# HYPERKÄHLER MANIFOLDS

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Abstract. These are personal notes for the course on hyperkähler manifolds taught by Alessio Bottini at Universität Bonn in the Winter 2025-2026 semester. Please email me at s37mbrai@uni-bonn.de if you notice any typo or mistake.

## Contents



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We first review some complex geometry.

**Definition 1.0.1.** A **complex manifold** is a locally ringed space $(X, \mathcal{O}_X)$ such that

- $X$ is Hausdorff and second countable (this part is to ensure we actually have a topological manifold);
- $(X, \mathcal{O}_X)$ is locally isomorphic to $(\Delta, \mathcal{O}_\Delta)$, where $\Delta \subset \mathbb{C}^n$ is the polydisc.

**Example 1.0.2.** Let $f_i, \ldots, f_d \in \mathbb{C}[z_1, \ldots, z_n]$ be complex polynomials such that the Jacobian of

$$f = (f_1, \ldots, f_d) : \mathbb{C}^n \to \mathbb{C}^d$$

has everywhere full rank on the vanishing set $V = V(f) \subset \mathbb{C}^n$. By the holomorphic implicit function theorem (regular value theorem), $V$ is a complex manifold.

**Example 1.0.3.** If $X$ is a smooth algebraic variety over $\mathbb{C}$, we may cover it by affines $V_i$ which are of the same form as in Example 1.0.2. We may consider the analytic topology $X^{an}$ obtained by gluing the different charts $V_i$. Similarly, we may define the sheaf $\mathcal{O}_X^{an}$ on $X^{an}$ by considering the sheaf of holomorphic functions on each $V_i$ (the transitions $V_i \to V_j$ are regular algebraic, hence holomorphic, so that this gluing makes sense). Then, $(X^{an}, \mathcal{O}_X^{an})$ is a complex manifold.

Note that in Example 1.0.3, we obtain a natural map of ringed spaces

$$\alpha : (X^{an}, \mathcal{O}_X^{an}) \to (X, \mathcal{O}_X)$$

since the analytic topology is finer than the Zariski topology, and regular functions are holomorphic. In particular, we obtain a functor between abelian categories:

$$\alpha^* : \mathcal{O}_X\text{-mod} \to \mathcal{O}_X^{an}\text{-mod}$$

restricting to

$$\alpha^* : \mathrm{Coh}(X) \to \mathrm{Coh}(X^{an}).$$

**Theorem 1.0.4** (Géométrie algébrique géométrie analytique; [Ser56])**.** *If $X$ is smooth*[1] *and proper functor $\alpha^* : Coh(X) \to Coh(X^{an})$ is an equivalence, which moreover induces an isomorphism of sheaf cohomology.*

## 1.1. Almost complex structures.

A complex manifold $(X, \mathcal{O}_X)$ has an underlying smooth manifold $(X, C_X^\infty)$, where $C_X^\infty$ denotes the sheaf of smooth functions on $X$; indeed, if $X$ has complex charts $U_i \subset \mathbb{C}^n$, the transitions are holomorphic, hence $C^\infty$.

**Notation 1.1.1.** Since the indices $i$ will be ubiquitous, $\iota$ shall denote the root of $-1$ for these notes (this spares the cumbersome $\sqrt{-1}$ alternative).

On each chart $U_i$, we have multiplication by $\iota$, but this does not globalise, as $\iota$ does not commute with holomorphic functions: in the Taylor expansion, we have terms which are of degree $m$ where $m \neq 1 \mod 4$. However, the differential of $\iota$ may be globalised, as we get rid of the higher order terms. In a local chart $U_i \subset \mathbb{C}^n$, the (real) tangent bundle has a local frame

$$T_\mathbb{R} U_i = \langle \partial_{x_j}, \partial_{y_j} : 1 \leq j \leq n \rangle,$$

on which $I := d\iota$ acts by

$$\begin{cases} \partial_{x_j} \mapsto \partial_{y_j} \\ \partial_{y_j} \mapsto -\partial_{x_j}. \end{cases}$$

**Definition 1.1.2.** An **almost complex structure** on a smooth manifold $X$ is an endomorphism $I \in \mathrm{End}(T_\mathbb{R} X)$ such that $I^2 = -1$. We say that $I$ is integrable if $X$ is a complex manifold and $I$ is obtained by locally differentiating $\iota$.

**Question 1.1.3.** Given $I$ an almost complex structure, when is it integrable?

Let us first set up some tools in order to address this question appropriately. Assume only for now that $X$ is a smooth manifold and $I$ is an almost complex structure. We can consider the complexified tangent bundle

$$T_\mathbb{C} X := T_\mathbb{R} X \otimes_\mathbb{R} \mathbb{C},$$

[1] This can be dropped by considering complex analytic spaces (rather than manifolds).

to which we can extend the action of $I$. Since $I^2 = -1$, the minimal polynomial of $I$ is $x^2+1$, which is separable over $\mathbb{C}$, meaning that $I$ is diagonalisable, with eigenvalues $\pm\iota$. The eigenspaces must have the same dimension as $I$ acts on the *real* tangent space. We thus obtain a decomposition

$$T_{\mathbb{C}}X = T^{1,0}X \oplus T^{0,1}X = T^{1,0}X \oplus \overline{T^{1,0}X}.$$

Note that we have

$$T^{1,0}X = \{(v - iIv) : v \in T_{\mathbb{C}}X\} \qquad T^{0,1}X = \{(v + iIv) : v \in T_{\mathbb{C}}X\}.$$

**Notation 1.1.4.** We will use the following notation

- $\mathcal{A}^0(X) := C^\infty_X$;
- $\mathcal{A}^k(X)$ denotes the sheaf of (smooth) degree $k$ real forms;
- $\mathcal{A}^k(X,\mathbb{C}) = \oplus_{p+q=k}\mathcal{A}^{p,q}(X)$ denotes the sheaf of sections of $\bigwedge^k T^*_{\mathbb{C}}X$ (i.e. smooth complex degree $k$ forms) and $\mathcal{A}^{p,q}(X)$ denotes the sheaf of sections of $\bigwedge^p T^{1,0}X \otimes \bigwedge^q T^{0,1}X$;
- $d : \mathcal{A}^k(X,\mathbb{C}) \to \mathcal{A}^{k+1}(X,\mathbb{C})$ denotes the complexification of the usual exterior derivative, and can be decomposed by types as $d = \partial + \bar\partial$, where $\partial$ denotes the part corresponding to the differentiation in holomorphic coordinates, and similarly $\bar\partial$ for anti-holomorphic coordinates.
- $A^k(X)$, $A^k(X,\mathbb{C})$, and $A^{p,q}(X)$ denotes the global sections of $\mathcal{A}^k(X), \mathcal{A}^k(X,\mathbb{C})$, and $\mathcal{A}^{p,q}(X)$ respectively.
- $\mathcal{T}_X$ denotes the sheaf of homolorphic vector fields, i.e. $\mathcal{T}_X := \mathcal{D}er(\mathcal{O}_X, \mathcal{O}_X)$;
- $\Omega_X := \mathcal{T}_X^*$ denotes the cotangent sheaf.

The following theorem answers Question 1.1.3.

**Theorem 1.1.5** (Newlander-Niremberg)**.** *$I$ is integrable if and only if $\bar\partial^2 = 0$.*

Note that this is equivalent to $T^{1,0}X$ being closed under the (complex) Lie bracket (this is much related to the Frobenius theorem of differential geometry), and also equivalent of the vanishing of a certain tensor $N_I$ called the *Nijenhuis* tensor.

1.2. **Metrics.** Let $E$ be a real vector bundle on $(X, C^\infty_X)$. A **Riemannian metric** $g$ on $E$ is a section of $\mathrm{Sym}^2 E^\vee$ such that for all $p \in X$, $g_p$ is positive definite. If $E$ is a complex bundle, a **Hermitian metric** $h$ is a map of sheaves $E \otimes \overline{E} \to C^\infty(X,\mathbb{C})$ such that each $h_p$ is Hermitian, i.e. $h_p(e,f) = \overline{h_p(f,e)}$ and $h_p(e,e) > 0$ for all $e, f \in E_p$. When $E = T_{\mathbb{R}}X$, we say that $g$ (resp. $h$) is a **Riemannian** (resp. **Hermitiam**) **metric on** $X$ (here, the almost complex structure $I$ is used to put a $\mathbb{C}$-structure on $T_{\mathbb{R}}X$).

If $X$ is a complex manifold and $h$ is a Hermitian metric, then we can write

$$h = g - i\omega$$

where $g = \mathfrak{Re}(h)$ and $\omega = -\mathfrak{Im}(h)$. We obtain that $g$ is a Riemannian metric, and $\omega$ is skew-symmetric since

$$\omega(X,Y) = \frac{\iota}{2}(h - \overline{h})$$

and $h$ is conjugate skew-symmetric. Thus, $\omega \in A^2(X)$.

**Definition 1.2.1.** $(X,h)$ is **Kähler** if $d\omega = 0$.

That $h$ is linear in the first variable and anti-linear in the second ensures that $h(I-, I-) = h(-,-)$, implying that $g(I-, I-) = g(-,-)$, a property that is sometimes called **compatibility** of the metric with $I$. We have

$$\omega(-,-) = \frac{\iota}{2}(h(-,-) - \overline{h}(-,-)) = \frac{1}{2}(h(I-,-) + \overline{h}(I-,-)) = g(I-,-),$$

which also implies

$$\omega(-, I-) = g(-,-).$$

**Definition 1.2.2.** A form $\omega \in A^2(X)$ is called **positive** if $\omega(u, Iu) > 0$ for all $u \in T_{\mathbb{R}}X$. We see that a de Rham cohomology class in $H^2(X,\mathbb{C})$ is **positive** if it can be represented by a positive form. If moreover $\omega$ is $I$-invariant (or equivalently, of type $(1,1)$ after embedding $A^2(X) \subset A(X,\mathbb{C})$), we say $\omega$ is **Kähler**.

If $\omega$ is Kähler, we may define the hermitian metric $h_\omega = \omega(-, I-) - i\omega$, and we have that $\omega$ is Kähler if and only if $(X, h_\omega)$ is Kähler.

**Example 1.2.3.** Let $X = \mathbb{P}^n$, with projective coordinates $Z_0, \ldots, Z_n$. Let $U_i$ be the $Z_i \neq 0$ chart, and define $z_j = \frac{Z_j}{Z_i}$. We may define on $U_i$ the metric

$$\omega_{FS} = \omega = i\partial\overline{\partial}\log\left(1 + \sum_j z_j \overline{z}_j\right),$$

and one checks that these glue to a global form, which we call the **Fubini-Study metric**. Written as a Kähler potential this way shows that it is a Kähler metric.

Note that if $(X, \omega)$ is Kähler, restricting the metric to a complex submanifold $Y$ preserves all properties of Definition 1.2.2, and so $(Y, \omega_Y)$ is Kähler. Thus, any projective manifold is Kähler.

1.3. **Connections.** Let $E$ be a complex (the real case is identical) vector bundle on $(X, C^\infty_X)$. A **complex connection** in $E$ is a $\mathbb{C}$-linear map

$$\nabla : \mathcal{A}^0(E, \mathbb{C}) \to \mathcal{A}^1(E, \mathbb{C}),$$

(here $\mathcal{A}^i(E, \mathbb{C}) = \mathcal{A}^i(X, \mathbb{C}) \otimes \Gamma(E)$) such that

$$\nabla(f \cdot s) = df \otimes s + f \cdot \nabla s$$

for all section $s$ of $E$ and $f \in C^\infty_X$.

If $E$ is a holomorphic bundle on a complex manifold, we can define the operator

$$\overline{\partial} : \mathcal{A}^0(E) \to \mathcal{A}^{0,1}(E)$$

as follows: if $\sigma_i$ is a local frame, and $s = s^i\sigma_i$ a section, we let

$$\overline{\partial}(s^i\sigma_i) := (\overline{\partial}s^i) \otimes \sigma_i. \tag{1.1}$$

Indeed, given another frame $\tau_j$ related by $\sigma_i = g_{ij}\tau_j$, we have

$$\overline{\partial}(s^i) \otimes \sigma_i = \overline{\partial}(s_i) \otimes g_{ij}\tau_j = \overline{\partial}(g_{ij}s^i) \otimes \tau_j$$

since the transitions $g_{ij}$ are holomorphic by assumption.

A (complex) connection being valued in $\mathcal{A}^1(E, \mathbb{C}) = \mathcal{A}^{1,0}(E) \oplus \mathcal{A}^{0,1}(E)$, we may split $\nabla = \nabla^{1,0} + \nabla^{0,1}$.

**Definition 1.3.1.** The complex connection $\nabla$ in $E$ is said to be **compatible** with the holomorphic structure if $\nabla^{0,1} = \overline{\partial}$. Suppose $E$ has a hermitian metric $h$. We say $\nabla$ is **compatible** with $h$ if for any sections $e, f$, we have equality of forms

$$d(h(e, f)) = h(\nabla e, f) + h(e, \nabla f).$$

More geometrically, this says that $h$ is parallel to the connection, i.e. constant along parallel transport, i.e. the connection has $U(n)$-holonomy. We say $\nabla$ is a **Chern connection** if it is both compatible with the holomorphic structure and the hermitian metric.

**Theorem 1.3.2** (Chern)**.** *There exists a unique Chern connection.*

When $E = T_{\mathbb{R}}X$, the Chern connection ought to be regarded as the complex geometric analogue of the Levi-Civita connection from Riemannian geometry. In fact this is more than an analogy. If $h$ is a hermitian metric, the Levi-Civita connection of $g = \mathfrak{Re}(h)$ can be complexified to a complex connection. It is a theorem that the Levi-Civita connection is the Chern connection if and only if $(X, h)$ is Kähler.

We can extend the connection $\nabla : \mathcal{A}^0(E, \mathbb{C}) \to \mathcal{A}^1(E, \mathbb{C})$ to a connection

$$\nabla : \mathcal{A}^p(E, \mathbb{C}) \to \mathcal{A}^{p+1}(E, \mathbb{C})$$

for all positive $p$ via

$$\nabla(\omega \otimes s) = d\omega \otimes s + (-1)^p \omega \wedge \nabla s,$$

where $\omega$ is a $p$-form and $s$ is a section of $E$.

**Remark 1.3.3.** Note that this is different from the usual extension of a connection to tensors since we are dealing with skew-symmetric forms. In particular, this satisfies a different Leibniz rule:

$$\nabla(fs) = df \wedge s \otimes f\nabla s.$$

**Definition 1.3.4.** We define the **curvature** of $\nabla$ to be the composition $\nabla^2 = \nabla \circ \nabla = F_\nabla$.

Note that

$$\begin{aligned}\nabla(\nabla f s) = \nabla(df \otimes s + f\nabla s) &= ddf - df \wedge \nabla s + \nabla(f\nabla s) \\ &= -df \wedge \nabla s + df \wedge \nabla s + f\nabla^2 s = f\nabla^2 s\end{aligned}$$

so that $F_\nabla$ is $C^\infty_X$-linear, that is a section of $\mathcal{A}^2(End(E), \mathbb{C})$.

We may also define

$$F_\nabla^k := \underbrace{F_\nabla \circ \cdots \circ F_\nabla}_{k} \in \mathcal{A}^{2k}(End(E), \mathbb{C}).$$

We define the $k$**th Chern character** of $\nabla$ to be

$$\mathrm{ch}_k(E, \nabla) := \mathrm{Tr}\left(\frac{1}{k!}\left(\frac{\iota}{2\pi}F_\nabla^k\right)\right) \in A^{2k}(X, \mathbb{C}).$$

**Theorem 1.3.5** (Chern-Weil)**.** *The following is true about the Chern character.*

(1) *$ch_k(E, \nabla)$ is closed;*
(2) *The cohomology class $ch_k(E) := [ch_k(E, \nabla)] \in H^{2k}_{dR}(X, \mathbb{C})$ is independent of $\nabla$;*
(3) *$ch_k(E)$ is real, i.e. in $H^{2k}_{dR}(X, \mathbb{R})$ (in fact, it is integral);*
(4) *The total Chern character $\sum_k ch_k(E)$ is equal to the cohomology class of $Tr(\exp(\frac{\iota}{2\pi}F_\nabla))$ (this one directly follows from developing the exponential).*

2.1. **Chern classes.** Let $V$ be a vector space over $\mathbb{C}$ of dimension $r$. Let $P \in \mathbb{C}[End(V)]$ be a homogeneous polynomial of degree $k$. Assume moreover $P$ is $GL(V)$ invariant, that is $P(A^{-1}BA) = P(B)$ for any $A \in GL(V)$.

Let now $E$ be a complex vector bundle and $\nabla$ a connection. By $GL(V)$ invariance, $P(\frac{\iota}{2\pi}F_\nabla)$ is well-defined, and lives in $A^{2k}(X, \mathbb{C})$.

**Fact 2.1.1** (Chern-Weil)**.** $P(\frac{\iota}{2\pi}F_\nabla)$ is closed, and the class $[P(\frac{\iota}{2\pi}F_\nabla)] \in H^{2k}(X, \mathbb{C})$ is independent of $\nabla$.

Consider now the $GL(V)$-invariant homogeneous polynomials $P_k$ returning the coefficients of the characteristic polynomials (i.e. $P_k$ $k$th elementary symmetric polynomial on the eigenvalues). We can explicitly define $P_k$ by the formula:

$$\det(I + tB) = \sum_k P_k(B)t^k.$$

We define the $k$**th Chern class** of $E$ to be the cohomology class of $P_k(\frac{i}{2\pi}F_\nabla)$, and the **total Chern class** of $E$ to be $c(E) := \sum_{i=0}^k c_k(E)$.

The Chern classes and characters satisfy certain properties:

- $c_0(E) = 1$ and $ch_0 = r$;
- $c_d = 0$ if $d > r$. In particular, if $L$ is a line bundle, $c(L) = 1 + c_1(L)$;
- $ch(L) = \exp(c_1(L)) := 1 + c_1(L) + \frac{c_1(L)^2}{2!} + \cdots \in H^{2\bullet}(X, \mathbb{C})$.

To derive further elementary properties of the Chern characters and classes, let us first observe how we can assemble connections into new connections

Let $E_1$ and $E_2$ be vector bundles with respective (complex) connections $\nabla_1$ and $\nabla_2$. Then,

- $\nabla_{E_1 \oplus E_2} := \nabla_1 \oplus \nabla_2$ is a connection on $E_1 \oplus E_2$, and $F_{\nabla_1 \oplus \nabla_2} = F_{\nabla_1} \oplus F_{\nabla_2}$, where this is seen as a block matrix
$$\begin{pmatrix} F_{\nabla_1} & \\ & F_{\nabla_2} \end{pmatrix};$$
- $\nabla_{E_1 \otimes E_2} := \nabla_1 \otimes \mathrm{id}_{E_2} + \mathrm{id}_{E_1} \otimes \nabla_2$ is a connection on $E_1 \otimes E_2$;
- the assignment
$$(\nabla^\vee \phi)(s) := d(\phi(s)) - \phi(\nabla(s))$$
where $s \in E$ and $\phi \in E^\vee$ defines a connection on $E^\vee$ (note that this is the usual way to extend connections on tensors). In other words, if $\langle -, - \rangle$ denotes the natural pairing on $E \otimes E^\vee$, the dual connection is defined by
$$d\langle s, \phi \rangle = \langle \nabla s, \phi \rangle + \langle s, \nabla^\vee \phi \rangle.$$
Let us try to compare the two curvatures. Given $s_i$ a frame of $E$, we consider the connection form $A = (A_i^j)$ defined $\nabla s_i = A_i^j \otimes s_j$. Let $s^i$ be the dual frame of $E^\vee$. We obtain
$$\begin{aligned} d\langle s_i, s^j \rangle = 0 &= \langle \nabla s_i, s^j \rangle + \langle s_i, \nabla^\vee s^j \rangle \\ &= \langle A_i^k \otimes s_k, s^j \rangle + \langle s_i, B^j_k \otimes s^k \rangle \\ &= A_i^j + B^j_i, \end{aligned}$$
where $B = (B^j_i)$ is the connection form of $\nabla^\vee$. And so we have $B = -A^t$ as sections of $\mathcal{A}^2(End(E), \mathbb{C}) = \mathcal{A}^2(End(E^\vee), \mathbb{C})$. Using Cartan's formula for the curvature of a connection, we conclude
$$F_{\nabla^\vee} = d(-A^t) + (-A^t) \wedge (-A^t) = -(dA + A \wedge A)^t = -F^t_\nabla.$$
- connections pull back, that is if $f : Y \to X$ is a smooth map and $E$ is a bundle on $X$ with connection $\nabla_E$, we may define the connection $\nabla_{f^*E}$ by *locally* demanding
$$\nabla_{f^*E}(f^*s) = f^*\nabla s.$$

**Corollary 2.1.2.** *Let $E_1, E_2$ be complex vector bundles on $X$. The following hold:*

1. $ch(E_1 \oplus E_2)) = ch(E_1) + ch(E_2)$ *and* $c(E_1 \oplus E_2) = c(E_1) \cup c(E_2)$;
2. $ch(E_1 \otimes E_2) = ch(E_1) \cup ch(E_2)$;
3. $c_k(E^\vee) = (-1)^k c_k(E)$
4. $ch_k(f^*E) = f^*(ch_k(E))$ *and and* $c_k(f^*E) = f^*(c_k(E))$.

