# HYPERKÄHLER MANIFOLDS

#### NOTES TAKEN BY MAXIM JEAN-LOUIS BRAIS

### 1. First lecture

We first review some complex geometry.

**Definition 1.0.1.** A complex manifold is a locally ringed space  $(X, \mathcal{O}_X)$  such that

- X is Hausdorff and second countable (this part is to ensure we actually have a topological manifold);
- $(X, \mathcal{O}_X)$  is locally isomorphic to  $(\Delta, \mathcal{O}_\Delta)$ , where  $\Delta \subset \mathbb{C}^n$  is the polydisc.

**Example 1.0.2.** Let  $f_i, \ldots, f_d \in \mathbb{C}[z_1, \ldots, z_n]$  be complex polynomials such that the Jacobian of

$$f = (f_1, \dots, f_d) : \mathbb{C}^n \to \mathbb{C}^d$$

has everywhere full rank on the vanishing set  $V = V(f) \subset \mathbb{C}^n$ . By the holomorphic implicit function theorem (regular value theorem), V is a complex manifold.

**Example 1.0.3.** If X is a smooth algebraic variety over  $\mathbb{C}$ , we may cover it by affines  $V_i$  which are of the same form as in Example 1.0.2. We may consider the analytic topology  $X^{an}$  obtained by gluing the different charts  $V_i$ . Similarly, we may define the sheaf  $\mathcal{O}_X^{an}$  on  $X^{an}$  by considering the sheaf of holomorphic functions on each  $V_i$  (the transitions  $V_i \to V_j$  are regular algebraic, hence holomorphic, so that this gluing makes sense). Then,  $(X^{an}, \mathcal{O}_X^{an})$  is a complex manifold.

Note that in Example 1.0.3, we obtain a natural map of ringed spaces

$$\alpha: (X^{an}, \mathcal{O}_X^{an}) \to (X, \mathcal{O}_X)$$

since the analytic topology is finer than the Zariski topology, and regular functions are holomorphic. In particular, we obtain a functor between abelian categories:

$$\alpha^*: \mathcal{O}_X\operatorname{-mod} \to \mathcal{O}_X^{an}\operatorname{-mod}$$

restricting to

$$\alpha^* : \operatorname{Coh}(X) \to \operatorname{Coh}(X^{an}).$$

**Theorem 1.0.4** (Géométrie algébrique géométrie analytique; [Ser56]). If X is smooth<sup>1</sup> and proper functor  $\alpha^*: Coh(X) \to Coh(X^{an})$  is an equivalence, therefore inducing an isomorphism.

1.1. Almost complex structures. A complex manifold  $(X, \mathcal{O}_X)$  has an underlying smooth manifold  $(X, C_X^{\infty})$ , where  $C_X^{\infty}$  denotes the sheaf of smooth functions on X; indeed, if X has complex charts  $U_i \subset \mathbb{C}^n$ , the transitions are holomorphic, hence  $C^{\infty}$ .

**Notation 1.1.1.** Since the indices i will be ubiquitous,  $\iota$  shall denote the root of -1 for these notes (this spares the cumbersome  $\sqrt{-1}$  alternative).

On each chart  $U_i$ , we have multiplication by  $\iota$ , but this does not globalise, as  $\iota$  does not commute with holomorphic functions: in the Taylor expansion, we have terms which are of degree m where  $m \neq 1 \mod 4$ . However, the differential of  $\iota$  may be globalised, as we get rid of the higher order terms. In a local chart  $U_i \subset \mathbb{C}^n$ , the (real) tangent bundle has a local frame

$$T_{\mathbb{R}}U_i = \langle \partial_{x_j}, \partial_{y_j} : 1 \le j \le n \rangle,$$

on which  $I := d\iota$  acts by

$$\begin{cases} \partial_{x_j} \mapsto \partial_{y_j} \\ \partial_{y_j} \mapsto -\partial_{x_j}. \end{cases}$$

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<sup>&</sup>lt;sup>1</sup>This can be dropped by considering complex analytic spaces (rather than manifolds).

**Definition 1.1.2.** An almost complex structure on a smooth manifold X is an endomorphism  $I \in$  $\operatorname{End}(T_{\mathbb{R}}X)$  such that  $I^2=-1$ . We say that I is integrable if X is a complex manifold and I is obtained by locally differentiating  $\iota$ .

Question 1.1.3. Given I an almost complex structure, when is it integrable?

