HYPERKÄHLER MANIFOLDS

NOTES TAKEN BY MAXIM JEAN-LOUIS BRAIS

1. First lecture

We first review some complex geometry.

Definition 1.0.1. A **complex manifold** is a locally ringed space $(X, \mathcal{O}_X)$ such that

- X is Hausdorff and second countable (this part is to ensure we actually have a topological manifold);
- $(X, \mathcal{O}_X)$ is locally isomorphic to $(\Delta, \mathcal{O}_\Delta)$, where $\Delta \subset \mathbb{C}^n$ is the polydisc.

Example 1.0.2. Let $f_i, \dots, f_d \in \mathbb{C}[z_1, \dots, z_n]$ be complex polynomials such that the Jacobian of

$$f = (f_1, \dots, f_d) : \mathbb{C}^n \to \mathbb{C}^d$$

has everywhere full rank on the vanishing set $V = V(f) \subset \mathbb{C}^n$. By the holomorphic implicit function theorem (regular value theorem), V is a complex manifold.

Example 1.0.3. If X is a smooth algebraic variety over $\mathbb{C}$, we may cover it by affines V_i which are of the same form as in Example 1.0.2. We may consider the analytic topology X^{an} obtained by gluing the different charts V_i. Similarly, we may define the sheaf $\mathcal{O}_X^{an}$ on X^{an} by considering the sheaf of holomorphic functions on each V_i (the transitions $V_i \to V_j$ are regular algebraic, hence holomorphic, so that this gluing makes sense). Then, $(X^{an}, \mathcal{O}_X^{an})$ is a complex manifold.

Note that in Example 1.0.3, we obtain a natural map of ringed spaces

$$\alpha : (X^{an}, \mathcal{O}_X^{an}) \to (X, \mathcal{O}_X)$$

since the analytic topology is finer than the Zariski topology, and regular functions are holomorphic. In particular, we obtain a functor between abelian categories:

$$\alpha^* : \mathcal{O}_X\text{-mod} \to \mathcal{O}_X^{an}\text{-mod}$$

restricting to

$$\alpha^* : \mathrm{Coh}(X) \to \mathrm{Coh}(X^{an}).$$

Theorem 1.0.4 (Géométrie algébrique géométrie analytique; [Ser56])**.** *If X is smooth*[1] *and proper functor $\alpha^* : Coh(X) \to Coh(X^{an})$ is an equivalence, therefore inducing an isomorphism.*

1.1. Almost complex structures.

A complex manifold $(X, \mathcal{O}_X)$ has an underlying smooth manifold (X, C_X^∞), where C_X^∞ denotes the sheaf of smooth functions on X; indeed, if X has complex charts $U_i \subset \mathbb{C}^n$, the transitions are holomorphic, hence C^∞.

Notation 1.1.1. Since the indices i will be ubiquitous, ι shall denote the root of -1 for these notes (this spares the cumbersome $\sqrt{-1}$ alternative).

On each chart U_i, we have multiplication by ι, but this does not globalise, as ι does not commute with holomorphic functions: in the Taylor expansion, we have terms which are of degree m where $m \neq 1 \mod 4$. However, the differential of ι may be globalised, as we get rid of the higher order terms. In a local chart $U_i \subset \mathbb{C}^n$, the (real) tangent bundle has a local frame

$$T_\mathbb{R} U_i = \langle \partial_{x_j}, \partial_{y_j} : 1 \leq j \leq n \rangle,$$

on which $I := d\iota$ acts by

$$\begin{cases} \partial_{x_j} \mapsto \partial_{y_j} \\ \partial_{y_j} \mapsto -\partial_{x_j}. \end{cases}$$

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[1] This can be dropped by considering complex analytic spaces (rather than manifolds).

Definition 1.1.2. An **almost complex structure** on a smooth manifold X is an endomorphism $I \in \mathrm{End}(T_{\mathbb{R}}X)$ such that $I^2 = -1$. We say that I is integrable if X is a complex manifold and I is obtained by locally differentiating ι.

Question 1.1.3. Given I an almost complex structure, when is it integrable?