Note that $c_k(E) \in H^{2k}(X, \mathbb{C})$ is in fact real: indeed, conjugation acts via $\overline{F_\nabla} = F_{\overline{\nabla}}$, and up to choosing a hermitian metric, we have $E^\vee \simeq \overline{E}$ and $F_{\overline{\nabla}} = -F_\nabla^t$, and in a Chern splitting, the eigenvalues $\omega_1, \dots, \omega_r$ are purely imaginary. We obtain

$$c_k(E) = P_k(\frac{\iota}{2\pi} F_\nabla) = \left[\frac{\iota^k}{(2\pi)^k} P_k(F_\nabla)\right] = \left[\frac{\iota^k}{(2\pi)^k} \sigma_k(\omega_1, \dots, \omega_r)\right]$$

while

$$\overline{c_k(E)} = \left[\frac{(-1)^k \iota^k}{(2\pi)^k} \sigma_k(-\omega_1, \dots, -\omega_r)\right] = \left[\frac{(-1)^{2k} \iota^k}{2\pi} \sigma_k(\omega_1, \dots, \omega_r)\right] = c_k(E)$$

where $\sigma_k$ denotes the $k$th standard symmetric polynomial, so that $c_k(E) \in H^{2k}(X, \mathbb{R})$. The $k$th Chern class is also $(1,1)$. Indeed, consider the Chern connection $\nabla = \nabla^{1,0} + \nabla^{0,1} = \nabla^{1,0} + \overline{\partial}$. From this decomposition, we see that the $(0,2)$-part of the curvature is $\overline{\partial}^2 = 0$. Similarly, one can use the fact that the hermitian metric is parallel to show that the $(2,0)$ part vanishes so that all $\omega_i$ are of type $(1,1)$, from which one obtains that $c_k(E)$ is of type $(k,k)$.

Let $D \subset X$ be a divisor. It is a fact that the fundamental class of $D$ is the Chern class of $\mathcal{O}(D)$, i.e.

$$[D] = c_1(\mathcal{O}(D)) \in H^2(X, \mathbb{C}),$$

showing that the first — and therefore any — Chern class is integral.

An important theorem relating to chern classes is Kodaira's embedding theorem.

**Theorem 2.1.3** ([Kod54]). *A (holomorphic) line bundle $L$ on a complex manifold $X$ is ample (i.e. induces an embedding in projective space) if and only if it admits a metric $h$ such that $\frac{\iota}{2\pi} F_{D_h}$ is a positive form.*

In particular, let $h_0$ be any hermitian metric on $L$, and let $\omega_0 = \frac{\iota}{2\pi} F_{D_{h_0}}$. Assuming $X$ is Kähler, if $c_1(L) = [\omega_0]$ is positive, i.e. if it has a positive form $\omega$ as representative of the cohomology class, then we can write $\omega = \omega_0 + \frac{\iota}{2\pi} \partial\overline{\partial}\phi$ for some function $\phi$ by the $\partial\overline{\partial}$-lemma. Then, one may compute that the metric $h := e^{-\phi} \cdot h_0$ satisfies $\frac{\iota}{2\pi} F_{D_h} = \omega$; indeed the $(1,0)$-part of the Chern of the connection is

$$h^{-1}\partial h = \partial \log h = \partial \log(e^{-\phi} h_0) = \partial \log h_0 + \partial(-\phi)$$

so that the curvature is given by

$$F_{D_{h_0}} + \overline{\partial}\partial(-\phi) = F_{D_{h_0}} + \partial\overline{\partial}\phi.$$

In particular, a line bundle $L$ is ample if and only if $c_1(L)$ is positive, i.e. is represented by a Kähler form. Now, on a compact manifold, slightly perturbing a Kähler form inside $H^{1,1}(X, \mathbb{R})$ still yields a Kähler form, since it preserves the positivity criterion. Thus, Kähler forms form an open positive cone $\mathcal{K}_X$ inside of $H^{1,1}(X, \mathbb{R})$ (scaling by a positive real preserves Kählerness). Moreover, by the Lefschetz theorem on $(1,1)$ classes, the Chern map $\mathrm{Pic}(X) \to H^{1,1}(X, \mathbb{Z})$ is surjective. Thus, we conclude that a compact complex manifold is projective if and only if $\mathcal{K}_X$ intersects with $H^{1,1}(X, \mathbb{Z})$ (or equivalently $H^{1,1}(X, \mathbb{Q})$) inside of $H^{1,1}(X, \mathbb{R})$.

## 2.2. Hirzebruch-Riemann-Roch.

**Definition 2.2.1.** Let $X$ be a compact complex manifold. Let $\nabla$ be a connection in the tangent bundle. We define the **Todd class** of $X$ to be

$$td(X) := \left[\det\left(\frac{\frac{\iota}{2\pi} F_\nabla}{1 - \exp(\frac{-\iota}{2\pi} F_\nabla)}\right)\right] \in H^{2\bullet}(X, \mathbb{C})$$

In terms of Chern roots $\omega_1, \dots, \omega_r$, we have that

$$td(X) = \prod_{i=1}^{r} \frac{\omega_i}{1 - e^{-\omega_i}}$$

It can be computed that we have

$$td_0(X) = 1; \quad td_1(X) = \frac{c_1}{2}; \quad td_2(X) = \frac{1}{12}(c_1^2 + c_2); \quad td_3(X) = \frac{c_1 c_2}{24} \quad td_4(X); = \frac{-c_1^4 + 4c_2 + c_1 c_3 + 3c_2^2 - c_4}{720}; \quad \cdots$$

where $td_k(X)$ denotes the $k$th homogeneous component of $td(X)$ and $c_i = c_i(X) := c_i(\mathcal{T}_X)$.

**Theorem 2.2.2** (Hirzebruch-Rieman-Roch). *Let $E$ be a holomorphic vector bundle on a compact complex manifold $X$. Then, we have equality*

$$\chi(X,E) := \sum_k (-1)^k h^i(X,E) = \int_X ch(E) \cup td(X). \tag{2.1}$$

Note that since we are integrating over $X$, we only need to consider the top degree parts.

**Example 2.2.3.** Let $X = C$ be a compact Riemann surface and $L$ be a line bundle on $X$. we have $ch(E) = 1 + c_1(L)$ and $td(1) = 1 + \frac{c_1(X)}{2}$. thus, we have

$$\chi(X,L) = \int_X 1 + c_1(L) + \frac{c_1(X)}{2} + \frac{c_1(L)c_1(X)}{2} = \int_X c_1(L) + \frac{c_1(X)}{2} = \deg(L) + \frac{\deg(\mathcal{T}_C)}{2}.$$

What is remarkable about this theorem is that the left-hand side of (2.1) is purely holomorphic (or algebraic) whilst the right-hand side is purely topological. Another similar theorem is the algebro-geometric Gauss-Bonnet theorem.

**Theorem 2.2.4.** *Let $X$ be a compact complex manifold of dimension $n$. Then,*

$$e(X) := \sum_i (-1)^i b_i(X) = \int_X c_n(X).$$

Recall the classical relation between the Euler characteristic $\chi_{top}$ and the genus $g$ of a topological surface: $e = 2 - 2g$. In particular, this implies for a compact Riemann surface $C$ as above, that

$$\int_X c_1(X) = \deg(\mathcal{T}_C) = 2 - 2g.$$

In particular, in light of what we found in Example 2.2.3, we recover the classical Riemann-Roch theorem:

$$\chi(X,L) = \deg(L) - g + 1.$$

### 2.3. **Kähler-Einstein manifolds.**

**Question 2.3.1.** When does a smooth projective variety over $\mathbb{C}$ admit a "canonical" metric?

**Definition 2.3.2.** Let $(X,\omega)$ be a compact Kähler manifold, and let $D_\omega$ be the corresponding Chern connection. We define the **Ricci form** $\mathrm{Ric}(\omega)$ of $\omega$ to be

$$\mathrm{Ric}(\omega) = i\mathrm{Tr}(F_{D_\omega}) \in A^2(X,\mathbb{C}).$$

We say that $(X,\omega)$ is **Kähler-Einstein** if $\mathrm{Ric}(\omega) = \lambda\omega$ for some constant $\lambda \in \mathbb{R}$.

**Remark 2.3.3.** We make the following comments.

(1) Recall that we argued earlier that all the Chern roots of $F_{D_\omega}$ were pure imaginary of type $(1,1)$ so that $\mathrm{Ric}(\omega)$ is real of type $(1,1)$.
(2) Note also that $D_\omega$ is invariant under rescaling $\omega$ by some $\lambda > 0$ (indeed, parallelness of $h$ is unaffected so we get the same connection). Thus, we may always assume that $\lambda = -1, 0, 1$.
(3) since $c_1(X) = [\frac{\iota}{2\pi}\mathrm{Tr}F_{D_\omega}]$ by definition, we have $[\mathrm{Ric}(\omega)] = 2\pi c_1(X) \in H^2(X,\mathbb{R})$.
(4) $\lambda$ is proportional to the scalar curvature, and so $(X,\omega)$ being Kähler-Einstein implies that the scalar curvature with respect to $g_\omega = \omega(I-,-)$ is constant.
(5) If $X$ is Kähler-Einstein, then we have

$$c_1(X) = \begin{cases} 0 \\ \pm \text{ positive form.} \end{cases}$$

**Definition 2.3.4.** We say that a complex manifold $X$ is **Calabi–Yau** if $c_1(X) = 0$, **Fano** if $c_1(X)$ is positive, of **general type** (or **canonically polarised**) if $-c_1(X)$ is positive.

Note that by Kodaira's embedding theorem, Fano and general type manifolds are projective.

**Caution 2.3.5.** It is not because a manifold fits in this trichotomy that it admits a Kähler-Einstein metric. In fact, there exist Fano varieties with no Kähler-Einstein metric. Whether a Fano variety admits such metric is equivalent to $K$-stability, a purely algebro-geometric notion. Nonetheless, Yau (cf. [Yau78]) proved that any Calabi–Yau manifold admits a Kähler-Einstein metric, and Aubin–Yau (cf. [Aub76; Yau78]) proved the same for general type manifolds.

Note also that not all manifolds fit in this trichotomy.

**Example 2.3.6** (Curves)**.** Let us see how these categories apply to curves.

- $g = 0$ gives only $\mathbb{P}^1$. Since it is diffeomorphic to a sphere, we have positive scalar curvature. And indeed, the Fubini-Study metric is Kähler-Einstein with $\lambda = 1$. Note also that $\mathbb{P}^1$ is Fano.
- The $g = 1$ case corresponds to elliptic curves. These are Ricci-flat and Calabi–Yau.
- The case $g > 1$ are of general type, and there exists a Kähler-Einstein metric with negative scalar curvature.

For Fano manifolds, here is a summary of the known classifications:

(1) In dimension 1 there is only the projective line.
(2) In dimension 2, they are called *del Pezzo* surfaces. There are 10 different deformation families. First, $\mathbb{P}^2$ and $\mathbb{P}^1 \times \mathbb{P}^1$ which are isolates. The other 8 families are obtained by blowing up $\mathbb{P}^2$ at d points in general position, where $1 \leq d \leq 8$.
(3) In dimension 3, Mukai proved there are 105 families.
(4) In dimension 4, we know there are finitely many families but it remains open to know how many.

For Calabi–Yau manifolds, there is the following structural theorem.

**Theorem 2.3.7** (Beauville–Bogomolov; Bog74 and Bea83)**.** *Let $X$ be Kähler and Calabi–Yau. Then, there exists an étale cover $\tilde{X} \to X$ such that*

$$\tilde{X} = T \times \prod_j X_j \times \prod_i V_i$$

*where $T$ is a torus, $X_j$ is* ***hyperkähler*** *for all $j$, and $V_i$ are* ***strict Calabi–Yau*** *for all $i$.*

We now define the terms.

**Definition 2.3.8.** A compact Kähler manifold $V$ is called **strict Calabi–Yau** if

- $K_V \simeq \mathcal{O}_V$ is trivial, where $K_V$ denotes the canonical bundle;
- $V$ is simply connected;
- $H^i(V, \mathcal{O}_V) = 0$ for all $0 < i < \dim V$.

A complex manifold $X$ is **hyperkähler** if

- it is simply connected;
- $H^0(X, \Omega_X^2) \simeq \mathbb{C}\sigma$ where $\sigma$ is holomorphic symplectic (in particular, it induces an isomorphism $\mathcal{T}_X \simeq \Omega_X$).

**Remark 2.3.9.** If $V$ is a strict Calabi–Yau of dimension greater than two, then $h^{2,0} = h^{0,2} = 0$. In particular, $H^{1,1}(X, \mathbb{C}) = H^2(X, \mathbb{C})$, and so $H^{1,1}(X, \mathbb{R}) = H^2(X, \mathbb{R}) = H^{1,1}(X, \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{R}$. Since the Kähler cone is not empty by assumption, we conclude that it intersects $H^{1,1}(X, \mathbb{Z})$, so that $V$ is projective by our discussion on Kodaira's embedding theorem. In dimension 2, non-projective K3 surfaces yield an example of non-projective strict Calabi–Yau manifolds.

**Remark 2.3.10.** Theorem 2.3.7 has an important corollarly. Namely, if $X$ is a Kähler, we have a degree $n$ étale map $f : \tilde{X} \to X$ with the canonical bunlde $K_{\tilde{X}}$ of $\tilde{X}$ trivial. Since $f$ is étale, $f^* K_X = K_{\tilde{X}}$ is trivial, and so by the projection formula,

$$f_* \mathcal{O}_{\tilde{X}} = f_* f^* K_X = K_X \otimes f_* \mathcal{O}_{\tilde{X}}.$$

Taking determinants, we have

$$\det(f_* \mathcal{O}_{\tilde{X}}) = K_X^n \otimes \det(f_* \mathcal{O}_{\tilde{X}}),$$

so that $K_X^n = \mathcal{O}_X$. Hence, the power of the canonical bundle of a Kähler Calabi–Yau is always trivial.

## 3. Third lecture: Hodge theory

In this lecture, we recollect Hodge theory.

### 3.1. **Linear algebra.**

Let us first explore the constructions of Hodge theory in the setting of linear algebra, our toy model.

Let $V$ be a real vector space of dimension $n$ and $\langle -,-\rangle$ be an inner product on $V$. The scalar product induces a scalar product on $\bigwedge^k V$ via declaring

$$\langle v_1 \wedge \cdots \wedge v_k, u_1 \wedge \cdots \wedge u_k\rangle := \det(\langle u_i, v_j\rangle)_{ij}.$$

Moreover, if $e_1, \ldots, e_n$ is an *ordered* orthonormal basis of $V$, the vectors $e_{i_1} \wedge \cdots \wedge e_{i_k}$ for $i_1 < \cdots < i_k$ form an orthonormal basis of $\bigwedge^k V$.

**Definition 3.1.1.** The **volume form** of $V$ (with respect to the chosen ordered basis) is

$$\mathrm{vol}_V = \mathrm{vol} := e_1 \wedge \cdots \wedge e_n.$$

For any $k \leq n$, we define the **Hodge operator** to be the map

$$\begin{aligned} * : \bigwedge^k V &\to \bigwedge^{n-k} V \\ e_{i_1} \wedge \cdots \wedge e_{i_k} &\mapsto \varepsilon e_{j_1} \wedge \cdots \wedge e_{j_{n-k}}, \end{aligned}$$

where $\{e_{j_1}, \ldots, e_{n-k}\}$ is the complement of $\{e_{i_1}, \ldots, e_{i_k}\}$ in the full basis $\{e_1, \ldots, e_n\}$ and this map is well-defined because of the permutation index $\varepsilon = \mathrm{sgn}(i_1, \ldots, i_k, j_1, \ldots, j_{n-k})$.

Let $\alpha = e_{i_1} \wedge \cdots \wedge e_{i_k}$ and $\beta = e_{j_1} \wedge \cdots \wedge e_{j_k}$ be elements of the basis of $\bigwedge^k V$. We have that

$$\alpha \wedge *\beta = \begin{cases} 0 & \text{if } \{i_1, \ldots, i_k\} \neq \{j_1, \ldots, j_k\} \\ \mathrm{vol} & \text{otherwise.} \end{cases}$$

In any case, we have that

$$\alpha \wedge *\beta = \langle \alpha, \beta\rangle \cdot \mathrm{vol}, \tag{3.1}$$

and by bilinearity of the left and right hand side, we see that (3.1) holds for any $\alpha, \beta \in \bigwedge^k V$.

Let $I$ be a complex structure on $V$, i.e. $I^2 = -\mathrm{id}_V$. Recall that it forces the dimension of $V$ to be even. Suppose furthermore that the inner product is compatible with $I$, i.e. $\langle I-, I-\rangle = \langle -,-\rangle$. Let $\langle -,-\rangle$ also denote the $\mathbb{C}$-sesquilinear extension of the inner product to a hermitian product on the complexified space $V_{\mathbb{C}}$.