Let us first set up some tools in order to address this question appropriately. Assume only for now that X is a smooth manifold and I is an almost complex structure. We can consider the complexified tangent bundle

$$T_{\mathbb{C}}X := T_{\mathbb{R}}X \otimes_{\mathbb{R}} \mathbb{C},$$

to which we can extend the action of I. Since  $I^2 = -1$ , the minimal polynomial of I is  $x^2 + 1$ , which is separable over  $\mathbb{C}$ , meaning that I is diagonalisable, with eigenvalues  $\pm \iota$ . The eigenspaces must have the same dimension as I acts on the real tangent space. We thus obtain a decomposition

$$T_{\mathbb{C}}X = T^{1,0}X \oplus T^{0,1}X = T^{1,0}X \oplus \overline{T^{1,0}X}.$$

Note that we have

$$T^{1,0}X = \{(v - iIv) : v \in T_{\mathbb{C}}X\}$$
 
$$T^{0,1}X = \{(v + iIv) : v \in T_{\mathbb{C}}X\}.$$

Notation 1.1.4. We will use the following notation

- $\bullet \ \mathcal{A}^0(X) := C_X^{\infty};$
- $\mathcal{A}^k(X)$  denotes the sheaf of (smooth) degree k real forms;
- $\mathcal{A}^k(X,\mathbb{C}) = \bigoplus_{p+q=k} \mathcal{A}^{p,q}(X)$  denotes the sheaf of sections of  $\bigwedge^k T^*_{\mathbb{C}}X$  (i.e. smooth complex degree k forms) and  $\mathcal{A}^{p,q}(X)$  denotes the sheaf of sections of  $\bigwedge^p T^{1,0}X \otimes \bigwedge^q T^{0,1}X$ ;
- $d: \mathcal{A}^k(X,\mathbb{C}) \to \mathcal{A}^{k+1}(X,\mathbb{C})$  denotes the complexification of the usual exterior derivative, and can be decomposed by types as  $d = \partial + \bar{\partial}$ , where  $\partial$  denotes the part corresponding to the differentiation in holomorphic coordinates, and similarly  $\bar{\partial}$  for anti-holomorphic coordinates.
- $A^k(X)$ ,  $A^k(X,\mathbb{C})$ , and  $A^{p,q}(X)$  denotes the global sections of respectively  $\mathcal{A}^k(X)$ ,  $\mathcal{A}^k(X,\mathbb{C})$ , and  $\mathcal{A}^{p,q}(X)$ .
- $\mathcal{T}_X$  denotes the sheaf of homolorphic vector fields, i.e.  $\mathcal{T}_X := \mathcal{D}er(\mathcal{O}_X, \mathcal{O}_X)$ ;
- $\Omega_X := \mathcal{T}_X^*$  denotes the cotangent sheaf.

The following theorem answers Question 1.1.3.

**Theorem 1.1.5** (Newlander-Niremberg). I is integrable if and only if  $\bar{\partial}^2 = 0$ .

Note that this is equivalent to  $T^{1,0}X$  being closed under the (complex) Lie bracket (this is much related to the Frobenius theorem of differential geometry), and also equivalent of the vanishing of a certain tensor  $N_I$  called the *Nijenhuis* tensor.

1.2. Metrics. Let E be a real vector bundle on  $(X, C_X^{\infty})$ . A Riemannian metric g on E is a section of Sym<sup>2</sup>  $E^{\vee}$  such that for all  $p \in X$ ,  $g_p$  is positive definite. If E is a complex bundle, a **Hermitian metric** h is a map of sheaves  $E \otimes \overline{E} \to C^{\infty}(X,\mathbb{C})$  such that each  $h_p$  is Hermitian, i.e.  $h_p(e,f) = \overline{h_p(f,e)}$  and  $h_p(e,e) > 0$ for all  $e, f \in E_p$ . When  $E = T_{\mathbb{R}}X$ , we say that g (resp. h) is a **Riemannian** (resp. **Hermitiam**) metric on X (here, the almost complex structure I is used to put a  $\mathbb{C}$ -structure on  $T_{\mathbb{R}}X$ ).

If X is a complex manifold and h is a Hermitian metric, then we can write

$$h = g - i\omega$$

where  $g = \Re \mathfrak{e}(h)$  and  $\omega = -\Im \mathfrak{m}(h)$ . We obtain that g is a Riemannian metric, and  $\omega$  is skew-symmetric since

$$\omega(X,Y) = \frac{\iota}{2}(h - \overline{h})$$

and h is conjugate skew-symmetric. Thus,  $\omega \in A^2(X)$ .

