HYPERKÄHLER MANIFOLDS

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Abstract. These are personal notes for the course on hyperkähler manifolds taught by Alessio Bottini at Universität Bonn in the Winter 2025-2026 semester. Please email me at s37mbrai@uni-bonn.de if you notice any typo.

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1. First lecture: complex and Kähler manifolds

We first review some complex geometry.

Definition 1.0.1. A **complex manifold** is a locally ringed space $(X, \mathcal{O}_X)$ such that

- X is Hausdorff and second countable (this part is to ensure we actually have a topological manifold);
- $(X, \mathcal{O}_X)$ is locally isomorphic to $(\Delta, \mathcal{O}_\Delta)$, where $\Delta \subset \mathbb{C}^n$ is the polydisc.

Example 1.0.2. Let $f_i, \dots, f_d \in \mathbb{C}[z_1, \dots, z_n]$ be complex polynomials such that the Jacobian of

$$f = (f_1, \dots, f_d) : \mathbb{C}^n \to \mathbb{C}^d$$

has everywhere full rank on the vanishing set $V = V(f) \subset \mathbb{C}^n$. By the holomorphic implicit function theorem (regular value theorem), V is a complex manifold.

Example 1.0.3. If X is a smooth algebraic variety over $\mathbb{C}$, we may cover it by affines V_i which are of the same form as in Example 1.0.2. We may consider the analytic topology X^{an} obtained by gluing the different charts V_i. Similarly, we may define the sheaf $\mathcal{O}_X^{an}$ on X^{an} by considering the sheaf of holomorphic functions on each V_i (the transitions $V_i \to V_j$ are regular algebraic, hence holomorphic, so that this gluing makes sense). Then, $(X^{an}, \mathcal{O}_X^{an})$ is a complex manifold.

Note that in Example 1.0.3, we obtain a natural map of ringed spaces

$$\alpha : (X^{an}, \mathcal{O}_X^{an}) \to (X, \mathcal{O}_X)$$

since the analytic topology is finer than the Zariski topology, and regular functions are holomorphic. In particular, we obtain a functor between abelian categories:

$$\alpha^* : \mathcal{O}_X\text{-mod} \to \mathcal{O}_X^{an}\text{-mod}$$

restricting to

$$\alpha^* : \mathrm{Coh}(X) \to \mathrm{Coh}(X^{an}).$$

Theorem 1.0.4 (Géométrie algébrique géométrie analytique; [Ser56])**.** *If X is smooth*[1] *and proper functor $\alpha^* : Coh(X) \to Coh(X^{an})$ is an equivalence, which moreover induces an isomorphism of sheaf cohomology.*

1.1. Almost complex structures.

A complex manifold $(X, \mathcal{O}_X)$ has an underlying smooth manifold (X, C_X^∞), where C_X^∞ denotes the sheaf of smooth functions on X; indeed, if X has complex charts $U_i \subset \mathbb{C}^n$, the transitions are holomorphic, hence C^∞.

Notation 1.1.1. Since the indices i will be ubiquitous, ι shall denote the root of -1 for these notes (this spares the cumbersome $\sqrt{-1}$ alternative).

On each chart U_i, we have multiplication by ι, but this does not globalise, as ι does not commute with holomorphic functions: in the Taylor expansion, we have terms which are of degree m where $m \neq 1 \mod 4$. However, the differential of ι may be globalised, as we get rid of the higher order terms. In a local chart $U_i \subset \mathbb{C}^n$, the (real) tangent bundle has a local frame

$$T_\mathbb{R} U_i = \langle \partial_{x_j}, \partial_{y_j} : 1 \leq j \leq n \rangle,$$

on which $I := d\iota$ acts by

$$\begin{cases} \partial_{x_j} \mapsto \partial_{y_j} \\ \partial_{y_j} \mapsto -\partial_{x_j}. \end{cases}$$

Definition 1.1.2. An **almost complex structure** on a smooth manifold X is an endomorphism $I \in \mathrm{End}(T_\mathbb{R} X)$ such that $I^2 = -1$. We say that I is integrable if X is a complex manifold and I is obtained by locally differentiating ι.

Question 1.1.3. Given I an almost complex structure, when is it integrable?

Let us first set up some tools in order to address this question appropriately. Assume only for now that X is a smooth manifold and I is an almost complex structure. We can consider the complexified tangent bundle

$$T_\mathbb{C} X := T_\mathbb{R} X \otimes_\mathbb{R} \mathbb{C},$$

[1] This can be dropped by considering complex analytic spaces (rather than manifolds).

to which we can extend the action of I. Since $I^2 = -1$, the minimal polynomial of I is x^2+1, which is separable over $\mathbb{C}$, meaning that I is diagonalisable, with eigenvalues $\pm\iota$. The eigenspaces must have the same dimension as I acts on the *real* tangent space. We thus obtain a decomposition

$$T_{\mathbb{C}}X = T^{1,0}X \oplus T^{0,1}X = T^{1,0}X \oplus \overline{T^{1,0}X}.$$

Note that we have

$$T^{1,0}X = \{(v - iIv) : v \in T_{\mathbb{C}}X\} \qquad T^{0,1}X = \{(v + iIv) : v \in T_{\mathbb{C}}X\}.$$

Notation 1.1.4. We will use the following notation

- $\mathcal{A}^0(X) := C^\infty_X$;
- $\mathcal{A}^k(X)$ denotes the sheaf of (smooth) degree k real forms;
- $\mathcal{A}^k(X,\mathbb{C}) = \oplus_{p+q=k}\mathcal{A}^{p,q}(X)$ denotes the sheaf of sections of $\bigwedge^k T^*_{\mathbb{C}}X$ (i.e. smooth complex degree k forms) and $\mathcal{A}^{p,q}(X)$ denotes the sheaf of sections of $\bigwedge^p T^{1,0}X \otimes \bigwedge^q T^{0,1}X$;
- $d : \mathcal{A}^k(X,\mathbb{C}) \to \mathcal{A}^{k+1}(X,\mathbb{C})$ denotes the complexification of the usual exterior derivative, and can be decomposed by types as $d = \partial + \bar\partial$, where ∂ denotes the part corresponding to the differentiation in holomorphic coordinates, and similarly $\bar\partial$ for anti-holomorphic coordinates.
- $A^k(X)$, $A^k(X,\mathbb{C})$, and $A^{p,q}(X)$ denotes the global sections of $\mathcal{A}^k(X), \mathcal{A}^k(X,\mathbb{C})$, and $\mathcal{A}^{p,q}(X)$ respectively.
- $\mathcal{T}_X$ denotes the sheaf of homolorphic vector fields, i.e. $\mathcal{T}_X := \mathcal{D}er(\mathcal{O}_X, \mathcal{O}_X)$;
- $\Omega_X := \mathcal{T}_X^*$ denotes the cotangent sheaf.

The following theorem answers Question 1.1.3.

Theorem 1.1.5 (Newlander-Niremberg)**.** *I is integrable if and only if $\bar\partial^2 = 0$.*

Note that this is equivalent to $T^{1,0}X$ being closed under the (complex) Lie bracket (this is much related to the Frobenius theorem of differential geometry), and also equivalent of the vanishing of a certain tensor N_I called the *Nijenhuis* tensor.

1.2. **Metrics.** Let E be a real vector bundle on (X, C^∞_X). A **Riemannian metric** g on E is a section of $\mathrm{Sym}^2 E^\vee$ such that for all $p \in X$, g_p is positive definite. If E is a complex bundle, a **Hermitian metric** h is a map of sheaves $E \otimes \overline{E} \to C^\infty(X,\mathbb{C})$ such that each h_p is Hermitian, i.e. $h_p(e,f) = \overline{h_p(f,e)}$ and $h_p(e,e) > 0$ for all $e, f \in E_p$. When $E = T_{\mathbb{R}}X$, we say that g (resp. h) is a **Riemannian** (resp. **Hermitiam**) **metric on** X (here, the almost complex structure I is used to put a $\mathbb{C}$-structure on $T_{\mathbb{R}}X$).

If X is a complex manifold and h is a Hermitian metric, then we can write

$$h = g - i\omega$$

where $g = \mathfrak{Re}(h)$ and $\omega = -\mathfrak{Im}(h)$. We obtain that g is a Riemannian metric, and ω is skew-symmetric since

$$\omega(X,Y) = \frac{\iota}{2}(h - \overline{h})$$

and h is conjugate skew-symmetric. Thus, $\omega \in A^2(X)$.

Definition 1.2.1. (X,h) is **Kähler** if $d\omega = 0$.

That h is linear in the first variable and anti-linear in the second ensures that $h(I-, I-) = h(-,-)$, implying that $g(I-, I-) = g(-,-)$, a property that is sometimes called **compatibility** of the metric with I. We have

$$\omega(-,-) = \frac{\iota}{2}(h(-,-) - \overline{h}(-,-)) = \frac{1}{2}(h(I-,-) + \overline{h}(I-,-)) = g(I-,-),$$

which also implies

$$\omega(-, I-) = g(-,-).$$

Definition 1.2.2. A form $\omega \in A^2(X)$ is called **positive** if $\omega(u, Iu) > 0$ for all $u \in T_{\mathbb{R}}X$. We see that a de Rham cohomology class in $H^2(X,\mathbb{C})$ is **positive** if it can be represented by a positive form. If moreover ω is I-invariant (or equivalently, of type $(1,1)$ after embedding $A^2(X) \subset A(X,\mathbb{C})$), we say ω is **Kähler**.

If ω is Kähler, we may define the hermitian metric $h_\omega = \omega(-, I-) - i\omega$, and we have that ω is Kähler if and only if (X, h_ω) is Kähler.

Example 1.2.3. Let $X = \mathbb{P}^n$, with projective coordinates $Z_0, \ldots, Z_n$. Let U_i be the $Z_i \neq 0$ chart, and define $z_j = \frac{Z_j}{Z_i}$. We may define on U_i the metric

$$\omega_{FS} = \omega = i\partial\overline{\partial} \log \left(1 + \sum_j z_j \overline{z}_j \right),$$

and one checks that these glue to a global form, which we call the **Fubini-Study metric**. Written as a Kähler potential this way shows that it is a Kähler metric.

Note that if (X, ω) is Kähler, restricting the metric to a complex submanifold Y preserves all properties of Definition 1.2.2, and so (Y, ω_Y) is Kähler. Thus, any projective manifold is Kähler.

1.3. **Connections.** Let E be a complex (the real case is identical) vector bundle on (X, C_X^∞). A **complex connection** in E is a $\mathbb{C}$-linear map

$$\nabla : \mathcal{A}^0(E, \mathbb{C}) \to \mathcal{A}^1(E, \mathbb{C}),$$

(here $\mathcal{A}^i(E, \mathbb{C}) = \mathcal{A}^i(X, \mathbb{C}) \otimes \Gamma(E)$) such that

$$\nabla(f \cdot s) = df \otimes s + f \cdot \nabla s$$

for all section s of E and $f \in C_X^\infty$.

If E is a holomorphic bundle on a complex manifold, we can define the operator

$$\overline{\partial} : \mathcal{A}^0(E) \to \mathcal{A}^{0,1}(E)$$

as follows: if σ_i is a local frame, and $s = s^i \sigma_i$ a section, we let

$$\overline{\partial}(s^i \sigma_i) := (\overline{\partial} s^i) \otimes \sigma_i. \tag{1.1}$$

Indeed, given another frame τ_j related by $\sigma_i = g_{ij}\tau_j$, we have

$$\overline{\partial}(s^i) \otimes \sigma_i = \overline{\partial}(s_i) \otimes g_{ij}\tau_j = \overline{\partial}(g_{ij}s^i) \otimes \tau_j$$

since the transitions g_{ij} are holomorphic by assumption.

