# HYPERKÄHLER MANIFOLDS

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Abstract. These are personal notes for the course on hyperkähler manifolds taught by Alessio Bottini at Universität Bonn in the Winter 2025-2026 semester. Please email me at s37mbrai@uni-bonn.de if you notice any typo.

*Date*: November 1, 2025.

We first review some complex geometry.

**Definition 1.0.1.** A **complex manifold** is a locally ringed space $(X, \mathcal{O}_X)$ such that

- $X$ is Hausdorff and second countable (this part is to ensure we actually have a topological manifold);
- $(X, \mathcal{O}_X)$ is locally isomorphic to $(\Delta, \mathcal{O}_\Delta)$, where $\Delta \subset \mathbb{C}^n$ is the polydisc.

**Example 1.0.2.** Let $f_i, \dots, f_d \in \mathbb{C}[z_1, \dots, z_n]$ be complex polynomials such that the Jacobian of

$$f = (f_1, \dots, f_d) : \mathbb{C}^n \to \mathbb{C}^d$$

has everywhere full rank on the vanishing set $V = V(f) \subset \mathbb{C}^n$. By the holomorphic implicit function theorem (regular value theorem), $V$ is a complex manifold.

**Example 1.0.3.** If $X$ is a smooth algebraic variety over $\mathbb{C}$, we may cover it by affines $V_i$ which are of the same form as in Example 1.0.2. We may consider the analytic topology $X^{an}$ obtained by gluing the different charts $V_i$. Similarly, we may define the sheaf $\mathcal{O}_X^{an}$ on $X^{an}$ by considering the sheaf of holomorphic functions on each $V_i$ (the transitions $V_i \to V_j$ are regular algebraic, hence holomorphic, so that this gluing makes sense). Then, $(X^{an}, \mathcal{O}_X^{an})$ is a complex manifold.

Note that in Example 1.0.3, we obtain a natural map of ringed spaces

$$\alpha : (X^{an}, \mathcal{O}_X^{an}) \to (X, \mathcal{O}_X)$$

since the analytic topology is finer than the Zariski topology, and regular functions are holomorphic. In particular, we obtain a functor between abelian categories:

$$\alpha^* : \mathcal{O}_X\text{-mod} \to \mathcal{O}_X^{an}\text{-mod}$$

restricting to

$$\alpha^* : \mathrm{Coh}(X) \to \mathrm{Coh}(X^{an}).$$

**Theorem 1.0.4** (Géométrie algébrique géométrie analytique; [Ser56])**.** *If $X$ is smooth*[1] *and proper functor $\alpha^* : Coh(X) \to Coh(X^{an})$ is an equivalence, which moreover induces an isomorphism of sheaf cohomology.*

## 1.1. Almost complex structures.

A complex manifold $(X, \mathcal{O}_X)$ has an underlying smooth manifold $(X, C_X^\infty)$, where $C_X^\infty$ denotes the sheaf of smooth functions on $X$; indeed, if $X$ has complex charts $U_i \subset \mathbb{C}^n$, the transitions are holomorphic, hence $C^\infty$.

**Notation 1.1.1.** Since the indices $i$ will be ubiquitous, $\iota$ shall denote the root of $-1$ for these notes (this spares the cumbersome $\sqrt{-1}$ alternative).

On each chart $U_i$, we have multiplication by $\iota$, but this does not globalise, as $\iota$ does not commute with holomorphic functions: in the Taylor expansion, we have terms which are of degree $m$ where $m \neq 1 \mod 4$. However, the differential of $\iota$ may be globalised, as we get rid of the higher order terms. In a local chart $U_i \subset \mathbb{C}^n$, the (real) tangent bundle has a local frame

$$T_\mathbb{R} U_i = \langle \partial_{x_j}, \partial_{y_j} : 1 \leq j \leq n \rangle,$$

on which $I := d\iota$ acts by

$$\begin{cases} \partial_{x_j} \mapsto \partial_{y_j} \\ \partial_{y_j} \mapsto -\partial_{x_j}. \end{cases}$$

**Definition 1.1.2.** An **almost complex structure** on a smooth manifold $X$ is an endomorphism $I \in \mathrm{End}(T_\mathbb{R} X)$ such that $I^2 = -1$. We say that $I$ is integrable if $X$ is a complex manifold and $I$ is obtained by locally differentiating $\iota$.