**Definition 1.2.1.** (X, h) is **Kähler** if  $d\omega = 0$ .

That h is linear in the first variable and anti-linear in the second ensures that h(I-,I) = h(-,-), implying that g(I-,I-)=g(-,-), a property that is sometimes called **compatibility** of the metric with I. We have

$$\omega(-,-) = \frac{\iota}{2}(h(-,-) - \overline{h}(-,-)) = \frac{1}{2}(h(I-,-) + \overline{h}(I-,-)) = g(I-,-),$$

which also implies

$$\omega(-, I-) = g(-, -).$$

**Definition 1.2.2.** A form  $\omega \in A^2(X)$  is called **positive** if  $\omega(u, Iu) > 0$  for all  $u \in T_{\mathbb{R}}X$ . We see that a de Rham cohomology class in  $H^2(X, \mathbb{C})$  is **positive** if it can be represented by a positive form. If moreover  $\omega$  is I-invariant (or equivalently, of type (1, 1) after embedding  $A^2(X) \subset A(X, \mathbb{C})$ ), we say  $\omega$  is **Kähler**.

If  $\omega$  is Kähler, we may define the hermitian metric  $h_{\omega} = \omega(-, I-) - i\omega$ , and we have that  $\omega$  is Kähler if and only if  $(X, h_{\omega})$  is Kähler.

**Example 1.2.3.** Let  $X = \mathbb{P}^n$ , with projective coordinates  $Z_0, \ldots, Z_n$ . Let  $U_i$  be the  $Z_i \neq 0$  chart, and define  $z_j = \frac{Z_j}{Z_i}$ . We may define on  $U_i$  the metric

$$\omega_{FS} = \omega = i\partial \overline{\partial} \log \left( 1 + \sum_{j} z_{j} \overline{z}_{j} \right),$$

and one checks that these glue to a global form, which we call the **Fubini-Study metric**. Written as a Kähler potential this way shows that it is a Kähler metric.

Note that if  $(X, \omega)$  is Kähler, restricting the metric to a complex submanifold Y preserves all properties of Definition 1.2.2, and so  $(Y, \omega_Y)$  is Kähler. Thus, any projective manifold is Kähler.

1.3. Connections. Let E be a complex (the real case is identical) vector bundle on  $(X, C_X^{\infty})$ . A complex connection in E is a  $\mathbb{C}$ -linear map

$$\nabla: \mathcal{A}^0(E, \mathbb{C}) \to \mathcal{A}^1(E, \mathbb{C}),$$

(here  $\mathcal{A}^i(E,\mathbb{C}) = \mathcal{A}^i(X,\mathbb{C}) \otimes \Gamma(E)$ ) such that

$$\nabla (f \cdot s) = df \otimes s + f \cdot \nabla s$$

for all section s of E and  $f \in C_X^{\infty}$ .

If E is a holomorphic bundle on a complex manifold, we can define the operator

$$\overline{\partial}: \mathcal{A}^0(E) \to \mathcal{A}^{0,1}(E)$$

as follows: if  $\sigma_i$  is a local frame, and  $s=s^i\sigma_i$  a section, we let

$$(1.1) \overline{\partial}(s^i\sigma_i) := (\overline{\partial}s^i) \otimes \sigma_i.$$

Indeed, given another frame  $\tau_j$  related by  $\sigma_i = g_{ij}\tau_j$ , we have

$$\overline{\partial}(s^i) \otimes \sigma_i = \overline{\partial}(s_i) \otimes g_{ij}\tau_j = \overline{\partial}(g_{ij}s^i) \otimes \tau_j$$

since the transitions  $g_{ij}$  are holomorphic by assumption.

A (complex) connection being valued in  $\mathcal{A}^1(E,\mathbb{C}) = \mathcal{A}^{1,0}(E) \oplus \mathcal{A}^{0,1}(E)$ , we may split  $\nabla = \nabla^{1,0} + \nabla^{0,1}$ .

**Definition 1.3.1.** The complex connection  $\nabla$  in E is said to be **compatible** with the holomorphic structure if  $\nabla^{0,1} = \overline{\partial}$ . Suppose E has a hermitian metric h. We say  $\nabla$  is **compatible** with h if for any sections e, f, we have equality of forms

$$d(h(e, f)) = h(\nabla e, f) + h(e, \nabla f).$$

More geometrically, this says that h is parallel to the connection, i.e. constant along parallel transport, i.e. the connection has U(n)-holonomy. We say  $\nabla$  is a **Chern connection** if it is both compatible with the holomorphic structure and the hermitian metric.