A (complex) connection being valued in $\mathcal{A}^1(E, \mathbb{C}) = \mathcal{A}^{1,0}(E) \oplus \mathcal{A}^{0,1}(E)$, we may split $\nabla = \nabla^{1,0} + \nabla^{0,1}$.

Definition 1.3.1. The complex connection ∇ in E is said to be **compatible** with the holomorphic structure if $\nabla^{0,1} = \overline{\partial}$. Suppose E has a hermitian metric h. We say ∇ is **compatible** with h if for any sections e, f, we have equality of forms

$$d(h(e, f)) = h(\nabla e, f) + h(e, \nabla f).$$

More geometrically, this says that h is parallel to the connection, i.e. constant along parallel transport, i.e. the connection has $U(n)$-holonomy. We say ∇ is a **Chern connection** if it is both compatible with the holomorphic structure and the hermitian metric.

Theorem 1.3.2 (Chern)**.** *There exists a unique Chern connection.*

When $E = T_{\mathbb{R}}X$, the Chern connection ought to be regarded as the complex geometric analogue of the Levi-Civita connection from Riemannian geometry. In fact this is more than an analogy. If h is a hermitian metric, the Levi-Civita connection of $g = \mathfrak{Re}(h)$ can be complexified to a complex connection. It is a theorem that the Levi-Civita connection is the Chern connection if and only if (X, h) is Kähler.

We can extend the connection $\nabla : \mathcal{A}^0(E, \mathbb{C}) \to \mathcal{A}^1(E, \mathbb{C})$ to a connection

$$\nabla : \mathcal{A}^p(E, \mathbb{C}) \to \mathcal{A}^{p+1}(E, \mathbb{C})$$

for all positive p via

$$\nabla(\omega \otimes s) = d\omega \otimes s + (-1)^p \omega \wedge \nabla s,$$

where ω is a p-form and s is a section of E.

Remark 1.3.3. Note that this is different from the usual extension of a connection to tensors since we are dealing with skew-symmetric forms. In particular, this satisfies a different Leibniz rule:

$$\nabla(fs) = df \wedge s \otimes d\nabla s.$$

Definition 1.3.4. We define the **curvature** of ∇ to be the composition $\nabla^2 = \nabla \circ \nabla = F_\nabla$.

Note that

$$\begin{aligned}\nabla(\nabla f s) = \nabla(df \otimes s + f\nabla s) &= ddf - df \wedge \nabla s + \nabla(f\nabla s) \\ &= -df \wedge \nabla s + df \wedge \nabla s + f\nabla^2 s = f\nabla^2 s\end{aligned}$$

so that F_∇ is C^∞_X-linear, that is a section of $\mathcal{A}^2(End(E), \mathbb{C})$.

We may also define

$$F^k_\nabla := \underbrace{F_\nabla \circ \cdots \circ F_\nabla}_{k} \in \mathcal{A}^{2k}(End(E), \mathbb{C}).$$

We define the k**th Chern character** of ∇ to be

$$\mathrm{ch}_k(E, \nabla) := \mathrm{Tr}\left(\frac{1}{k!}\left(\frac{\iota}{2\pi}F^k_\nabla\right)\right) \in A^{2k}(X, \mathbb{C}).$$

Theorem 1.3.5 (Chern-Weil)**.** *The following is true about the Chern character.*

(1) *$ch_k(E, \nabla)$ is closed;*
(2) *The cohomology class $ch_k(E) := [ch_k(E, \nabla)] \in H^{2k}_{dR}(X, \mathbb{C})$ is independent of ∇;*
(3) *$ch_k(E)$ is real, i.e. in $H^{2k}_{dR}(X, \mathbb{R})$ (in fact, it is integral);*
(4) *The total Chern character $\sum_k ch_k(E)$ is equal to the cohomology class of $Tr(\exp(\frac{\iota}{2\pi}F_\nabla))$ (this one directly follows from developing the exponential).*

2. Second Lecture: characteristic classes

2.1. **Chern classes.** Let V be a vector space over $\mathbb{C}$ of dimension r. Let $P \in \mathbb{C}[End(V)]$ be a homogeneous polynomial of degree k. Assume moreover P is $GL(V)$ invariant, that is $P(A^{-1}BA) = P(B)$ for any $A \in GL(V)$.

Let now E be a complex vector bundle and ∇ a connection. By $GL(V)$ invariance, $P(\frac{\iota}{2\pi}F_\nabla)$ is well-defined, and lives in $A^{2k}(X, \mathbb{C})$.

Fact 2.1.1 (Chern-Weil)**.** $P(\frac{\iota}{2\pi}F_\nabla)$ is closed, and the class $[P(\frac{\iota}{2\pi}F_\nabla)] \in H^{2k}(X, \mathbb{C})$ is independent of ∇.

Consider now the $GL(V)$-invariant homogeneous polynomials P_k returning the coefficients of the characteristic polynomials (i.e. P_k kth elementary symmetric polynomial on the eigenvalues). We can explicitly define P_k by the formula:

$$\det(I + tB) = \sum_k P_k(B)t^k.$$

We define the **kth Chern class** of E to be the cohomology class of $P_k(\frac{i}{2\pi}F_\nabla)$, and the **total Chern class** of E to be $c(E) := \sum_{i=0}^k c_k(E)$.

The Chern classes and characters satisfy certain properties:

- $c_0(E) = 1$ and $ch_0 = r$;
- $c_d = 0$ if $d > r$. In particular, if L is a line bundle, $c(L) = 1 + c_1(L)$;
- $ch(L) = \exp(c_1(L)) := 1 + c_1(L) + \frac{c_1(L)^2}{2!} + \cdots \in H^{2\bullet}(X, \mathbb{C})$.

To derive further elementary properties of the Chern characters and classes, let us first observe how we can assemble connections into new connections

Let E_1 and E_2 be vector bundles with respective (complex) connections ∇_1 and ∇_2. Then,

- $\nabla_{E_1 \oplus E_2} := \nabla_1 \oplus \nabla_2$ is a connection on $E_1 \oplus E_2$, and $F_{\nabla_1 \oplus \nabla_2} = F_{\nabla_1} \oplus F_{\nabla_2}$, where this is seen as a block matrix
$$\begin{pmatrix} F_{\nabla_1} & \\ & F_{\nabla_2} \end{pmatrix};$$
- $\nabla_{E_1 \otimes E_2} := \nabla_1 \otimes \mathrm{id}_{E_2} + \mathrm{id}_{E_1} \otimes \nabla_2$ is a connection on $E_1 \otimes E_2$;
- the assignment
$$(\nabla^\vee \phi)(s) := d(\phi(s)) - \phi(\nabla(s))$$
where $s \in E$ and $\phi \in E^\vee$ defines a connection on $E^\vee$ (note that this is the usual way to extend connections on tensors). In other words, if $\langle -, - \rangle$ denotes the natural pairing on $E \otimes E^\vee$, the dual connection is defined by
$$d\langle s, \phi \rangle = \langle \nabla s, \phi \rangle + \langle s, \nabla^\vee \phi \rangle.$$
Let us try to compare the two curvatures. Given s_i and t_j frames of E, we consider the connection form $A = (A_i^{\ j})$ satisfying $\nabla s_i = A_i^{\ j} \otimes t_j$. Let s^i and t^j be the dual frames. We obtain
$$\begin{aligned} d\langle s_i, t^j \rangle = 0 &= \langle \nabla s_i, t^j \rangle + \langle s_i, \nabla^\vee t^j \rangle \\ &= \langle A_i^{\ k} \otimes t_k, t^j \rangle + \langle s_i, B^j_{\ k} \otimes s^k \rangle \\ &= A_i^{\ j} + B^j_{\ i}, \end{aligned}$$
where $B = (B^j_{\ i})$ is the connection form of $\nabla^\vee$. And so we have $B = -A^t$ as sections of $\mathcal{A}^2(End(E), \mathbb{C}) = \mathcal{A}^2(End(E^\vee), \mathbb{C})$. Using Cartan's formula for the curvature of a connection, we conclude
$$F_{\nabla^\vee} = d(-A^t) + (-A^t) \wedge (-A^t) = -(dA + A \wedge A)^t = -F_\nabla^t.$$
- connections pull back, that is if $f : Y \to X$ is a smooth map and E is a bundle on X with connection ∇_E, we may define the connection ∇_{f^*E} by *locally* demanding
$$\nabla_{f^*E}(f^*s) = f^*\nabla s.$$

Corollary 2.1.2. *Let E_1, E_2 be complex vector bundles on X. The following hold:*

1. $ch(E_1 \oplus E_2)) = ch(E_1) + ch(E_2)$ *and* $c(E_1 \oplus E_2) = c(E_1) \cup c(E_2)$*;*
2. $ch(E_1 \otimes E_2) = ch(E_1) \cup ch(E_2)$*;*
3. $c_k(E^\vee) = (-1)^k c_k(E)$
4. $ch_k(f^*E) = f^*(ch_k(E))$ *and and* $c_k(f^*E) = f^*(c_k(E))$.

Note that $c_k(E) \in H^{2k}(X, \mathbb{C})$ is in fact real: indeed, conjugation acts via $\overline{F_\nabla} = F_{\overline{\nabla}}$, and up to choosing a hermitian metric, we have $E^\vee \simeq \overline{E}$ and $F_{\overline{\nabla}} = -F_\nabla^t$, and in a Chern splitting, the eigenvalues $\omega_1, \ldots, \omega_r$ are purely imaginary. We obtain

$$c_k(E) = P_k(\frac{\iota}{2\pi} F_\nabla) = \left[\frac{\iota^k}{(2\pi)^k} P_k(F_\nabla)\right] = \left[\frac{\iota^k}{(2\pi)^k} \sigma_k(\omega_1, \ldots, \omega_r)\right]$$

while

$$\overline{c_k(E)} = \left[\frac{(-1)^k \iota^k}{(2\pi)^k} \sigma_k(-\omega_1, \ldots, -\omega_r)\right] = \left[\frac{(-1)^{2k} \iota^k}{2\pi} \sigma_k(\omega_1, \ldots, \omega_r)\right] = c_k(E)$$

where σ_k denotes the kth standard symmetric polynomial, so that $c_k(E) \in H^{2k}(X, \mathbb{R})$. The kth Chern class is also $(1,1)$. Indeed, consider the Chern connection $\nabla = \nabla^{1,0} + \nabla^{0,1} = \nabla^{1,0} + \overline{\partial}$. From this decomposition, we see that the $(0,2)$-part of the curvature is $\overline{\partial}^2 = 0$. Similarly, one can use the fact that the hermitian metric is parallel to show that the $(2,0)$ part vanishes so that all ω_i are of type $(1,1)$, from which one obtains that $c_k(E)$ is of type (k,k).

Let $D \subset X$ be a divisor. It is a fact that the fundamental class of D is the Chern class of $\mathcal{O}(D)$, i.e.

$$[D] = c_1(\mathcal{O}(D)) \in H^2(X, \mathbb{C}),$$

showing that the first — and therefore any — Chern class is integral.

An important theorem relating to chern classes is Kodaira's embedding theorem.

