathbf{F}_\theta, \boldsymbol{\alpha})$ is defined as

$$\mathcal{LP}(\mathbf{F}_\theta, \boldsymbol{\alpha}) := \pi(\theta|\boldsymbol{\alpha}) = t(\boldsymbol{\alpha}) \prod_{i=0}^K f_i(\theta)^{\alpha_i}, \quad (1)$$

where $t(\boldsymbol{\alpha}) = \int_{\Theta} \prod_{i=0}^K f_i(\theta)^{\alpha_i} d\theta$. This pooling method enjoys several desirable properties and yields tractable distributions for a large class of distribution families (Genest et al., 1984, 1986).

Definition 1. Relative propensity consistency (Genest et al., 1984). Taking \mathbf{F}_X as a set of expert opinions with support on a space \mathcal{X} , define $\boldsymbol{\xi} = \{\mathbf{F}_X, a, b\}$ for arbitrary $a, b \in \mathcal{X}$. Let \mathcal{T} be a pooling operator and define two functions U and V such that

$$U(\boldsymbol{\xi}) := \left(\frac{f_0(a)}{f_0(b)}, \frac{f_1(a)}{f_1(b)}, \dots, \frac{f_K(a)}{f_K(b)} \right) \text{ and} \quad (2)$$

$$V(\boldsymbol{\xi}) := \frac{\mathcal{T}_{\mathbf{F}_X}(a)}{\mathcal{T}_{\mathbf{F}_X}(b)}. \quad (3)$$

We then say that \mathcal{T} enjoys relative propensity consistency (RPC) if and only if

$$U(\boldsymbol{\xi}_1) \geq U(\boldsymbol{\xi}_2) \implies V(\boldsymbol{\xi}_1) \geq V(\boldsymbol{\xi}_2), \quad (4)$$

for all $\boldsymbol{\xi}_1, \boldsymbol{\xi}_2$.

Informally, this property says that if all experts consider a particular event A more probable than another event B , then the pooled opinion should be consistent with these relative judgments.

Lemma 1. Uniqueness of LP for RPC (Genest et al., 1984). Logarithmic pooling is the *only* pooling operator that enjoys RPC.

We refer the reader to Genest et al. (1984) for a proof.

Lemma 2. Representation of a pooling operator with RPC (Genest et al., 1984, eq. 3.1). The only relative propensity consistent operator can always be represented by

$$\mathcal{T}(\mathbf{F}_\theta)(\theta) = \mathbf{B}(\mathbf{F}_\theta) c(\theta) \prod_{i=0}^K [f_i(\theta)]^{w_i},$$

with $\mathbf{B}(\mathbf{F}_\theta) > 0$, $c(\theta) > 0$ and $w_0, w_1, \dots, w_K \geq 0$ arbitrary.

Again, see Genest et al. (1984) for a proof.

The result

Now we can state and prove the result (Remark 1).

Remark 1. Log-concavity. *If \mathbf{F}_θ is a set of log-concave distributions, then $\pi(\theta \mid \boldsymbol{\alpha})$ is also log-concave. Moreover, logarithmic pooling is the only pooling operator that will always produce a log-concave density.*

Proof. First, we will show by direct calculation that logarithmic pooling (LP) leads to a log-concave distribution. Notice that each f_i can be written as $f_i(\theta) \propto e^{\nu_i(\theta)}$, where $\nu_i(\cdot)$ is a concave function. We can then write

$$\begin{aligned}\pi(\theta \mid \boldsymbol{\alpha}) &\propto \prod_{i=0}^K [\exp(\nu_i(\theta))]^{\alpha_i}, \\ &\propto \exp(\nu^*(\theta)),\end{aligned}$$

where $\nu^*(\theta) = \sum_{i=0}^K \alpha_i \nu_i(\theta)$ is a concave function because it is a linear combination of concave functions.

We will now show that LP is the only operator that guarantees log-concavity when \mathbf{F}_θ is a set of concave distributions. First, recall that LP is the only pooling operator that enjoys RPC (Lemma 1). Then, with the goal of obtaining a contradiction, suppose that there exists a pooling operator \mathcal{T} that is log-concave but does not enjoy RPC. From Lemma 2, we know that \mathcal{T} cannot be represented as $\mathbf{B}(\mathbf{F}_\theta)c(\theta) \prod_{i=0}^K f_i(\theta)^{w_i}$. Every non-negative log-concave function $g(\theta)$ can be represented as

$$g(\theta) = a \cdot c(\theta) \cdot h(\theta), \tag{5}$$

with $a \geq 0$ and $c(\theta)$ and $h(\theta)$ non-negative and log-concave. But under the assumptions on \mathbf{F}_θ , we have that $h(\theta) := \prod_{i=0}^K f_i(\theta)^{w_i}$ is non-negative and log-concave and therefore \mathcal{T} can in fact be represented in the form of (5) and thus the form of Lemma 2, a contradiction. \square

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References

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