**Theorem 1.3.2** (Chern). There exists a unique Chern connection.

When  $E = T_{\mathbb{R}}X$ , the Chern connection ought to be regarded as the complex geometric analogue of the Levi-Civita connection from Riemannian geoemtry. In fact this is more than an analogy. If h is a hermitian metric, the Levi-Civita connection of  $g = \Re \mathfrak{e}(h)$  can be complexified to a complex connection. It is a theorem that the Levi-Civita connection is the Chern connection if and only if (X, h) is Kähler.

We can extend the connection  $\nabla: \mathcal{A}^0(E,\mathbb{C}) \to \mathcal{A}^1(E,\mathbb{C})$  to a connection

$$\nabla: \mathcal{A}^p(E,\mathbb{C}) \to \mathcal{A}^{p+1}(E,\mathbb{C})$$

for all positive p via

$$\nabla(\omega \otimes s) = d\omega \otimes s + (-1)^p \omega \wedge \nabla s,$$

where  $\omega$  is a p-form and s is a section of E.

**Remark 1.3.3.** Note this different to the usual extension of a connection to tensors since we are dealing with skew-symmetric forms. In particular, this satisfied a different Leibniz rule:

$$\nabla(fs) = df \wedge s \otimes d\nabla s.$$

**Definition 1.3.4.** We define the **curvature** of  $\nabla$  to be the composition  $\nabla^2 = \nabla \circ \nabla = F_{\nabla}$ .

Note that

$$\nabla(\nabla f s) = \nabla(df \otimes s + f \nabla s) = ddf - df \wedge \nabla s + \nabla(f \nabla s)$$
$$= -df \wedge \nabla s + df \wedge \nabla s + f \nabla^2 s = f \nabla^2 s$$

so that  $F_{\nabla}$  is  $C_X^{\infty}$ -linear, that is a section of  $\mathcal{A}^2(End(E), \mathbb{C})$ .

We may also define

$$F_{\nabla}^{k} := \underbrace{F_{\nabla} \circ \cdots \circ F_{\nabla}}_{k} \in \mathcal{A}^{2k}(End(E), \mathbb{C}).$$

We define the kth Chern character of  $\nabla$  to be

$$\operatorname{ch}_k(E, \nabla) := \operatorname{Tr}\left(\frac{1}{k!} \left(\frac{\iota}{2\pi} F_{\nabla}^k\right)\right) \in A^{2k}(X, \mathbb{C}).$$

**Theorem 1.3.5** (Chern-Weil). The following is true about the Chern character.

- (1)  $ch_k(E, \nabla)$  is closed;
- (2) The cohomology class  $ch_k(E) := [ch_k(E, \nabla)] \in H^{2k}_{dR}(X, \mathbb{C})$  is independent of  $\nabla$ ;
- (3)  $ch_k(E)$  is real, i.e. in  $H^{2k}_{dR}(X,\mathbb{R})$  (in fact, it is integral);
- (4) The total Chern character  $\sum_k ch_k(E)$  is equal to the cohomology class of  $Tr(\exp(\frac{\iota}{2\pi}F_{\nabla}))$  (this one directly follows from developing the exponential).

#### 2. Second Lecture

2.1. Chern classes. Let V be a vector space over  $\mathbb{C}$  of dimension r. Let  $P \in \mathbb{C}[End(V)]$  be a homogeneous polynomial of degree k. Assume moreover P is GL(V) invariant, that is  $P(A^{-1}BA) = P(B)$  for any  $A \in GL(V)$ .

Let now E be a complex vector bundle and  $\nabla$  a connection. By GL(V) invariance,  $P(\frac{\iota}{2\pi}F_{\nabla})$  is well-defined, and lives in  $A^{2k}(X,\mathbb{C})$ .

Fact 2.1.1 (Chern-Weil).  $P(\frac{\iota}{2\pi}F_{\nabla})$  is closed, and the class  $[P(\frac{\iota}{2\pi}F_{\nabla})] \in H^{2k}(X,\mathbb{C})$  is independent of  $\nabla$ .