Let us first set up some tools in order to address this question appropriately. Assume only for now that X is a smooth manifold and I is an almost complex structure. We can consider the complexified tangent bundle

$$T_{\mathbb{C}}X := T_{\mathbb{R}}X \otimes_{\mathbb{R}} \mathbb{C},$$

to which we can extend the action of I. Since $I^2 = -1$, the minimal polynomial of I is $x^2 + 1$, which is separable over $\mathbb{C}$, meaning that I is diagonalisable, with eigenvalues $\pm\iota$. The eigenspaces must have the same dimension as I acts on the *real* tangent space. We thus obtain a decomposition

$$T_{\mathbb{C}}X = T^{1,0}X \oplus T^{0,1}X = T^{1,0}X \oplus \overline{T^{1,0}X}.$$

Note that we have

$$T^{1,0}X = \{(v - iIv) : v \in T_{\mathbb{C}}X\} \qquad\qquad T^{0,1}X = \{(v + iIv) : v \in T_{\mathbb{C}}X\}.$$

Notation 1.1.4. We will use the following notation

- $\mathcal{A}^0(X) := C^\infty_X$;
- $\mathcal{A}^k(X)$ denotes the sheaf of (smooth) degree k real forms;
- $\mathcal{A}^k(X,\mathbb{C}) = \oplus_{p+q=k}\mathcal{A}^{p,q}(X)$ denotes the sheaf of sections of $\bigwedge^k T^*_{\mathbb{C}}X$ (i.e. smooth complex degree k forms) and $\mathcal{A}^{p,q}(X)$ denotes the sheaf of sections of $\bigwedge^p T^{1,0}X \otimes \bigwedge^q T^{0,1}X$;
- $d : \mathcal{A}^k(X,\mathbb{C}) \to \mathcal{A}^{k+1}(X,\mathbb{C})$ denotes the complexification of the usual exterior derivative, and can be decomposed by types as $d = \partial + \bar\partial$, where ∂ denotes the part corresponding to the differentiation in holomorphic coordinates, and similarly $\bar\partial$ for anti-holomorphic coordinates.
- $A^k(X)$, $A^k(X,\mathbb{C})$, and $A^{p,q}(X)$ denotes the global sections of respectively $\mathcal{A}^k(X), \mathcal{A}^k(X,\mathbb{C})$, and $\mathcal{A}^{p,q}(X)$.
- $\mathcal{T}_X$ denotes the sheaf of homolorphic vector fields, i.e. $\mathcal{T}_X := \mathcal{D}er(\mathcal{O}_X, \mathcal{O}_X)$;
- $\Omega_X := \mathcal{T}^*_X$ denotes the cotangent sheaf.

The following theorem answers Question 1.1.3.

Theorem 1.1.5 (Newlander-Niremberg)**.** *I is integrable if and only if $\bar\partial^2 = 0$.*

Note that this is equivalent to $T^{1,0}X$ being closed under the (complex) Lie bracket (this is much related to the Frobenius theorem of differential geometry), and also equivalent of the vanishing of a certain tensor N_I called the *Nijenhuis* tensor.

1.2. **Metrics.** Let E be a real vector bundle on (X, C^∞_X). A **Riemannian metric** g on E is a section of $\mathrm{Sym}^2 E^\vee$ such that for all $p \in X$, g_p is positive definite. If E is a complex bundle, a **Hermitian metric** h is a map of sheaves $E \otimes \overline{E} \to C^\infty(X,\mathbb{C})$ such that each h_p is Hermitian, i.e. $h_p(e,f) = \overline{h_p(f,e)}$ and $h_p(e,e) > 0$ for all $e, f \in E_p$. When $E = T_{\mathbb{R}}X$, we say that g (resp. h) is a **Riemannian** (resp. **Hermitiam**) **metric on** X (here, the almost complex structure I is used to put a $\mathbb{C}$-structure on $T_{\mathbb{R}}X$).

If X is a complex manifold and h is a Hermitian metric, then we can write

$$h = g - i\omega$$

where $g = \mathfrak{Re}(h)$ and $\omega = -\mathfrak{Im}(h)$. We obtain that g is a Riemannian metric, and ω is skew-symmetric since

$$\omega(X,Y) = \frac{\iota}{2}(h - \overline{h})$$

and h is conjugate skew-symmetric. Thus, $\omega \in A^2(X)$.