The decomposition $V = V^{1,0} \oplus V^{0,1}$ into $\pm\iota$ eigenspaces is orthodonal for $\langle -,-\rangle$, indeed, for $v \in V^{1,0}$ and $u \in V^{0,1}$, we have

$$\langle v, u\rangle = \langle Iv, Iu\rangle = \langle \iota v, \iota u\rangle = -\langle v, u\rangle.$$

By extension, this shows that the decomposition

$$\bigwedge^k V_{\mathbb{C}} = \bigoplus_{p+q=k} \bigwedge^{p,q} V \qquad \text{where } \bigwedge^{p,q} V := \bigwedge^p V^{1,0} \otimes \bigwedge^q V^{0,1} \tag{3.2}$$

is orthogonal. We may extend the hodge star operator $\mathbb{C}$-linearly to

$$* : \bigwedge^k V_{\mathbb{C}} \to \bigwedge^{n-k} V_{\mathbb{C}},$$

and by looking at an orthonormal basis for (3.2), we see that the operator restricts to

$$* : \bigwedge^{p,q} V \to \bigwedge^{n-q,n-p} V.$$

Note that the equality (3.1) now becomes

$$\alpha \wedge *\overline{\beta} = \langle \alpha, \beta\rangle \mathrm{vol}_{V_{\mathbb{C}}}.$$

3.2. **Harmonic forms.** From now on, $(X, h)$ is a Kähler manifold and $g = \mathfrak{Re}(h)$ is the compatible Riemannian metric associated to $h$. We will write $\langle -, - \rangle$ for the hermitian metric induced by $h$ on $T^*_{\mathbb{R}}X$. We have the real volume form

$$\mathrm{vol} \in A^{2n}(X).$$

The decomposition

$$\mathcal{A}^k(X, \mathbb{C}) = \bigoplus_{p+q=k} \mathcal{A}^{p,q}(X)$$

is orthogonal with respect to $\langle -, - \rangle$. As before, we have the hodge operator

$$* : \mathcal{A}^{p,q}(X) \to \mathcal{A}^{n-q,n-p}.$$

Recall also our three different exterior derivatives:

$$\begin{aligned} d &: \mathcal{A}^k(X) \to \mathcal{A}^{k+1}(X) \\ \partial &: \mathcal{A}^{p,q}(X) \to \mathcal{A}^{p+1,q}(X) \\ \overline{\partial} &: \mathcal{A}^{p,q}(X) \to \mathcal{A}^{p,q+1}(X). \end{aligned}$$

We use the hodge operator to define other operators:

$$\begin{aligned} d^* &:= (-1)^k *^{-1} \circ d \circ * : \mathcal{A}^k(X) \to \mathcal{A}^{k-1}(X) \\ \partial^* &:= - * \circ \overline{\partial} \circ * : \mathcal{A}^{p,q}(X) \to \mathcal{A}^{p-1,q}(X) \\ \overline{\partial}^* &:= - * \circ \partial \circ * : \mathcal{A}^{p,q}(X) \to \mathcal{A}^{p,q-1}(X). \end{aligned}$$

The mismatch in the sign is a convention, and we note that $*^2 = (-1)^{k(n-k)}$. Extending $d^*$ $\mathbb{C}$-linearly, these operators satisfy

$$d^* = \partial^* + \overline{\partial}^*$$

We use these operators to define three different Laplacians:

$$\begin{aligned} \Delta_d &:= dd^* + d^*d : \mathcal{A}^k(X) \to \mathcal{A}^k(X) \\ \Delta_\partial &:= \partial\partial^* + \partial^*\partial : \mathcal{A}^{p,q}(X) \to \mathcal{A}^{p,q}(X) \\ \Delta_{\overline{\partial}} &:= \overline{\partial}\,\overline{\partial}^* + \overline{\partial}^*\overline{\partial} : \mathcal{A}^{p,q}(X) \to \mathcal{A}^{p,q}(X). \end{aligned}$$

A $k$-form $\alpha$ is $d$**-harmonic** if $\Delta_d\alpha = 0$, and a $(p, q)$-form $\beta$ is $\overline{\partial}$**-harmonic** if $\Delta_{\overline{\partial}}\beta = 0$. We will write $\mathcal{H}^k(X) := \ker \Delta_d$ for the space of $d$-harmonic $k$-forms (note, we can do this either over $\mathbb{R}$ or $\mathbb{C}$ and do not specify), and $\mathcal{H}^{p,q}(X) := \ker \Delta_{\overline{\partial}}$ for the space of $\overline{\partial}$-harmonic $(p, q)$-forms.

We have an $L^2$ inner product

$$\begin{aligned} A^k(X) \otimes A^k(X) &\to \mathbb{R} \\ (\alpha, \beta) &\mapsto \int_X \langle \alpha, \beta \rangle \mathrm{vol} = \int_X \alpha \wedge *\beta. \end{aligned}$$

From the right hand side, we compute that for $\alpha \in A^k(X)$ and $\beta \in A^{k+1}(X)$,

$$(d\alpha, \beta) = \int_X d\alpha \wedge *\beta = \int_X \alpha \wedge (-1)^{k+1} d(*\beta) = \int_X \alpha \wedge *(-1)^{k+1} *^{-1} d(*\beta) = \int_X \alpha \wedge *d^*\beta = (\alpha, d^*\beta),$$

so that $d$ and $d^*$ are formal adjoints with respect to $(-, -)$. From this, we see that

$$(\alpha, \Delta_d\alpha) = (\alpha, dd^*\alpha + d^*d\alpha) = (d^*\alpha, d^*\alpha) + (d\alpha, d\alpha),$$

and so by positive definiteness, $\Delta\alpha = 0$ if and only if $d\alpha = d^*\alpha = 0$. This yields a map

$$\mathcal{H}^k(X, \mathbb{R}) \to H^k(X, \mathbb{R}).$$

**Theorem 3.2.1** (Hod41)**.** *This map is an isomorphism.*

**Remark 3.2.2.** Note that this works equally well over $\mathbb{C}$ witht the hermitian product

$$(\alpha, \beta) \mapsto \int_X \langle \alpha, \beta \rangle \mathrm{vol} = \int_X \alpha \wedge *\overline{\beta}$$

3.3. **The Dolbeault side.** We can get a similar theorem by taking into account the $(p,q)$ type of forms. First note that for any $p$, we have a complex

$$0 \to \Omega_X^p \to \mathcal{A}^{p,0}(X) \to \mathcal{A}^{p,1}(X) \to \cdots \to \mathcal{A}^{p,n}(X) \to 0.$$

The $\overline{\partial}$-Poincaré lemma (also called the Dolbeault–Grothendieck lemma) says that this complex is exact, i.e. that analytically locally, a $\overline{\partial}$-closed form is $\overline{\partial}$-exact. Moreover, since each $\mathcal{A}^{p,q}(X)$ admits partitions of unity (these are smooth sections), this is an acyclic resolution. As corollary, we obtain an isomorphism

$$H^q(X, \Omega_X^p) = \mathbb{H}^q(\mathcal{A}^{p,0}(X) \to \cdots \to \mathcal{A}^{p,n}(X) \to 0) = \frac{\ker\left(\overline{\partial} : A^{p,q}(X) \to A^{p,q+1}(X)\right)}{\operatorname{im}\left(\overline{\partial} : A^{p-1,q}(X) \to A^{p,q}(X)\right)} =: H^{p,q}(X), \tag{3.3}$$

and we call the right-hand side the $(p,q)$ Dolbeault cohomology of $X$.

As before, we have an inner product on $(p,q)$-forms:

$$(\alpha, \beta) := \int_X \alpha \wedge *\overline{\beta},$$

and the same computation shows that $\partial^*$ and $\overline{\partial}^*$ are formal adjoints. Therefore, $\Delta_{\overline{\partial}}\alpha = 0$ if and only if $\overline{\partial}\alpha = \overline{\partial}^*\alpha = 0$. This yields a map

$$\mathcal{H}^{p,q}(X) \to H^{p,q}(X).$$

**Theorem 3.3.1.** *This map is an isomorphism.*

It is worth noting that Theorem 3.3.1 does not require $X$ to be Kähler.

**Proposition 3.3.2.** *If $X$ is compact Kähler, then*

$$\Delta_d = 2\Delta_{\overline{\partial}} = 2\Delta_{\partial}.$$

**Corollary 3.3.3** (Hodge decomposition)**.** *Let $X$ be a compact Kähler manifold. We have a decomposition*

$$H^k(X, \mathbb{C}) = \oplus_{p+q=k} H^{p,q}(X)$$

*such that $H^{p,q}(X) = \overline{H^{q,p}(X)}$ and $H^{p,q}(X) \simeq H^q(X, \Omega_X^p)$.*

*Proof.* Note that $\Delta_{\overline{\partial}}$ preserves the $(p,q)$-type, and thus so does $\Delta_d$ by Proposition 3.3.2. Therefore, we have

$$\bigoplus_{p+q=k} H^{p,q}(X) \simeq \bigoplus_{p+q=k} \mathcal{H}^{p,q}(X) = \mathcal{H}^k(X, \mathbb{C}) \xrightarrow{\sim} H^k(X, \mathbb{C}). \tag{3.4}$$

It remains to show that $H^{p,q}(X) = \overline{H^{q,p}(X)}$, and this follows from the fact that $\Delta_d(\overline{\alpha}) = \overline{\Delta_d(\alpha)}$, as $\Delta_d$ is a real operator. □

**Remark 3.3.4.** It is worthwile to observe that along the isomorphism (3.4), $H^{p,q}(X)$ is identified with the subset $\{[\alpha] : d\alpha = 0 \text{ and } \alpha \in A^{p,q}\} \subset H^k(X, \mathbb{C})$. Note moreover that the Hodge decomposition is compatible witht the wedge product (that is, wedging $(p,q)$ $\overline{\partial}$-closed forms yields the same algebra structure as that on $H^\bullet(X, \mathbb{C})$ along (3.4)). However, there is no clear algebra structure on

$$\bigoplus_{p,q \geq 0} \mathcal{H}^{p,q}(X),$$

as wedging two harmonic forms need not yield a harmonic form (this comes from the fact that $d^*$ does not satisfy a Leibniz rule).

**Corollary 3.3.5.** *Let $X$ be a compact Kähler manifold. If $k$ is odd, then the $k$th Betti number $b_k(X)$ is even.*

**Remark 3.3.6.** For any Kähler form $\omega$,

$$\int_X \omega^{\dim X} = n! \cdot \operatorname{vol}(X) > 0,$$

implying that $\omega$ is not $d$-exact (assuming $X$ compact), i.e. that $b_k > 0$ for $k$ odd.

**Remark 3.3.7** (Hodge diamond)**.** We can make the Hodge numbers fit in what is called the Hodge diamond:

$$\begin{array}{ccccccccc}
 & & & & h^{0,0} & & & & \\
 & & & h^{1,0} & & h^{0,1} & & & \\
 & & h^{2,0} & & h^{1,1} & & h^{0,2} & & \\
 & \iddots & & & & & & \ddots & \\
h^{n,0} & & \cdots & & \circlearrowleft & & \cdots & & h^{0,n} \\
 & \ddots & & & & & & \iddots & \\
 & & h^{n,n-2} & & h^{n-1,n-1} & & h^{n-2,n} & & \\
 & & & h^{n,n-1} & & h^{n-1,n} & & & \\
 & & & & h^{n,n} & & & & \\
 & & & & \longleftrightarrow & & & &
\end{array}$$

where $n = \dim_{\mathbb{C}} X$, and these are the only non-zero Hodge numbers. Moreover, there are some symmetries. We already saw Hodge symmetry, which implies $h^{p,q} = h^{q,p}$, which is represented by the arrow in the bottom.

Recalll that some version of Poincaré duality says that there is a non-degenerate pairing

$$\begin{aligned} H^k(X,\mathbb{C}) \times H^{2n-k}(X,\mathbb{C}) &\to \mathbb{C} \\ (\alpha,\beta) &\mapsto \int_X \alpha\wedge\beta, \end{aligned} \tag{3.5}$$

yielding an isomorphism $H^k(X,\mathbb{C})^\vee = H_k(X,\mathbb{C}) \simeq H^{2n-k}(X,\mathbb{C})$. This pairing restricts to a non-degenerate pairing

$$H^{p,q}(X) \times H^{n-p,n-q}(X) \to \mathbb{C};$$

indeed, if a $d$-closed form of type $(p,q)$ is non-zero, we know by (3.5) that we may pair it with a form $\beta$ with $\alpha\wedge\beta \neq 0$, but this forces the type of $\beta$ to be $(n-p, n-q)$. Therefore, we get an isomorphism

$$H^{p,q}(X) = H^{n-p,n-q}(X)^\vee. \tag{3.6}$$

In particular, our diamond has the symmetry $h^{p,q} = h^{n-p,n-q}$, which is represented by the central circling arrow in the Hodge diamond. Note that (3.6) can also be seen using Serre duality:

$$H^{p,q}(X) = H^q(X,\Omega_X^p) = H^{n-q}(X,(\Omega_X^p)^\vee\otimes K_X)^\vee = H^{n-q}(X,\Omega_X^{n-p})^\vee = H^{n-p,n-q}(X)^\vee,$$

where the isomorphism $(\Omega_X^p)^\vee \otimes K_X = \bigwedge^p \mathcal{T}_X \otimes K_X \simeq \Omega^{n-p}$ comes from contraction of vector fields:

$$\begin{aligned} \bigwedge^p \mathcal{T}_X \otimes K_X &\xrightarrow{\sim} \Omega^{n-p} \\ X_1\wedge\cdots\wedge X_p\otimes\alpha &\mapsto \alpha(X_1,\dots,X_p,-,\dots,-). \end{aligned}$$

Note that there is also a way to see this duality with the Hodge star operator.

The Hodge diamond also satisfies a unimodal condition. Namely, in each row (hence each column by Poincaré/Serre-duality), the Hodge numbers increase before reaching half, then decrease (the latter follows from the former by Hodge symmetry).

3.4. **Lefschetz theorems.** Let $X$ be compact Kähler manifold with Kähler form $\omega$. As $\omega$ is a $(1,1)$ real form, we obtain an operator

$$\begin{aligned} L_\omega : A^k(X) &\to A^{k+2}(X) \\ \alpha &\mapsto \omega\wedge\alpha. \end{aligned}$$

In the complexification, this restrict to

$$L_\omega : A^{p,q}(X) \to A^{p+1,q+1}(X),$$

and these operators descend to cohomology by definition. We define the operator

$$\Lambda_\omega := *^{-1}\circ L_\omega \circ *,$$

and the degree operator

$$h : H^*(X) \to H^*(X)$$

where $h|_{H^k(X)} = (k-n)\mathrm{id}|_{H^k(X)}$. Here, we do not specify the coefficients, as we want to work over either $\mathbb{R}$ or $\mathbb{C}$.

**Theorem 3.4.1.** *On cohomology, we have $[L_\omega, \Lambda_\omega] = h$, $[h, L_\omega] = 2L_\omega$ and $[2, \Lambda_\omega] = -2\Lambda_\omega$, that is, $L_\omega, \Lambda_\omega$ and $h$ form an $\mathfrak{sl}_2$-triple. Moreover, we have $[L_\omega, \Delta_d] = 0 = [\Lambda_\omega, \Delta_d]$.*

From this $\mathfrak{sl}_2$-representation, one can deduce the following.

**Theorem 3.4.2** (Hard Lefschetz)**.** *Let $k \leq n$.*

(1) *the map*

$$L_\omega^{n-k} : H^k(X) \to H^{2n-k}(X)$$

*is an isomorphism.*

(2) *Let $H^k(X)_{prim} \subset H^k(X)$ be the kernel of $L_\omega^{n-k+1}$. We have a* ***Lefschetz decomposition***

$$H^k(X) = \bigoplus_{2r \leq kr} L_\omega^r H^{k-2r}(X)_{prim}.$$

*Moreover, over $\mathbb{C}$, this decomposition is compatible with the Hodge decomposition; i.e, if we define*

$$H^{p,q}(X)_{prim} := \ker\left(L_\omega^{2n-p-q+1} : H^{p,q}(X) \to H^{n-p+1,n-q+1}(X)\right),$$

*we have*

$$H^{p,q}(X) = \bigoplus_{2r \leq p+q} L_\omega^r H^{p-r,q-r}(X)_{prim},$$

*and*

$$H^k(X, \mathbb{C})_{prim} = \bigoplus_{p+q=k} H^{p,q}(X)_{prim}.$$

**Remark 3.4.3.** Note that when $X$ is projective, we may choose $\omega$ to be integral and the Lefschetz decomposition also holds over $\mathbb{Q}$.

**Example 3.4.4.** Let us study the consequences of these theorems on the cohomology of a surface $X$ with Kähler form $\omega$. We have the diagram

$$\begin{tikzcd}
H^0(X) \arrow[r, equal] \arrow[dddd, bend right=60, "L_\omega^2"'] & H^0(X)_{\mathrm{prim}} \\
H^1(X) \arrow[r, equal] \arrow[dd, bend right, "L_\omega"'] & H^1(X)_{\mathrm{prim}} \\
H^2(X) = H^2(X)_{\mathrm{prim}} \oplus LH^0(X) & \\
H^3(X) & \\
H^4(X), &
\end{tikzcd}$$

where $H^0(X)_{\mathrm{prim}} = H^0(X)$ and $H^1(X)_{\mathrm{prim}} = H^1(X)$ since the primitive parts are defined as the kernel of maps to a cohomology groups that vanish for dimension reasons. Since $H^0(X)$ is generated by the identity element in the cohomology ring, we obtain (over say $\mathbb{R}$ coefficients) $H^2(X, \mathbb{R}) = \mathbb{R}[\omega] \oplus H^2(X, \mathbb{R})_{\mathrm{prim}}$. As we soon shall see, this decomposition is orthogonal with respect to a certain intersection pairing, so that we may write $H^2(X, \mathbb{R}) = \mathbb{R}[\omega] \oplus [\omega]^\perp$.

3.5. **Hodge index theorem.** Let $X$ be compact and $\omega$ be a Kähler form. Consider the complex Poincaré pairing

$$H^k(X, \mathbb{C}) \times H^{2n-k}(X, \mathbb{C}) \to \mathbb{C}$$
$$(\alpha, \beta) \mapsto \int_X \alpha \wedge \beta.$$

This pairing is skew symmetric for $k$ odd and symmetric for $k$ even. We use the polarisation $\omega$ to turn this into a pairing on $H^k(X, \mathbb{C})$ for $k \leq n$:

$$Q_k : H^k(X, \mathbb{C}) \times H^k(X, \mathbb{C}) \to \mathbb{C}$$
$$(\alpha, \beta) \mapsto \int_X \alpha \wedge L_\omega^{n-k}(\beta).$$

We now define the hermitian pairing $H_k(\alpha, \beta) := \iota^k Q_k(\alpha, \overline{\beta})$ called the **Hodge–Riemann bilinear form**.

**Theorem 3.5.1** (Hodge–Riemann bilinear relations)**.** *The following hold true*

(1) *The Hodge decomposition is orthogonal with respect to* $H_k$*;*
(2) *The Lefschetz decomposition is orthogonal with respect to* $H_k$*, and for* $\alpha, \beta \in H^{k-2r}(X, \mathbb{C})_{prim}$*, we have*
$$H_k(L^r(\alpha), L^r(\beta)) = (-1)^{k+r} H_{k-2r}(\alpha, \beta);$$
(3) *The form*
$$(-1)^{\frac{k(k-1)}{2}} \iota^{p-q-k} H_k$$
*is positive definite on* $H^{p,q}(X)_{prim}$*.*

*Proof.* (1) Let $\alpha \in H^{p,q}(X)$ and $\beta \in H^{p',q'}(X)$ with $p+q = p'+q'$. By definition, we have
$$H_k(\alpha, \beta) = \iota^k \int_X \alpha \wedge \omega^{n-k} \wedge \overline{\beta},$$
but the form $\alpha \omega^{n-k} \wedge \wedge \overline{\beta}$ is of degree $2n$ but not of type $(n,n)$. Hence it vanishes, implying that the integral vanishes.

(2) Let $\alpha' = L^r_\omega(\alpha)$ for $\alpha \in H^{k-2r}(X)_{\text{prim}}$ and $\beta' = L^s_\omega(\beta)$ for $\beta \in H^{k-2s}(X)_{\text{prim}}$. Without loss of generality, assume $r > s$. We have that
$$H_k(\alpha', \beta') = \iota^k \int_X \alpha' \wedge L^{n-k+s}_\omega(\overline{\beta'}) = \iota^k \int_X \omega^r \alpha \wedge \omega^{n-k+s} \wedge \overline{\beta} = (-1)^k \iota^k \int_X \alpha \wedge L^{n-k+r+s}_\omega(\overline{\beta}) = 0$$
since $n-k+r+s > n-k+2s$, so that primitiveness of $\beta$ ensures it is in the kernel of $L^{n-k+r+s}_\omega$.