**Question 1.1.3.** Given $I$ an almost complex structure, when is it integrable?

Let us first set up some tools in order to address this question appropriately. Assume only for now that $X$ is a smooth manifold and $I$ is an almost complex structure. We can consider the complexified tangent bundle

$$T_\mathbb{C} X := T_\mathbb{R} X \otimes_\mathbb{R} \mathbb{C},$$

[1] This can be dropped by considering complex analytic spaces (rather than manifolds).

to which we can extend the action of $I$. Since $I^2 = -1$, the minimal polynomial of $I$ is $x^2+1$, which is separable over $\mathbb{C}$, meaning that $I$ is diagonalisable, with eigenvalues $\pm\iota$. The eigenspaces must have the same dimension as $I$ acts on the *real* tangent space. We thus obtain a decomposition

$$T_{\mathbb{C}}X = T^{1,0}X \oplus T^{0,1}X = T^{1,0}X \oplus \overline{T^{1,0}X}.$$

Note that we have

$$T^{1,0}X = \{(v - iIv) : v \in T_{\mathbb{C}}X\} \qquad\qquad T^{0,1}X = \{(v + iIv) : v \in T_{\mathbb{C}}X\}.$$

**Notation 1.1.4.** We will use the following notation

- $\mathcal{A}^0(X) := C^\infty_X$;
- $\mathcal{A}^k(X)$ denotes the sheaf of (smooth) degree $k$ real forms;
- $\mathcal{A}^k(X,\mathbb{C}) = \oplus_{p+q=k}\mathcal{A}^{p,q}(X)$ denotes the sheaf of sections of $\bigwedge^k T^*_{\mathbb{C}}X$ (i.e. smooth complex degree $k$ forms) and $\mathcal{A}^{p,q}(X)$ denotes the sheaf of sections of $\bigwedge^p T^{1,0}X \otimes \bigwedge^q T^{0,1}X$;
- $d : \mathcal{A}^k(X,\mathbb{C}) \to \mathcal{A}^{k+1}(X,\mathbb{C})$ denotes the complexification of the usual exterior derivative, and can be decomposed by types as $d = \partial + \bar\partial$, where $\partial$ denotes the part corresponding to the differentiation in holomorphic coordinates, and similarly $\bar\partial$ for anti-holomorphic coordinates.
- $A^k(X)$, $A^k(X,\mathbb{C})$, and $A^{p,q}(X)$ denotes the global sections of $\mathcal{A}^k(X), \mathcal{A}^k(X,\mathbb{C})$, and $\mathcal{A}^{p,q}(X)$ respectively.
- $\mathcal{T}_X$ denotes the sheaf of homolorphic vector fields, i.e. $\mathcal{T}_X := \mathcal{D}er(\mathcal{O}_X, \mathcal{O}_X)$;
- $\Omega_X := \mathcal{T}_X^*$ denotes the cotangent sheaf.

The following theorem answers Question 1.1.3.

**Theorem 1.1.5** (Newlander-Niremberg)**.** *$I$ is integrable if and only if $\bar\partial^2 = 0$.*

Note that this is equivalent to $T^{1,0}X$ being closed under the (complex) Lie bracket (this is much related to the Frobenius theorem of differential geometry), and also equivalent of the vanishing of a certain tensor $N_I$ called the *Nijenhuis* tensor.

1.2. **Metrics.** Let $E$ be a real vector bundle on $(X, C^\infty_X)$. A **Riemannian metric** $g$ on $E$ is a section of $\mathrm{Sym}^2 E^\vee$ such that for all $p \in X$, $g_p$ is positive definite. If $E$ is a complex bundle, a **Hermitian metric** $h$ is a map of sheaves $E \otimes \overline{E} \to C^\infty(X,\mathbb{C})$ such that each $h_p$ is Hermitian, i.e. $h_p(e,f) = \overline{h_p(f,e)}$ and $h_p(e,e) > 0$ for all $e, f \in E_p$. When $E = T_{\mathbb{R}}X$, we say that $g$ (resp. $h$) is a **Riemannian** (resp. **Hermitiam**) **metric on** $X$ (here, the almost complex structure $I$ is used to put a $\mathbb{C}$-structure on $T_{\mathbb{R}}X$).