Definition 1.2.1. (X,h) is **Kähler** if $d\omega = 0$.

That h is linear in the first variable and anti-linear in the second ensures that $h(I-, I) = h(-,-)$, implying that $g(I-, I-) = g(-,-)$, a property that is sometimes called **compatibility** of the metric with I. We have

$$\omega(-,-) = \frac{\iota}{2}(h(-,-) - \overline{h}(-,-)) = \frac{1}{2}(h(I-,-) + \overline{h}(I-,-)) = g(I-,-),$$

which also implies

$$\omega(-, I-) = g(-,-).$$

Definition 1.2.2. A form $\omega \in A^2(X)$ is called **positive** if $\omega(u, Iu) > 0$ for all $u \in T_{\mathbb{R}}X$. We see that a de Rham cohomology class in $H^2(X, \mathbb{C})$ is **positive** if it can be represented by a positive form. If moreover ω is I-invariant (or equivalently, of type $(1,1)$ after embedding $A^2(X) \subset A(X, \mathbb{C})$), we say ω is **Kähler**.

If ω is Kähler, we may define the hermitian metric $h_\omega = \omega(-, I-) - i\omega$, and we have that ω is Kähler if and only if (X, h_ω) is Kähler.

Example 1.2.3. Let $X = \mathbb{P}^n$, with projective coordinates $Z_0, \ldots, Z_n$. Let U_i be the $Z_i \neq 0$ chart, and define $z_j = \frac{Z_j}{Z_i}$. We may define on U_i the metric

$$\omega_{FS} = \omega = i\partial\overline{\partial} \log \left(1 + \sum_j z_j \overline{z}_j\right),$$

and one checks that these glue to a global form, which we call the **Fubini-Study metric**. Written as a Kähler potential this way shows that it is a Kähler metric.

Note that if (X, ω) is Kähler, restricting the metric to a complex submanifold Y preserves all properties of Definition 1.2.2, and so (Y, ω_Y) is Kähler. Thus, any projective manifold is Kähler.

1.3. Connections.

Let E be a complex (the real case is identical) vector bundle on (X, C_X^∞). A **complex connection** in E is a $\mathbb{C}$-linear map

$$\nabla : \mathcal{A}^0(E, \mathbb{C}) \to \mathcal{A}^1(E, \mathbb{C}),$$

(here $\mathcal{A}^i(E, \mathbb{C}) = \mathcal{A}^i(X, \mathbb{C}) \otimes \Gamma(E)$) such that

$$\nabla(f \cdot s) = df \otimes s + f \cdot \nabla s$$

for all section s of E and $f \in C_X^\infty$.

If E is a holomorphic bundle on a complex manifold, we can define the operator

$$\overline{\partial} : \mathcal{A}^0(E) \to \mathcal{A}^{0,1}(E)$$

as follows: if σ_i is a local frame, and $s = s^i \sigma_i$ a section, we let

$$\overline{\partial}(s^i \sigma_i) := (\overline{\partial} s^i) \otimes \sigma_i. \tag{1.1}$$

Indeed, given another frame τ_j related by $\sigma_i = g_{ij}\tau_j$, we have

$$\overline{\partial}(s^i) \otimes \sigma_i = \overline{\partial}(s_i) \otimes g_{ij}\tau_j = \overline{\partial}(g_{ij}s^i) \otimes \tau_j$$

since the transitions g_{ij} are holomorphic by assumption.

A (complex) connection being valued in $\mathcal{A}^1(E, \mathbb{C}) = \mathcal{A}^{1,0}(E) \oplus \mathcal{A}^{0,1}(E)$, we may split $\nabla = \nabla^{1,0} + \nabla^{0,1}$.

Definition 1.3.1. The complex connection ∇ in E is said to be **compatible** with the holomorphic structure if $\nabla^{0,1} = \overline{\partial}$. Suppose E has a hermitian metric h. We say ∇ is **compatible** with h if for any sections e, f, we have equality of forms

$$d(h(e, f)) = h(\nabla e, f) + h(e, \nabla f).$$

More geometrically, this says that h is parallel to the connection, i.e. constant along parallel transport, i.e. the connection has $U(n)$-holonomy. We say ∇ is a **Chern connection** if it is both compatible with the holomorphic structure and the hermitian metric.