Now if $\alpha', \beta'$ are chosen as above but $r = s$, we have
$$\begin{aligned} H_k(\alpha', \beta') &= \iota^k \int_X \alpha \wedge L^{n-k+r}(\overline{\beta'}) = \iota^k \int_X \omega^r \wedge \alpha \wedge \omega^{n-k+r} \wedge \overline{\beta} \\ &= (-1)^k \iota^{2r} \iota^{k-2r} \int_X \alpha \wedge \omega^{n-k+2r} \wedge \overline{\beta} = (-1)^{k+r} H_{k-2r}(\alpha, \beta). \end{aligned}$$
(3) For $\alpha \in H^{p,q}(X)_{\text{prim}}$. For such form, on Kähler manifolds, we have
$$*\alpha = \iota^{p-q} (-1)^{\frac{k(k-1)}{2}} \frac{\omega^{n-k}}{(n-k)!} \wedge \alpha;$$
see [Voi98, Proposition 6.29]. Therefore, we obtain that
$$\begin{aligned} H_k(\alpha, \alpha) &= \iota^k \int_X \alpha \wedge \omega^{n-k} \wedge \overline{\alpha} = (n-k)!(-1)^{\frac{k(k-1)}{2}} \iota^{k-p+q} \int_X \alpha \wedge *\overline{\alpha} \\ &= (n-k)!(-1)^{\frac{k(k-1)}{2}} \iota^{k-p+q} \int_X \langle \alpha, \alpha \rangle \mathrm{vol}_{\mathbb{C}} \end{aligned}$$
so that
$$(-1)^{\frac{k(k-1)}{2}} \iota^{p-q+k} H_k$$
is positive-definite. □

**Corollary 3.5.2** (Hodge index theorem)**.** *Let $X$ be a compact Kähler surface. Then, the signature of the Poincaré intersection pairing $Q_2$ on $H^2(X, \mathbb{R})$ is*
$$(2h^{2,0} + 1, h^{1,1} - 1).$$

*Proof.* Let $\alpha \in H^2(X, \mathbb{R})$. We may decompose into types: $\alpha = \alpha^{2,0} + \alpha^{1,1} + \alpha^{0,2}$. The fact that $\alpha = \overline{\alpha}$ forces $\alpha^{1,1} \in H^{1,1}(X, \mathbb{R})$, and $\alpha^{2,0} = \overline{\alpha^{2,0}}$. We thus have the decomposition
$$H^2(X, \mathbb{R}) = ((H^{2,0}(X) \oplus H^{0,2}) \cap H^2(X, \mathbb{R})) \oplus H^{1,1}(X, \mathbb{R}),$$
which we know to be orthogonal with respect to the Poincaré pairing. Any $\alpha \in (H^{2,0}(X) \oplus H^{0,2}(X)) \cap H^2(X, \mathbb{R})$ is primitive for degree reasons: taking the cup product with $\omega$ yields types $(3,1)$ or $(1,3)$. Thus, we have
$$Q_2(\alpha, \alpha) = \int_X \alpha \wedge \alpha = \int_X \alpha^{2,0} \wedge \overline{\alpha^{2,0}},$$
which is positive by Theorem 3.5.1(3). Now, we have the Lefschetz decomposition
$$H^{1,1}(X, \mathbb{R}) = H^{1,1}(X, \mathbb{R})_{\text{prim}} \oplus L_\omega H^0(X, \mathbb{R}) = H^{1,1}(X, \mathbb{R})_{\text{prim}} \oplus \mathbb{R}[\omega].$$

This decomposition is orthogonal: if $\alpha \in H^{1,1}(X,\mathbb{R})_{\text{prim}}$, then

$$Q_2(\omega,\alpha) = \int_X \omega \wedge \alpha = 0$$

as $\omega \wedge \alpha = 0$ by definition of primitive cohomology. We have

$$Q_2(\omega,\omega) = \int_X \omega^2 = 2 \cdot \mathrm{vol}(X) > 0.$$

There remains to compute $Q_2$ on real $(1,1)$ primitive classes. But by Theorem 3.5.1,

$$\int_X \alpha^2 < 0,$$

and so we obtain the right count for the index of the pairing. $\square$

3.6. **(1,1) classes.** Consider now the exponential exact sequence

$$0 \to \mathbb{Z} \to \mathcal{O}_X \xrightarrow{exp(2\pi i -)} \mathcal{O}_X^* \to 0.$$

It is a fact that the induced connecting homomorphism $\delta$ in the long exact cohomological sequence

$$\cdots \to H^1(X,\mathbb{Z}) \to H^1(X,\mathcal{O}_X) \to \mathrm{Pic}(X) \xrightarrow{\delta} H^2(X,\mathbb{Z}) \to H^2(X,\mathcal{O}_X) \to \cdots$$

is related to the Chern map $c_1 : \mathrm{Pic}(X) \to H^2(X,\mathbb{C})$: after composing $\delta$ with the map $H^2(X,\mathbb{Z}) \to H^2(X,\mathbb{C})$ (which kills torsion), we have $\delta(\mathcal{L}) = -c_1(\mathcal{L})$, where $\mathcal{L} \in \mathrm{Pic}(X)$.

**Theorem 3.6.1** (Lefschetz theorem on $(1,1)$ classes)**.** *If $X$ is compact Kähler, then the first chern map $c_1$ composed with $H^2(X,\mathbb{Z}) \to H^2(X,\mathbb{C})$ is surjective onto $H^{1,1}(X,\mathbb{Z})$.*

*Proof.* Note that the compositio is indeed valued in $H^{1,1}(X,\mathbb{Z})$ by definition of the Chern class (indeed, the $(2,0)$ and $(0,2)$ part of the curvature of the Chern connection vanish). The map $H^2(X,\mathbb{Z}) \to H^2(X,\mathcal{O}_X)$ factors as

$$H^2(X,\mathbb{Z}) \to H^2(X,\mathbb{C}) \to H^{0,2}(X,\mathbb{C}) \simeq H^2(X,\mathcal{O}_X),$$

where the middle arrow is the projection onto the $(2,0)$ part. This can be seen as follows: the map $\mathbb{Z} \to \mathcal{O}_X$ factors as $\mathbb{Z} \to \mathbb{C} \to \mathcal{O}_X$, and the map in cohomology $H^2(\mathbb{C},X) \to H^2(X,\mathcal{O}_X)$ may be computed by considering the map between the resolutions

$$\begin{tikzcd}
0 \arrow[r] & \mathbb{C} \arrow[r] \arrow[d] & \mathcal{A}^1(\mathbb{C}) \arrow[r, "d"] \arrow[d] & \mathcal{A}^2(\mathbb{C}) \arrow[r, "d"] \arrow[d] & \cdots \\
0 \arrow[r] & \mathcal{O}_X \arrow[r] & \mathcal{A}^{0,1} \arrow[r, "\bar\partial"] & \mathcal{A}^{0,2} \arrow[r, "\bar\partial"] & \cdots ,
\end{tikzcd}$$

where the vertical arrows are projections. Thus, the map $H^2(X,\mathbb{Z}) \to H^2(X,\mathcal{O}_X)$ vanishes on $(1,1)$ classes, so that $(1,1)$ classes are in the kernel of this map, and equivalently in the image of $c_1 = -\delta$. $\square$

**Definition 3.6.2.** We define the **Néron-Severi** group of $X$, denoted $\mathrm{NS}(X)$, to be the image of $c_1$ in $H^2(X,\mathbb{Z})$. We note that $\mathrm{NS}(X)/\text{torsion} = H^{1,1}(X,\mathbb{Z})$, as $H^{1,1}(X,\mathbb{Z})$ is defined to be the intersection of $H^{1,1}(X)$ with the image of $H^2(X,\mathbb{Z})$ in $H^2(X,\mathbb{C})$.

We define the rank $\rho(X)$ of $\mathrm{NS}(X)$ (or equivalently of $H^{1,1}(X,\mathbb{Z})$) to be the **Picard number**. To motivate this terminology, note that there is an exact sequence

$$0 \to \mathrm{Pic}^0(X) \to \mathrm{Pic}(X) \to \mathrm{NS}(X) \to 0,$$

where $\mathrm{Pic}^0(X)$ is the connected component of the identity.

**Remark 3.6.3.** The integral Hodge conjecture, which has been proven to be false, states that the map

$$\{\text{algebraic } \mathbb{Z}\text{-cycles of dimension } k\} \to H^{(k,k)}(X,\mathbb{Z})$$
$$Z \mapsto [Z]$$

is surjective. Nevertheless, the Lefschetz theorem on $(1,1)$ classes shows that it is true for $k=1$. The rational Hodge conjecture, notoriously unsolved, asks whether taking this map over $\mathbb{Q}$-coefficients is surjective.

## 4. Fourth Lecture: K3 surfaces

4.1. **K3 Surfaces.** We now focus on K3 surfaces, which are the fundamental examples in dimension 2 arising from the Beauville-Bogomolov decomposition theorem.

**Definition 4.1.1** (Strong definition)**.** A **K3 surface** is a compact connected Kähler surface $S$ such that

(1) The canonical bundle is trivial, $K_S \simeq \mathcal{O}_S$;
(2) $S$ is simply connected, i.e. $\pi_1(S) = \{e\}$.

K3 surfaces are exactly the strict Calabi–Yau manifolds of dimension 2. To show this, we introduce a second definition of K3 surfaces, which we will show is equivalent to the one above.

**Definition 4.1.2** (Weak definition)**.** A K3 surface is a compact, connected Kähler surface $S$ such that

(1) $K_S \simeq \mathcal{O}_S$;
(2) $H^1(S, \mathcal{O}_S) = 0$.

**Proposition 4.1.3.** *The two definitions (Definition 4.1.1 and Definition 4.1.2) are equivalent. In particular, K3 surfaces are exactly the two dimensional strict Calabi–Yau manifolds.*

*Proof.* (4.1.1 $\implies$ 4.1.2): Assume $S$ satisfies the strong definition. We need to show $H^1(S, \mathcal{O}_S) = 0$. Since $S$ is simply connected, $H_1(S, \mathbb{C})$ and hence $H^1(S, \mathbb{C}) = H_1(S, \mathbb{C})^\vee$ vanish. Since $S$ is Kähler, the Hodge decomposition gives $H^1(S, \mathbb{C}) = H^{1,0}(S) \oplus H^{0,1}(S)$. Thus $H^{0,1}(S) \simeq H^1(S, \mathcal{O}_S) = 0$.

(4.1.2 $\implies$ 4.1.1): Assume $S$ satisfies the weak definition. A deep theorem proved by Siu in [Siu83] that any such surface is Kähler. We need to prove that $S$ is simply connected.

First, we compute the holomorphic Euler characteristic:

$$\begin{aligned}\chi(S, \mathcal{O}_S) &= h^0(S, \mathcal{O}_S) - h^1(S, \mathcal{O}_S) + h^2(S, \mathcal{O}_S) \\ &= 1 - 0 + h^0(S, K_S) \quad \text{(by Serre duality)} \\ &= 1 - 0 + 1 = 2.\end{aligned}$$

We apply the Beauville-Bogomolov decomposition theorem (Theorem 2.3.7) to conclude by dimension reasons that there exists an étale cover $\pi : \tilde{S} \to S$ with $\tilde{S}$ either a complex torus, a Hyperkähler surface, a strict Calabi–Yau surface, or a product of two elliptic curves. Suppose first that $\tilde{S}$ is either a product of two elliptic curves or a complex torus. Then,

$$\chi(\tilde{S}, \mathcal{O}_{\tilde{S}}) = \int_{\tilde{S}} ch(\mathcal{O}_S) \cup td(S) = \int_{\tilde{S}} \frac{c_1^2(S) + c_2(S)}{12} = 0$$

since $\tilde{S}$ is flat. But by Hirzebruch-Riemann-Roch again, we have $\chi(\tilde{S}, \mathcal{O}_{\tilde{S}}) = \deg(\pi)\chi(S, \mathcal{O}_S) = 2\deg(\pi)$, which forces $\deg(\pi) = 0$, contradicting that $\pi$ is a covering.

Therefore, $\tilde{S}$ is either a strict Calabi–Yau or a Hyperkähler manifold (we will see *a posteriori* that these two conditions coincide for surfaces), and in particular is simply connected. Thus, $\deg(\pi) = 1$ and $S = \tilde{S}$ is simply connected. □

**Remark 4.1.4.** Importantly, it is also true that a two dimensional Hyperkähler manifold is nothing but a K3 surface, but we shall see this later.

4.2. **Cohomology and Picard group of K3 surfaces.**

**Theorem 4.2.1.** *Let $S$ be a K3 surface. Then:*

(1) $H^0(S, \mathbb{Z}) \simeq H^4(S, \mathbb{Z}) \simeq \mathbb{Z}$.
(2) $H^1(S, \mathbb{Z}) = H^3(S, \mathbb{Z}) = 0$.
(3) $H^2(S, \mathbb{Z}) \simeq \mathbb{Z}^{22}$ *and is torsion-free.*
(4) *The intersection pairing (cup product)*

$$(-, -) : H^2(S, \mathbb{Z}) \times H^2(S, \mathbb{Z}) \to \mathbb{Z}, \quad (\alpha, \beta) \mapsto \int_S \alpha \cup \beta$$

*is symmetric, bilinear, and unimodular (i.e. induces an isomorphism $H^2(S, \mathbb{Z}) \to H^2(S, \mathbb{Z})^\vee$).*
(5) *The signature of the pairing is* $(3, 19)$.
(6) *The pairing is even, i.e.,* $\alpha^2 := (\alpha, \alpha) \equiv 0 \pmod 2$ *for all* $\alpha \in H^2(S, \mathbb{Z})$.

*Proof.* We use the strong definition.

(1): $S$ is compact and oriented.

(2): Since $S$ is simply connected, $H_1(S,\mathbb{Z})=0$. By the universal coefficient theorem, $H^1(S,\mathbb{Z})=0$ (there is no torsion in $H_0(S,\mathbb{Z})$). By Poincaré duality, $H^3(S,\mathbb{Z})\simeq H_1(S,\mathbb{Z})^\vee=0$.

(3) By the universal coefficient theorem, the torsion subgroup of $H^2(S,\mathbb{Z})$ comes from the torsion of $H_1(S,\mathbb{Z})=0$. So $H^2(S,\mathbb{Z})$ is torsion-free.

To compute the rank $b_2(S)$, we use the topological Euler characteristic $e(S)$.

$$e(S)=\sum(-1)^i b_i(S)=1-0+b_2(S)-0+1=2+b_2(S).$$

By the Gauss-Bonnet theorem, we have

$$e(S)=\int_S c_2(S).$$

However, Hirzebruch-Riemann-Roch theorem gives us

$$\chi(S,\mathcal{O}_S)=\int_S\frac{c_1(S)^2+c_2(S)}{12}=\int_S\frac{c_2(S)}{12},$$

since $S$ is Ricci-flat by definition. Since $\chi(S,\mathcal{O}_S)=2$, we conclude that $b_2(S)=2\cdot 12-2=22$.

(4) Unimodularity follows from Poincaré duality over $\mathbb{Z}$. Symmetry just follows from the degree in cohomology.

(5) This is a direct consequence of the Hodge index theorem (Corollary 3.5.2) and the fact that $h^{1,1}(S)=20$ since $h^{2,0}=h^0(S,\mathcal{O}_S)=1$.

(6) For algebraic classes $\alpha=c_1(L)\in NS(S)$, we use Hirzebruch-Riemann-Roch:

$$\chi(S,L)=\frac{c_1(L)^2}{2}+\chi(S,\mathcal{O}_S)=\frac{\alpha^2}{2}+2.$$

Since $\chi(S,L)\in\mathbb{Z}$, $\alpha^2$ must be even.

For the general $\alpha\in H^2(S,\mathbb{Z})$, this follows from Wu's formula. □

**Remark 4.2.2** (Hodge Diamond)**.** The Hodge diamond of a K3 surface is the following

$$\begin{array}{ccccc} & & 1 & & \\ & 0 & & 0 & \\ 1 & & 20 & & 1 \\ & 0 & & 0 & \\ & & 1 & & \end{array}$$

**Proposition 4.2.3.** *For a K3 surface $S$, the Picard group is isomorphic to the Néron-Severi group.*

*Proof.* Consider the exponential sequence:

$$\cdots\to H^1(S,\mathcal{O}_S)\to \mathrm{Pic}(S)\xrightarrow{-c_1} H^2(S,\mathbb{Z})\to\cdots$$

Since $H^1(S,\mathcal{O}_S)=0$, the map $c_1$ is injective. Thus $\mathrm{Pic}(S)\simeq \mathrm{im}\ (c_1)=\mathrm{NS}(S)$. □

**Remark 4.2.4.** This implies that a line bundle on a K3 surface is determined by its first Chern class.

The Picard number satisfies $\rho(S)\le h^{1,1}(S)=20$. If $S$ is algebraic, $\rho(S)\ge 1$ by Kodaira's embedding theorem. For the very general[2] K3 surface, $\rho(S)=0$.

4.3. **Examples of K3 surfaces.** We now discuss examples of how K3 surfaces may be constructed.

**Example 4.3.1** (Quartic surface in $\mathbb{P}^3$)**.** Let $S=\{f=0\}\subset\mathbb{P}^3$ be a smooth hypersurface defined by a homogeneous polynomial $f$ of degree 4. We verify that $S$ is a K3 surface.

By adjunction formula, we have

$$K_S=(K_{\mathbb{P}^3}\otimes\mathcal{O}_{\mathbb{P}^3}(4))|_S=(\mathcal{O}_{\mathbb{P}^3}(-4)\otimes\mathcal{O}_{\mathbb{P}^3}(4))|_S=\mathcal{O}_S.$$

There remains to show $H^1(S,\mathcal{O}_S)=0$: Consider the short exact sequence defining $S$:

$$0\to\mathcal{O}_{\mathbb{P}^3}(-4)\xrightarrow{\cdot f}\mathcal{O}_{\mathbb{P}^3}\to\mathcal{O}_S\to 0.$$

[2]The notion of "very general" is to be distinguished from "generic": a property holds for the very general K3 surface if, in the appropriate moduli space, it holds outside of a *countable* union of (analytic) Zariski-closed sets.

The long exact sequence in cohomology gives:

$$\cdots \to H^1(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}) \to H^1(S, \mathcal{O}_S) \to H^2(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(-4)) \to \cdots.$$

Since $H^1(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}) = H^2(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(-4)) = 0$, we conclude $H^1(X, \mathcal{O}_X) = 0$.

To construct further examples, we review the cyclic covering trick, used to construct branched covers. If we are given a covering $f : X \to Y$, the ramification divisor $R$ is that where the rank of the differential drops, i.e. the zero locus of

$$\det(df) : f^*K_Y \to K_X.$$

This shows that $R \in |f^*K_Y^\vee \otimes K_X|$. Quite tautologically, the Hurwitz formula follows:

$$K_X = f^*K_Y \otimes \mathcal{O}(R).$$

We say that $f$ is ramified over $f(R)_{\mathrm{red}} = B$. Conversely, given the choice of an effective divisor $B \subset Y$, we want to describe ways to construct finite coverings of $Y$ that are ramified over $B$.

**Construction 4.3.2** (Cyclic covering trick)**.** Assume for simplicity that $Y$ is algebraic, the analytic case shall be discussed in Remark 4.3.7. Let $B \subset Y$ be an effective reduced divisor, and suppose that $\mathcal{O}(B)$ is a $m$th power for some $m \geq 2$, that is, there exists a line bundle $\mathcal{L} \in \mathrm{Pic}(Y)$ with $\mathcal{L}^m = \mathcal{O}(B)$, and let $s \in \mathcal{O}(B)$ be a defining section for $D$.