Theorem 2.1.3 ([Kod54])**.** *A (holomorphic) line bundle L on a complex manifold X is ample (i.e. induces an embedding in projective space) if and only if it admits a metric h such that $\frac{\iota}{2\pi} F_{D_h}$ is a positive form.*

In particular, let h_0 be any hermitian metric on L, and let $\omega_0 = \frac{\iota}{2\pi} F_{D_{h_0}}$. Assuming X is Kähler, if $c_1(L) = [\omega_0]$ is positive, i.e. if it has a positive form ω as representative of the cohomology class, then we can write $\omega = \omega_0 + \frac{\iota}{2\pi} \partial\overline{\partial}\phi$ for some function ϕ by the $\partial\overline{\partial}$-lemma. Then, one may compute that the metric $h := e^{-\phi} \cdot h_0$ satisfies $\frac{\iota}{2\pi} F_{D_h} = \omega$; indeed the $(1,0)$-part of the Chern of the connection is

$$h^{-1}\partial h = \partial \log h = \partial \log(e^{-\phi} h_0) = \partial \log h_0 + \partial(-\phi)$$

so that the curvature is given by

$$F_{D_{h_0}} + \overline{\partial}\partial(-\phi) = F_{D_{h_0}} + \partial\overline{\partial}\phi.$$

In particular, a line bundle L is ample if and only if $c_1(L)$ is positive, i.e. is represented by a Kähler form. Now, on a compact manifold, slightly perturbing a Kähler form inside $H^{1,1}(X, \mathbb{R})$ still yields a Kähler form, since it preserves the positivity criterion. Thus, Kähler forms form an open positive cone $\mathcal{K}_X$ inside of $H^{1,1}(X, \mathbb{R})$ (scaling by a positive real preserves Kählerness). Moreover, by the Lefschetz theorem on $(1,1)$ classes, the Chern map $\mathrm{Pic}(X) \to H^{1,1}(X, \mathbb{Z})$ is surjective. Thus, we conclude that a compact complex manifold is projective if and only if $\mathcal{K}_X$ intersects with $H^{1,1}(X, \mathbb{Z})$ (or equivalently $H^{1,1}(X, \mathbb{Q})$) inside of $H^{1,1}(X, \mathbb{R})$.

2.2. **Hirzebruch-Riemann-Roch.**

Definition 2.2.1. Let X be a compact complex manifold. Let ∇ be a connection in the tangent bundle. We define the **Todd class** of X to be

$$td(X) := \left[\det\left(\frac{\frac{\iota}{2\pi} F_\nabla}{1 - \exp(\frac{-\iota}{2\pi} F_\nabla)}\right)\right] \in H^{2\bullet}(X, \mathbb{C})$$

In terms of Chern roots $\omega_1, \ldots, \omega_r$, we have that

$$td(X) = \prod_{i=1}^r \frac{\omega_i}{1 - e^{-\omega_i}}$$

It can be computed that we have

$$td_0(X) = 1; \quad td_1(X) = \frac{c_1}{2}; \quad td_2(X) = \frac{1}{12}(c_1^2 + c_2); \quad td_3(X) = \frac{c_1 c_2}{24} \quad td_4(X);= \frac{-c_1^4 + 4c_2 + c_1 c_3 + 3c_2^2 - c_4}{720}; \quad \cdots$$

where $td_k(X)$ denotes the kth homogeneous component of $td(X)$ and $c_i = c_i(X) := c_i(\mathcal{T}_X)$.

Theorem 2.2.2 (Hirzebruch-Rieman-Roch). *Let E be a holomorphic vector bundle on a compact complex manifold X. Then, we have equality*

$$\chi(X,E) := \sum_k (-1)^k h^i(X,E) = \int_X ch(E) \cup td(X). \tag{2.1}$$

Note that since we are integrating over X, we only need to consider the top degree parts.

Example 2.2.3. Let $X = C$ be a compact Riemann surface and L be a line bundle on X. we have $ch(E) = 1 + c_1(L)$ and $td(1) = 1 + \frac{c_1(X)}{2}$. thus, we have

$$\chi(X,L) = \int_X 1 + c_1(L) + \frac{c_1(X)}{2} + \frac{c_1(L)c_1(X)}{2} = \int_X c_1(L) + \frac{c_1(X)}{2} = \deg(L) + \frac{\deg(\mathcal{T}_C)}{2}.$$

What is remarkable about this theorem is that the left-hand side of (2.1) is purely holomorphic (or algebraic) whilst the right-hand side is purely topological. Another similar theorem is the algebro-geometric Gauss-Bonnet theorem.

Theorem 2.2.4. *Let X be a compact complex manifold of dimension n. Then,*

$$e(X) := \sum_i (-1)^i b_i(X) = \int_X c_n(X).$$

Recall the classical relation between the Euler characteristic χ_{top} and the genus g of a topological surface: $e = 2 - 2g$. In particular, this implies for a compact Riemann surface C as above, that

$$\int_X c_1(X) = \deg(\mathcal{T}_C) = 2 - 2g.$$

In particular, in light of what we found in Example 2.2.3, we recover the classical Riemann-Roch theorem:

$$\chi(X,L) = \deg(L) - g + 1.$$

2.3. Kähler-Einstein manifolds.

Question 2.3.1. When does a smooth projective variety over $\mathbb{C}$ admit a "canonical" metric?

Definition 2.3.2. Let (X,ω) be a compact Kähler manifold, and let D_ω be the corresponding Chern connection. We define the **Ricci form** $\mathrm{Ric}(\omega)$ of ω to be

$$\mathrm{Ric}(\omega) = i\mathrm{Tr}(F_{D_\omega}) \in A^2(X,\mathbb{C}).$$

We say that (X,ω) is **Kähler-Einstein** if $\mathrm{Ric}(\omega) = \lambda\omega$ for some constant $\lambda \in \mathbb{R}$.

Remark 2.3.3. We make the following comments.

(1) Recall that we argued earlier that all the Chern roots of F_{D_ω} were pure imaginary of type $(1,1)$ so that $\mathrm{Ric}(\omega)$ is real of type $(1,1)$.
(2) Note also that D_ω is invariant under rescaling ω by some $\lambda > 0$ (indeed, parallelness of h is unaffected so we get the same connection). Thus, we may always assume that $\lambda = -1, 0, 1$.
(3) since $c_1(X) = [\frac{\iota}{2\pi}\mathrm{Tr}F_{D_\omega}]$ by definition, we have $[\mathrm{Ric}(\omega)] = 2\pi c_1(X) \in H^2(X,\mathbb{R})$.
(4) λ is proportional to the scalar curvature, and so (X,ω) being Kähler-Einstein implies that the scalar curvature with respect to $g_\omega = \omega(I-,-)$ is constant.
(5) If X is Kähler-Einstein, then we have

$$c_1(X) = \begin{cases} 0 \\ \pm \text{ positive form.} \end{cases}$$

Definition 2.3.4. We say that a complex manifold X is **Calabi-Yau** if $c_1(X) = 0$, **Fano** if $c_1(X)$ is positive, of **general type** (or **canonically polarised**) if $-c_1(X)$ is positive.

Note that by Kodaira's embedding theorem, Fano and general type manifolds are projective.

Caution 2.3.5. It is not because a manifold fits in this trichotomy that it admits a Kähler-Einstein metric. In fact, there exist Fano varieties with no Kähler-Einstein metric. Whether a Fano variety admits such metric is equivalent to K-stability, a purely algebro-geometric notion. Nonetheless, Yau (cf. [Yau78]) proved that any Calabi-Yau manifold admits a Kähler-Einstein metric, and Aubin–Yau (cf. [Aub76; Yau78]) proved the same for general type manifolds.

Note also that not all manifolds fit in this trichotomy.

Example 2.3.6 (Curves)**.** Let us see how these categories apply to curves.

- $g = 0$ gives only $\mathbb{P}^1$. Since it is diffeomorphic to a sphere, we have positive scalar curvature. And indeed, the Fubini-Study metric is Kähler-Einstein with $\lambda = 1$. Note also that $\mathbb{P}^1$ is Fano.
- The $g = 1$ case corresponds to elliptic curves. These are Ricci-flat and Calabi-Yau.
- The case $g > 1$ are of general type, and there exists a Kähler-Einstein metric with negative scalar curvature.

For Fano manifolds, here is a summary of the known classifications:

(1) In dimension 1 there is only the projective line.
(2) In dimension 2, they are called *del Pezzo* surfaces. There are 10 different deformation families. First, $\mathbb{P}^2$ and $\mathbb{P}^1 \times \mathbb{P}^1$ which are isolates. The other 8 families are obtained by blowing up $\mathbb{P}^2$ at d points in general position, where $1 \leq d \leq 8$.
(3) In dimension 3, Mukai proved there are 105 families.
(4) In dimension 4, we know there are finitely many families but it remains open to know how many.

For Calabi-Yau manifolds, there is the following structural theorem.

Theorem 2.3.7 (Beauville–Bogomolov; Bog74 and Bea83)**.** *Let X be Kähler and Calabi-Yau. Then, there exists an étale cover $\tilde{X} \to X$ such that*

$$\tilde{X} = T \times \prod_j X_j \times \prod_i V_i$$

where T is a torus, X_j is ***hyperkähler*** *for all j, and V_i are* ***strict Calabi-Yau*** *for all i.*

We now define the terms.

Definition 2.3.8. A compact Kähler manifold V is called **strict Calabi-Yau** if

- $K_V \simeq \mathcal{O}_V$ is trivial, where K_V denotes the canonical bundle;
- V is simply connected;
- $H^i(V, \mathcal{O}_V) = 0$ for all $0 < i < \dim V$.

A complex manifold X is **hyperkähler** if

- it is simply connected;
- $H^0(X, \Omega_X^2) \simeq \mathbb{C}\sigma$ where σ is holomorphic symplectic (in particular, it induces an isomorphism $\mathcal{T}_X \simeq \Omega_X$).

Remark 2.3.9. If V is a strict Calabi-Yau of dimension greater than two, then $h^{2,0} = h^{0,2} = 0$. In particular, $H^{1,1}(X, \mathbb{C}) = H^2(X, \mathbb{C})$, and so $H^{1,1}(X, \mathbb{R}) = H^2(X, \mathbb{R}) = H^{1,1}(X, \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{R}$. Since the Kähler cone is not empty by assumption, we conclude that it intersects $H^{1,1}(X, \mathbb{Z})$, so that V is projective by our discussion on Kodaira's embedding theorem. In dimension 2, non-projective K3 surfaces yield an example of non-projective strict Calabi-Yau manifolds.

Remark 2.3.10. Theorem 2.3.7 has an important corollarly. Namely, if X is a Kähler, we have a degree n étale map $f : \tilde{X} \to X$ with the canonical bunlde $K_{\tilde{X}}$ of $\tilde{X}$ trivial. Since f is étale, $f^* K_X = K_{\tilde{X}}$ is trivial, and so by the projection formula,

$$f_* \mathcal{O}_{\tilde{X}} = f_* f^* K_X = K_X \otimes f_* \mathcal{O}_{\tilde{X}}.$$

Taking determinants, we have

$$\det(f_* \mathcal{O}_{\tilde{X}}) = K_X^n \otimes \det(f_* \mathcal{O}_{\tilde{X}}),$$

so that $K_X^n = \mathcal{O}_X$. Hence, the power of the canonical bundle of a Kähler Calabi-Yau is always trivial.

3. Third lecture: Hodge theory

In this lecture, we recollect Hodge theory.

3.1. **Linear algebra.** Let us first explore the constructions of Hodge theory in the setting of linear algebra, our toy model.

Let V be a real vector space of dimension n and $\langle -,-\rangle$ be an inner product on V. The scalar product induces a scalar product on $\bigwedge^k V$ via declaring

$$\langle v_1 \wedge \cdots \wedge v_k, u_1 \wedge \cdots \wedge u_k\rangle := \det(\langle u_i, v_j\rangle)_{ij}.$$

Moreover, if $e_1, \ldots, e_n$ is an *ordered* orthonormal basis of V, the vectors $e_{i_1} \wedge \cdots \wedge e_{i_k}$ for $i_1 < \cdots < i_k$ form an orthonormal basis of $\bigwedge^k V$.