Theorem 1.3.2 (Chern)**.** *There exists a unique Chern connection.*

When $E = T_{\mathbb{R}}X$, the Chern connection ought to be regarded as the complex geometric analogue of the Levi-Civita connection from Riemannian geoemtry. In fact this is more than an analogy. If h is a hermitian metric, the Levi-Civita connection of $g = \mathfrak{Re}(h)$ can be complexified to a complex connection. It is a theorem that the Levi-Civita connection is the Chern connection if and only if (X, h) is Kähler.

We can extend the connection $\nabla : \mathcal{A}^0(E, \mathbb{C}) \to \mathcal{A}^1(E, \mathbb{C})$ to a connection

$$\nabla : \mathcal{A}^p(E, \mathbb{C}) \to \mathcal{A}^{p+1}(E, \mathbb{C})$$

for all positive p via

$$\nabla(\omega \otimes s) = d\omega \otimes s + (-1)^p \omega \wedge \nabla s,$$

where ω is a p-form and s is a section of E.

Remark 1.3.3. Note this different to the usual extension of a connection to tensors since we are dealing with skew-symmetric forms. In particular, this satisfied a different Leibniz rule:

$$\nabla(fs) = df \wedge s \otimes d\nabla s.$$

Definition 1.3.4. We define the **curvature** of ∇ to be the composition $\nabla^2 = \nabla \circ \nabla = F_\nabla$.

Note that

$$\begin{aligned}\nabla(\nabla fs) = \nabla(df \otimes s + f\nabla s) &= ddf - df \wedge \nabla s + \nabla(f\nabla s)\\ &= -df \wedge \nabla s + df \wedge \nabla s + f\nabla^2 s = f\nabla^2 s\end{aligned}$$

so that F_∇ is C_X^∞-linear, that is a section of $\mathcal{A}^2(End(E), \mathbb{C})$.

We may also define

$$F_\nabla^k := \underbrace{F_\nabla \circ \cdots \circ F_\nabla}_{k} \in \mathcal{A}^{2k}(End(E), \mathbb{C}).$$

We define the k**th Chern character** of ∇ to be

$$\mathrm{ch}_k(E, \nabla) := \mathrm{Tr}\left(\frac{1}{k!}\left(\frac{\iota}{2\pi}F_\nabla^k\right)\right) \in A^{2k}(X, \mathbb{C}).$$

Theorem 1.3.5 (Chern-Weil)**.** *The following is true about the Chern character.*

1. *$ch_k(E, \nabla)$ is closed;*
2. *The cohomology class $ch_k(E) := [ch_k(E, \nabla)] \in H_{dR}^{2k}(X, \mathbb{C})$ is independent of ∇;*
3. *$ch_k(E)$ is real, i.e. in $H_{dR}^{2k}(X, \mathbb{R})$ (in fact, it is integral);*
4. *The total Chern character $\sum_k ch_k(E)$ is equal to the cohomology class of $Tr(\exp(\frac{\iota}{2\pi}F_\nabla)$ (this one directly follows from developing the exponential).*

2. Second Lecture

2.1. **Chern classes.** Let V be a vector space over $\mathbb{C}$ of dimension r. Let $P \in \mathbb{C}[End(V)]$ be a homogeneous polynomial of degree k. Assume moreover P is $GL(V)$ invariant, that is $P(A^{-1}BA) = P(B)$ for any $A \in GL(V)$.

Let now E be a complex vector bundle and ∇ a connection. By $GL(V)$ invariance, $P(\frac{\iota}{2\pi}F_\nabla)$ is well-defined, and lives in $A^{2k}(X, \mathbb{C})$.

Fact 2.1.1 (Chern-Weil)**.** $P(\frac{\iota}{2\pi}F_\nabla)$ is closed, and the class $[P(\frac{\iota}{2\pi}F_\nabla)] \in H^{2k}(X, \mathbb{C})$ is independent of ∇.