If $X$ is a complex manifold and $h$ is a Hermitian metric, then we can write

$$h = g - i\omega$$

where $g = \mathfrak{Re}(h)$ and $\omega = -\mathfrak{Im}(h)$. We obtain that $g$ is a Riemannian metric, and $\omega$ is skew-symmetric since

$$\omega(X,Y) = \frac{\iota}{2}(h - \overline{h})$$

and $h$ is conjugate skew-symmetric. Thus, $\omega \in A^2(X)$.

**Definition 1.2.1.** $(X,h)$ is **Kähler** if $d\omega = 0$.

That $h$ is linear in the first variable and anti-linear in the second ensures that $h(I-, I-) = h(-,-)$, implying that $g(I-,I-) = g(-,-)$, a property that is sometimes called **compatibility** of the metric with $I$. We have

$$\omega(-,-) = \frac{\iota}{2}(h(-,-) - \overline{h}(-,-)) = \frac{1}{2}(h(I-,-) + \overline{h}(I-,-)) = g(I-,-),$$

which also implies

$$\omega(-,I-) = g(-,-).$$

**Definition 1.2.2.** A form $\omega \in A^2(X)$ is called **positive** if $\omega(u, Iu) > 0$ for all $u \in T_{\mathbb{R}}X$. We see that a de Rham cohomology class in $H^2(X,\mathbb{C})$ is **positive** if it can be represented by a positive form. If moreover $\omega$ is $I$-invariant (or equivalently, of type $(1,1)$ after embedding $A^2(X) \subset A(X,\mathbb{C})$), we say $\omega$ is **Kähler**.

If $\omega$ is Kähler, we may define the hermitian metric $h_\omega = \omega(-,I-) - i\omega$, and we have that $\omega$ is Kähler if and only if $(X, h_\omega)$ is Kähler.

**Example 1.2.3.** Let $X = \mathbb{P}^n$, with projective coordinates $Z_0, \dots, Z_n$. Let $U_i$ be the $Z_i \neq 0$ chart, and define $z_j = \frac{Z_j}{Z_i}$. We may define on $U_i$ the metric

$$\omega_{FS} = \omega = i\partial\overline{\partial} \log \left( 1 + \sum_j z_j \overline{z}_j \right),$$

and one checks that these glue to a global form, which we call the **Fubini-Study metric**. Written as a Kähler potential this way shows that it is a Kähler metric.

Note that if $(X, \omega)$ is Kähler, restricting the metric to a complex submanifold $Y$ preserves all properties of Definition 1.2.2, and so $(Y, \omega_Y)$ is Kähler. Thus, any projective manifold is Kähler.

1.3. **Connections.** Let $E$ be a complex (the real case is identical) vector bundle on $(X, C^\infty_X)$. A **complex connection** in $E$ is a $\mathbb{C}$-linear map

$$\nabla : \mathcal{A}^0(E, \mathbb{C}) \to \mathcal{A}^1(E, \mathbb{C}),$$

(here $\mathcal{A}^i(E, \mathbb{C}) = \mathcal{A}^i(X, \mathbb{C}) \otimes \Gamma(E)$) such that

$$\nabla(f \cdot s) = df \otimes s + f \cdot \nabla s$$

for all section $s$ of $E$ and $f \in C^\infty_X$.

If $E$ is a holomorphic bundle on a complex manifold, we can define the operator

$$\overline{\partial} : \mathcal{A}^0(E) \to \mathcal{A}^{0,1}(E)$$

as follows: if $\sigma_i$ is a local frame, and $s = s^i \sigma_i$ a section, we let

$$\overline{\partial}(s^i \sigma_i) := (\overline{\partial} s^i) \otimes \sigma_i. \tag{1.1}$$

Indeed, given another frame $\tau_j$ related by $\sigma_i = g_{ij}\tau_j$, we have

$$\overline{\partial}(s^i) \otimes \sigma_i = \overline{\partial}(s_i) \otimes g_{ij}\tau_j = \overline{\partial}(g_{ij}s^i) \otimes \tau_j$$

since the transitions $g_{ij}$ are holomorphic by assumption.