Definition 3.1.1. The **volume form** of V (with respect to the chosen ordered basis) is

$$\mathrm{vol}_V = \mathrm{vol} := e_1 \wedge \cdots \wedge e_n.$$

For any $k \leq n$, we define the **Hodge operator** to be the map

$$\begin{aligned} * : \bigwedge^{k} V &\to \bigwedge^{n-k} V \\ e_{i_1} \wedge \cdots \wedge e_{i_k} &\mapsto \varepsilon e_{j_1} \wedge \cdots \wedge e_{j_{n-k}}, \end{aligned}$$

where $\{e_{j_1}, \ldots, e_{n-k}\}$ is the complement of $\{e_{i_1}, \ldots, e_{i_k}\}$ in the full basis $\{e_1, \ldots, e_n\}$ and this map is well-defined because of the permutation index $\varepsilon = \mathrm{sgn}(i_1, \ldots, i_k, j_1, \ldots, j_{n-k})$.

Let $\alpha = e_{i_1} \wedge \cdots \wedge e_{i_k}$ and $\beta = e_{j_1} \wedge \cdots \wedge e_{j_k}$ be elements of the basis of $\bigwedge^k V$. We have that

$$\alpha \wedge *\beta = \begin{cases} 0 & \text{if } \{i_1, \ldots, i_k\} \neq \{j_1, \ldots, j_k\} \\ \mathrm{vol} & \text{otherwise.} \end{cases}$$

In any case, we have that

$$\alpha \wedge *\beta = \langle \alpha, \beta\rangle \cdot \mathrm{vol}, \tag{3.1}$$

and by bilinearity of the left and right hand side, we see that (3.1) holds for any $\alpha, \beta \in \bigwedge^k V$.

Let I be a complex structure on V, i.e. $I^2 = -\mathrm{id}_V$. Recall that it forces the dimension of V to be even. Suppose furthermore that the inner product is compatible with I, i.e. $\langle I-, I-\rangle = \langle -,-\rangle$. Let $\langle -,-\rangle$ also denote the $\mathbb{C}$-sesquilinear extension of the inner product to a hermitian product on the complexified space $V_{\mathbb{C}}$.

The decomposition $V = V^{1,0} \oplus V^{0,1}$ into $\pm\iota$ eigenspaces is orthodonal for $\langle -,-\rangle$, indeed, for $v \in V^{1,0}$ and $u \in V^{0,1}$, we have

$$\langle v, u\rangle = \langle Iv, Iu\rangle = \langle \iota v, \iota u\rangle = -\langle v, u\rangle.$$

By extension, this shows that the decomposition

$$\bigwedge^{k} V_{\mathbb{C}} = \bigoplus_{p+q=k} \bigwedge^{p,q} V \qquad \text{where } \bigwedge^{p,q} V := \bigwedge^{p} V^{1,0} \otimes \bigwedge^{q} V^{0,1} \tag{3.2}$$

is orthogonal. We may extend the hodge star operator $\mathbb{C}$-linearly to

$$* : \bigwedge^{k} V_{\mathbb{C}} \to \bigwedge^{n-k} V_{\mathbb{C}},$$

and by looking at an orthonormal basis for (3.2), we see that the operator restricts to

$$* : \bigwedge^{p,q} V \to \bigwedge^{n-q,n-p} V.$$

Note that the equality (3.1) now becomes

$$\alpha \wedge *\overline{\beta} = \langle \alpha, \beta\rangle \mathrm{vol}_{V_{\mathbb{C}}}.$$

3.2. **Harmonic forms.** From now on, (X, h) is a Kähler manifold and $g = \mathfrak{Re}(h)$ is the compatible Riemannian metric associated to h. We will write $\langle -, - \rangle$ for the hermitian metric induced by h on $T^*_{\mathbb{R}}X$. We have the real volume form

$$\mathrm{vol} \in A^{2n}(X).$$

The decomposition

$$\mathcal{A}^k(X, \mathbb{C}) = \bigoplus_{p+q=k} \mathcal{A}^{p,q}(X)$$

is orthogonal with respect to $\langle -, - \rangle$. As before, we have the hodge operator

$$* : \mathcal{A}^{p,q}(X) \to \mathcal{A}^{n-q,n-p}.$$

Recall also our three different exterior derivatives:

$$\begin{aligned} d &: \mathcal{A}^k(X) \to \mathcal{A}^{k+1}(X) \\ \partial &: \mathcal{A}^{p,q}(X) \to \mathcal{A}^{p+1,q}(X) \\ \overline{\partial} &: \mathcal{A}^{p,q}(X) \to \mathcal{A}^{p,q+1}(X). \end{aligned}$$

We use the hodge operator to define other operators:

$$\begin{aligned} d^* &:= (-1)^k *^{-1} \circ d \circ * : \mathcal{A}^k(X) \to \mathcal{A}^{k-1}(X) \\ \partial^* &:= - * \circ \overline{\partial} \circ * : \mathcal{A}^{p,q}(X) \to \mathcal{A}^{p-1,q}(X) \\ \overline{\partial}^* &:= - * \circ \partial \circ * : \mathcal{A}^{p,q}(X) \to \mathcal{A}^{p,q-1}(X). \end{aligned}$$

The mismatch in the sign is a convention, and we note that $*^2 = (-1)^{k(n-k)}$. Extending d^* $\mathbb{C}$-linearly, these operators satisfy

$$d^* = \partial^* + \overline{\partial}^*$$

We use these operators to define three different Laplacians:

$$\begin{aligned} \Delta_d &:= dd^* + d^*d : \mathcal{A}^k(X) \to \mathcal{A}^k(X) \\ \Delta_\partial &:= \partial\partial^* + \partial^*\partial : \mathcal{A}^{p,q}(X) \to \mathcal{A}^{p,q}(X) \\ \Delta_{\overline{\partial}} &:= \overline{\partial}\,\overline{\partial}^* + \overline{\partial}^*\overline{\partial} : \mathcal{A}^{p,q}(X) \to \mathcal{A}^{p,q}(X). \end{aligned}$$

A k-form α is **d-harmonic** if $\Delta_d \alpha = 0$, and a (p,q)-form β is **$\overline{\partial}$-harmonic** if $\Delta_{\overline{\partial}} \beta = 0$. We will write $\mathcal{H}^k(X) := \ker \Delta_d$ for the space of d-harmonic k-forms (note, we can do this either over $\mathbb{R}$ or $\mathbb{C}$ and do not specify), and $\mathcal{H}^{p,q}(X) := \ker \Delta_{\overline{\partial}}$ for the space of $\overline{\partial}$-harmonic (p,q)-forms.

We have an L^2 inner product

$$\begin{aligned} A^k(X) \otimes A^k(X) &\to \mathbb{R} \\ (\alpha, \beta) &\mapsto \int_X \langle \alpha, \beta \rangle \mathrm{vol} = \int_X \alpha \wedge *\beta. \end{aligned}$$

From the right hand side, we compute that for $\alpha \in A^k(X)$ and $\beta \in A^{k+1}(X)$,

$$(d\alpha, \beta) = \int_X d\alpha \wedge *\beta = \int_X \alpha \wedge (-1)^{k+1} d(*\beta) = \int_X \alpha \wedge *(-1)^{k+1} *^{-1} d(*\beta) = \int_X \alpha \wedge *d^*\beta = (\alpha, d^*\beta),$$

so that d and d^* are formal adjoints with respect to $(-,-)$. From this, we see that

$$(\alpha, \Delta_d \alpha) = (\alpha, dd^*\alpha + d^*d\alpha) = (d^*\alpha, d^*\alpha) + (d\alpha, d\alpha),$$

and so by positive definiteness, $\Delta\alpha = 0$ if and only if $d\alpha = d^*\alpha = 0$. This yields a map

$$\mathcal{H}^k(X, \mathbb{R}) \to H^k(X, \mathbb{R}).$$

Theorem 3.2.1 (Hod41)**.** *This map is an isomorphism.*

Remark 3.2.2. Note that this works equally well over $\mathbb{C}$ witht the hermitian product

$$(\alpha, \beta) \mapsto \int_X \langle \alpha, \beta \rangle \mathrm{vol} = \int_X \alpha \wedge *\overline{\beta}$$

3.3. **The Dolbeault side.** We can get a similar theorem by taking into account the (p,q) type of forms. First note that for any p, we have a complex

$$0 \to \Omega_X^p \to \mathcal{A}^{p,0}(X) \to \mathcal{A}^{p,1}(X) \to \cdots \to \mathcal{A}^{p,n}(X) \to 0.$$

The $\overline{\partial}$-Poincaré lemma (also called the Dolbeault–Grothendieck lemma) says that this complex is exact, i.e. that analytically locally, a $\overline{\partial}$-closed form is $\overline{\partial}$-exact. Moreover, since each $\mathcal{A}^{p,q}(X)$ admits partitions of unity (these are smooth sections), this is an acyclic resolution. As corollary, we obtain an isomorphism

$$H^q(X, \Omega_X^p) = \mathbb{H}^q(\mathcal{A}^{p,0}(X) \to \cdots \to \mathcal{A}^{p,n}(X) \to 0) = \frac{\ker\left(\overline{\partial} : A^{p,q}(X) \to A^{p,q+1}(X)\right)}{\operatorname{im}\left(\overline{\partial} : A^{p-1,q}(X) \to A^{p,q}(X)\right)} =: H^{p,q}(X), \tag{3.3}$$

and we call the right-hand side the (p,q) Dolbeault cohomology of X.

As before, we have an inner product on (p,q)-forms:

$$(\alpha, \beta) := \int_X \alpha \wedge *\overline{\beta},$$

and the same computation shows that ∂^* and $\overline{\partial}^*$ are formal adjoints. Therefore, $\Delta_{\overline{\partial}}\alpha = 0$ if and only if $\overline{\partial}\alpha = \overline{\partial}^*\alpha = 0$. This yields a map

$$\mathcal{H}^{p,q}(X) \to H^{p,q}(X).$$

Theorem 3.3.1. *This map is an isomorphism.*

It is worth noting that Theorem 3.3.1 does not require X to be Kähler.

Proposition 3.3.2. *If X is compact Kähler, then*

$$\Delta_d = 2\Delta_{\overline{\partial}} = 2\Delta_{\partial}.$$

Corollary 3.3.3 (Hodge decomposition)**.** *Let X be a compact Kähler manifold. We have a decomposition*

$$H^k(X, \mathbb{C}) = \oplus_{p+q=k} H^{p,q}(X)$$

such that $H^{p,q}(X) = \overline{H^{q,p}(X)}$ and $H^{p,q}(X) \simeq H^q(X, \Omega_X^p)$.

Proof. Note that $\Delta_{\overline{\partial}}$ preserves the (p,q)-type, and thus so does Δ_d by Proposition 3.3.2. Therefore, we have

$$\bigoplus_{p+q=k} H^{p,q}(X) \simeq \bigoplus_{p+q=k} \mathcal{H}^{p,q}(X) = \mathcal{H}^k(X, \mathbb{C}) \xrightarrow{\sim} H^k(X, \mathbb{C}). \tag{3.4}$$

It remains to show that $H^{p,q}(X) = \overline{H^{q,p}(X)}$, and this follows from the fact that $\Delta_d(\overline{\alpha}) = \overline{\Delta_d(\alpha)}$, as Δ_d is a real operator. □

Remark 3.3.4. It is worthwile to observe that along the isomorphism (3.4), $H^{p,q}(X)$ is identified with the subset $\{[\alpha] : d\alpha = 0 \text{ and } \alpha \in A^{p,q}\} \subset H^k(X, \mathbb{C})$. Note moreover that the Hodge decomposition is compatible witht the wedge product (that is, wedging (p,q) $\overline{\partial}$-closed forms yields the same algebra structure as that on $H^\bullet(X, \mathbb{C})$ along (3.4)). However, there is no clear algebra structure on

$$\bigoplus_{p,q \geq 0} \mathcal{H}^{p,q}(X),$$

as wedging two harmonic forms need not yield a harmonic form (this comes from the fact that d^* does not satisfy a Leibniz rule).