Consider now the $GL(V)$-invariant homogeneous polynomials P_k returning the coefficients of the characteristic polynomials (i.e. P_k kth elementary symmetric polynomial on the eigenvalues). We can explicitely define P_k by the formula:

$$\det(I + tB) = \sum_k P_k(B)t^k.$$

We define the k**th Chern class** of E to be the cohomology class of $P_k(\frac{i}{2\pi}F_\nabla)$, and the **total Chern class** of E to be $c(E) := \sum_{i=0}^k c_k(E)$.

The Chern classes and characters satisfy certain properties:

- $c_0(E) = 1$ and $ch_0 = r$;
- $c_d = 0$ if $d > r$. In particular, if L is a line bundle, $c(L) = 1 + c_1(L)$;
- $ch(L) = \exp(c_1(L)) := 1 + c_1(L) + \frac{c_1(L)^2}{2!} + \cdots \in H^{2\bullet}(X, \mathbb{C})$.

To derive further elementary properties of the Chern characters and classes, let us first observe how we can assemble connections into new connections

Let E_1 and E_2 be vector bundles with respective (complex) connections ∇_1 and ∇_2. Then,

- $\nabla_{E_1 \oplus E_2} := \nabla_1 \oplus \nabla_2$ is a connection on $E_1 \oplus E_2$, and $F_{\nabla_1 \oplus \nabla_2} = F_{\nabla_1} \oplus F_{\nabla_2}$, where this is seen as a block matrix
$$\begin{pmatrix} F_{\nabla_1} & \\ & F_{\nabla_2}\end{pmatrix};$$
- $\nabla_{E_1 \otimes E_2} := \nabla_1 \otimes \mathrm{id}_{E_2} + \mathrm{id}_{E_1} \otimes \nabla_2$ is a connection on $E_1 \otimes E_2$;

- the assignment
$$(\nabla^\vee \phi)(s) := d(\phi(s)) - \phi(\nabla(s))$$
where $s \in E$ and $\phi \in E^\vee$ defines a connection on $E^\vee$ (note that this is the usual way to extend connections on tensors). In other words, if $\langle -, - \rangle$ denotes the natural pairing on $E \otimes E^\vee$, the dual connection is defined by
$$d\langle s, \phi \rangle = \langle \nabla s, \phi \rangle + \langle s, \nabla^\vee \phi \rangle.$$
Let us try to compare the two curvature. Given s_i ant t_j frames of E, we consider the connexion form $A = (A_i^{\ j})$ satisfying $\nabla s_i = A_i^{\ j} \otimes t_j$. Let s^i and t^j be the dual frames. We obtain
$$\begin{aligned} d\langle s_i, t^j \rangle = 0 &= \langle \nabla s_i, t^j \rangle + \langle s_i, \nabla^\vee t^j \rangle \\ &= \langle A_i^{\ k} \otimes t_k, t^j \rangle + \langle s_i, B^j_{\ k} \otimes s^k \rangle \\ &= A_i^{\ j} + B^j_{\ i}, \end{aligned}$$
where $B = (B^j_{\ i})$ is the connection form of $\nabla^\vee$. And so we have $B = -A^t$ as sections of $\mathcal{A}^2(End(E), \mathbb{C}) = \mathcal{A}^2(End(E^\vee), \mathbb{C})$. Using Cartan's formula for the curvature of a connection, we conclude
$$F_{\nabla^\vee} = d(-A^t) + (-A^t) \wedge (-A^t) = -(dA + A \wedge A)^t = -F_\nabla^t.$$
- connections pull back, that is if $f : Y \to X$ is a smooth map and E is a bundle on X with connection ∇_E, we may define the connection ∇_{f^*E} by *locally* demanding
$$\nabla_{f^*E}(f^* s) = f^* \nabla s.$$

Corollary 2.1.2. *Let E_1, E_2 be complex vector bundles on X. The following hold:*
1. $ch(E_1 \oplus E_2)) = ch(E_1) + ch(E_2)$ *and* $c(E_1 \oplus E_2) = c(E_1) \cup c(E_2)$;
2. $ch(E_1 \otimes E_2) = ch(E_1) \cup ch(E_2)$;
3. $c_k(E^\vee) = (-1)^k c_k(E)$
4. $ch_k(f^*E) = f^*(ch_k(E))$ *and and* $c_k(f^*E) = f^*(c_k(E))$.