Let $\mathbb{V}(\mathcal{L})$ be the total space of $\mathcal{L}$. We have

$$\mathbb{V}(\mathcal{L}) = \mathrm{Spec}(\mathrm{Sym}^\bullet \mathcal{L}^\vee).$$

Let $\pi : \mathbb{V}(\mathcal{L}) \to Y$ be the projection. We have a tautological section

$$\tau \in H^0(\mathbb{V}(\mathcal{L}), \pi^*\mathcal{L}) = \mathcal{L} \otimes \bigoplus_{i \leq 0} \mathcal{L}^i$$

given by the identity element of $\mathcal{L}^\vee \otimes \mathcal{L}$ and whose zero locus coincides with that of the zero section $Y \subset \mathbb{V}(\mathcal{L})$. Consider the variety $X$ defined as the zero locus of the section $\tau^m - \pi^* s \in \pi^*\mathcal{L}^m$, i.e.

$$X := Z(\tau^m - \pi^* s) \subset \mathbb{V}(\mathcal{L}).$$

the map $f : X \hookrightarrow \mathbb{V}(\mathcal{L}) \xrightarrow{\pi} Y$ is finite, and it is ramified over $B$

**Remark 4.3.3.** By using the Jacobian criterion on the local equations for $X$, it is obvious that $X$ is smooth if and only if $B$ is.

**Lemma 4.3.4.** *Let $\pi : X \to Y$ be the $m$-cyclic cover defined by $B$ as above.*

1. *The pushforward of the structure sheaf is $f_*\mathcal{O}_X \simeq \bigoplus_{j=0}^{m-1} \mathcal{L}^{-j}$.*
2. *The canonical bundle is $K_X \simeq f^*(K_Y \otimes \mathcal{L}^{m-1})$.*

*Proof.* (1): Consider the short exact sequence defining $X$ in $\mathbb{V}(\mathcal{L})$:

$$0 \to \mathcal{O}_{\mathbb{V}(\mathcal{L})}(-X) \xrightarrow{\tau^m - \pi^* s} \mathcal{O}_{\mathbb{V}(\mathcal{L})} \to \mathcal{O}_X \to 0.$$

We have by definition $\mathcal{O}_{\mathbb{V}(\mathcal{L})}(-X) \simeq \pi^*(\mathcal{L}^{-m})$. We push-forward via $\pi$, which preserves exactness since $\pi$ is affine:

$$0 \to \pi_*(\pi^* L^{-m}) = \mathcal{L}^{-m} \otimes \mathrm{Sym}^\bullet \mathcal{L}^\vee \xrightarrow{-s} \mathrm{Sym}^\bullet \mathcal{L}^\vee \to f_*\mathcal{O}_X \to 0, \tag{4.1}$$

where the equality is obtained from the projection formula. Since $s$ has degree $m$, this multiplication map identifies a section in $\mathrm{Sym}^\bullet \mathcal{L}^\vee$ of degree $a + mb$ (where $a, b \leq 0$, $a > -m$) with a section of degree $a$. Therefore, as $\mathcal{O}_Y$-modules, we have

$$f_*\mathcal{O}_X \simeq \bigoplus_{j=0}^{m-1} \mathcal{L}^{-j}.$$

(2): By the adjunction formula, we have

$$K_X \simeq \left(K_{\mathbb{V}(\mathcal{L})} \otimes \mathcal{O}_{\mathbb{V}(\mathcal{L})}(X)\right)|_X = \left(\pi^*(K_Y \otimes \mathcal{L}^{-1}) \otimes \pi^*\mathcal{L}^m\right)|_X = f^*(K_Y \otimes \mathcal{L}^{m-1}).$$

Alternatively, looking at the defining equation, the ramification divisor of $X \to Y$ is $(m-1)Z(\tau) \subset X$, and so we have by the Hurwitz formula

$$K_X = f^*(K_Y \otimes \mathcal{L}^{m-1}).$$ □

**Remark 4.3.5.** Such a cyclic cover has a $\mu_m$-action, acting transitively on the fibers, which can be seen from the $\mu_m$-graded structure on $f_*\mathcal{O}_X$. Conversely, it can be shown that any cover that has such a $\mu_m$-action arises as a cyclic covering as constructed above.

**Remark 4.3.6.** Note that not every finite covering $f : X \to Y$ of degree $m$ is a cyclic covering. However, in characteristic coprime to $m$, we can say the following. The exact sequence

$$0 \to \mathcal{O}_Y \to f_*\mathcal{O}_X \to \mathrm{coker} \to 0$$

of $\mathcal{O}_Y$-modules splits as we have a retraction given by $\frac{1}{m}\mathrm{Tr}$, where $\mathrm{Tr} : \pi_*\mathcal{O}_X \to \mathcal{O}_Y$ is the trace, defined because we can view elements of $f_*\mathcal{O}_X$ as acting on $f_*\mathcal{O}_X$ by multiplication. Since $\pi_*\mathcal{O}_X$ is locally free of rank $m$, this implies that $\mathcal{T}^\vee := \mathrm{coker}$ is also locally free, of rank $m-1$. The splitting also ensures that we have a section $\mathcal{T}^\vee \to f_*\mathcal{O}_X$, and the universal property of the symmetric algebra yields a map of $\mathcal{O}_Y$-algebras

$$\mathrm{Sym}^\bullet \mathcal{T}^\vee \to f_*\mathcal{O}_X,$$

which is obviously surjective. Since $f$ is affine, this map comes from a closed immersion

$$X \hookrightarrow \mathbb{V}(\mathcal{T})$$

over $Y$. $\mathcal{T}$ is called the *Tschirnhausen* bundle, and we have shown that any finite map (under characteristic assumptions) factors through its Tschirnhausen bundle. It is to be expected that this bundle would have rank $m-1$. Indeed, if we take $m$ points in a very big vector space, the affine space they span has dimension $m-1$. In particule, any degree 2 covering factors through (the total space of) a line bundle, and one sees readily that this recovers the cyclic covering trick.

**Remark 4.3.7.** The only subtlety in the analytic case is that the total space of the line bundle $\mathcal{L}$ has more functions than $\mathrm{Sym}^\bullet\mathcal{L}^\vee$. But we can circumvent this e.g. by constructing directly the $\mathcal{O}_Y$-algebra structure on

$$\mathcal{O}_Y \oplus \mathcal{L}^{-1} \oplus \cdots \oplus \mathcal{L}^{m-1},$$

and showing afterwards it corresponds to $Z(\tau^m - \pi^* s) \subset \mathbb{V}(\mathcal{L})$. Alternatively, one may also show that the analytic counterpart of the exact sequence (4.1) yields the desired $\mathcal{O}_Y$-algebra structure in the same way, e.g. after locally injecting holomorphic functions into formal functions and taking Taylor expansions in local coordinates.

**Example 4.3.8** (Double cover of $\mathbb{P}^2$ branched over a sextic)**.** Let $Y = \mathbb{P}^2$. Let $B \subset \mathbb{P}^2$ be a smooth curve of degree 6. We have $\mathcal{O}(B) = \mathcal{O}(6)$. We take $m = 2$ and $\mathcal{L} = \mathcal{O}(3)$. Let $f : X \to \mathbb{P}^2$ be the double cyclic cover branched along $B$. We check the K3 conditions.

1. Canonical bundle: Using Lemma 4.3.4 (2),

$$K_X = f^*(K_{\mathbb{P}^2} \otimes \mathcal{L}) = f^*(\mathcal{O}(-3) \otimes \mathcal{O}(3)) = \mathcal{O}_X.$$

2. Vanishing of $H^1(X, \mathcal{O}_X)$: Using Lemma 4.3.4 (1),

$$f_*\mathcal{O}_X = \mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{L}^{-1} = \mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}(-3).$$

Since $f$ is finite,

$$H^1(X, \mathcal{O}_X) \simeq H^1(\mathbb{P}^2, f_*\mathcal{O}_X) = H^1(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}) \oplus H^1(\mathbb{P}^2, \mathcal{O}(-3)) = 0.$$

Since $B$ was chosen to be smooth, so is $X$, which is also connected since it is smooth and a ramified cover. Thus, $X$ is a K3 surface.

**Example 4.3.9** (Kummer surfaces)**.** Let $A$ be a complex torus of dimension 2. Consider the involution $i : A \to A$, $x \mapsto -x$.

The fixed locus of $i$ is the set of 2-torsion points $A[2]$, which consists of 16 points. These points induce 16 singularities in $A/i$. So we first blow up.

Let $p : \tilde{A} \to A$ be the blow-up of $A$ at the 16 fixed points. The exceptional locus is a disjoint union of 16 smooth rational curves $\tilde{E}_i$. The involution $i$ lifts to an involution $\tilde{i} : \tilde{A} \to \tilde{A}$ by the universal property of the blow-up: indeed, we have the commutative square

$$\begin{tikzcd} \tilde{A} \arrow[r, dotted, "\tilde{i}"] \arrow[d] \arrow[dr] & \tilde{A} \arrow[d] \\ A \arrow[r, "i"] & A, \end{tikzcd}$$

and so the diagonal arrow lifts to $\tilde{i}$ since the total transform of the 16 points under the diagonal arrow is a Cartier divisor. Incidentally, the fixed locus of $\tilde{i}$ is exactly the union of the exceptional divisors $\bigcup E_i$. This implies that the quotient $X = \tilde{A}/\tilde{i}$, which we define to be the **Kummer surface** of $A$, is smooth because the fixed locus has codimension 1. $X$ is Kähler since it is the quotient of a Kähler surface by a finite group. In order to check that it is a K3 surface, there are two conditions to check: that the canonical bundle is trivial, and that $H^1(X, \mathcal{O}_X) = 0$.

We first show that $H^1(X, \mathcal{O}_X) = 0$. Let $f : \tilde{A} \to \tilde{A}/\tilde{i}$ be the quotient map. Since $f$ is finite, we have that the pullback $f^* : H^1(X, \mathbb{C}) \to H^1(\tilde{A}, \mathbb{C})$ is injective; indeed this is seen easily as for a given $\alpha \in H^1(X, \mathbb{C})$, the projection formula (singular cohomology version) gives

$$f_* f^* \alpha = \alpha \cup f_* 1_{\tilde{A}} = 2\alpha,$$

and so $f_* f^*$ is injective. Thus, we may identify

$$H^1(X, \mathbb{C}) = H^1(\tilde{A}, \mathbb{C})^{\mu_2} = H^1(A, \mathbb{C})^{\mu_2},$$

where the superscript indicates taking the invariants under $i$, and the last equality holds because blowing up a point does not affect singular cohomology[3]. But now, on $H^1(A, \mathbb{C}) = \mathbb{C}^4$, the involution acts as multiplication by $-1$; this can be seen directly by looking at how it acts on the generators $dz_1, dz_2, d\bar{z}_1$ and $d\bar{z}_2$. Therefore, $H^1(A, \mathbb{C})^{\mu_2} = H^1(X, \mathbb{C}) = 0$, implying that $H^1(X, \mathcal{O}_X) = 0$ by the Hodge decomposition.

We now show that $K_X = \mathcal{O}_X$. Let $b : \tilde{A} \to A$ be the blow-down. By the Hurwitz formula, we have

$$K_{\tilde{A}} \simeq f^* K_X \otimes \mathcal{O}\left(\sum \tilde{E}_i\right),$$

and similarly, by Hurwitz, we have

$$K_{\tilde{A}} \simeq b^* K_A \otimes \mathcal{O}\left(\sum \tilde{E}_i\right) = \mathcal{O}\left(\sum \tilde{E}_i\right),$$

so that $f^* K_X = \mathcal{O}_{\tilde{A}}$. Using the projection formula, we obtain

$$f_* \mathcal{O}_{\tilde{A}} \simeq f_* f^* K_X \simeq f_* \mathcal{O}_{\tilde{A}} \otimes K_X, \tag{4.2}$$

implying, after taking determinants, that $K_X$ is 2-torsion. Now, $f$ is a $\mu_2$-covering, and so by our previous discussion on the cyclic covering trick, we have

$$f_* \mathcal{O}_{\tilde{A}} \simeq \mathcal{O}_X \oplus \mathcal{L}^\vee,$$

where $f^* \mathcal{L}^2 = \mathcal{O}\left(\sum \tilde{E}_i\right)$. Thus, we have

$$K_X \oplus \left(K_X \otimes \mathcal{L}^\vee\right) \simeq \mathcal{O}_X \oplus \mathcal{L}^\vee.$$

This isomorphism has to be given by a matrix of the form

$$\begin{pmatrix} \mathrm{Hom}(K_X, \mathcal{O}_X) & \mathrm{Hom}(K_X, \mathcal{L}^\vee) \\ \mathrm{Hom}(K_X \otimes \mathcal{L}^\vee, \mathcal{O}_X) & \mathrm{Hom}(K_X \otimes \mathcal{L}^\vee, \mathcal{L}^\vee) \end{pmatrix} = \begin{pmatrix} H^0(X, K_X^\vee) & H^0(X, (K_X \otimes \mathcal{L})^\vee) \\ H^0(X, K_X^\vee \otimes \mathcal{L}) & H^0(X, K_X^\vee). \end{pmatrix} \tag{4.3}$$

Suppose for the purpose of contradiction that $K_X \neq O_X$. Then, since $K_X = K_X^\vee$, it cannot have a non-zero global section, and so (4.3) is of the form

$$\begin{pmatrix} 0 & H^0(X, (K_X \otimes \mathcal{L})^\vee) \\ H^0(X, K_X^\vee \otimes \mathcal{L}) & 0, \end{pmatrix}$$

implying that we must have an isomorphism $K_X \simeq \mathcal{L}^\vee$, so that $\mathcal{L}$ is 2-torsion. But this is impossible, as $f^* \mathcal{L}^2 = \mathcal{O}\left(\sum \tilde{E}_i\right) \not\simeq \mathcal{O}_{\tilde{A}}$. Thus, $K_X \simeq \mathcal{O}_X$.

[3] In general, if $\tilde{Y} \to Y$ is the blow-up at a complex submanifold $Z$, we have the formula

$$H^\bullet(\tilde{Y}) = H^\bullet(Y) \oplus \bigoplus_{i=1}^{\mathrm{codim}(Z)-1} H^{\bullet - 2i}(Z).$$

5.1. **The Hilbert square of a K3 surface.** Let $S$ be a K3 surface. We will construct a new manifold from $S$, called the Hilbert scheme of two points, denoted $S^{[2]}$. This is the first example of a hyperkähler manifold of dimension more than 2 that we shall see.

We start with the product $S^2 := S \times S$. Let $\Delta \subset S^2$ be the diagonal. We consider the blow-up $\tilde{S^2} := \mathrm{Bl}_\Delta(S^2)$.

**Remark 5.1.1** (Blow-ups in the holomorphic category)**.** This is a good place to recall how to deal with blow-ups in the holomorphic category. Let $Z$ be a subvariety regularly embedded[4] in $X$ which is defined by an ideal sheaf $\mathcal{I}$. By definition of a regular embedding, we can locally on an open set $U \subset X$, find generators $\mathcal{I} = \langle f_1, \dots, f_k \rangle$. We define a map

$$\begin{aligned} \underline{f} : U \setminus (U \cap Z) &\to \mathbb{P}^{k-1} \\ x &\mapsto [f_1(x) : \cdots : f_k(x)]. \end{aligned}$$

The blow-up $\mathrm{Bl}_{Z\cap U}(U)$ is defined as the closure of the graph $\Gamma(\underline{f})$ inside $U \times \mathbb{P}^{k-1}$. This picture is local, but naturally glues to the blow-up $\mathrm{Bl}_Z(X)$ of $X$ at $Z$. It satisfies the same universal property as in the algebraic case: any map to $X$ whose pre-image of $Z$ is a divisor[5] factors through the blow-up.

Let $\iota : S^2 \to S^2$ be the involution $\iota(x, y) = (y, x)$. The **symmetric product** is the quotient $S^{(2)} := S^2/\iota$.

**Remark 5.1.2.** The quotient $S^{(2)}$ is singular along the image of the diagonal since the action has stabilisers on the diagonal, which has codimension more than 1. Indeed, locally around the diagonal, we can use the dimension of the fixed locus to see that $S^{(2)}$ is of the form

$$\mathbb{C}^2 \times \mathbb{C}^2/\pm .$$

Now, $\mathbb{C}^2/\pm = \mathrm{Spec}\ \mathbb{C}[x^2, xy, y^2] = \mathrm{Spec}\ \frac{\mathbb{C}[u,w,z]}{uw-z^2}$, so that the singularities are of the form of those of a cone times $\mathbb{C}^2$. Since blowing up the cone at the origin resolves the singularities, similarly, blowing up $S^{(2)}$ at the image of the diagonal resolves all singularities.

We define the **Hilbert Square** of $S$ to be the blow-up $S^{[2]} \xrightarrow{h} S^{(2)}$ at the image of the diagonal, which is smooth by Remark 5.1.2. The map $h$ is an example of the so-called *Hilbert–Chow* morphism, i.e. the map $\mathrm{Hilb}^n(X) \to \mathrm{Sym}^n X$. It is a theorem by Fogarty (see [Fog68]) that the Hilbert scheme of $n$ points of a surface is smooth, and so that the Hilbert–Chow morphism is a resolution of singularities in those instances.

**Remark 5.1.3.** We can also define the Hilbert scheme of $n$ points of $X$ in a functorial way: a map $T \to X^{[n]}$ corresponds to a flat map

$$X \times T \supset Y \to T,$$

whose fibres $Y_t$ are all zero-dimensional schemes of length $n$ (i.e. embedded deformations of $n$ points in $X$).

**Remark 5.1.4.** The involution $i$ lifts to an involution $\tilde{i}$ on the blow-up $\tilde{S^2}$ of $S^2$ at the diagonal, and a careful analysis of the local form of this action and of the singularities of $S^{(2)}$ shows that $S^{[2]}$ is also the quotient of $\tilde{S^2}$ by $\tilde{i}$. In particular, we have a Cartesian diagram

$$\begin{tikzcd} \tilde{i} \circlearrowright \tilde{S^2} \arrow[r, "q"] \arrow[d, "\pi"'] & S^{[2]} \arrow[d, "h"] \\ i \circlearrowright S^2 \arrow[r, "p"] & S^{(2)} \end{tikzcd}$$

Let $E \subset S^{[2]}$ be the exceptional divisor of $h$. The map $q$ is a double cover branched along $E$. In particular, we have a divisor class $\delta$ such that $\mathcal{O}(\delta)$ is the Tschirnhausen line bundle of this cover. This implies that we have $2\delta = E$.

[4]This means locally as many equations as codimension. If for example $Z$ is smooth, this condition is always satisfied.

[5]If $X$ is not smooth, one must replace the word "divisor" by "Cartier divisor".

**Proposition 5.1.5.** *Let $S$ be a K3 surface. Then $S^{[2]}$ is a 4-dimensional hyperkähler manifold. Moreover, we have an isomorphism*

$$H^2(S^{[2]}, \mathbb{Z}) \cong H^2(S, \mathbb{Z}) \oplus \mathbb{Z}\delta,$$

*compatible witth the Hodge structure.*

**Remark 5.1.6.** Taking $S^{[n]}$ also yields a hyperkähler manifold of dimension $2n$. However, it is false that given any hyperkähler manifold $X$, its Hilbert scheme of $n$-points is also Hyperkähler.

We sketch the proof of Proposition 5.1.5, starting with the fundamental group.

**Lemma 5.1.7.** *The map $q$ induces a surjection $q_* : \pi_1(\tilde{S^2}) \to \pi_1(S^{[2]})$.*

*Proof.* This is a property of ramified double covers and more generally of ramified covers such that the preimage of some point in the branched locus is set theoretically one point.

We can compute the fundamental group by taking loops based at some point $p \in E$. Given any such path $\gamma : [0,1] \to S^{[2]}$, we can deform it so that only $\gamma(0) = \gamma(1) = p$ are in $E$. Since $\tilde{S}^2 \setminus q^{-1}(E) \to S^{[2]} \setminus E$ is étale, we can lift the path $\gamma : (0,1) \to S^{[2]} \setminus E$ to a path $\tilde{\gamma} : (0,1) \to \tilde{S}^2 \setminus q^{-1}(E)$, and by continuity, the limits $\tilde{\gamma}(0)$ and $\tilde{\gamma}(1)$ must lie in the (set-theoretical) preimage of $p$. But since $q$ is a double cover, this set-theoretical image is only one point, and so $\gamma$ lifts to a path $\tilde{\gamma}$. □

We shall also make use of the following fact.