Consider now the GL(V)-invariant homogeneous polynomials  $P_k$  returning the coefficients of the characteristic polynomials (i.e.  $P_k$  kth elementary symmetric polynomial on the eigenvalues). We can explicitly define  $P_k$  by the formula:

$$\det(I + tB) = \sum_{k} P_k(B)t^k.$$

We define the kth Chern class of E to be the cohomology class of  $P_k(\frac{i}{2\pi}F_{\nabla})$ , and the total Chern class of E to be  $c(E) := \sum_{i=0}^{k} c_k(E)$ . The Chern classes and characters satisfy certain properties:

- $c_0(E) = 1$  and  $ch_0 = r$ ;
- $c_d = 0$  if d > r. In particular, if L is a line bundle,  $c(L) = 1 + c_1(L)$ ;  $ch(L) = \exp(c_1(L)) := 1 + c_1(L) + \frac{c_1(L)^2}{2!} + \dots \in H^{2\bullet}(X, \mathbb{C})$ .

To derive further elementary properties of the Chern characters and classes, let us first observe how we can assemble connections into new connections

Let  $E_1$  and  $E_2$  be vector bundles with respective (complex) connections  $\nabla_1$  and  $\nabla_2$ . Then,

•  $\nabla_{E_1 \oplus E_2} := \nabla_1 \oplus \nabla_2$  is a connection on  $E_1 \oplus E_2$ , and  $F_{\nabla_1 \oplus \nabla_2} = F_{\nabla_1} \oplus F_{\nabla_2}$ , where this is seen as a

$$\begin{pmatrix} F_{\nabla_1} & \\ & F_{\nabla_2} \end{pmatrix};$$

•  $\nabla_{E_1 \otimes E_2} := \nabla_1 \otimes \mathrm{id}_{E_2} + \mathrm{id}_{E_1} \otimes \nabla_2$  is a connection on  $E_1 \otimes E_2$ ;

• the assignment

$$(\nabla^{\vee}\phi)(s) := d(\phi(s)) - \phi(\nabla(s))$$

where  $s \in E$  and  $\phi \in E^{\vee}$  defines a connection on  $E^{\vee}$  (note that this is the usual way to extend connections on tensors). In other words, if  $\langle -, - \rangle$  denotes the natural pairing on  $E \otimes E^{\vee}$ , the dual connection is defined by

$$d\langle s, \phi \rangle = \langle \nabla s, \phi \rangle + \langle s, \nabla^{\vee} \phi \rangle.$$

Let us try to compare the two curvature. Given  $s_i$  and  $t_j$  frames of E, we consider the connexion form  $A = (A_i^j)$  satisfying  $\nabla s_i = A_i^j \otimes t_j$ . Let  $s^i$  and  $t^j$  be the dual frames. We obtain

$$\begin{split} d\langle s_i, t^j \rangle &= 0 = \langle \nabla s_i, t^j \rangle + \langle s_i, \nabla^{\vee} t^j \rangle \\ &= \langle A_i^{\ k} \otimes t_k, t^j \rangle + \langle s_i, B_k^j \otimes s^k \rangle \\ &= A_i^{\ j} + B_i^j, \end{split}$$

where  $B=(B_i^j)$  is the connection form of  $\nabla^\vee$ . And so we have  $B=-A^t$  as sections of  $\mathcal{A}^2(End(E),\mathbb{C})=\mathcal{A}^2(End(E^\vee),\mathbb{C})$ . Using Cartan's formula for the curvature of a connection, we conclude

$$F_{\nabla^{\vee}} = d(-A^t) + (-A^t) \wedge (-A^t) = -(dA + A \wedge A)^t = -F_{\nabla}^t$$

• connections pull back, that is if  $f: Y \to X$  is a smooth map and E is a bundle on X with connection  $\nabla_E$ , we may define the connection  $\nabla_{f^*E}$  by locally demanding

$$\nabla_{f^*E}(f^*s) = f^*\nabla s.$$

Corollary 2.1.2. Let  $E_1, E_2$  be complex vector bundles on X. The following hold:

- (1)  $ch(E_1 \oplus E_2) = ch(E_1) + ch(E_2)$  and  $c(E_1 \oplus E_2) = c(E_1) \cup c(E_2)$ ;
- (2)  $ch(E_1 \otimes E_2) = ch(E_1) \cup ch(E_2);$
- (3)  $c_k(E^{\vee}) = (-1)^k c_k(E)$
- (4)  $ch_k(f^*E) = f^*(ch_k(E))$  and and  $c_k(f^*E) = f^*(c_k(E))$ .