Note that $c_k(E) \in H^{2k}(X, \mathbb{C})$ is in fact real: indeed, conjugation acts on via $\overline{F_\nabla} = F_{\overline{\nabla}}$, and up to choosing a hermitian metric, we have $E^\vee \simeq \overline{E}$ and $F_{\overline{\nabla}} = -F_\nabla^t$, and in a Chern splitting, the eigenvalues $\omega_1, \dots, \omega_r$ are purely imaginary. We obtain
$$c_k(E) = P_k(\frac{\iota}{2\pi} F_\nabla) = \left[\frac{\iota^k}{(2\pi)^k} P_k(F_\nabla) \right] = \left[\frac{\iota^k}{(2\pi)^k} \sigma_k(\omega_1, \dots, \omega_r) \right]$$
while
$$\overline{c_k(E)} = \left[\frac{(-1)^k \iota^k}{(2\pi)^k} \sigma_k(-\omega_1, \dots, -\omega_r) \right] = \left[\frac{(-1)^{2k} \iota^k}{2\pi} \sigma_k(\omega_1, \dots, \omega_r) \right] = c_k(E)$$
where σ_k denotes the kth standard symmetric polynomial, so that $c_k(E) \in H^{2k}(X, \mathbb{R})$. The kth Chern class is also $(1,1)$. Indeed, consider the Chern connection $\nabla = \nabla^{1,0} + \nabla^{0,1} = \nabla^{1,0} + \overline{\partial}$. From this decomposition, we see that the $(0,2)$-part of the curvature is $\overline{\partial}^2 = 0$. Similarly, one can use the fact that the hermitian metric is parallel to show that the $(2,0)$ part vanishes so that all ω_i are of type $(1,1)$, from which one obtains that $c_k(E)$ is of type (k,k).

Let $D \subset X$ be a divisor. It is a fact that the fundamental class of D is the Chern class of $\mathcal{O}(D)$, i.e.
$$[D] = c_1(\mathcal{O}(D)) \in H^2(X, \mathbb{C}),$$
showing that the first — and therefore fore any — Chern class is integral.

An important theorem relating to chern classes is Kodaira's embedding theorem.

Theorem 2.1.3 (Kodaira). *A (holomorphic) line bundle L on a complex manifold X is ample (i.e. induces an embedding in projective space) if and only if it admits a metric h such that $\frac{\iota}{2\pi} F_{D_h}$ is a positive form.*

In particular, let h_0 be any hermitian metric on L, and let $\omega_0 = \frac{\iota}{2\pi} F_{D_{h_0}}$. If $c_1(L) = [\omega_0]$ is positive, i.e. has a positive form ω as representative of the cohomology class, then we can write $\omega = \omega_0 + \partial\overline{\partial}\phi$ for some function ϕ by the $\partial\overline{\partial}$-lemma. Then, one may compute that the metric $h := e^\phi \cdot h_0$ satisfies $\frac{\iota}{2\pi} F_{D_h} = \omega$ (this is because the $(1,0)$-part of the Chern connection D_h is $h^{-1}\partial h = \partial \log(h)$). In particular, a line bundle L is ample if and only if $c_1(L)$ is positive, i.e. is represented by a Kähler form. Now, on a compact manifold, slightly perturbing a Kähler form inside $H^{1,1}(X, \mathbb{R})$ still yields a Kähler form, since it preserves the positivity criterion. Thus, Kähler forms form an open positive cone $\mathcal{K}_X$ inside of $H^{1,1}(X, \mathbb{R})$ (scaling by a positive real preserves Kählerness). Moreover, by the Lefschetz theorem on $(1,1)$ classes, the map Chern map $\mathrm{Pic}(X) \to H^{1,1}(X,\mathbb{Z})$ is surjective. Thus, we conclude that a compact complex manifold is projective if and only if $\mathcal{K}_X$ intersects with $H^{1,1}(X,\mathbb{Z})$ inside of $H^{1,1}(X,\mathbb{R})$.