Corollary 3.3.5. *Let X be a compact Kähler manifold. If k is odd, then the kth Betti number $b_k(X)$ is even.*

Remark 3.3.6. For any Kähler form ω,

$$\int_X \omega^{\dim X} = n! \cdot \operatorname{vol}(X) > 0,$$

implying that ω is not d-exact (assuming X compact), i.e. that $b_k > 0$ for k odd.

Remark 3.3.7 (Hodge diamond)**.** We can make the Hodge numbers fit in what is called the Hodge diamond:

$$\begin{array}{ccccccccc}
 & & & & h^{0,0} & & & & \\
 & & & h^{1,0} & & h^{0,1} & & & \\
 & & h^{2,0} & & h^{1,1} & & h^{0,2} & & \\
 & \iddots & & & & & & \ddots & \\
h^{n,0} & & \cdots & & \circlearrowleft & & \cdots & & h^{0,n} \\
 & \ddots & & & & & & \iddots & \\
 & & h^{n,n-2} & & h^{n-1,n-1} & & h^{n-2,n} & & \\
 & & & h^{n,n-1} & & h^{n-1,n} & & & \\
 & & & & h^{n,n} & & & & \\
 & & & & \longleftrightarrow & & & &
\end{array}$$

where $n = \dim_{\mathbb{C}} X$, and these are the only non-zero Hodge numbers. Moreover, there are some symmetries. We already saw Hodge symmetry, which implies $h^{p,q} = h^{q,p}$, which is represented by the arrow in the bottom.

Recalll that some version of Poincaré duality says that there is a non-degenerate pairing

$$\begin{aligned} H^k(X, \mathbb{C}) \times H^{2n-k}(X, \mathbb{C}) &\to \mathbb{C} \\ (\alpha, \beta) &\mapsto \int_X \alpha \wedge \beta, \end{aligned} \tag{3.5}$$

yielding an isomorphism $H^k(X, \mathbb{C})^\vee = H_k(X, \mathbb{C}) \simeq H^{2n-k}(X, \mathbb{C})$. This pairing restricts to a non-degenerate pairing

$$H^{p,q}(X) \times H^{n-p,n-q}(X) \to \mathbb{C};$$

indeed, if a d-closed form of type (p, q) is non-zero, we know by (3.5) that we may pair it with a form β with $\alpha \wedge \beta \neq 0$, but this forces the type of β to be $(n - p, n - q)$. Therefore, we get an isomorphism

$$H^{p,q}(X) = H^{n-p,n-q}(X)^\vee. \tag{3.6}$$

In particular, our diamond has the symmetry $h^{p,q} = h^{n-p,n-q}$, which is represented by the central circling arrow in the Hodge diamond. Note that (3.6) can also be seen using Serre duality:

$$H^{p,q}(X) = H^q(X, \Omega^p_X) = H^{n-q}(X, (\Omega^p_X)^\vee \otimes K_X)^\vee = H^{n-q}(X, \Omega^{n-p}_X)^\vee = H^{n-p,n-q}(X)^\vee,$$

where the isomorphism $(\Omega^p_X)^\vee \otimes K_X = \bigwedge^p \mathcal{T}_X \otimes K_X \simeq \Omega^{n-p}$ comes from contraction of vector fields:

$$\begin{aligned} \bigwedge^p \mathcal{T}_X \otimes K_X &\xrightarrow{\sim} \Omega^{n-p} \\ X_1 \wedge \cdots \wedge X_p \otimes \alpha &\mapsto \alpha(X_1, \ldots, X_p, -, \ldots, -). \end{aligned}$$

Note that there is also a way to see this duality with the Hodge star operator.

The Hodge diamond also satisfies a unimodal condition. Namely, in each row (hence each column by Poincaré/Serre-duality), the Hodge numbers increase before reaching half, then decrease (the latter follows from the former by Hodge symmetry).

3.4. **Lefschetz theorems.** Let X be compact Kähler manifold with Kähler form ω. As ω is a $(1, 1)$ real form, we obtain an operator

$$\begin{aligned} L_\omega : A^k(X) &\to A^{k+2}(X) \\ \alpha &\mapsto \omega \wedge \alpha. \end{aligned}$$

In the complexification, this restrict to

$$L_\omega : A^{p,q}(X) \to A^{p+1,q+1}(X),$$

and these operators descend to cohomology by definition. We define the operator

$$\Lambda_\omega := *^{-1} \circ L_\omega \circ *,$$

and the degree operator

$$h : H^*(X) \to H^*(X)$$

where $h|_{H^k(X)} = (k - n)\mathrm{id}|_{H^k(X)}$. Here, we do not specify the coefficients, as we want to work over either $\mathbb{R}$ or $\mathbb{C}$.

Theorem 3.4.1. *On cohomology, we have $[L_\omega, \Lambda_\omega] = h$, $[h, L_\omega] = 2L_\omega$ and $[2, \Lambda_\omega] = -2\Lambda_\omega$, that is, L_ω, Λ_ω and h form an $\mathfrak{sl}_2$-triple. Moreover, we have $[L_\omega, \Delta_d] = 0 = [\Lambda_\omega, \Delta_d]$.*

From this $\mathfrak{sl}_2$-representation, one can deduce the following.

Theorem 3.4.2 (Hard Lefschetz)**.** *Let $k \leq n$.*

(1) *the map*

$$L_\omega^{n-k} : H^k(X) \to H^{2n-k}(X)$$

is an isomorphism.

(2) *Let $H^k(X)_{prim} \subset H^k(X)$ be the kernel of L_ω^{n-k+1}. We have a **Lefschetz decomposition***

$$H^k(X) = \bigoplus_{2r \leq kr} L_\omega^r H^{k-2r}(X)_{prim}.$$

Moreover, over $\mathbb{C}$, this decomposition is compatible with the Hodge decomposition; i.e, if we define

$$H^{p,q}(X)_{prim} := \ker\left(L_\omega^{2n-p-q+1} : H^{p,q}(X) \to H^{n-p+1,n-q+1}(X)\right),$$

we have

$$H^{p,q}(X) = \bigoplus_{2r \leq p+q} L_\omega^r H^{p-r,q-r}(X)_{prim},$$

and

$$H^k(X, \mathbb{C})_{prim} = \bigoplus_{p+q=k} H^{p,q}(X)_{prim}.$$

Remark 3.4.3. Note that when X is projective, we may choose ω to be integral and the Lefschetz decomposition also holds over $\mathbb{Q}$.

Example 3.4.4. Let us study the consequences of these theorems on the cohomology of a surface X with Kähler form ω. We have the diagram

$$\begin{tikzcd}
H^0(X) \arrow[r, equal] \arrow[dddd, bend right=60, leftrightarrow, "L_\omega^2"'] & H^0(X)_{\mathrm{prim}} \\
H^1(X) \arrow[r, equal] \arrow[dd, bend right=60, leftrightarrow, "L_\omega"'] & H^1(X)_{\mathrm{prim}} \\
H^2(X) = H^2(X)_{\mathrm{prim}} \oplus LH^0(X) & \\
H^3(X) & \\
H^4(X), &
\end{tikzcd}$$

where $H^0(X)_{\mathrm{prim}} = H^0(X)$ and $H^1(X)_{\mathrm{prim}} = H^1(X)$ since the primitive parts are defined as the kernel of maps to a cohomology groups that vanish for dimension reasons. Since $H^0(X)$ is generated by the identity element in the cohomology ring, we obtain (over say $\mathbb{R}$ coefficients) $H^2(X, \mathbb{R}) = \mathbb{R}[\omega] \oplus H^2(X, \mathbb{R})_{\mathrm{prim}}$. As we soon shall see, this decomposition is orthogonal with respect to a certain intersection pairing, so that we may write $H^2(X, \mathbb{R}) = \mathbb{R}[\omega] \oplus [\omega]^\perp$.

3.5. **Hodge index theorem.** Let X be compact and ω be a Kähler form. Consider the complex Poincaré pairing

$$H^k(X, \mathbb{C}) \times H^{2n-k}(X, \mathbb{C}) \to \mathbb{C}$$
$$(\alpha, \beta) \mapsto \int_X \alpha \wedge \beta.$$

This pairing is skew symmetric for k odd and symmetric for k even. We use the polarisation ω to turn this into a pairing on $H^k(X, \mathbb{C})$ for $k \leq n$:

$$Q_k : H^k(X, \mathbb{C}) \times H^k(X, \mathbb{C}) \to \mathbb{C}$$
$$(\alpha, \beta) \mapsto \int_X \alpha \wedge L_\omega^{n-k}(\beta).$$

We now define the hermitian pairing $H_k(\alpha, \beta) := \iota^k Q_k(\alpha, \overline{\beta})$ called the **Hodge–Riemann bilinear form**.

Theorem 3.5.1 (Hodge–Riemann bilinear relations)**.** *The following hold true*

(1) *The Hodge decomposition is orthogonal with respect to H_k;*
(2) *The Lefschetz decomposition is orthogonal with respect to H_k, and for $\alpha, \beta \in H^{k-2r}(X, \mathbb{C})_{prim}$, we have*
$$H_k(L^r(\alpha), L^r(\beta)) = (-1)^{k+r} H_{k-2r}(\alpha, \beta);$$
(3) *The form*
$$(-1)^{\frac{k(k-1)}{2}} \iota^{p-q-k} H_k$$
is positive definite on $H^{p,q}(X)_{prim}$.

Proof. (1) Let $\alpha \in H^{p,q}(X)$ and $\beta \in H^{p',q'}(X)$ with $p+q = p'+q'$. By definition, we have
$$H_k(\alpha, \beta) = \iota^k \int_X \alpha \wedge \omega^{n-k} \wedge \overline{\beta},$$
but the form $\alpha\omega^{n-k} \wedge \wedge\overline{\beta}$ is of degree $2n$ but not of type (n, n). Hence it vanishes, implying that the integral vanishes.

(2) Let $\alpha' = L^r_\omega(\alpha)$ for $\alpha \in H^{k-2r}(X)_{\text{prim}}$ and $\beta' = L^s_\omega(\beta)$ for $\beta \in H^{k-2s}(X)_{\text{prim}}$. Without loss of generality, assume $r > s$. We have that
$$H_k(\alpha', \beta') = \iota^k \int_X \alpha' \wedge L^{n-k+s}_\omega(\overline{\beta'}) = \iota^k \int_X \omega^r \alpha \wedge \omega^{n-k+s} \wedge \overline{\beta} = (-1)^k \iota^k \int_X \alpha \wedge L^{n-k+r+s}_\omega(\overline{\beta}) = 0$$
since $n-k+r+s > n-k+2s$, so that primitiveness of β ensures it is in the kernel of $L^{n-k+r+s}_\omega$.