Note that  $c_k(E) \in H^{2k}(X,\mathbb{C})$  is in fact real: indeed, conjugation acts on via  $\overline{F_{\nabla}} = F_{\overline{\nabla}}$ , and up to choosing a hermitian metric, we have  $E^{\vee} \simeq \overline{E}$  and  $F_{\overline{\nabla}} = -F_{\nabla}^t$ , and in a Chern splitting, the eigenvalues  $\omega_1, \ldots, \omega_r$ are purely imaginary. We obtain

$$c_k(E) = P_k(\frac{\iota}{2\pi}F_{\nabla}) = \left[\frac{\iota^k}{(2\pi)^k}P_k(F_{\nabla})\right] = \left[\frac{\iota^k}{(2\pi)^k}\sigma_k(\omega_1,\dots,\omega_r)\right]$$

while

$$\overline{c_k(E)} = \left\lceil \frac{(-1)^k \iota^k}{(2\pi)^k} \sigma_k(-\omega_1, \dots, -\omega_r) \right\rceil = \left\lceil \frac{(-1)^{2k} \iota^k}{2\pi} \sigma_k(\omega_1, \dots, \omega_r) \right\rceil = c_k(E)$$

where  $\sigma_k$  denotes the kth standard symmetric polynomial, so that  $c_k(E) \in H^{2k}(X,\mathbb{R})$ . The kth Chern class is also (1,1). Indeed, consider the Chern connection  $\nabla = \nabla^{1,0} + \nabla^{0,1} = \nabla^{1,0} + \overline{\partial}$ . From this decomposition, we see that the (0,2)-part of the curvature is  $\overline{\partial}^2 = 0$ . Similarly, one can use the fact that the hermitian metric is parallel to show that the (2,0) part vanishes so that all  $\omega_i$  are of type (1,1), from which one obtains that  $c_k(E)$  is of type (k, k).

Let  $D \subset X$  be a divisor. It is a fact that the fundamental class of D is the Chern class of  $\mathcal{O}(D)$ , i.e.

$$[D] = c_1(\mathcal{O}(D)) \in H^2(X, \mathbb{C}),$$

showing that the first—and therefore fore any—Chern class is integral.

An important theorem relating to chern classes is Kodaira's embedding theorem.

**Theorem 2.1.3** ([Kod54]). A (holomorphic) line bundle L on a complex manifold X is ample (i.e. induces an embedding in projective space) if and only if it admits a metric h such that  $\frac{\iota}{2\pi}F_{D_h}$  is a positive form.

In particular, let  $h_0$  be any hermitian metric on L, and let  $\omega_0 = \frac{\iota}{2\pi} F_{D_{h_0}}$ . Assuming X is Kähler, if  $c_1(L) = [\omega_0]$  is positive, i.e. if it has a positive form  $\omega$  as representative of the cohomology class, then we can write  $\omega = \omega_0 + \frac{\iota}{2\pi} \partial \partial \phi$  for some function  $\phi$  by the  $\partial \partial$ -lemma. Then, one may compute that the metric  $h := e^{-\phi} \cdot h_0$  satisfies  $\frac{\iota}{2\pi} F_{D_h} = \omega$ ; indeed the (1,0)-part of the Chern of the connection is

$$h^{-1}\partial h = \partial \log h = \partial \log(e^{-\phi}h_0) = \partial \log h_0 + \partial(-\phi)$$

so that the curvature is given by

$$F_{D_{h_0}} + \overline{\partial}\partial(-\phi) = F_{D_{h_0}} + \partial\overline{\partial}\phi.$$

In particular, a line bundle L is ample if and only if  $c_1(L)$  is positive, i.e. is represented by a Kähler form. Now, on a compact manifold, slightly perturbing a Kähler form inside  $H^{1,1}(X,\mathbb{R})$  still yields a Kähler form, since it preserves the positivity criterion. Thus, Kähler forms form an open positive cone  $\mathcal{K}_X$  inside of  $H^{1,1}(X,\mathbb{R})$  (scaling by a positive real preserves Kählerness). Moreover, by the Lefschetz theorem on (1,1)classes, the map Chern map  $Pic(X) \to H^{1,1}(X,\mathbb{Z})$  is surjective. Thus, we conclude that a compact complex manifold is projective if and only if  $\mathcal{K}_X$  intersects with  $H^{1,1}(X,\mathbb{Z})$  inside of  $H^{1,1}(X,\mathbb{R})$ .

