Homework 21 - MATH 3440-1

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November 2021

1 Introduction

This is a LaTeX writing assignment focusing on proofs and *mathematical induction*. It was assigned by Professor Ken Takata and is due on 11/15/21.

Definition 1 *Mathematical induction is a mathematical proof technique. It is essentially used to prove that a statement* $P(n)$ *holds for every natural number* $n = 0, 1, 2, 3, ...$ *(to be further explained in section 3.1).*

2 Write up in LaTeX a proof regarding the sum $\sum_{i=1}^{n} i = 1 + 2 + 3 + 4 + ... + n$ and find a closed form that requires at most $O(1)$ arithmetic operations (constant time).

Definition 2 *An algorithm is said to be O(1) (or constant time) if the value of T(n) is bounded by a value that does not depend on the size of the input. Examples of this may include accessing the index of an item in a list or array.*

2.1 Closed Form

A closed form of this equation that requires at most $O(1)$ arithmetic operations is:

$$\frac{n(n+1)}{2} \tag{1}$$

2.2 Proof

Let us assign the following equation to S.

$$1 + 2 + 3 + ... + n = S \tag{2}$$

and add the following equation to it (this equation is the previous equation reversed, starting with n and adding $(n - 1)$ until we reach 1).

$$n + (n-1) + (n-2) + ... + 1 = S \tag{3}$$

This step can be better visualized if we stack the components of the addition

	1 +	2 +	3 +	...	+ n = S
+	n +	(n-1) +	(n-2) +	...	+ 1 = S
	(n+1) +	(n+1) +	(n+1) +	...	+ (n+1) = 2S

We then get a sum of

$$(n+1) + (n+1) + (n+1)... + (n+1) = 2S \tag{4}$$

This is equivalent to n amounts of $(n+1)$, so we can rewrite this as

$$n(n+1) = 2S \tag{5}$$

then divide both sides by 2 to isolate S

$$\frac{n(n+1)}{2} = S \tag{6}$$

and we arrive at our closed form.

3 Now prove your formula using mathematical induction.

3.1 Intro

The principle of mathematical induction is to prove that $P(n)$ is true for all positive integers, n, where $P(n)$ is a propositional function. Two steps must be completed in mathematical induction:

- **Basis Step (or atomic step, base/atomic case):** Verify that $P(1)$ is true.

- **Inductive Step (or inductive case):** Show that the conditional statement $P(k) \rightarrow P(k+1)$ is true for all positive integers k.

3.2 Base/Atomic Case

In section 2 we proved that the closed form for the following

$$\sum_{i=1}^{n} i = 1 + 2 + 3 + 4 + ... + n \tag{7}$$

is equivalent to

$$\frac{n(n+1)}{2} \tag{8}$$

This can be written as

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2} \tag{9}$$

As 3.1 states, the base case must verify that $P(1)$ is true.

$$P(n) \equiv True \text{ means } \sum_{i=1}^{n} i = \frac{n(n+1)}{2} = 1 \tag{10}$$

$$P(1) \text{ means } \sum_{i=1}^{1} i = \frac{1(2)}{2} = 1 \tag{11}$$

This is true, and our base case is satisfied.

3.3 Inductive Case

Now we must show that the conditional statement $P(k) \rightarrow P(k+1)$ is true for all positive integers k.

$$\text{Assume } P(k)\text{: } \sum_{i=1}^{k} i = \frac{k(k+1)}{2} \tag{12}$$

$$\text{Show } P(k+1)\text{: } \sum_{i=1}^{k+1} i = \frac{(k+1)[(k+1)+1]}{2} \tag{13}$$

$P(1)$ was true, now we must test the inductive case with $P(k+1)$. We will test $P(1+1)$

$$\text{Show } P(1+1)\text{: } \sum_{i=1}^{1+1} i = \frac{(1+1)[(1+1)+1]}{2} = 3 \tag{14}$$

This proves that closed form for $P(1) \rightarrow P(1+1)$. We can programmatically test this mathematical induction by calculating the equation for more integers to satisfy $P(k+1)$ for k integers.

4 Bonus - Code

```
def eqn(n):
  sum = 0
  for i in range(1,n+1):
    sum+=i
  return sum

def clsd_frm(n):
  return (n*(n+1)) / 2

def test(n):
  bool_array = []
  for i in range(1,n+1):
    bool_array.append(eqn(i) == clsd_frm(i))
  return all(bool_array)

>>> test(10000)
True
```

Listing 1: This script tests if the equation and closed form are equal for a range of a given number. A boolean for each iteration is passed to an list. If all booleans in the list are true we know our test has passed.