Now if α', β' are chosen as above but $r = s$, we have
$$\begin{aligned} H_k(\alpha', \beta') &= \iota^k \int_X \alpha \wedge L^{n-k+r}(\overline{\beta'}) = \iota^k \int_X \omega^r \wedge \alpha \wedge \omega^{n-k+r} \wedge \overline{\beta} \\ &= (-1)^k \iota^{2r} \iota^{k-2r} \int_X \alpha \wedge \omega^{n-k+2r} \wedge \overline{\beta} = (-1)^{k+r} H_{k-2r}(\alpha, \beta). \end{aligned}$$
(3) For $\alpha \in H^{p,q}(X)_{\text{prim}}$. For such form, on Kähler manifolds, we have
$$*\alpha = \iota^{p-q}(-1)^{\frac{k(k-1)}{2}} \frac{\omega^{n-k}}{(n-k)!} \wedge \alpha;$$
see [Voi98, Proposition 6.29]. Therefore, we obtain that
$$\begin{aligned} H_k(\alpha, \alpha) &= \iota^k \int_X \alpha \wedge \omega^{n-k} \wedge \overline{\alpha} = (n-k)!(-1)^{\frac{k(k-1)}{2}} \iota^{k-p+q} \int_X \alpha \wedge *\overline{\alpha} \\ &= (n-k)!(-1)^{\frac{k(k-1)}{2}} \iota^{k-p+q} \int_X \langle \alpha, \alpha \rangle \mathrm{vol}_{\mathbb{C}} \end{aligned}$$
so that
$$(-1)^{\frac{k(k-1)}{2}} \iota^{p-q+k} H_k$$
is positive-definite. □

Corollary 3.5.2 (Hodge index theorem)**.** *Let X be a compact Kähler surface. Then, the signature of the Poincaré intersection pairing Q_2 on $H^2(X, \mathbb{R})$ is*
$$(2h^{2,0} + 1, h^{1,1} - 1).$$

Proof. Let $\alpha \in H^2(X, \mathbb{R})$. We may decompose into types: $\alpha = \alpha^{2,0} + \alpha^{1,1} + \alpha^{0,2}$. The fact that $\alpha = \overline{\alpha}$ forces $\alpha^{1,1} \in H^{1,1}(X, \mathbb{R})$, and $\alpha^{2,0} = \overline{\alpha^{2,0}}$. We thus have the decomposition
$$H^2(X, \mathbb{R}) = ((H^{2,0}(X) \oplus H^{0,2}) \cap H^2(X, \mathbb{R})) \oplus H^{1,1}(X, \mathbb{R}),$$
which we know to be orthogonal with respect to the Poincaré pairing. Any $\alpha \in (H^{2,0}(X) \oplus H^{0,2}(X)) \cap H^2(X, \mathbb{R})$ is primitive for degree reasons: taking the cup product with ω yields types $(3, 1)$ or $(1, 3)$. Thus, we have
$$Q_2(\alpha, \alpha) = \int_X \alpha \wedge \alpha = \int_X \alpha^{2,0} \wedge \overline{\alpha^{2,0}},$$
which is positive by Theorem 3.5.1(3). Now, we have the Lefschetz decomposition
$$H^{1,1}(X, \mathbb{R}) = H^{1,1}(X, \mathbb{R})_{\text{prim}} \oplus L_\omega H^0(X, \mathbb{R}) = H^{1,1}(X, \mathbb{R})_{\text{prim}} \oplus \mathbb{R}[\omega].$$

This decomposition is orthogonal: if $\alpha \in H^{1,1}(X, \mathbb{R})_{\text{prim}}$, then

$$Q_2(\omega, \alpha) = \int_X \omega \wedge \alpha = 0$$

as $\omega \wedge \alpha = 0$ by definition of primitive cohomology. We have

$$Q_2(\omega, \omega) = \int_X \omega^2 = 2 \cdot \mathrm{vol}(X) > 0.$$

There remains to compute Q_2 on real $(1,1)$ primitive classes. But by Theorem 3.5.1,

$$\int_X \alpha^2 < 0,$$

and so we obtain the right count for the index of the pairing. $\square$

3.6. **(1,1) classes.** Consider now the exponential exact sequence

$$0 \to \mathbb{Z} \to \mathcal{O}_X \xrightarrow{exp(2\pi i -)} \mathcal{O}_X^* \to 0.$$

It is a fact that the induced connecting homomorphism δ in the long exact cohomological sequence

$$\cdots \to H^1(X, \mathbb{Z}) \to H^1(X, \mathcal{O}_X) \to \mathrm{Pic}(X) \xrightarrow{\delta} H^2(X, \mathbb{Z}) \to H^2(X, \mathcal{O}_X) \to \cdots$$

is related to the Chern map $c_1 : \mathrm{Pic}(X) \to H^2(X, \mathbb{C})$: after composing δ with the map $H^2(X, \mathbb{Z}) \to H^2(X, \mathbb{C})$ (which kills torsion), we have $\delta(\mathcal{L}) = -c_1(\mathcal{L})$, where $\mathcal{L} \in \mathrm{Pic}(X)$.

Theorem 3.6.1 (Lefschetz theorem on $(1,1)$ classes)**.** *If X is compact Kähler, then the first chern map c_1 composed with $H^2(X, \mathbb{Z}) \to H^2(X, \mathbb{C})$ is surjective onto $H^{1,1}(X, \mathbb{Z})$.*

Proof. Note that the compositio is indeed valued in $H^{1,1}(X, \mathbb{Z})$ by definition of the Chern class (indeed, the $(2,0)$ and $(0,2)$ part of the curvature of the Chern connection vanish). The map $H^2(X, \mathbb{Z}) \to H^2(X, \mathcal{O}_X)$ factors as

$$H^2(X, \mathbb{Z}) \to H^2(X, \mathbb{C}) \to H^{0,2}(X, \mathbb{C}) \simeq H^2(X, \mathcal{O}_X),$$

where the middle arrow is the projection onto the $(2,0)$ part. This can be seen as follows: the map $\mathbb{Z} \to \mathcal{O}_X$ factors as $\mathbb{Z} \to \mathbb{C} \to \mathcal{O}_X$, and the map in cohomology $H^2(\mathbb{C}, X) \to H^2(X, \mathcal{O}_X)$ may be computed by considering the map between the resolutions

$$\begin{tikzcd}
0 \arrow[r] & \mathbb{C} \arrow[r] \arrow[d] & \mathcal{A}^1(\mathbb{C}) \arrow[r, "d"] \arrow[d] & \mathcal{A}^2(\mathbb{C}) \arrow[r, "d"] \arrow[d] & \cdots \\
0 \arrow[r] & \mathcal{O}_X \arrow[r] & \mathcal{A}^{0,1} \arrow[r, "\bar{\partial}"] & \mathcal{A}^{0,2} \arrow[r, "\bar{\partial}"] & \cdots ,
\end{tikzcd}$$

where the vertical arrows are projections. Thus, the map $H^2(X, \mathbb{Z}) \to H^2(X, \mathcal{O}_X)$ vanishes on $(1,1)$ classes, so that $(1,1)$ classes are in the kernel of this map, and equivalently in the image of $c_1 = -\delta$. $\square$

Definition 3.6.2. We define the **Néron-Severi** group of X, denoted $\mathrm{NS}(X)$, to be the image of c_1 in $H^2(X, \mathbb{Z})$. We note that $\mathrm{NS}(X)/\text{torsion} = H^{1,1}(X, \mathbb{Z})$, as $H^{1,1}(X, \mathbb{Z})$ is defined to be the intersection of $H^{1,1}(X)$ with the image of $H^2(X, \mathbb{Z})$ in $H^2(X, \mathbb{C})$.

We define the rank $\rho(X)$ of $\mathrm{NS}(X)$ (or equivalently of $H^{1,1}(X, \mathbb{Z})$) to be the **Picard number**. To motivate this terminology, note that there is an exact sequence

$$0 \to \mathrm{Pic}^0(X) \to \mathrm{Pic}(X) \to \mathrm{NS}(X) \to 0,$$

where $\mathrm{Pic}^0(X)$ is the connected component of the identity.

Remark 3.6.3. The integral Hodge conjecture, which has been proven to be false, states that the map

$$\begin{aligned} \{\text{algebraic } \mathbb{Z}\text{-cycles of dimension } k\} &\to H^{(k,k)}(X, \mathbb{Z}) \\ Z &\mapsto [Z] \end{aligned}$$

is surjective. Nevertheless, the Lefschetz theorem on $(1,1)$ classes shows that it is true for $k = 1$. The rational Hodge conjecture, notoriously unsolved, asks whether taking this map over $\mathbb{Q}$-coefficients is surjective.

4.1. **K3 Surfaces.** We now focus on K3 surfaces, which are the fundamental examples in dimension 2 arising from the Beauville-Bogomolov decomposition theorem.

Definition 4.1.1 (Strong definition)**.** A **K3 surface** is a compact connected Kähler surface S such that

(1) The canonical bundle is trivial, $K_S \simeq \mathcal{O}_S$;
(2) S is simply connected, i.e. $\pi_1(S) = \{e\}$.

K3 surfaces are exactly the strict Calabi-Yau manifolds of dimension 2. To show this, we introduce a second definition of K3 surfaces, which we will show is equivalent to the one above.

Definition 4.1.2 (Weak definition)**.** A K3 surface is a compact, connected Kähler surface S such that

(1) $K_S \simeq \mathcal{O}_S$;
(2) $H^1(S, \mathcal{O}_S) = 0$.

Proposition 4.1.3. *The two definitions (Definition 4.1.1 and Definition 4.1.2) are equivalent. In particular, K3 surfaces are exactly the two dimensional strict Calabi-Yau manifolds.*

Proof. (4.1.1 $\implies$ 4.1.2): Assume S satisfies the strong definition. We need to show $H^1(S, \mathcal{O}_S) = 0$. Since S is simply connected, $H_1(S, \mathbb{C})$ and hence $H^1(S, \mathbb{C}) = H_1(S, \mathbb{C})^\vee$ vanish. Since S is Kähler, the Hodge decomposition gives $H^1(S, \mathbb{C}) = H^{1,0}(S) \oplus H^{0,1}(S)$. Thus $H^{0,1}(S) \simeq H^1(S, \mathcal{O}_S) = 0$.

(4.1.2 $\implies$ 4.1.1): Assume S satisfies the weak definition. A deep theorem proved by Siu in [Siu83] that any such surface is Kähler. We need to prove that S is simply connected.

First, we compute the holomorphic Euler characteristic:

$$\begin{aligned}\chi(S, \mathcal{O}_S) &= h^0(S, \mathcal{O}_S) - h^1(S, \mathcal{O}_S) + h^2(S, \mathcal{O}_S) \\ &= 1 - 0 + h^0(S, K_S) \quad \text{(by Serre duality)} \\ &= 1 - 0 + 1 = 2.\end{aligned}$$

We apply the Beauville-Bogomolov decomposition theorem (Theorem 2.3.7) to conclude by dimension reasons that there exists an étale cover $\pi : \tilde{S} \to S$ with $\tilde{S}$ either a complex torus, a Hyperkähler surface, a strict Calabi-Yau surface, or a product of two elliptic curves. Suppose first that $\tilde{S}$ is either a product of two elliptic curves or a complex torus. Then,

$$\chi(\tilde{S}, \mathcal{O}_{\tilde{S}}) = \int_{\tilde{S}} ch(\mathcal{O}_S) \cup td(S) = \int_{\tilde{S}} \frac{c_1^2(S) + c_2(S)}{12} = 0$$

since $\tilde{S}$ is flat. But by Hirzebruch-Riemann-Roch again, we have $\chi(\tilde{S}, \mathcal{O}_{\tilde{S}}) = \deg(\pi)\chi(S, \mathcal{O}_S) = 2\deg(\pi)$, which forces $\deg(\pi) = 0$, contradicting that π is a covering.