**Lemma 5.1.8.** *Let $X$ be a real manifold and $Z \subset X$ a subvariety of (real) codimension $c$. The map $\pi_1(X \setminus Z) \to \pi_1(X)$ induced by inclusion is surjective if $c \geq 1$ and an isomorphism if $c \geq 2$.*

*Proof.* If $c \geq 1$, we can move loops to avoid $Z$, and if $c \geq 2$, we can also move homotopies between those loops to avoid $Z$. □

**Proposition 5.1.9.** *$S^{[2]}$ is simply connected.*

*Proof.* Since $S$ is simply connected, $S^2$ is also simply connected. Let $\Delta \subset S^2$ be the diagonal. By Lemma 5.1.8, $S^2 \setminus \Delta \simeq \tilde{S}^2 \setminus q^{-1}(\Delta)$ is also simply connected, and by Lemma 5.1.8 again, $\tilde{S}^2$ is simply connected (the exceptional divisor has real codimension 2). Thus, by Lemma 5.1.7, we conclude that $S^{[2]}$ is simply connected. □

**Lemma 5.1.10.** *$S^{[2]}$ is Kähler.*

*Proof.* $S^2$ is Kähler, and the blow-up of $\tilde{S}^2$ is again Kähler; see [Voi98, Proposition 3.24]. Let $\omega$ be a Kähler form on $\tilde{S}^2$. We claim the form $q_*\omega$ is Kähler. Indeed, by a criterion of Demailly and Paun (see [DP04]), to show that $q_*\omega$ is Kähler, it suffices to show that for all analytic subvariety $Y \subset S^{[2]}$, we have

$$\int_Y q_*\omega^{\dim Y} = \int_{S^{[2]}} q_*\omega^{\dim Y} \cup [Y] > 0,$$

and that $-\omega$ is not Kähler.[6] But by the projection formula, we have that

$$\int_{S^{[2]}} q_*\omega^{\dim Y} \cup [Y] = \int_{\tilde{S}^2} \omega^{\dim f^{-1}(Y)} \cup [f^{-1}(Y)] = \int_{f^{-1}(Y)} \omega^{\dim(f^{-1}(Y))} > 0$$

since $\omega$ is Kähler. Since $S^{[2]}$ has a divisor $E$, it is clear that $-\omega$ is not Kähler, as

$$\int_E (-\omega)^3 = -\int_E \omega^3 < 0$$

by what we have shown. □

**Proposition 5.1.11.** *$S^{[2]}$ has trivial canonical bundle*

*Proof.* Since $K_{S^2}$ is trivial, we have

$$K_{\tilde{S}^2} = \mathcal{O}(\tilde{E}).$$

Similarly, using Hurwitz formula, we have

$$K_{\tilde{S}^2} = \mathcal{O}(E) \otimes q^* K_{S^{[2]}},$$

[6] This second condition is only present to avoid pathologies with complex manifolds that have no odd-dimensional complex subvarieties.

so that $q^*K_{S^{[2]}} = \mathcal{O}_{\tilde{S}^2}$. Pushing forward, we see that $K_{S^{[2]}}$ is 2-torsion, and using that $q$ is a cyclic covering, we have moreover an isomorphism

$$K_{S^{[2]}} \oplus K_{S^{[2]}}(-\delta) \simeq \mathcal{O}_{S^{[2]}} \oplus \mathcal{O}(-\delta).$$

This isomorphism must be given by a matrix of the type

$$\begin{pmatrix} H^0(K^\vee_{S^{[2]}}) & H^0(K^\vee_{S^{[2]}}(\delta)) \\ H^0(K^\vee_{S^{[2]}}(-\delta)) & H^0(K^\vee_{S^{[2]}}). \end{pmatrix}$$

Now, if $K_{S^{[2]}} \not\simeq \mathcal{O}_{S^{[2]}}$, we have $H^0(K^\vee_{S^{[2]}}) = 0$ since $K_{S^{[2]}} = K^\vee_{S^{[2]}}$. This implies we must have an isomorphism $K^\vee_{S^{[2]}}(-\delta) \simeq \mathcal{O}_{S^{[2]}}$, i.e. that $K_{S^{[2]}} \simeq \mathcal{O}(\delta)$, and so taking the square, $\mathcal{O}_{S^{[2]}} \simeq \mathcal{O}(E)$, which is impossible since $E$ is an effective divisor. Thus, we must have $K_{S^{[2]}} \simeq \mathcal{O}_{S^{[2]}}$. $\square$

We are now ready to prove that $S^{[2]}$ is hyperkähler.

*Proof of Proposition 5.1.5.* By the Künneth formula, we have that

$$H^2(S^2, \mathbb{Q}) = p_1^* H^2(S, \mathbb{Q}) \oplus p_2^* H^2(S, \mathbb{Q}) \simeq H^2(S, \mathbb{Q}) \oplus H^2(S, \mathbb{Q}),$$

where $p_i : S^2 \to S$ are the projections. The involution $i^*$ acts on this space by switching both summands, and so we see that

$$H^2(S^2, \mathbb{Q})^{\mu_2} = \{p_1^*\alpha + p_2^*\alpha : \alpha \in H^2(S, \mathbb{Q})\},$$

and so we have an isomorphism

$$\begin{aligned} H^2(S, \mathbb{Q}) &\xrightarrow{\sim} H^2(S^2, \mathbb{Q})^{\mu_2} \\ \alpha &\mapsto p_1^*\alpha + p_2^*\alpha =: \tilde{\alpha}. \end{aligned} \tag{5.1}$$

We now recall some facts from topology. First, over $\mathbb{Q}$ the cohomology of the quotient by a free action is given exactly by the invariant cohomology classes. Therefore, we have an isomorphism

$$H^2(S^{[2]} \setminus E, \mathbb{Q}) = H^2(S^{(2)} \setminus p(\Delta), \mathbb{Q}) = H^2(S^2 \setminus \Delta, \mathbb{Q})^{\mu_2}.$$

Now, from the long exact sequence in relative cohomology, and the identification $H^k(S^2, S^2 \setminus \Delta; \mathbb{Q}) = H^{k-4}(\Delta, \mathbb{Q})$ (since $\Delta$ has codiemension 4) we have

$$\cdots \to H^{-2}(\Delta, \mathbb{Q}) \to H^2(S^2, \mathbb{Q}) \to H^2(S^2 \setminus \Delta, \mathbb{Q}) \to H^{-1}(\Delta, \mathbb{Q}) \to \cdots$$

Thus, the left and right terms vanish, and so pullback gives an isomorphism $H^2(S^2, \mathbb{Q}) \simeq H^2(S^2 \setminus \Delta, \mathbb{Q})$, and in particular,

$$H^2(S, \mathbb{Q}) \simeq H^2(S^2, \mathbb{Q})^{\mu_2} \simeq H^2(S^2 \setminus \Delta, \mathbb{Q})^{\mu_2} \simeq H^2(S^{[2]} \setminus E, \mathbb{Q}).$$

The exact same argument also shows that

$$H^2(S^{(2)}, \mathbb{Q}) \simeq H^2(S^{(2)} \setminus p(\Delta), \mathbb{Q}) = H^2(S^2 \setminus \Delta, \mathbb{Q})^{\mu_2} = H^2(S^2, \mathbb{Q})^{\mu_2}.$$

We now use the long exact sequence in relative cohomology for the pair $(S^{[2]}, S^{[2]} \setminus E)$:

$$\cdots \to H^1(S^{[2]} \setminus E, \mathbb{Q}) \to H^0(E, \mathbb{Q}) \to H^2(S^{[2]}, \mathbb{Q}) \to H^2(S^{[2]} \setminus E, \mathbb{Q}) \to H^1(E, \mathbb{Q}) \to \cdots$$

Recall that $E$ is the projectivisation of the normal bundle of $\Delta \subset S \times S$, i.e. the projectivisation of the tangent bundle $\mathcal{T}_S$. We have a formula for the cohomology of such projectivisation:

$$H^\bullet(E, \mathbb{Q}) = H^\bullet(\mathbb{P}(\mathcal{T}_S), \mathbb{Q}) = \frac{H^\bullet(S, \mathbb{Q})[\zeta]}{(\xi^2 + c_1(\mathcal{T}_S)\zeta + c_2(\mathcal{T}_S))},$$

where $\zeta$ is the first Chern class of $\mathcal{O}_{\mathbb{P}(\mathcal{T}_X)}(1)$, and so has degree 2. Since $S$ is simply connected, we see from the formula that $H^1(E, \mathbb{Q}) = 0$. Moreover, by Lemma 5.1.8, $S^{[2]} \setminus E$ is simply connected, and therefore $H^1(S^{[2]} \setminus E, \mathbb{Q}) = 0$ by the universal coefficient theorem. Thus, we have a short exact sequence

$$0 \to H^0(E, \mathbb{Q}) \to H^2(S^{[2]}, \mathbb{Q}) \to H^2(S^{[2]} \setminus E, \mathbb{Q}) \to 0, \tag{5.2}$$

Or equivalently by what we shown,

$$0 \to H^0(E, \mathbb{Q}) \to H^2(S^{[2]}, \mathbb{Q}) \to H^2(S^{(2)}, \mathbb{Q}) \to 0.$$

Since we are over $\mathbb{Q}$, this sequence splits, ad we have

$$H^2(S^{[2]}, \mathbb{Q}) = H^2(S^{(2)}, \mathbb{Q}) \oplus [E]\mathbb{Q} = H^2(S, \mathbb{Q}) \oplus \delta\mathbb{Q}.$$

Now suppose we have an integral form

$$\omega \in H^2(S^{[2]}, \mathbb{Z}).$$

Since $S^{[2]}$ is simply connected, $H^2(S^{[2]}, \mathbb{Z})$ has no torsion. We can therefore split $\omega = a\tilde{\alpha} + b\delta$ where $\alpha \in H^2(S, \mathbb{Z})$ is assumed not to be divisible and $a, b \in \mathbb{Q}$. Let $s \in S$ and consider the rational line $l$ lying over $s \in S \simeq p(\Delta) \subset S^{(2)}$ in the exceptional divisor. Here, we must recall $\tilde{E} \simeq E$ and that $\tilde{E} = \mathbb{P}(\mathcal{N}_{\Delta/S^2}) = \mathbb{P}(\mathcal{T}_S)$ as $\tilde{S}^2 \to S^2$ is a smooth blow-up. Note that

$$\int_l \tilde{\alpha} = 0$$

since $l$ is a fibre and $\tilde{\alpha}$ is pulledback from $H^2(S^{(2)}, \mathbb{Z})$. On the other hand, we have

$$2\int_l \delta = \int_{f^{-1}(l)} \tilde{E}.$$

But $f^{-1}(l) = 2l'$, where $l'$ is the reduced pre-image of $l$ along $q$, and so

$$\int_l \delta = \int_{l'} \tilde{E}.$$

Since $\tilde{S}^2 \to S^2$ is a blow-up, we have $\mathcal{O}(\tilde{E})|_{\tilde{E}} = \mathcal{O}_{\mathbb{P}(\mathcal{T}_S)}(-1)$ (this generalises the well-known case of the blow-up of a point and is seen directly by looking at the Rees algebra), and so we have $\mathcal{O}(\tilde{E})|_{l'} = \mathcal{O}_{l'}(-1)$, and thus

$$\int_l \delta = -1.$$

Therefore, we have

$$\int_l \omega = a\int_l \tilde{\alpha} + b\int_l \delta = -b.$$

Since $[l]$ is an integral cochomogy class, we conclude that $b \in \mathbb{Z}$. This implies that $b\delta$ is integral, and so that $a\tilde{\alpha}$ is integral. Now, for any $\gamma \in H^6(S^{(2)}, \mathbb{Z})$, we have

$$\int_{S^{(2)}} a\tilde{\alpha} \cup \gamma = \int_{S^{[2]}} a\tilde{\alpha} \cup q^*\gamma$$

since $h$ is birational. Since $q^*\gamma$ is integral, this quantity is an integer. Since the pairing is unimodular and $\tilde{\alpha}$ is not divisible by assumption, we may pick $\gamma$ so that

$$\int_{S^{(2)}} \tilde{\alpha} \cup \gamma = 1,$$

so that $a \in \mathbb{Z}$. Thus, we have shown that

$$H^2(S^{[2]}, \mathbb{Z}) = \delta\mathbb{Z} \oplus H^2(S, \mathbb{Z}). \tag{5.3}$$

From (5.1), the map $H^2(S, \mathbb{Z}) \to H^2(S^2, \mathbb{Z})^{\mu_2} = H^2(S, \mathbb{Z})$ is induced by sums and pullback, and so respects the Hodge structure. Moreover, $\delta$ is a pure $(1,1)$-class, and so (5.3) respects the Hodge structure.

There remains only to show that $H^0(S^{[2]}, \Omega^2_{S^{[2]}})$ is spanned by a symplectic form. Since (5.3) respects the Hodge decomposition, we know $H^0(S^{[2]}, \Omega^2_{S^{[2]}}) \simeq H^0(S, \Omega^2_S)$ is one dimensional, and so it suffices to show that there exists a holomoprhic symplectic form on $S^{[2]}$. Let $\sigma$ be a holomorphic symplectic form on $S$. Then, we know there exists a holomorphic 2-form $\tilde{\sigma}$ on $S^{[2]}$ whose pullback to $S^2 \setminus \Delta$ is of the form

$$p_1^*\sigma + p_2^*\sigma.$$

Now, $\tilde{\sigma}$ is symplectic if and only if $\tilde{\sigma}^2 \in H^0(S^{[2]}, \Omega^4_{S^{[2]}})$ is a non vanishing section. Since $\Omega^4_{S^{[2]}} = K_{S^{[2]}} = \mathcal{O}_{S^{[2]}}$, it suffices to show that $\tilde{\sigma}^2$ does not vanish at a single point, i.e. that $\tilde{\sigma}$ is symplectic at one point. Thus, we may check this away from $E$, and since $S^2 \setminus \Delta \to S^{[2]} \setminus E$ is étale, it suffices to check this for the form $p_1^*\sigma + p_2^*\sigma$ on $S^2 \setminus \Delta$, which is trivial (the direct sum of two symplectic forms is symplectic). ☐

5.2. **Local systems.** We introduce local systems and holonomy, important concepts in differential geometry.

**Definition 5.2.1.** Let $B$ be a connected manifold and let $A$ be a ring (e.g., $\mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{C}$). A **local system** of $A$-modules on $B$ is a sheaf $\mathbb{H}$ of $A$-modules which is locally isomorphic to a constant sheaf with fiber $V$, where $V$ is an $A$-module.

Given a local system $\mathbf{H}$ (over $A = \mathbb{R}$ or $\mathbb{C}$), we can associate a vector bundle

$$\mathcal{H} := \mathbf{H} \otimes_A \mathcal{A}^0(B, A),$$

where $\mathcal{A}^0(B)$ is the sheaf of smooth functions. This bundle has a canonical flat connection

$$\begin{tikzcd} \mathcal{H} \arrow[r, "\nabla"] \arrow[d, equal] & \mathcal{H} \otimes_{\mathcal{A}^0(B,A)} \mathcal{A}^1(B, A) \arrow[d, equal] \\ \mathbf{H} \otimes_A \mathcal{A}^0(B, A) \arrow[r] & \mathbf{H} \otimes_A \mathcal{A}^1(B, A) \end{tikzcd}$$

defined by $\nabla := \mathrm{id}_{\mathbf{H}} \otimes_A d$. Indeed, this is a connection: we can write locally a section $s \in \mathcal{H}$ as

$$s = e_i \otimes s^i$$

where the elements $e_i$ form a basis of $V$, and we have

$$\nabla(f \cdot s) = e_i \otimes_A d(fs^i) = e_i \otimes_A (fds^i + (df)s^i) = df \otimes_{\mathcal{A}^0(B,A)} s + f\nabla s$$

This connection is flat; indeed, we have

$$\nabla(\nabla(e_i \otimes s^i)) = \nabla(e_i \otimes ds^i) = e_i \otimes dds^i + ds^i \wedge \nabla e_i = 0$$

**Theorem 5.2.2.** *There is a bijective correspondence between (real or complex) local systems of rank $r$ on $B$ and pairs $(\mathcal{E}, \nabla)$ where $\mathcal{E}$ is a (real of complex) vector bundle of rank $r$ on $B$ and $\nabla$ is a flat connection.*

*Sketch of proof (see [Voi98, Chapitre 9]).* We have already shown that a local system yields a vector bundle and a flat connection. Conversely, one may show that given a vector bundle and a flat connection, the sheaf of parallel sections yields a local system. $\square$

Let $(E, \nabla)$ be a vector bundle with a connection on some manifold $X$. Let $\gamma : [0, 1] \to X$ be a path from $x = \gamma(0)$ to $y = \gamma(1)$. The pullback bundle $(\gamma^*E, \gamma^*\nabla)$ on $[0, 1]$ is flat for dimension reasons (recall that the curvature is a matrix of $E$-valued 2-forms). Thus, $(\gamma^*E, \gamma^*\nabla)$ corresponds to a local system $H_\gamma$ on $[0, 1]$. Since $[0, 1]$ is contractible, any vector bundle is topologically trivial, and therefore this local system is trivial, which provides a canonical isomorphism $E_x \xrightarrow{\sim} E_y$ between the fibers.

**Definition 5.2.3.** The isomorphism $E_x \xrightarrow{\sim} E_y$ is called the **parallel transport** along $\gamma$.

If $x = y$, we consider the space of loops based at $x$, $\Omega(x)$. The parallel transport defines a map

$$\rho : \Omega(x) \to \mathrm{Aut}(E_x).$$

**Definition 5.2.4.** The **holonomy group** $\mathrm{Hol}(E, \nabla) \subseteq \mathrm{GL}(E_x)$ of the connection $\nabla$ is the image of $\Omega(x)$ under $\rho$. The **reduced holonomy group** $\mathrm{Hol}_0(E, \nabla)$ is the image of the subgroup $\Omega_0(x)$ of contractible loops.

**Remark 5.2.5.** If $X$ is connected, $\mathrm{Hol}(E, \nabla)$ does not depend on the choice of $x$ up to isomorphism. Indeed, any path $x \mapsto y$ gives an isomorphism between the holonomy groups at $x$ and at $y$. Note however that this isomorphism is path-dependent. Nonetheless, if there is no reduced holonomy, then this identification is only dependent on paths up to homotopy. Moreover, note that if the connection $\nabla$ is flat, the reduced holonomy is trivial: indeed, any contractible loop lies in a contractible neighbourhood of $x$, which is in particular a neighbourhood where the local system is trivial. Conversely, it is not hard to see that if the reduced holonomy is trivial, then the connection is flat: on a contractible neighbourhood, every fibre is canonically isomorphic. Thus, the connection is flat if and only if the reduced holonomy is trivial. In this case, the morphism $\rho$ descends to a representation

$$\overline{\rho} : \pi_1(X, x) \to \mathrm{GL}(E_x)$$

which is called the **monodromy** of the flat connection $\nabla$ (or of the associated local system).

If $(X, g)$ is a Riemannian manifold, we consider the Levi-Civita connection $\nabla$ on the tangent bundle $T_{\mathbb{R}}X$. Recall that this connection is uniquely determined by two conditions

- $\nabla g = 0$ (i.e. $g$ is parallel, that is, the holonomy commutes with $g$)[7]
- The connection has no torsion, i.e. $\nabla_X Y - \nabla_Y X = [X, Y]$, where the bracket is the Lie bracket.

An important theorem (see [Bal06, Theorem 4.17]) states that a complex manifold is Kähler if and only if the (complexification of the) Levi-Civita connection coincides with the Chern connection.[8]

We will denote $\mathrm{Hol}(X, g) := \mathrm{Hol}(T_{\mathbb{R}}X, \nabla)$. A fundamental phenomenon of the theory of connections is the following.