2.2. Hirzebruch-Riemann-Roch.

Definition 2.2.1. Let X be a compact complex manifold. Let ∇ be a connection in the tangent bundle. We define the **Todd class** of X to be

$$td(X) := \left[\det\left(\frac{\frac{\iota}{2\pi}F_\nabla}{1-\exp(\frac{-\iota}{2\pi}F_\nabla)}\right)\right] \in H^{2\bullet}(X,\mathbb{C})$$

In terms of Chern roots $\omega_1,\dots,\omega_r$, we have that

$$td(X) = \prod_{i=1}^{r} \frac{\omega_i}{1-e^{-\omega_i}}$$

It can be computed that we have

$$td_0(X) = 1;\quad td_1(X) = \frac{c_1}{2};\quad td_2(X) = \frac{1}{12}(c_1^2+c_2);\quad td_3(X) = \frac{c_1c_2}{24}\quad td_4(X); = \frac{-c_1^4+4c_2+c_1c_3+3c_2^2-c_4}{720};\quad \cdots$$

where $td_k(X)$ denotes the kth homogeneous component of $td(X)$ and $c_i = c_i(X) := c_i(\mathcal{T}_X)$.

Theorem 2.2.2 (Hirzebruch-Rieman-Roch)**.** *Let E be a holomorphic vector bundle on a compact complex manifold X. Then, we have equality*

$$\chi(X,E) := \sum_k (-1)^k h^i(X,E) = \int_X ch(E)\cup td(X). \tag{2.1}$$

Note that since we are integrating over X, we only need to consider the top degree parts.

Example 2.2.3. Let $X = C$ be a compact Riemann surface and L be a line bundle on X. we have $ch(E) = 1 + c_1(L)$ and $td(1) = 1 + \frac{c_1(X)}{2}$. thus, we have

$$\chi(X,L) = \int_X 1 + c_1(L) + \frac{c_1(X)}{2} + \frac{c_1(L)c_1(X)}{2} = \int_X c_1(L) + \frac{c_1(X)}{2} = \deg(L) + \frac{\deg(\mathcal{T}_C)}{2}.$$

What is remarkable about this theorem is that the left-hand side of (2.1) is purely holomorphic (or algebraic) whilst the right-hand side is purely topological. Another similar theorem is the algebro-geometric Gauss-Bonnet theorem.

Theorem 2.2.4. *Let X be a compact complex dimension of dimension n. Then,*

$$\chi_{top}(X) := \sum_i (-1)^i b_i(X) = \int_X c_n(X).$$

Recall the classical relation between the Euler characteristic e and the genus g of a topological surface: $\chi_{top} = 2-2g$. In particular, this implies that for a compact Riemann surface C as above, that

$$\int_X c_1(X) = \deg(\mathcal{T}_C) = 2-2g.$$

In particular, in light of what we found in Example 2.2.3, we recover the classical Riemann-Roch theorem:

$$\chi(X,L) = \deg(L) - g + 1.$$

2.3. Kähler-Einstein manifolds.

Question 2.3.1. When does a smooth projective variety over $\mathbb{C}$ admit a "canonical" metric?

Definition 2.3.2. Let (X,ω) be a compact Kähler manifold, and let D_ω be the corresponding Chern connection. We define the **Ricci form** $\mathrm{Ric}(\omega)$ of ω to be

$$\mathrm{Ric}(\omega) = i\mathrm{Tr}(F_{D_\omega}) \in A^2(X,\mathbb{C}).$$

We say that (X,ω) is **Kähler-Einstein** if $\mathrm{Ric}(\omega) = \lambda\omega$ for some constant $\lambda \in \mathbb{R}$.

Remark 2.3.3. We make the following comments.