Therefore, $\tilde{S}$ is either a strict Calabi-Yau or a Hyperkähler manifold (we will see *a posteriori* that these two conditions coincide for surfaces), and in particular is simply connected. Thus, $\deg(\pi) = 1$ and $S = \tilde{S}$ is simply connected. ☐

Remark 4.1.4. Importantly, it is also true that a two dimensional Hyperkähler manifold is nothing but a K3 surface, but we shall see this later.

4.2. **Cohomology and Picard group of K3 surfaces.**

Theorem 4.2.1. *Let S be a K3 surface. Then:*

(1) $H^0(S, \mathbb{Z}) \simeq H^4(S, \mathbb{Z}) \simeq \mathbb{Z}$.
(2) $H^1(S, \mathbb{Z}) = H^3(S, \mathbb{Z}) = 0$.
(3) $H^2(S, \mathbb{Z}) \simeq \mathbb{Z}^{22}$ *and is torsion-free.*
(4) *The intersection pairing (cup product)*

$$(-,-) : H^2(S, \mathbb{Z}) \times H^2(S, \mathbb{Z}) \to \mathbb{Z}, \quad (\alpha, \beta) \mapsto \int_S \alpha \cup \beta$$

is symmetric, bilinear, and unimodular (i.e. induces an isomorphism $H^2(S, \mathbb{Z}) \to H^2(S, \mathbb{Z})^\vee$).
(5) *The signature of the pairing is* $(3, 19)$.
(6) *The pairing is even, i.e.,* $\alpha^2 := (\alpha, \alpha) \equiv 0 \pmod 2$ *for all* $\alpha \in H^2(S, \mathbb{Z})$.

Proof. We use the strong definition.

(1): S is compact and oriented.

(2): Since S is simply connected, $H_1(S,\mathbb{Z})=0$. By the universal coefficient theorem, $H^1(S,\mathbb{Z})=0$ (there is no torsion in $H_0(S,\mathbb{Z})$). By Poincaré duality, $H^3(S,\mathbb{Z})\simeq H_1(S,\mathbb{Z})^\vee=0$.

(3) By the universal coefficient theorem, the torsion subgroup of $H^2(S,\mathbb{Z})$ comes from the torsion of $H_1(S,\mathbb{Z})=0$. So $H^2(S,\mathbb{Z})$ is torsion-free.

To compute the rank $b_2(S)$, we use the topological Euler characteristic $e(S)$.

$$e(S)=\sum(-1)^i b_i(S)=1-0+b_2(S)-0+1=2+b_2(S).$$

By the Gauss-Bonnet theorem, we have

$$e(S)=\int_S c_2(S).$$

However, Hirzebruch-Riemann-Roch theorem gives us

$$\chi(S,\mathcal{O}_S)=\int_S\frac{c_1(S)^2+c_2(S)}{12}=\int_S\frac{c_2(S)}{12},$$

since S is Ricci-flat by definition. Since $\chi(S,\mathcal{O}_S)=2$, we conclude that $b_2(S)=2\cdot 12-2=22$.

(4) Unimodularity follows from Poincaré duality over $\mathbb{Z}$. Symmetry just follows from the degree in cohomology.

(5) This is a direct consequence of the Hodge index theorem (Corollary 3.5.2) and the fact that $h^{1,1}(S)=20$ since $h^{2,0}=h^0(S,\mathcal{O}_S)=1$.

(6) For algebraic classes $\alpha=c_1(L)\in NS(S)$, we use Hirzebruch-Riemann-Roch:

$$\chi(S,L)=\frac{c_1(L)^2}{2}+\chi(S,\mathcal{O}_S)=\frac{\alpha^2}{2}+2.$$

Since $\chi(S,L)\in\mathbb{Z}$, α^2 must be even.

For the general $\alpha\in H^2(S,\mathbb{Z})$, this follows from Wu's formula. □

Remark 4.2.2 (Hodge Diamond)**.** The Hodge diamond of a K3 surface is the following

$$\begin{array}{ccccc} & & 1 & & \\ & 0 & & 0 & \\ 1 & & 20 & & 1 \\ & 0 & & 0 & \\ & & 1 & & \end{array}$$

Proposition 4.2.3. *For a K3 surface S, the Picard group is isomorphic to the Néron-Severi group.*

Proof. Consider the exponential sequence:

$$\cdots\to H^1(S,\mathcal{O}_S)\to \mathrm{Pic}(S)\xrightarrow{-c_1} H^2(S,\mathbb{Z})\to\cdots$$

Since $H^1(S,\mathcal{O}_S)=0$, the map c_1 is injective. Thus $\mathrm{Pic}(S)\simeq \mathrm{im}\ (c_1)=\mathrm{NS}(S)$. □

Remark 4.2.4. This implies that a line bundle on a K3 surface is determined by its first Chern class.

The Picard number satisfies $\rho(S)\le h^{1,1}(S)=20$. If S is algebraic, $\rho(S)\ge 1$ by Kodaira's embedding theorem. For the very general[2] K3 surface, $\rho(S)=0$.

4.3. **Examples of K3 surfaces.** We now discuss examples of how K3 surfaces may be constructed.

Example 4.3.1 (Quartic surface in $\mathbb{P}^3$)**.** Let $S=\{f=0\}\subset\mathbb{P}^3$ be a smooth hypersurface defined by a homogeneous polynomial f of degree 4. We verify that S is a K3 surface.

By adjunction formula, we have

$$K_S=(K_{\mathbb{P}^3}\otimes\mathcal{O}_{\mathbb{P}^3}(4))|_S=(\mathcal{O}_{\mathbb{P}^3}(-4)\otimes\mathcal{O}_{\mathbb{P}^3}(4))|_S=\mathcal{O}_S.$$

There remains to show $H^1(S,\mathcal{O}_S)=0$: Consider the short exact sequence defining S:

$$0\to\mathcal{O}_{\mathbb{P}^3}(-4)\xrightarrow{\cdot f}\mathcal{O}_{\mathbb{P}^3}\to\mathcal{O}_S\to 0.$$

[2] The notion of "very general" is to be distinguished from "generic": a property holds for the very general K3 surface if, in the appropriate moduli space, it holds outside of a *countable* union of (analytic) Zariski-closed sets.

The long exact sequence in cohomology gives:

$$\cdots \to H^1(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}) \to H^1(S, \mathcal{O}_S) \to H^2(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(-4)) \to \cdots .$$

Since $H^1(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}) = H^2(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(-4)) = 0$, we conclude $H^1(X, \mathcal{O}_X) = 0$.

To construct further examples, we review the cyclic covering trick, used to construct branched covers. If we are given a covering $f : X \to Y$, the ramification divisor R is that where the rank of the differential drops, i.e. the zero locus of

$$\det(df) : f^*K_Y \to K_X.$$

This shows that $R \in |f^*K_Y^\vee \otimes K_X|$. Quite tautologically, the Hurwitz formula follows:

$$K_X = f^*K_Y \otimes \mathcal{O}(R).$$

We say that f is ramified over $f(R)_{\text{red}} = B$. Conversely, given the choice of an effective divisor $B \subset Y$, we want to describe ways to construct finite coverings of Y that are ramified over B.

Construction 4.3.2 (Cyclic covering trick)**.** Assume for simplicity that Y is algebraic, the analytic case shall be discussed in Remark 4.3.7. Let $B \subset Y$ be an effective reduced divisor, and suppose that $\mathcal{O}(B)$ is a mth power for some $m \geq 2$, that is, there exists a line bundle $\mathcal{L} \in \mathrm{Pic}(Y)$ with $\mathcal{L}^m = \mathcal{O}(B)$, and let $s \in \mathcal{O}(B)$ be a defining section for D.

Let $\mathbb{V}(\mathcal{L})$ be the total space of $\mathcal{L}$. We have

$$\mathbb{V}(\mathcal{L}) = \mathrm{Spec}(\mathrm{Sym}^\bullet \mathcal{L}^\vee).$$

Let $\pi : \mathbb{V}(\mathcal{L}) \to Y$ be the projection. We have a tautological section

$$\tau \in H^0(\mathbb{V}(\mathcal{L}), \pi^*\mathcal{L}) = \mathcal{L} \otimes \bigoplus_{i \leq 0} \mathcal{L}^i$$

given by the identity element of $\mathcal{L}^\vee \otimes \mathcal{L}$ and whose zero locus coincides with that of the zero section $Y \subset \mathbb{V}(\mathcal{L})$. Consider the variety X defined as the zero locus of the section $\tau^m - \pi^*s \in \pi^*\mathcal{L}^m$, i.e.

$$X := Z(\tau^m - \pi^*s) \subset \mathbb{V}(\mathcal{L}).$$

the map $f : X \hookrightarrow \mathbb{V}(\mathcal{L}) \xrightarrow{\pi} Y$ is finite, and it is ramified over B

Remark 4.3.3. By using the Jacobian criterion on the local equations for X, it is obvious that X is smooth if and only if B is.

Lemma 4.3.4. *Let $\pi : X \to Y$ be the m-cyclic cover defined by B as above.*

(1) *The pushforward of the structure sheaf is $f_*\mathcal{O}_X \simeq \bigoplus_{j=0}^{m-1} \mathcal{L}^{-j}$.*
(2) *The canonical bundle is $K_X \simeq f^*(K_Y \otimes \mathcal{L}^{m-1})$.*

Proof. (1): Consider the short exact sequence defining X in $\mathbb{V}(\mathcal{L})$:

$$0 \to \mathcal{O}_{\mathbb{V}(\mathcal{L})}(-X) \xrightarrow{\tau^m - \pi^*s} \mathcal{O}_{\mathbb{V}(\mathcal{L})} \to \mathcal{O}_X \to 0.$$

We have by definition $\mathcal{O}_{\mathbb{V}(\mathcal{L})}(-X) \simeq \pi^*(\mathcal{L}^{-m})$. We push-forward via π, which preserves exactness since π is affine:

$$0 \to \pi_*(\pi^*L^{-m}) = \mathcal{L}^{-m} \otimes \mathrm{Sym}^\bullet \mathcal{L}^\vee \xrightarrow{-s} \mathrm{Sym}^\bullet \mathcal{L}^\vee \to f_*\mathcal{O}_X \to 0, \tag{4.1}$$

where the equality is obtained from the projection formula. Since s has degree m, this multiplication map identifies a section in $\mathrm{Sym}^\bullet \mathcal{L}^\vee$ of degree $a + mb$ (where $a, b \leq 0$, $a > -m$) with a section of degree a. Therefore, as $\mathcal{O}_Y$-modules, we have

$$f_*\mathcal{O}_X \simeq \bigoplus_{j=0}^{m-1} \mathcal{L}^{-j}.$$

(2): By the adjunction formula, we have

$$K_X \simeq \left(K_{\mathbb{V}(\mathcal{L})} \otimes \mathcal{O}_{\mathbb{V}(\mathcal{L})}(X)\right)|_X = \left(\pi^*(K_Y \otimes \mathcal{L}^{-1}) \otimes \pi^*\mathcal{L}^m\right)|_X = f^*(K_Y \otimes \mathcal{L}^{m-1}).$$

Alternatively, looking at the defining equation, the ramification divisor of $X \to Y$ is $(m-1)Z(\tau) \subset X$, and so we have by the Hurwitz formula

$$K_X = f^*(K_Y \otimes \mathcal{L}^{m-1}).$$ □

Remark 4.3.5. Such a cyclic cover has a μ_m-action, acting transitively on the fibers, which can be seen from the μ_m-graded structure on $f_*\mathcal{O}_X$. Conversely, it can be shown that any cover that has such a μ_m-action arises as a cyclic covering as constructed above.