**Proposition 5.2.6** (Holonomy Principle). *Let $(X, g)$ be a Riemannian manifold. Then, the space of tensors parallel to the Levi-Civita connection*

$$A_{par}(TX^{\otimes a} \otimes T^*X^{\otimes b}) := \{\alpha \in \Gamma(TX^{\otimes a} \otimes T^*X^{\otimes b}) \mid \nabla\alpha = 0\}$$

*is equal to the space of tensors $\alpha$ such that for any $p \in U$, the restriction map*

$$A_{par}(TU^{\otimes a} \otimes T^*U^{\otimes b}) \to T_pU^{\otimes a} \otimes T_p^*U^{\otimes b}$$

*is an isomorphism onto the space of holonomy invariant tensors.*

**Example 5.2.7.** Since $g$ is by definition parallel to the Levi-Civita connection, the holonomy commutes with the metric. In other words, $\mathrm{Hol}(X, g) \subset O(n)$. If $X$ is oriented, the natural orientation

$$\sqrt{\det g_{ij}}dx^1 \wedge \cdots \wedge x^n$$

can be computed to be parallel, and so $\mathrm{Hol}_0(X, g) \subseteq SO(n)$.

If $X$ is Kähler, one can show that the almost complex structure $I$ is parallel. This implies $Hol(X, g) \subseteq U(m)$, where $m = \dim_{\mathbb{C}} X$. It is an important theorem (see [Bal06, Theorem 4.17]) that a Riemannian manifold is Kähler if and only if its holonomy lies in $U(m)$. In other words, Kähler manifolds are exactly those oriented Riemannian manifolds whose parallel transport is $\mathbb{C}$-linear.

If $(X, \omega)$ is Kähler and Calabi–Yau of complex dimension $m$ in the stronger sense that $K_X = \mathcal{O}_X$, then by definition, the tangent bundle is Ricci-flat (with respect to the Chern connection of some Kâhler metric, by the now proved Calabi conjecture). One can compute that this implies that the induced connection on $\det TM$, and dually on $K_X$, is flat. Since the canonical bundle is trivial, the corresponding local system is trivial. We have therefore a global parallel $(m, 0)$-form (i.e. a "holomorphic orientation"), so that using the holonomy principle, we find that the holonomy lies in $SU(m)$.

Conversely, suppose a Riemannian manifold $X$ has global $SU(m)$ holonomy. One can compute that the parallel transport on the determinant $\det TX$ is given by the (complex) determinant of the parallel transport on $TX$, implying that there is no (global) holonomy. Thus, $\nabla_{\det TX}$ is flat and $\det TX$ moreover corresponds to the trivial local system, and so is topologically trivial. Taking a global parallel smooth $(m, 0)$-form $\Omega$, and using that the determinant of the Chern connection is the Chern connection of the determinant, we find that $\nabla^{0,1}_{\det TX} = \overline{\partial}_{\det TX}$. By considering types, we must have $\nabla^{1,0}_{\det TX}\Omega = 0$. Thus, $\overline{\partial}_{\det TX}\Omega = 0$, i.e. $\Omega$ is holomorphic. This implies that $\det TX$ is holomorphically trivial, and so that $X$ is Calabi–Yau in the stronger sense.

In some sense, this shows that Kähler manifolds are the complex analog of Riemannian manifolds, and that Calabi–Yau manifolds are the complex analog of oriented Riemannian manifolds. We will show that this analogy extends to hyperkähler manifolds, which are the complex analog of symplectic manifolds.

After this discussion on Calabi–Yau manifolds, it is a good point to introduce the Bochner principle.

**Proposition 5.2.8** (Bochner's Principle). *If $(X, \omega)$ is a compact Kähler manifold with $Ric(\omega) = 0$, then all holomorphic tensors are parallel.*

---

[7] Recall/learn that given a connection on a bundle $E$ (in our case the tangent bundle), we may extend the connection to a $(m, n)$-tensor $\alpha$ by declaring

$$\begin{aligned}(\nabla_Z\alpha)(X_1, \ldots, X_m, \omega_1, \ldots, \omega_n) := Z(\alpha(X_1, \ldots, X_m, \omega_1, \ldots, \omega_m)) &- \sum_{i=1}^m \alpha(X_1, \ldots, \nabla_Z X_i, \ldots, X_m, \omega_1, \ldots, \omega_n) \\ &- \sum_{i=1}^n \alpha(X_1, \ldots, X_n, \omega_1, \ldots, \nabla_Z\omega_i, \ldots, \omega_n),\end{aligned}$$

where $Z, X_1, \ldots, X_m$ are vector fields and $\omega_1, \ldots, \omega_n$ are forms. In our situation, we obtain that $(\nabla_Z g)(X_1, X_2) = Z(g(X_1, X_2)) - g(\nabla_Z X_1, X_2) - g(X_1, \nabla_Z X_2)$. Thus, $\nabla g = 0$ translates to $Z(g(X_1, X_2)) = g(\nabla_Z X_1, X_2) + g(X_1, \nabla_Z X_2)$, and this condition is often called compatibiiity of $g$ with the connection.

[8] One ought to be precise here: complexifying the Levi-Civita connection to $T_{\mathbb{C}}X$ and then restricting to $T^{1,0}X$ yields the Chern connection.

**Remark 5.2.9.** Note that by the (proven) Calabi–Yau conjecture, if $X$ is Calabi–Yau (i.e. $c_1(X) = 0$), then it is possible to find a Kähler metric whose Ricci-curvature is trivial. Therefore, by Bochner's principle, it is always possible to find a Kähler metric on a Calabi–Yau manifold such that all holomorphic tensors are parallel to the corresponding Chern/Levi-Civita connection.

We use this principle to give a purely Riemannian geometric characterisation of hyperkähler manifolds. Recall that the compact symplectig group $Sp(m) \subset U(2m) \subset SO(4m)$ is that of unitary endomorphisms preserving a given *complex* symplectic form.

**Theorem 5.2.10.** *Let $(X, g)$ be a compact Riemannian manifold. Then $X$ is hyperkähler*[9] *if and only if $Hol(X, g) = Sp(m)$.*

*Proof.* ($\Leftarrow$) Assume that $\mathrm{Hol}(X, g) = Sp(m)$. By definition, we can find a holonomy-invariant symplectic complex two form at a point $p \in X$. By the holonomy principle, this extends to a global parallel $(2, 0)$-form $\sigma$ on $X$ since $X$ is simply connected. By the definition of the extension of the Levi-Civita connection to forms and using that the Levi-Civita connection is torsion-free, we have

$$(d\sigma)(X_1, X_2, X_3) = (\nabla_{X_1}\sigma)(X_2, X_3) - (\nabla_{X_2}\sigma)(X_1, X_3) + (\nabla_{X_3}\sigma)(X_1, X_2) = 0,$$

so that $\sigma$ is closed. But using $d\sigma = \partial\sigma + \overline{\partial}\sigma$ and comparing types, we must have $\overline{\partial}\sigma = 0$, i.e. $\sigma \in H^0(X, \Omega_X^2)$ is holomorphic.

We must verify that $H^0(X, \Omega_X^2)$ is spanned by $\sigma$. By Bochner's principle, any holomorphic 2-form $\sigma'$ must be parallel, and using the holonomy principle, this implies that the space of holomorphic 2-forms is contained in the space of $Sp(m)$ invariant (complex) symplectic forms at a point $p$. But this space is known to be of one (complex) dimension, so that $\sigma' = c\sigma$ for some constant $c$. Moreover, by Example 5.2.7, we know that $X$ is Kähler and $K_X = \mathcal{O}_X$, so that we conclude $X$ is hyperkähler.

($\Rightarrow$) Assume $X$ is hyperkähler, and choose a Ricci-flat Kähler metric $h$. $X$ admits a holomorphic symplectic form $\sigma$ which is parallel by Bochner's principle. By the Holonomy Principle, $\mathrm{Hol}(X, g)$ preserves $\sigma$, so $\mathrm{Hol}(X, g) \subseteq Sp(m)$. One can then use Berger's classification of holonomy groups to show that in fact we have the equality $\mathrm{Hol}x(X, g) = Sp(m)$. ☐

**Corollary 5.2.11.** *Let $X$ be a hyperkähler manifold of dimension $2m$. Then*

$$H^0(X, \Omega_X^k) = \begin{cases} 0 & \text{if } k \text{ is odd} \\ \mathbb{C} \cdot \sigma^{k/2} & \text{if } k \text{ is even.} \end{cases}$$

*In particular, the holomorphic Euler characteristic is $\chi(X, \mathcal{O}_X) = m + 1$.*

*Proof.* By Bochner's principle, holomorphic forms are parallel. By the Holonomy Principle and Theorem 5.2.10, they correspond to $Sp(m)$-invariant elements in $\bigwedge^k V^*$. It is a fact from representation theory that the space of $Sp(V, \sigma)$-invariants in $\bigwedge^k V^*$ is spanned by powers of $\sigma$ if $k$ is even, and is zero if $k$ is odd.

The Euler characteristic is $\chi(X, \mathcal{O}_X) = \sum_k (-1)^k h^k(X, \mathcal{O}_X)$. By Hodge symmetry, $h^k(X, \mathcal{O}_X) = h^{0,k}(X) = h^{k,0}(X) = h^0(X, \Omega_X^k)$, and so that $\chi(X, \mathcal{O}_X) = m + 1$ follows directly. ☐

Here is a table to remember the different holonomies of the different types of Riemannian manifolds we have discussed. For each row, the condition on holonomy can be given as a purely Riemannian definition of this type of manifold.

[9] Or rather can be endowed with a complex structure making it a hyperkähler manifold.

| Manifold | Real dimension | Holonomy | Reason |
|---|---|---|---|
| Riemannian | $m$ | contained in $O(m)$ | preserves metric |
| Oriented Riemannian | $m$ | contained in $SO(m)$ | preserves orientation |
| Symplectic | $2m$ | contained[10] in $Sp(m,\mathbb{R}) \subset SO(2m)$ | preserves the symplectic form |
| Kähler | $2m$ | contained in $U(m)$ | preserves the complex structure |
| Calabi–Yau ($K_X = \mathcal{O}_X$) | $2m$ | contained in $SU(m)$ | preserves the "complex orientation" (top holomorphic form) |
| Hyperkähler | $4m$ | equal to $Sp(m) \subset SU(2m)$ | preserves the holomorphic symplectic form |

These groups are part of Berger's classification of reduced holonomy groups.[11] In fact, Berger proved that In the case of a non-flat reduced (in the Riemannian sense)[12] Kähler manifold, the reduced holonomy has to be *exactly* $U(m), SU(m)$ or $Sp(m)$. In particular, a non-hyperkähler and non-flat Calabi–Yau manifold has reduced holonomy *exactly* $SU(m)$, and a non Calabi–Yau Kähler manifold has reduced holonomy *exactly* $U(m)$.

Note also that the classification of these reduced holonomies is precisely how Beauville–Bogomolov decomposition arose: flat corresponds to complex tori, $SU(m)$ corresponds to strict Calabi–Yau manifolds, and $Sp(m)$ to hyperkähler manifolds.

[10]Here, it must be said that contrary to the other manifolds in the list, the holonomy is taken with respect to a "symplectic" connection, which is in general not the same as the Levi-Civita connection.

[11]The only other possible reduced holonomy groups of an irreducible Riemannian manifold that is not locally a symmetric space are $Sp(n), U(1)$, $Spin(7)$ and $G_2$.

[12]i.e. not a product as a Riemannian manifold.

Today, we will set up the necessary constructions to tackle the deformation theory of hyperkähler manifolds. To motivate this, let us first look at an crucial example, *twistor lines*.

6.1. **Twistor lines.** Recall that the **quaternion algebra** $\mathbb{H}$ are a non-commutative algebra over $\mathbb{R}$. In terms of generators, we have

$$\mathbb{H} = \frac{\mathbb{R}\langle 1, i, j, k\rangle}{\langle i^2 = j^2 = k^2 = -1, ijk = -1\rangle},$$

i.e. the quaternion algebra are the group ring of the quaternion group $Q_8$. For a hyperkähler manifold $(X, g)$, we claim there is a natural action of $\mathbb{H}$ on the tangent bundle. Let us see how this arises.

Recall that as a Riemannian manifold, a hyperkähler manifold is characterised by having $Sp(n)$ holonomy. Let $(V, I)$ be a real vector space of dimension $4n$ with $I$ a complex structure. Let $h$ be a hermitian metric on $(V, I)$ and let $\Omega$ be a complex non degenerate symplectic form on $(V, I)$.

So far, we have defined

$$Sp(n) = U(2n) \cap Sp(2n, \mathbb{C}),$$

that is, as the real endomorphisms of $V$ which preserve $I$, $h$ and $\Omega$.

We can decompose

$$\Omega = \alpha + i\beta$$

into real and complex parts. Let $\omega = -\mathfrak{Im}(h)$ be real $(1,1)$ on $V$ induced by $h$. Since $\omega$ is non-degenerate, we may define the real endomorphisms $J$ and $K$ of $V$ via asking

$$\omega(u, Jv) = \alpha(u, v) \qquad \text{and} \qquad \omega(u, Kv) = \beta(u, v)$$

for all $u, v \in V$. Let $A \in Sp(n)$. Since $\Omega(u, v) = \Omega(Au, Av)$, we have that $\alpha(u, v) = \alpha(Au, Av)$. This gives

$$\alpha(u, Jv) = \omega(Au, JAv),$$

and since $A$ preserves $\omega$, we also have

$$\omega(u, Jv) = \omega(Au, AJv).$$

Since $\omega$ is non degenerate, this equality holding for all $v \in V$ implies that $JA = AJ$, and the same argument shows $AK = KA$, so that elements of $Sp(n)$ preserve all of the three endomorphisms $I, J, K$.

Since $\Omega$ is complex, we have $\Omega(Iu, Iv) = -\Omega(u, v)$. We compute

$$\omega(u, JIv) + i\omega(u, KIv) = \Omega(u, Iv) = i\Omega(u, v) = -\omega(u, Kv) + i\omega(u, Jv).$$

which gives the relations

$$JI = -K \qquad \text{and} \qquad KI = J.$$

Similarly, computing

$$-\omega(u, IJv) - i\omega(u, IKv) = \omega(Iu, Jv) + i\omega(Iu, Kv) = \Omega(Iu, v) = i\Omega(u, v) = -\omega(u, Kv) + i\omega(u, Jv)$$

gives the relations

$$IJ = K \qquad \text{and} \qquad IK = -J.$$

Since we already know $I^2 = -1$, in order to verify that $I, J, K$ satisfy the quaternionic relations, it remains to show

$$J^2 = K^2 = -\mathrm{id},$$

which is the hardest, since these equality do not hold yet and we will need to rescale $J$ and $K$.

First, notice that since $\Omega(u, v) = -\Omega(v, u)$, we have

$$\omega(u, Jv) = -\omega(v, Ju) = \omega(Ju, v), \tag{6.1}$$

Moreover, we have

$$K^2 = (IJ)(IJ) = I(-K)J = J^2,$$

and this endomorphism commutes with $I$:

$$K^2 I = -KIK = IK^2.$$

Letting $g = \mathfrak{Re}(h) = \omega(I-, -)$, we see that $J^2 = K^2$ is self adjoint for the inner product $g$ by using (6.1) and that it commutes with $I$. By the spectral theorem, $J^2 = K^2$ must be diagonalisable. Since $Sp(n)$ commutes with the endomorphism $J^2 = K^2$, it preserves the corresponding decomposition into eigenspaces. On the other hand, one can show using the symplectic form $\Omega$ that this representation of $Sp(n)$ is irreducible, and so there is a unique eigenspace, implying that

$$J^2 = K^2 = \mu \cdot \mathrm{id},$$

for some $0 \neq \mu \in \mathbb{R}$ (here, $J$ and $K$ are viewed as real operators on $V$). Suppose $\mu > 0$. Since $J^2 = K^2$ has non-zero eigenvalues, we know that $J$ and $K$ are diagonalisable with $\pm\sqrt{\mu}$ eigenspaces (here, we use that $\pm\sqrt{\mu}$ is real). But using again that we have an irreducible representation, this implies that $J, K = \pm\mu \cdot \mathrm{id}$. But whatever choice we make for $J$ and $K$, this contradicts the relations that we have found, as $J$ and $K$ anti-commute:

$$JK = KIIJ = -KJ.$$

Thus, $\mu < 0$. Up to rescaling $\Omega$ by $\sqrt{\frac{1}{-\mu}}$, we obtain $J^2 = K^2 = -\mathrm{id}$.

This way, we see $Sp(n)$ commutes with the action of the three complex structures $I, J, K$ on $V$ which, together, induce an action of $\mathbb{H}$ on $V$. One can also show that we can go the other way, given an action of $\mathbb{H}$ on a vector space $V$ which preserves a quaternionic hermitian metric, which is equivalent to an inner product $g$ which is compatible with the three complex structures:

$$g(-,-) = g(I-, I-) = g(J-, J-) = g(K-, K-),$$

then we can recover the complex hermitian metric on $V$ by defining

$$h(-,-) = g(-,-) + ig(-, I-),$$

and the symplectic two form via

$$\Omega(-,-) = g(I-, J-) + ig(I-, K-).$$

These tensors being defines using $g$ and the complex structures, they are all preserved by $Sp(n)$. Therefore, $Sp(n)$ may equally be defined as the endomorphisms preserving a prescribed quaternionic structure on a vector space of (real) dimension $4n$.

Coming back to the hyperkähler situation, given a hyperkähler manifold $(X, g, I)$ — here we assume that our choice of metric $g$ induces $Sp(n)$ holonomy — there is an iduced action of $\mathbb{H}$ on the tangent bundle by the holonomy principle. In other words, we have two other complex structures $J, K$ that satisfy the quaternionic relations together with $I$. By construction, these three complex structures are all parallel, and so $(X, g)$ may be seen as a Kähler manifold with respect of each one. Moreover, let

$$L = aI + bJ + cK,$$

where $a^2 + b^2 + c^2 = 1$ and $a, b, c \in \mathbb{R}$. Then, using the quaternionic relations,

$$L^2 = (a^2 I + b^2 J + c^2 K) = -\mathrm{id},$$

so that $L$ is also a parallel complex structure, meaning that $(X, g)$ is equally a Kähler manifold with respect to $L$ (in particular $L$ is integrable). In this situation, the Kähler form is given by $\omega_L(-,-) = g(L-,-)$.

We thus get a sphere $\mathbb{S}^2 \simeq \mathbb{P}^1$ of integrable complex structure, called a **twistor line**.

**Remark 6.1.1.** This is a good place to make precise another terminology found in the literature, that of an *irreducible symplectic manifold.* This terminology is sometimes use to refer to hyperkähler manifolds seen as a complex manifold, i.e. as the object $(X, I)$, while the term *hyperkähler manifold* is sometimes reserved to the Riemannian view point, i.e. to the object $(X, g)$ — again, our choice of $g$ here must give $Sp(n)$ holonomy.

Starting from the Riemannian manifold $(X, g)$, and using the $Sp(n)$ holonomy as above, we find a full sphere $\mathbb{S}^2$ worth of parallel complex structures. By choosing different complex structures in this sphere, one gets non-isomorphic complex manifolds.

Conversely, if we start with the complex manifold $(X, I)$, we do not yet have a Riemannian metric $g$, and so we do not yet have an associated twistor line. In fact, choosing a Riemannian metric $g$ such that $(X, g)$ has $Sp(n)$ holonomy amounts to choosing a Ricci-flat Kähler metric, as Bochner's principle ensures that the non-degenerate holomorphic-symplectic 2-form is parallel to the induced Chern (Levi-Civita) connection. By Yau's theorem (Calabi's conjecture), there is a unique such Kähler metric in every Kähler class, so that the Kähler cone $\mathcal{K}_X \subset H^{1,1}(X, \mathbb{R})$ parametrises twistor lines "passing through" $(X, I)$. We shall use this fact at a later point to show that the deformation theory of hyperkähler manifolds is unobstructed (i.e. that the Kuranishi (deformation) space is smooth).

