Logarithmic pooling and log-concavity

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Abstract

In this brief note I claim to show that logarithmic pooling is the *only* pooling operator that will *always* produce a log-concave opinion when all expert opinions are also log-concave.

Key-words: logarithmic pooling; log-concavity; uniqueness.

Background

Logarithmic pooling is a popular method for combining opinions on an agreed quantity, specially when these opinions can be framed as probability distributions. Let $\theta \in \Theta \subseteq \mathbb{R}$, $|\Theta| \geq 3$, be a quantity of interest and let $\mathbf{F}_\theta := \{f_0(\theta), f_1(\theta), \ldots, f_K(\theta)\}$ be a set of distributions representing the opinions of $K+1$ experts and let $\boldsymbol{\alpha} := \{\alpha_0, \alpha_1, \ldots, \alpha_K\} \in \mathcal{S}^K$ be the vector of weights, such that $\alpha_i > 0\, \forall i$ and $\sum_{i=0}^K \alpha_i = 1$, i.e., $\mathcal{S}^{K+1}$ is the space of all open simplices of dimension $K+1$. The **logarithmic pooling operator** $\mathcal{LP}(\mathbf{F}_\theta, \boldsymbol{\alpha})$ is defined as

$$\mathcal{LP}(\mathbf{F}_\theta, \boldsymbol{\alpha}) := \pi(\theta|\boldsymbol{\alpha}) = t(\boldsymbol{\alpha}) \prod_{i=0}^{K} f_i(\theta)^{\alpha_i}, \tag{1}$$

where $t(\boldsymbol{\alpha}) = \int_{\boldsymbol{\Theta}} \prod_{i=0}^K f_i(\theta)^{\alpha_i} d\theta$. This pooling method enjoys several desirable properties and yields tractable distributions for a large class of distribution families (Genest et al., 1984, 1986).

Another desirable property of the logarithmic pooling operator is log-concavity. Log-concavity of the pooled prior may be important to consider in order to guarantee unimodality and certain conditions on tail behaviour (Bagnoli and Bergstrom, 2005).

Definition 1. ***Relative propensity consistency (Genest et al., 1984).*** *Taking $\boldsymbol{F}_X$ as a set of expert opinions with support on a space $\mathcal{X}$, define $\boldsymbol{\xi} = \{\boldsymbol{F}_X, a, b\}$ for arbitrary $a, b \in \mathcal{X}$. Let $\mathcal{T}$ be a pooling operator and define two functions U and V such that*

$$U(\boldsymbol{\xi}) := \left(\frac{f_0(a)}{f_0(b)}, \frac{f_1(a)}{f_1(b)}, \ldots, \frac{f_K(a)}{f_K(b)}\right) \text{ and} \tag{2}$$

$$V(\boldsymbol{\xi}) := \frac{\mathcal{T}_{\boldsymbol{F}_X}(a)}{\mathcal{T}_{\boldsymbol{F}_X}(b)}. \tag{3}$$

We then say that $\mathcal{T}$ enjoys relative propensity consistency (RPC) if and only if

$$U(\boldsymbol{\xi}_1) \geq U(\boldsymbol{\xi}_2) \implies V(\boldsymbol{\xi}_1) \geq V(\boldsymbol{\xi}_2), \tag{4}$$

for all $\boldsymbol{\xi}_1, \boldsymbol{\xi}_2$.

Informally, this property says that if all experts consider a particular event A more probable than another event B, then the pooled opinion should be consistent with these relative judgments.

Lemma 1. ***Representation of a pooling operator with RPC*** *(Genest et al., 1984, eq. 1.4). When $|\Theta| \geq 3$, a relative propensity consistent operator is always of the form*

$$\mathcal{T}(\boldsymbol{F}_\theta)(\theta) = \boldsymbol{B}(\boldsymbol{F}_\theta)\, c(\theta) \prod_{i=0}^{K} [f_i(\theta)]^{w_i},$$

with $\boldsymbol{B}(\boldsymbol{F}_\theta) > 0$, $c(\theta) > 0$ and $w_0, w_1, \ldots, w_K \geq 0$ arbitrary.

We refer the reader to Genest et al. (1984) for a proof.

Corollary 1. ***Uniqueness of RPC for LP.*** *Logarithmic pooling is the* ***only*** *pooling operator that enjoys RPC.*

Proof. The result follows trivially from the general definition of the logarithmic pool (see e.g. Genest and Zidek (1986), eq. 1.7) and Lemma 1. ∎

The result

Now we can state and prove the following result.

Remark 1. ***Log-concavity.*** *If $\mathbf{F}_\theta$ is a set of log-concave distributions, then $\pi(\theta \mid \boldsymbol{\alpha})$ is also log-concave. Moreover, logarithmic pooling is the only pooling operator to preserve log-concavity.*

Proof. First, we will show by direct calculation that logarithmic pooling (LP) leads to a log-concave distribution. Notice that each f_i can be written as $f_i(\theta) \propto e^{\nu_i(\theta)}$, where $\nu_i(\cdot)$ is a concave function. We can then write

$$\begin{aligned}\pi(\theta \mid \boldsymbol{\alpha}) &\propto \prod_{i=0}^{K}[\exp(\nu_i(\theta))]^{\alpha_i},\\ &\propto \exp(\nu^*(\theta)),\end{aligned}$$

where $\nu^*(\theta) = \sum_{i=0}^{K} \alpha_i \nu_i(\theta)$ is a concave function because it is a linear combination of concave functions.

We will now show that LP is the only operator that guarantees log-concavity when $\boldsymbol{F}_\theta$ is a set of concave distributions. First, recall that LP is the only pooling operator that enjoys RPC as implied by Corollary 1. Then, with the goal of obtaining a contradiction, suppose that there exists a pooling operator $\mathcal{T}$ that is log-concave but does not enjoy RPC. From Lemma 1, we know that $\mathcal{T}$ cannot be represented as $\boldsymbol{B}(\boldsymbol{F}_\theta)c(\theta)\prod_{i=0}^{K} f_i(\theta)^{w_i}$. Every non-negative log-concave function $g(\theta)$ can be represented as

$$g(\theta) = a \cdot c(\theta) \cdot h(\theta), \tag{5}$$

with $a \geq 0$ and $c(\theta)$ and $h(\theta)$ non-negative and log-concave. But under the assumptions on $\boldsymbol{F}_\theta$, we have that $h(\theta) := \prod_{i=0}^{K} f_i(\theta)^{w_i}$ is non-negative and log-concave and therefore $\mathcal{T}$ can in fact be represented in the form of (5) and thus the form of Lemma 1, a contradiction. □

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