A (complex) connection being valued in $\mathcal{A}^1(E, \mathbb{C}) = \mathcal{A}^{1,0}(E) \oplus \mathcal{A}^{0,1}(E)$, we may split $\nabla = \nabla^{1,0} + \nabla^{0,1}$.

**Definition 1.3.1.** The complex connection $\nabla$ in $E$ is said to be **compatible** with the holomorphic structure if $\nabla^{0,1} = \overline{\partial}$. Suppose $E$ has a hermitian metric $h$. We say $\nabla$ is **compatible** with $h$ if for any sections $e, f$, we have equality of forms

$$d(h(e, f)) = h(\nabla e, f) + h(e, \nabla f).$$

More geometrically, this says that $h$ is parallel to the connection, i.e. constant along parallel transport, i.e. the connection has $U(n)$-holonomy. We say $\nabla$ is a **Chern connection** if it is both compatible with the holomorphic structure and the hermitian metric.

**Theorem 1.3.2** (Chern)**.** *There exists a unique Chern connection.*

When $E = T_\mathbb{R} X$, the Chern connection ought to be regarded as the complex geometric analogue of the Levi-Civita connection from Riemannian geometry. In fact this is more than an analogy. If $h$ is a hermitian metric, the Levi-Civita connection of $g = \mathfrak{Re}(h)$ can be complexified to a complex connection. It is a theorem that the Levi-Civita connection is the Chern connection if and only if $(X, h)$ is Kähler.

We can extend the connection $\nabla : \mathcal{A}^0(E, \mathbb{C}) \to \mathcal{A}^1(E, \mathbb{C})$ to a connection

$$\nabla : \mathcal{A}^p(E, \mathbb{C}) \to \mathcal{A}^{p+1}(E, \mathbb{C})$$

for all positive $p$ via

$$\nabla(\omega \otimes s) = d\omega \otimes s + (-1)^p \omega \wedge \nabla s,$$

where $\omega$ is a $p$-form and $s$ is a section of $E$.

**Remark 1.3.3.** Note that this is different from the usual extension of a connection to tensors since we are dealing with skew-symmetric forms. In particular, this satisfies a different Leibniz rule:

$$\nabla(fs) = df \wedge s \otimes d\nabla s.$$

**Definition 1.3.4.** We define the **curvature** of $\nabla$ to be the composition $\nabla^2 = \nabla \circ \nabla = F_\nabla$.

Note that

$$\begin{aligned}\nabla(\nabla f s) = \nabla(df \otimes s + f\nabla s) &= ddf - df \wedge \nabla s + \nabla(f\nabla s) \\ &= -df \wedge \nabla s + df \wedge \nabla s + f\nabla^2 s = f\nabla^2 s\end{aligned}$$

so that $F_\nabla$ is $C^\infty_X$-linear, that is a section of $\mathcal{A}^2(End(E), \mathbb{C})$.

We may also define

$$F^k_\nabla := \underbrace{F_\nabla \circ \cdots \circ F_\nabla}_{k} \in \mathcal{A}^{2k}(End(E), \mathbb{C}).$$

We define the $k$**th Chern character** of $\nabla$ to be

$$\mathrm{ch}_k(E, \nabla) := \mathrm{Tr}\left(\frac{1}{k!}\left(\frac{\iota}{2\pi}F^k_\nabla\right)\right) \in A^{2k}(X, \mathbb{C}).$$

**Theorem 1.3.5** (Chern-Weil)**.** *The following is true about the Chern character.*

1. $ch_k(E, \nabla)$ *is closed;*
2. *The cohomology class* $ch_k(E) := [ch_k(E, \nabla)] \in H^{2k}_{dR}(X, \mathbb{C})$ *is independent of* $\nabla$*;*
3. $ch_k(E)$ *is real, i.e. in* $H^{2k}_{dR}(X, \mathbb{R})$ *(in fact, it is integral);*
4. *The total Chern character* $\sum_k ch_k(E)$ *is equal to the cohomology class of* $Tr(\exp(\frac{\iota}{2\pi}F_\nabla))$ *(this one directly follows from developing the exponential).*