Given a hyperkähler manifold $(X, I, \omega)$, we have a $C^\infty$-submersion

$$X \times \mathbb{S}^2 \to \mathbb{S}^2$$

given by projection onto the second factor. Since $\mathbb{S}^2 \simeq \mathbb{P}^1$ parametrises the complex structures compatible with $g = \omega(-, I-)$, we can endow these spaces of complex structures making this map holomorphic. In particular, we endow $X \times \mathbb{S}^2$ with an almost complex structure by defining

$$\mathbb{I} \in \mathrm{End}(T_\mathbb{R} X \oplus T_\mathbb{R}\mathbb{P}^1)$$
$$\mathbb{I}_{(x,t)}(u, v) = (I_t(u), I_{\mathbb{P}^1}(v)),$$

where $I_t$ is the complex structure on $X$ represented by $t \in \mathbb{S}^2 \simeq \mathbb{P}^1$, and $I_{\mathbb{P}^1}$ is the complex structure on $\mathbb{P}^1$. One may use the Nijenhuis tensor coupled with the quaternionic relations to show that this almost complex structure is integrable. We will denote this complex manifold by $T_\omega(X) := (X \times \mathbb{S}^2, \mathbb{I})$, and call it the **twistor space**. The subscript $\omega$ comes from the fact that, as explained in Remark 6.1.1, given the complex manifold $(X, I)$ , the choice of a twistor line is equivalent to the choice of a Kähler class. By looking at our complex structure $\mathbb{I}$, it is evident that the proper and submersive map

$$T_\omega(X) \to \mathbb{P}^1$$

is holomorphic, in other words, a deformation of $(X, I)$.

One must be careful: while the fibers $(X, I_t)$ are all Kähler ($I_t$ is parallel), the twistor space $T_\omega(X)$ is not Kähler. This is easy to see using parallel transport. As a Riemannian manifold $T_\omega(X) = X \times \mathbb{S}^2$ is just a product. Therefore, the parallel transport from $(x, 1)$ to $(x, -1)$ takes the complex structure $\mathbb{I}_{(x,1)} = (I, I_{\mathbb{P}^1})$ to $(I, I_{\mathbb{P}^1}) = -\mathbb{I}_{(x,-1)}$, meaning that $\mathbb{I}$ is not parallel. Thus, $T_\omega(X)$ is not Kähler.

6.2. **Deformation theory of complex manifolds.** In algebraic geometry, one typically studies deformation of schemes using deformation functors. Given a scheme $X$ over $\mathbb{C}$, its deformation functor is of the form

$$\mathcal{D}ef_X : \mathbf{Art}_\mathbb{C} \to \mathbf{Set}$$
$$A \longmapsto \left\{ \left( \begin{array}{c} \mathcal{X} \\ \big\downarrow f \\ \mathrm{Spec} A \end{array}, \phi \right) : \begin{array}{c} f \text{ is proper and flat} \\ \text{and } \phi : \mathcal{X}_0 \to X \\ \text{is an isomorphism} \end{array} \right\} / \sim,$$

where $\mathbf{Art}_\mathbb{C}$ is the category of local Artinian algebras over $\mathbb{C}$, and $\mathcal{X}_0$ denotes the special fiber. Here, the equivalence $\sim$ identifies two families $(\mathcal{X}, \phi)$ and $(\mathcal{X}', \phi')$ if they are isomorphic over $\mathrm{Spec}A$ in such a way that the restriction to the special fibre commutes with $\phi$ and $\phi'$. The functor $\mathcal{D}ef_X$ is **representable** whenever there exists a scheme $\mathrm{Spec}A_{univ}$ such that $\mathcal{D}ef_X$ is naturally isomorphic to the Hom functor $\mathrm{Hom}(A_{univ}, -)$.

This transports well to the analytic setting. Instead of working with local artinian algebras, we work with germs of complex analytic spaces. Note that complex analytic spaces generalise complex manifold as they may — very importantly — have singularities and even be non-reduced. The **germ** $(B, 0)$ of a complex manifold $B$ at a point $0$ is an equivalence class of all complex manifold $(B', p')$ such that $p'$ as a neighbourhood isomorphic to a neighbourhood of $0$ in $B$. This is a way to speak of the local ring without having to pass to the algebra side. If $X$ is a comlex manifold, the anaytic deformation functor is the following:

$$\mathcal{D}ef_X^{an} : (B, 0) \longmapsto \left\{ \left( \begin{array}{c} \mathcal{X} \\ \big\downarrow f \\ B \end{array}, \phi \right) : \begin{array}{c} f \text{ is proper and flat} \\ \text{and } \phi : \mathcal{X}_0 \to X \\ \text{is an isomorphism} \end{array} \right\} / \sim,$$

where two objects $(f : \mathcal{X} \to B, \phi)$, $(f' : \mathcal{X}' \to B, \phi')$ are equivalent if, after changing $B$ for a small enough neighbourhood of $0$, there exists an isomorphism $\tilde{X} \to \tilde{X}'$ over $B$ which commutes with $\phi$ and $\phi'$ after restricting to $0$.

**Definition 6.2.1.** We say that $\mathcal{D}ef_X^{an}$ is **representable** if there exists a germ $(B_{univ}, 0)$ such that $\mathcal{D}ef_X^{an}$ is naturally isomorphic to the Hom functor $\mathrm{Hom}(-, (B_{univ}, 0))$ (this functor has source in the category of germs of complex analytic spaces).

Note that if we have such a germ $(B_{univ}, 0)$, then the identity $(B_{univ}, 0) \to (B_{univ}, 0)$ represents a **universal** family

$$\begin{array}{ccc} \mathcal{X}_{univ} & \longrightarrow & B_{univ} \\ \uparrow & & \uparrow \\ X & \longrightarrow & \{0\}. \end{array}$$

With a bit of diagram chasing, we see that if an element $e \in \mathcal{D}ef_X^{an}((B, 0))$ represents the morphism $f : (B, 0) \to (B_{univ}, )$, then the corresponnding family over $(B, 0)$ is $f^*\mathcal{X}_{univ}$.

Sometimes, we do not have a universal family at our disposal, but other intermediate notions exist. A family $\mathcal{X} \to (B, 0)$ is **versal** if any deformation $\mathcal{X}' \to (B', 0)$ arises from a map $f : (B', 0) \to (B, 0)$, and is **semi-universal** if moreover, while $f$ is not necessarily unique, its differenital at 0 is.

We are now ready to state the crucial theorem at the basis of deformation of complex manifolds. We recommend [Cat11] for details. We stop specifying every time, but all the deformation spaces and families considered are germs of analytic spaces.

**Theorem 6.2.2** (Kuranishi)**.** *Let $X$ be a complex manifold. There exists an open neighbourhood $U \subset H^1(X, \mathcal{T}_X)$ and a holomorphic map*

$$K : U \to H^2(X, \mathcal{T}_X),$$

*called the **Kuranishi map**[13] such that the **Kuranishi space** $(\mathcal{B}_X, 0) := K^{-1}(0)$ admits a family $\mathcal{X}_{kur} \to (\mathcal{B}_X, 0)$, called the **Kuranishi family**, which satisfies the following properties.*

- *The differential of $K$ vanishes at the origin; in particular $T_0\mathcal{B}_X = H^1(X, \mathcal{T}_X)$;*
- *The Kuranishi family is semi-universal;*
- *If $h^0(X, \mathcal{T}_X) = 0$ (i.e. if there is no infinitesimal automorphisms), then $\mathcal{X}_{kur}$ is universal;*
- *(Wavrik) if $\mathcal{B}_X$ is reduced and $h^0(\mathcal{X}_{kur,t}, \mathcal{T}_{\mathcal{X}_{kur,t}})$ is constant in a neighbourhood of the origin, then the Kuranishi family is universal.*

**Remark 6.2.3.** It is conceptually no surprise why $h^0(X, \mathcal{T}_X)$ is an obstruction to the universality of the Kuranishi family: recall that it is a recurring theme in moduli theory that automorphisms implies the existence of non-isomorphic families nevertheless with isomorphic fibers, preventing the possibility of a fine moduli space, i.e. of a universal family. Since we are considering germs, only infinitesimal automorphisms pose a problem.

**Remark 6.2.4.** There is no hope for scheme theory to detect the Kuranishi space. Indeed, the Kuranishi map is holomorphic and not algebraic. In the algebraic setting, the deformation functor $\mathcal{D}ef_X$ is typically not representable, but rather *pro-representable.*

Note that in the hyperkähler case, $h^0(X, \mathcal{T}_X) = h^0(X, \Omega_X) = h^{1,0} = 0$ since $X$ is simply connected, and so the Kuranishi family is indeed universal in this case.

**Definition 6.2.5.** Assume the Kuranishi family is universal. Let $\mathcal{X} \to (B, 0)$ be a deformation. It corresponds to a unique map $(B, 0) \to (\mathcal{B}_X, 0)$. The **Kodaira–Spencer** map of $\mathcal{X} \to (B, 0)$ is its differential $T_0B \to H^1(X, \mathcal{T}_X)$ at the origin.

**Remark 6.2.6.** There are many ways to view the Kodaira–Spencer map; we sketch some of the view-points here. Assume $B$ is smooth. Since $\mathcal{X} \to (B, 0)$ is flat, it is submersive (i.e. smooth), as we consider germs. Therefore, $\mathcal{X}$ is a germ of a smooth variety, and so $X \subset \mathcal{X}$ is regularly embedded, so that we may consider the normal exact sequence:

$$0 \to \mathcal{T}_X \to \mathcal{T}_{\mathcal{X}|_X} \to \mathcal{N}_{X/\mathcal{X}} \to 0.$$

Since we are considering germs, we have $\mathcal{N}_{X/\mathcal{X}} = T_0B \otimes \mathcal{O}_X$. Thus, the connecting homomorphis in cohomology gives a map

$$H^0(X, \mathcal{N}_{X/\mathcal{X}}) = T_0B \to H^1(X, \mathcal{T}_X),$$

which coincides with the Kodaira–Spencer map.

Similarly, one may show with explicit Čech cocycles that an infinitesimal deformation $\mathcal{X} \to \mathrm{Spec}\mathbb{C}[\epsilon]/\epsilon^2$ corresponds to a cohomology class $H^1(X, \mathcal{T}_X)$, which is the image of the Kodaira–Spencer map.

Using the Dolbeault–Čech correspondence, the resolution

$$0 \to \mathcal{T}_X \to \mathcal{A}^0(\mathcal{T}_X) \xrightarrow{\overline{\partial}} \mathcal{A}^{0,1}(\mathcal{T}_X) \xrightarrow{\overline{\partial}} \mathcal{A}^{0,2}(\mathcal{T}_X) \xrightarrow{\overline{\partial}} \cdots,$$

[13] An explicit Taylor series for this map may be obtaiend by considering the Schouten bracket.

gives an identification

$$H^1(X, \mathcal{T}_X) = \frac{\{\text{closed } (0,1) \text{ forms valued in } \mathcal{T}_X = T^{1,0}X\}}{\{\text{exact } (0,1) \text{ forms valued in } T^{1,0}X\}},$$

and a natural question is to ask for the description of the Kodaira–Spencer map in this setting. Here is the answer. Let $f : \mathcal{X} \to (B, 0)$ be a deformation, which we assume to be small enough so that $f$ is submersive. In particular, this implies that $f$ is a trivial $C^\infty$-fibration (this is Ehresmann's theorem), i.e., that it is the map $X \times B \to (B, 0)$ if we forget the complex structures. Let $b \in B$ be a point in the base. By our assumtions, we have a canonical idenitfication $T_{\mathbb{C}}X_b = T_{\mathbb{C}}X$. In particular, considering the complex structure at $b$ gives a subspace

$$T_b^{0,1}X \subset T_{\mathbb{C}}X.$$

Now, we have a projection map $P_b^{0,1} : TX \to T_b^{0,1}X$ obtained from the decomposition

$$T_{\mathbb{C}}X = T_b^{1,0}X \oplus T_b^{0,1}X.$$

We can restrict this projection $P_b^{0,1}$ to $T^{0,1}X$. Post-composing with the map $P^{1,0}$, we get a map of sheaves

$$\alpha_b = P^{1,0} \circ P_b^{0,1} : T^{0,1}X \to T^{1,0}X.$$

Now let $v$ be a tangent vector in $T_0B$ and $v(b)$ be a curve $[0, 1] \to B$ integrating this vector. It turns out that the association

$$v \mapsto \frac{d}{db}\alpha_{v(b)}|_{b=0}$$

is the Kodaira–Spencer map. We can get a better expression. Note that

$$P_b^{0,1} = \frac{1}{2}(\mathrm{id} + iI_b),$$

where $I_b$ is the complex structure at the fiber at $b$. Taking the derivative as above, we see that the Kodaira–Spencer map is the association

$$v \to \frac{i}{2}\left[\frac{d}{db}I_{v(b)}|_{b=0}\right]^{1,0},$$

that is, the $(1, 0)$ part of the infinitesimal deformation of the complex structure.

**Example 6.2.7** (Kodaira–Spencer for twistor lines)**.** We follow [LeB17] and adapt to our conventions. Let $X$ be hyperkähler and let $T_\omega(X) \to \mathbb{P}^1$ be the twistor space induced by a Ricci-flat metric $\omega$. We put a coordinate $t$ on $\mathbb{P}^1$, and consider the tangent vector $\frac{\partial}{\partial t}$. We want to understand how the Kodaira–Spencer map act on this vector.

We will let $I_t$ denote the the complex structure corresponding to $t \in \mathbb{P}^1$, and let $I = I_0$, $I_1 = J$ and $I_i = K$. Let us also put a real parametrisation for the sphere $\mathbb{S}^2 \simeq \mathbb{P}^1$ via $a^2 + b^2 + c^2 = 1$. Here, we assume that 0 corresponds to $(1, 0, 0)$, 1 corresponds to $(0, 1, 0)$ and $i$ corresponds to $(0, 0, 1)$.

We first want to conpute

$$\frac{d}{dt}I_t|_{t=0}$$

By considering the standard stereographic coordinates $t = \zeta + i\xi$, where $\zeta = b/(1 + a)$ and $\xi = c/(1 + a)$, we find that

$$\frac{d}{d\zeta}I_\zeta = 2J \qquad \frac{d}{d\xi}I_\xi = 2K,$$

and so

$$\frac{d}{dt}I_t = J - iK.$$

Let $u \in T^{0,1}X$. We compute the image of $v$ along the Kodaira–Spencer map to be that which associates to $u$

$$\begin{aligned}\frac{i}{2}\left[\frac{d}{dt}I_t(u)\right]^{1,0} &= \frac{i}{2}[(J - iK)(u)]^{1,0} \\ &= \frac{i}{2}[J + iJI(u)]^{1,0} \\ &= \frac{i}{2}[J + i(-i)J(u)]^{1,0} \\ &= i[J(u)]^{1,0} = iJ(u),\end{aligned}$$

where the last inequality holds because $J$ anti-commutes with $i$, ans so switches its eigenspaces.

Note that on $(1,0)$ vectors, we have that $\sigma := \omega_J + i\omega_K = i(\omega_I(-, J-) + i\omega_I(-, K-)) = i\Omega$, where $\Omega$ is a holomorphic symplectic form defined as in Section 6.1. Thus, $\sigma$ is a non-degenerate holomorphic symplectic form, and it defines an isomorphism

$$\sigma : \mathcal{T}_X \to \Omega_X,$$

by contraction and so induces an isomorphism

$$H^1(X, \mathcal{T}_X) \simeq H^1(X, \Omega_X). \tag{6.2}$$

Let us investigate what this isomorphism does to the Kodaira–Spencer map. we must conpute the contraction

$$\begin{aligned} iJ \lrcorner\, \sigma &= ig((J + iK)J-, -) \\ &= ig((-\mathrm{id} - iI)-, -) \\ &= g(I-, -) = \omega_I(-, -), \end{aligned}$$

where the last line comes from the fact that $ig(-id-, -)$ is applied via our identifications to a $(0,1)$ vector, so that it does not contribute to the $(1,1)$ form that we get along (6.2).

In other words, with our choice of symplectic form for the contraction, the image along the Kodaira–Spencer map of the tangent vector of the twistor line yields the Kähler class that induces the twistor line. This has the following important corrolary

**Corollary 6.2.8.** *Let $X$ be a hyperkähler manifold. Then its Kuranishi space is smooth.*

*Proof.* Note that clearly, if a complex analytic space $\mathcal{B}$ has a set of smooth complex curves passing through some point $p$ such that the corresponding tangent lines span the tangent space at $p$, then $X$ is smooth at $p$. For $X$ hyperkähler, by what we have shown, the line tangent to the twistor line in the Kuranishi space of $X$ is spanned by the corresponding Kähler class along the identification $T_0\mathcal{B}_X = H^{1,1}(X)$ induced by our choice of symplectic form. Since all the Kähler classes induce a twistor line (recall this is discussed in Section 6.1), the tangent space spanned by the twistor lines contains the Kähler cone, which is open in $H^{1,1}(X, \mathbb{R})$. Thus, the subspace of $H^{1,1}(X)$ spanned by the twistor lines contains $H^{1,1}(X, \mathbb{R})$. Note that $\dim_{\mathbb{R}} H^{1,1}(X, \mathbb{R}) = \dim_{\mathbb{C}} H^{1,1}(X)$, since conjugation acts on $H^{1,1}(X)$. Thus, $H^{1,1}(X, \mathbb{R})$ spans $H^{1,1}(X)$ (i.e., contains a basis), and so the twistor lines span $H^{1,1}(X) \simeq T_0\mathcal{B}_X$. □

We also note here that a more general statement holds true.

**Theorem 6.2.9** (Bogomolov–Tian–Todorov)**.** *Let $X$ be a compact Calabi–Yau manifold (in the stricter sense that $K_X$ is trivial). Then, its Kuranishi space is smooth.*

*Proof idea.* The original proof uses the Maurer–Cartan equations, which we have not introduced; see [Tia87] for the original proof and [Huy05, Chapter 6] for a nice exposition.

We will give an idea of why this is true, assuming that the Kuranishi space is reduced — which is certainly a non-trivial assumption, it is in general a hard question whether a given moduli space is reduced.

Let $\mathcal{X} \to \mathcal{B}_X$ be the Kuranishi family and suppose $\mathcal{B}_X$ is reduced. Note that for any $b$ in any neighbourhood of $0 \in \mathcal{B}_X$ (here $\mathcal{X}_0 = X$), we have $T_b\mathcal{B}_X = H^1(\mathcal{X}_b, \mathcal{T}_{\mathcal{X}_b})$.

Note also that by considering the isomorphism

$$\mathcal{T}_X = \mathcal{T}_X \otimes \omega_X \xrightarrow{\sim} \Omega_X^{\dim X - 1},$$

$h^1(X, \mathcal{T}_X) = h^{1, \dim X - 1}$ is a Hodge number. It is a standard fact (though hard to prove) that small deformations of a Kähler manifold are Kähler; see [Voi98, Théorème 9.23]. Therefore, in a suitable neighobourhood of 0, all the manifolds $\mathcal{X}_b$ are Kähler, so that Hodge decomposition holds. Since $X$ is smooth and we consider germs, the Kuranishi family is submersive (i.e. smooth), and so the Betti number

$$b_{\dim X}(\mathcal{X}_b) = \sum_{p+q = \dim X} h^{p,q}(\mathcal{X}_b) \tag{6.3}$$

is constant (this follows from Ehresmann theorems on submersive maps). By the upper-semicontinuity theorem, the hodge numbers in (6.3) can only go down in a neighbourhood of 0. But this cannot happen, since the Betti number is constant. Thus, $h^{1, \dim X}$ is constant in a suitable neighbourhood of 0, and so — since we assume its reduceness — $\mathcal{B}_X$ is smooth. □

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