(1) Recall that we argued earlier that all the Chern roots of F_{D_ω} were pure imaginary of type $(1,1)$ so that $\mathrm{Ric}(\omega)$ is real of type $(1,1)$.
(2) Note also that D_ω is invariant under rescaling ω by some $\lambda > 0$ (indeed, parallelness of h is unaffected so we get the same connection). Thus, we may always assume that $\lambda = -1, 0, 1$.
(3) since $c_1(X) = [\frac{\iota}{2\pi}\mathrm{Tr}F_{D_\omega}]$ by definition, we have $[\mathrm{Ric}(\omega)] = 2\pi c_1(X) \in H^2(X, \mathbb{R})$.
(4) λ is proportional to the scalar curvature, and so (X, ω) being Kähler-Einstain implies that the scalar curvature with respect to $g_\omega = \omega(I-, -)$ is constant.
(5) If X is Kähler-Einstein, then we have

$$c_1(X) = \begin{cases} 0 \\ \pm \text{ positive form.} \end{cases}$$

Definition 2.3.4. We say that a complex manifold X is **Calabi-Yau** if $c_1(X) = 0$, **Fano** if $c_1(X)$ is positive, of **general type** (or **canonically polarised**) if $-c_1(X)$ is positive.

Note that by Kodaira's embedding theorem, Fano and general type manifolds are projective.

Caution 2.3.5. It is not because a manifold fits in this trichotomy that it admits a Kähler-Einstein metric. In fact, there exist Fano varieties with no Kähler-Einstein metric. Whether a Fano variety admits such metric is equivalent to K-stability, a purely algebro-geometric notion. Nonetheless, Yau proved that any Calabi-Yau manifodl admits a Kähler-Einstein metric, and Aubin–Yau proved the same for general type manifolds.

Note also that not all manifolds fit in this trichotomy.

Example 2.3.6 (Curves)**.** Let us see how these categories apply to curves.

- $g = 0$ gives only $\mathbb{P}^1$. Since it is diffeomorphic to a sphere, we have positive scalar curvature. And indeed, the Fubini-Study metric is Kähler-Einstein with $\lambda = 1$. Note also that $\mathbb{P}^1$ is Fano.
- The $g = 1$ case corresponds to elliptic curves. These are Ricci-flat and Calabi-Yau.
- The case $g > 1$ are of general type, and there exists a Kähler-Einstein metric with negative scalar curvature.

For Fano manifolds, here is a summary of the known classifications:

(1) In dimension 1 there is only the projective line.
(2) In dimension 2, they are called *del Pezzo* surfaces. There are 10 different deformation families. First, $\mathbb{P}^2$ and $\mathbb{P}^1 \times \mathbb{P}^1$ which are isolates. The other 8 families are obtained by blowing up $\mathbb{P}^2$ at d points in general position, where $1 \leq d \leq 8$.
(3) In dimension 3, Mukai proved there are 105 families.
(4) In dimension 4, we know there are finitely many families but it remains open to know how many.

For Calabi-Yau manifolds, there is the following structural theorem.

Theorem 2.3.7 (Beauville–Bogomolov)**.** *Let X be Kähler and Calabi-Yau. Then, there exists an étale cover $\tilde{X} \to X$ such that*

$$\tilde{X} = T \times \prod_j X_j \times \prod_i V_i$$

*where T is a torus, X_j is **hyperkähler** for all j, and V_i are **strict Calabi-Yau** for all i.*

We now define the terms.

Definition 2.3.8. A compact Kähler manifold V is called **strict Calabi-Yau** if

- $K_V \simeq \mathcal{O}_V$ is trivial, where K_V denotes the canonical bundle;
- V is simply connected;
- $H^i(V, \mathcal{O}_V) = 0$ for all $i < i < \dim V$.

A complex manifold X is **hyperkähler** if

- it is simply connected;
- $H^0(X, \Omega_X^2) \simeq \mathbb{C}\sigma$ where σ is holomorphic symplectic (in particular, it induces an isomorphism $\mathcal{T}_X \simeq \Omega_X$).

Remark 2.3.9. If V is a strict Calabi-Yau of dimension greater than two, then $h^{2,0} = h^{0,2} = 0$. In particular, $H^{1,1}(X, \mathbb{C}) = H^2(X, \mathbb{C})$, and so $H^{1,1}(X, \mathbb{R}) = H^2(X, \mathbb{R}) = H^{1,1}(X, \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{R}$. Since the Kähler cone is not empty by assumption, we conclude that it intersects $H^{1,1}(X, \mathbb{Z})$, so that V is projective by our discussion on Kodaira's embedding theorem. In dimension 2, non-projective K3 surfaces yield an example of a non-projective strict Calabi-Yau manifold.

References

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