### 2.2. Hirzebruch-Riemann-Roch.

**Definition 2.2.1.** Let X be a compact complex manifold. Let  $\nabla$  be a connection in the tangent bundle. We define the **Todd class** of X to be

$$td(X) := \left[ \det \left( \frac{\frac{\iota}{2\pi} F_{\nabla}}{1 - \exp(\frac{-\iota}{2\pi} F_{\nabla})} \right) \right] \in H^{2\bullet}(X, \mathbb{C})$$

In terms of Chern roots  $\omega_1, \ldots, \omega_r$ , we have that

$$td(X) = \prod_{i=1}^{r} \frac{\omega_i}{1 - e^{-\omega_i}}$$

It can be computed that we have

$$td_0(X) = 1;$$
  $td_1(X) = \frac{c_1}{2};$   $td_2(X) = \frac{1}{12}(c_1^2 + c_2);$   $td_3(X) = \frac{c_1c_2}{24}$   $td_4(X); = \frac{-c_1^4 + 4c_2 + c_1c_3 + 3c_2^2 - c_4}{720};$   $\cdots$ 

where  $td_k(X)$  denotes the kth homogeneous component of td(X) and  $c_i = c_i(X) := c_i(\mathcal{T}_X)$ .

**Theorem 2.2.2** (Hirzebruch-Rieman-Roch). Let E be a holomorphic vector bundle on a compact complex manifold X. Then, we have equality

(2.1) 
$$\chi(X,E) := \sum_{k} (-1)^k h^i(X,E) = \int_X ch(E) \cup td(X).$$

Note that since we are integrating over X, we only need to consider the top degree parts.

**Example 2.2.3.** Let X = C be a compact Riemann surface and L be a line bundle on X. we have  $ch(E) = 1 + c_1(L)$  and  $td(1) = 1 + \frac{c_1(X)}{2}$ . thus, we have

$$\chi(X,L) = \int_X 1 + c_1(L) + \frac{c_1(X)}{2} + \frac{c_1(L)c_1(X)}{2} = \int_X c_1(L) + \frac{c_1(X)}{2} = \deg(L) + \frac{\deg(\mathcal{T}_C)}{2}.$$

What is remarkable about this theorem is that the left-hand side of (2.1) is purely holomorphic (or algebraic) whilst the right-hand side is purely topological. Another similar theorem is the algebro-geometric Gauss-Bonnet theorem.

**Theorem 2.2.4.** Let X be a compact complex dimension of dimension n. Then,

$$\chi_{top}(X) := \sum_{i} (-1)^{i} b_{i}(X) = \int_{X} c_{n}(X).$$

Recall the classical relation between the Euler characteristic  $\chi_{top}$  and the genus g of a topological surface:  $\chi_{top} = 2 - 2g$ . In particular, this implies that for a compact Riemann surface C as above, that

$$\int_X c_1(X) = \deg(\mathcal{T}_C) = 2 - 2g.$$

In particular, in light of what we found in Example 2.2.3, we recover the classical Riemann-Roch theorem:

$$\chi(X, L) = \deg(L) - g + 1.$$

## 2.3. Kähler-Einstein manifolds.

Question 2.3.1. When does a smooth projective variety over  $\mathbb{C}$  admit a "canonical" metric?

**Definition 2.3.2.** Let  $(X,\omega)$  be a compact Kähler manifold, and let  $D_{\omega}$  be the corresponding Chern connection. We define the **Ricci form**  $Ric(\omega)$  of  $\omega$  to be

$$\operatorname{Ric}(\omega) = i\operatorname{Tr}(F_{D_{\omega}}) \in A^2(X, \mathbb{C}).$$

We say that  $(X, \omega)$  is **Kähler-Einstein** if  $Ric(\omega) = \lambda \omega$  for some constant  $\lambda \in \mathbb{R}$ .

**Remark 2.3.3.** We make the following comments.

- (1) Recall that we argued earlier that all the Chern roots of  $F_{D_{ij}}$  were pure imaginary of type (1,1) so that  $Ric(\omega)$  is real of type (1,1).
- (2) Note also that  $D_{\omega}$  is invariant under rescaling  $\omega$  by some  $\lambda > 0$  (indeed, parallelness of h is unaffected so we get the same connection). Thus, we may always assume that  $\lambda = -1, 0, 1$ .
- (3) since  $c_1(X) = \left[\frac{\iota}{2\pi} \operatorname{Tr} F_{D_{\omega}}\right]$  by definition, we have  $\left[\operatorname{Ric}(\omega)\right] = 2\pi c_1(X) \in H^2(X,\mathbb{R})$ .
- (4)  $\lambda$  is proportional to the scalar curvature, and so  $(X,\omega)$  being Kähler-Einstein implies that the scalar curvature with respect to  $g_{\omega} = \omega(I-, -)$  is constant.
- (5) If X is Kähler-Einstein, then we have

