Homework 22 - MATH 3440-1

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1 Introduction

This is a LaTeX writing assignment focusing on proofs and *mathematical induction*. It was assigned by Professor Ken Takata and is due on 11/17/21.

Definition 1 *Mathematical induction is a mathematical proof technique. It is essentially used to prove that a statement* $P(n)$ *holds for every natural number* $n = 0, 1, 2, 3, ...$ *(to be further explained in section 3.1).*

2 Prove that $n^2 < 2^n$ for $n > 4$

2.1 Intro

The principle of mathematical induction is to prove that $P(n)$ is true for all positive integers, n, where $P(n)$ is a propositional function. Two steps must be completed in mathematical induction:

- **Basis Step (or atomic step, base/atomic case):** Verify that $P(1)$ is true.

- **Inductive Step (or inductive case):** Show that the conditional statement $P(k) \to P(k+1)$ is true for all positive integers k.

2.2 Base/Atomic Case

Our base case will be $P(5)$ (or $P(4+1)$) since 5 is the first case that is greater than 4 which satisfies the conditions of the problem.

$$P(n, where\ n > 4) \equiv True \text{ means } n^2 < 2^n \tag{1}$$

$$P(5) \text{ means } 5^2 < 2^5 \Longrightarrow 25 < 32 \tag{2}$$

This is true, and our base case is satisfied.

2.3 Inductive Case

Now we must show that the conditional statement $P(k) \rightarrow P(k+1)$ is true for all positive integers k.

$$\text{Assume } P(k, where\ k > 4)\text{: } k^2 < 2^k \tag{3}$$

$$\text{Show } P(k+1)\text{: } (n+1)^2 < 2^{n+1} \rightarrow n^2 + 2n + 1 < 2 \times 2^n \tag{4}$$

Assuming our integer is greater than 4, we can conclude that

$$2^{n+1} = n \times 2^n > 2 \times n^2 > (n+1)^2 \tag{5}$$

The left side of the equation is following the induction step by adding one to the base case. As for the right side

$$(n-1)^2 \geq 4^2 > 2 \tag{6}$$

Since $n \geq 5$, the inequality $(n-1)^2 > 2$ can be expanded using basic algebraic rules

$$n^2 - 2n - 1 > 2 \tag{7}$$

$$n^2 - 2n - 1 > 0 \tag{8}$$

$$2n^2 - 2n - 1 > n^2 \tag{9}$$

$$2n^2 > n^2 + 2n + 1 \tag{10}$$

Which can be expressed as

$$2n^2 > (n+1)^2 \tag{11}$$

Which matches the second inequality in equation (5). Thus, we have proved the problem via mathematical induction. We can programmatically test this mathematical induction with the following code.

3 Bonus - Code

```
def proof(n):
  if n <= 4:
    print("False. n <= 4 doesn't satisfy this equation")

  else:
    bool_array = []
    for i in range(5,n+1):
      bool_array.append(i**2 < 2**i)
    return all(bool_array)

>>> proof(10000)
True
```

Listing 1: This script tests if $n^2 > 2^n$ for a range of a given number greater than 4. A boolean for each iteration is passed to an list. If all booleans in the list are true we know our test has passed.