Remark 4.3.6. Note that not every finite covering $f : X \to Y$ of degree m is a cyclic covering. However, in characteristic coprime to m, we can say the following. The exact sequence

$$0 \to \mathcal{O}_Y \to f_*\mathcal{O}_X \to \operatorname{coker} \to 0$$

of $\mathcal{O}_Y$-modules splits as we have a retraction given by $\frac{1}{m}\mathrm{Tr}$, where $\mathrm{Tr} : \pi_*\mathcal{O}_X \to \mathcal{O}_Y$ is the trace, defined because we can view elements of $f_*\mathcal{O}_X$ as acting on $f_*\mathcal{O}_X$ by multiplication. Since $\pi_*\mathcal{O}_X$ is locally free of rank m, this implies that $\mathcal{T}^\vee := \operatorname{coker}$ is also locally free, of rank $m-1$. The splitting also ensures that we have a section $\mathcal{T}^\vee \to f_*\mathcal{O}_X$, and the universal property of the symmetric algebra yields a map of $\mathcal{O}_Y$-algebras

$$\mathrm{Sym}^\bullet \mathcal{T}^\vee \to f_*\mathcal{O}_X,$$

which is obviously surjective. Since f is affine, this map comes from a closed immersion

$$X \hookrightarrow \mathbb{V}(\mathcal{T})$$

over Y. $\mathcal{T}$ is called the *Tschirnhausen* bundle, and we have shown that any finite map (under characteristic assumptions) factors through its Tschirnhausen bundle. It is to be expected that this bundle would have rank $m-1$. Indeed, if we take m points in a very big vector space, the affine space they span has dimension $m-1$. In particule, any degree 2 covering factors through (the total space of) a line bundle, and one sees readily that this recovers the cyclic covering trick.

Remark 4.3.7. The only subtlety in the analytic case is that the total space of the line bundle $\mathcal{L}$ has more functions than $\mathrm{Sym}^\bullet\mathcal{L}^\vee$. But we can circumvent this e.g. by constructing directly the $\mathcal{O}_Y$-algebra structure on

$$\mathcal{O}_Y \oplus \mathcal{L}^{-1} \oplus \cdots \oplus \mathcal{L}^{m-1},$$

and showing afterwards it corresponds to $Z(\tau^m - \pi^* s) \subset \mathbb{V}(\mathcal{L})$. Alternatively, one may also show that the analytic counterpart of the exact sequence (4.1) yields the desired $\mathcal{O}_Y$-algebra structure in the same way, e.g. after locally injecting holomorphic functions into formal functions and taking Taylor expansions in local coordinates.

Example 4.3.8 (Double cover of $\mathbb{P}^2$ branched over a sextic)**.** Let $Y = \mathbb{P}^2$. Let $B \subset \mathbb{P}^2$ be a smooth curve of degree 6. We have $\mathcal{O}(B) = \mathcal{O}(6)$. We take $m = 2$ and $\mathcal{L} = \mathcal{O}(3)$. Let $f : X \to \mathbb{P}^2$ be the double cyclic cover branched along B. We check the K3 conditions.

1. Canonical bundle: Using Lemma 4.3.4 (2),

$$K_X = f^*(K_{\mathbb{P}^2} \otimes \mathcal{L}) = f^*(\mathcal{O}(-3) \otimes \mathcal{O}(3)) = \mathcal{O}_X.$$

2. Vanishing of $H^1(X, \mathcal{O}_X)$: Using Lemma 4.3.4 (1),

$$f_*\mathcal{O}_X = \mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{L}^{-1} = \mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}(-3).$$

Since f is finite,

$$H^1(X, \mathcal{O}_X) \simeq H^1(\mathbb{P}^2, f_*\mathcal{O}_X) = H^1(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}) \oplus H^1(\mathbb{P}^2, \mathcal{O}(-3)) = 0.$$

Since B was chosen to be smooth, so is X, which is also connected since it is smooth and a ramified cover. Thus, X is a K3 surface.

Example 4.3.9 (Kummer surfaces)**.** Let A be a complex torus of dimension 2. Consider the involution $i : A \to A$, $x \mapsto -x$.

The fixed locus of i is the set of 2-torsion points $A[2]$, which consists of 16 points. These points induce 16 singularities in A/i. So we first blow up.

Let $p : \tilde{A} \to A$ be the blow-up of A at the 16 fixed points. The exceptional locus is a disjoint union of 16 smooth rational curves $\tilde{E}_i$. The involution i lifts to an involution $\tilde{i} : \tilde{A} \to \tilde{A}$ by the universal property of the blow-up: indeed, we have the commutative square

$$\begin{tikzcd} \tilde{A} \arrow[r, dotted, "\tilde{i}"] \arrow[d] \arrow[dr] & \tilde{A} \arrow[d] \\ A \arrow[r, "i"] & A, \end{tikzcd}$$

and so the diagonal arrow lifts to $\tilde{i}$ since the total transform of the 16 points under the diagonal arrow is a Cartier divisor. Incidentally, the fixed locus of $\tilde{i}$ is exactly the union of the exceptional divisors $\bigcup E_i$. This implies that the quotient $X = \tilde{A}/\tilde{i}$, which we define to be the **Kummer surface** of A, is smooth because the fixed locus has codimension 1. X is Kähler since it is the quotient of a Kähler surface by a finite group. In order to check that it is a K3 surface, there are two conditions to check: that the canonical bundle is trivial, and that $H^1(X, \mathcal{O}_X) = 0$.

We first show that $H^1(X, \mathcal{O}_X) = 0$. Let $f : \tilde{A} \to \tilde{A}/\tilde{i}$ be the quotient map. Since f is finite, we have that the pullback $f^* : H^1(X, \mathbb{C}) \to H^1(\tilde{A}, \mathbb{C})$ is injective; indeed this is seen easily as for a given $\alpha \in H^1(X, \mathbb{C})$, the projection formula (singular cohomology version) gives

$$f_* f^* \alpha = \alpha \cup f_* 1_{\tilde{A}} = 2\alpha,$$

and so $f_* f^*$ is injective. Thus, we may identify

$$H^1(X, \mathbb{C}) = H^1(\tilde{A}, \mathbb{C})^{\mu_2} = H^1(A, \mathbb{C})^{\mu_2},$$

where the superscript indicates taking the invariants under i, and the last equality holds because blowing up a point does not affect singular cohomology[3]. But now, on $H^1(A, \mathbb{C}) = \mathbb{C}^4$, the involution acts as multiplication by -1; this can be seen directly by looking at how it acts on the generators $dz_1, dz_2, d\overline{z}_1$ and $d\overline{z}_2$. Therefore, $H^1(A, \mathbb{C})^{\mu_2} = H^1(X, \mathbb{C}) = 0$, implying that $H^1(X, \mathcal{O}_X) = 0$ by the Hodge decomposition.

We now show that $K_X = \mathcal{O}_X$. Let $b : \tilde{A} \to A$ be the blow-down. By the Hurwitz formula, we have

$$K_{\tilde{A}} \simeq f^* K_X \otimes \mathcal{O}\left(\sum \tilde{E}_i\right),$$

and similarly, by Hurwitz, we have

$$K_{\tilde{A}} \simeq b^* K_A \otimes \mathcal{O}\left(\sum \tilde{E}_i\right) = \mathcal{O}\left(\sum \tilde{E}_i\right),$$

so that $f^* K_X = \mathcal{O}_{\tilde{A}}$. Using the projection formula, we obtain

$$f_* \mathcal{O}_{\tilde{A}} \simeq f_* f^* K_X \simeq f_* \mathcal{O}_{\tilde{A}} \otimes K_X, \tag{4.2}$$

implying, after taking determinants, that K_X is 2-torsion. Now, f is a μ_2-covering, and so by our previous discussion on the cyclic covering trick, we have

$$f_* \mathcal{O}_{\tilde{A}} \simeq \mathcal{O}_X \oplus \mathcal{L}^\vee,$$

where $f^* \mathcal{L}^2 = \mathcal{O}\left(\sum \tilde{E}_i\right)$. Thus, we have

$$K_X \oplus \left(K_X \otimes \mathcal{L}^\vee\right) \simeq \mathcal{O}_X \oplus \mathcal{L}^\vee.$$

This isomorphism has to be given by a matrix of the form

$$\begin{pmatrix} \operatorname{Hom}(K_X, \mathcal{O}_X) & \operatorname{Hom}(K_X, \mathcal{L}^\vee) \\ \operatorname{Hom}(K_X \otimes \mathcal{L}^\vee, \mathcal{O}_X) & \operatorname{Hom}(K_X \otimes \mathcal{L}^\vee, \mathcal{L}^\vee) \end{pmatrix} = \begin{pmatrix} H^0(X, K_X^\vee) & H^0(X, (K_X \otimes \mathcal{L})^\vee) \\ H^0(X, K_X^\vee \otimes \mathcal{L}) & H^0(X, K_X^\vee). \end{pmatrix} \tag{4.3}$$

Suppose for the purpose of contradiction that $K_X \neq O_X$. Then, since $K_X = K_X^\vee$, it cannot have a non-zero global section, and so (4.3) is of the form

$$\begin{pmatrix} 0 & H^0(X, (K_X \otimes \mathcal{L})^\vee) \\ H^0(X, K_X^\vee \otimes \mathcal{L}) & 0, \end{pmatrix}$$

implying that we must have an isomorphism $K_X \simeq \mathcal{L}^\vee$, so that $\mathcal{L}$ is 2-torsion. But this is impossible, as $f^* \mathcal{L}^2 = \mathcal{O}\left(\sum \tilde{E}_i\right) \not\simeq \mathcal{O}_{\tilde{A}}$. Thus, $K_X \simeq \mathcal{O}_X$. ∎

Remark 4.3.10. Note that the map $q : A \to A/i$ is flat. Let Z be the singular locus of A/i, comprising of 16 points. Recall that the blow-up (i.e. Rees) algebra of A at Z is

$$\mathcal{O}_{A/i}[\mathcal{I}_Z t],$$

[3] In general, if $\tilde{Y} \to Y$ is the blow-up at a complex submanifold Z, we have the formula

$$H^\bullet(\tilde{Y}) = H^\bullet(Y) \oplus \bigoplus_{i=1}^{\operatorname{codim}(Z)-1} H^{\bullet - 2i}(Z).$$

where $\mathcal{I}_Z$ is the ideal at Z. We can equally recover the blow-up from the Rees algebra in the analytic setting, where there exists an analytic version of the projective spectrum; see [AHV18]. Now, since q is flat,

$$q^*\mathcal{O}_{A/i}[\mathcal{I}_Z t] = \mathcal{O}_A[(q^*\mathcal{I}_Z)t] = \mathcal{O}_A[\mathcal{I}_{Z'}t],$$

where $Z' = q^{-1}(Z)$, that is, Z' consists of the 16 points fixed by i. The projective spectrum commutes with base change, and so if $\widetilde{A/i}$ denotes the blow up of A/i at Z, we have a cartesian diagram

$$\begin{tikzcd} \tilde{A} \arrow[r] \arrow[d] & \widetilde{A/i} \arrow[d] \\ A \arrow[r, "q"] & A/i \end{tikzcd}$$

now the quotient $q : A \to A/i$ may locally be defined as a GIT quotient, (we are quotienting by a finite group), and so it is universal under base change, hence $\tilde{A} \to \widetilde{A/i}$ is also the quotient of $\tilde{A}$ by the lifted action. But this implies that $\tilde{A}/\tilde{i} = \widetilde{A/i} = X$. That is, when constructing X, we may equally have taken the quotient first, and resolve the singularities afterwards.

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