$$c_1(X) = \begin{cases} 0 \\ \pm \text{ positive form.} \end{cases}$$

**Definition 2.3.4.** We say that a complex manifold X is Calabi-Yau if  $c_1(X) = 0$ , Fano if  $c_1(X)$  is positive, of general type (or canonically polarised) if  $-c_1(X)$  is positive.

Note that by Kodaira's embedding theorem, Fano and general type manifolds are projective.

Caution 2.3.5. It is not because a manifold fits in this trichotomy that it admits a Kähler-Einstein metric. In fact, there exist Fano varieties with no Kähler-Einstein metric. Whether a Fano variety admits such metric is equivalent to K-stability, a purely algebro-geometric notion. Nonetheless, Yau (cf. [Yau78]) proved that any Calabi-Yau manifodl admits a Kähler-Einstein metric, and Aubin-Yau (cf. [Aub76; Yau78]) proved the same for general type manifolds.

Note also that not all manifolds fit in this trichotomy.

**Example 2.3.6** (Curves). Let us see how these categories apply to curves.

- g=0 gives only  $\mathbb{P}^1$ . Since it is diffeomorphic to a sphere, we have positive scalar curvature. And indeed, the Fubini-Study metric is Kähler-Einstein with  $\lambda = 1$ . Note also that  $\mathbb{P}^1$  is Fano.
- The g=1 case corresponds to elliptic curves. These are Ricci-flat and Calabi-Yau.
- The case q>1 are of general type, and there exists a Kähler-Einstein metric with negative scalar curvature.

For Fano manifolds, here is a summary of the known classifications:

- (1) In dimension 1 there is only the projective line.
- (2) In dimension 2, they are called del Pezzo surfaces. There are 10 different deformation families. First,  $\mathbb{P}^2$  and  $\mathbb{P}^1 \times \mathbb{P}^1$  which are isolates. The other 8 families are obtained by blowing up  $\mathbb{P}^2$  at d points in general position, where  $1 \le d \le 8$ .
- (3) In dimension 3, Mukai proved there are 105 families.
- (4) In dimension 4, we know there are finitely many families but it remains open to know how many.

For Calabi-Yau manifolds, there is the following structural theorem.

Theorem 2.3.7 (Beauville-Bogomolov). Let X be Kähler and Calabi-Yau. Then, there exists an étale cover  $\tilde{X} \to X$  such that

$$\tilde{X} = T \times \prod_{i} X_{i} \times \prod_{i} V_{i}$$

 $\tilde{X} = T \times \prod_{j} X_{j} \times \prod_{i} V_{i}$  where T is a torus,  $X_{j}$  is **hyperkähler** for all j, and  $V_{i}$  are **strict Calabi-Yau** for all i.

We now define the terms.

**Definition 2.3.8.** A compact Kähler manifold V is called **strict Calabi-Yau** if

•  $K_V \simeq \mathcal{O}_V$  is trivial, where  $K_V$  denotes the canonical bundle;

- V is simply connected;
- $H^i(V, \mathcal{O}_V) = 0$  for all  $i < i < \dim V$ .

A complex manifold X is **hyperkähler** if

- it is simply connected;
- $H^0(X, \Omega_X^2) \simeq \mathbb{C}\sigma$  where  $\sigma$  is holomorphic symplectic (in particular, it induces an isomorphism  $\mathcal{T}_X \simeq \Omega_X$ ).

Remark 2.3.9. If V is a strict Calabi-Yau of dimension greater than two, then  $h^{2,0} = h^{0,2} = 0$ . In particular,  $H^{1,1}(X,\mathbb{C}) = H^2(X,\mathbb{C})$ , and so  $H^{1,1}(X,\mathbb{R}) = H^2(X,\mathbb{R}) = H^{1,1}(X,\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{R}$ . Since the Kähler cone is not empty by assumption, we conclude that it intersects  $H^{1,1}(X,\mathbb{Z})$ , so that V is projective by our discussion on Kodaira's embedding theorem. In dimension 2, non-projective K3 surfaces yield an example of a non-projective strict Calabi-Yau manifold.

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