2.1. **Chern classes.** Let $V$ be a vector space over $\mathbb{C}$ of dimension $r$. Let $P \in \mathbb{C}[End(V)]$ be a homogeneous polynomial of degree $k$. Assume moreover $P$ is $GL(V)$ invariant, that is $P(A^{-1}BA) = P(B)$ for any $A \in GL(V)$.

Let now $E$ be a complex vector bundle and $\nabla$ a connection. By $GL(V)$ invariance, $P(\frac{\iota}{2\pi}F_\nabla)$ is well-defined, and lives in $A^{2k}(X, \mathbb{C})$.

**Fact 2.1.1** (Chern-Weil)**.** $P(\frac{\iota}{2\pi}F_\nabla)$ is closed, and the class $[P(\frac{\iota}{2\pi}F_\nabla)] \in H^{2k}(X, \mathbb{C})$ is independent of $\nabla$.

Consider now the $GL(V)$-invariant homogeneous polynomials $P_k$ returning the coefficients of the characteristic polynomials (i.e. $P_k$ $k$th elementary symmetric polynomial on the eigenvalues). We can explicitly define $P_k$ by the formula:

$$\det(I + tB) = \sum_k P_k(B)t^k.$$

We define the $k$**th Chern class** of $E$ to be the cohomology class of $P_k(\frac{i}{2\pi}F_\nabla)$, and the **total Chern class** of $E$ to be $c(E) := \sum_{i=0}^k c_k(E)$.

The Chern classes and characters satisfy certain properties:

- $c_0(E) = 1$ and $ch_0 = r$;
- $c_d = 0$ if $d > r$. In particular, if $L$ is a line bundle, $c(L) = 1 + c_1(L)$;
- $ch(L) = \exp(c_1(L)) := 1 + c_1(L) + \frac{c_1(L)^2}{2!} + \cdots \in H^{2\bullet}(X, \mathbb{C})$.

To derive further elementary properties of the Chern characters and classes, let us first observe how we can assemble connections into new connections

Let $E_1$ and $E_2$ be vector bundles with respective (complex) connections $\nabla_1$ and $\nabla_2$. Then,

- $\nabla_{E_1 \oplus E_2} := \nabla_1 \oplus \nabla_2$ is a connection on $E_1 \oplus E_2$, and $F_{\nabla_1 \oplus \nabla_2} = F_{\nabla_1} \oplus F_{\nabla_2}$, where this is seen as a block matrix
$$\begin{pmatrix} F_{\nabla_1} & \\ & F_{\nabla_2} \end{pmatrix};$$
- $\nabla_{E_1 \otimes E_2} := \nabla_1 \otimes \mathrm{id}_{E_2} + \mathrm{id}_{E_1} \otimes \nabla_2$ is a connection on $E_1 \otimes E_2$;
- the assignment
$$(\nabla^\vee \phi)(s) := d(\phi(s)) - \phi(\nabla(s))$$
where $s \in E$ and $\phi \in E^\vee$ defines a connection on $E^\vee$ (note that this is the usual way to extend connections on tensors). In other words, if $\langle -, - \rangle$ denotes the natural pairing on $E \otimes E^\vee$, the dual connection is defined by
$$d\langle s, \phi \rangle = \langle \nabla s, \phi \rangle + \langle s, \nabla^\vee \phi \rangle.$$
Let us try to compare the two curvatures. Given $s_i$ and $t_j$ frames of $E$, we consider the connection form $A = (A_i^{\ j})$ satisfying $\nabla s_i = A_i^{\ j} \otimes t_j$. Let $s^i$ and $t^j$ be the dual frames. We obtain
$$\begin{aligned} d\langle s_i, t^j \rangle = 0 &= \langle \nabla s_i, t^j \rangle + \langle s_i, \nabla^\vee t^j \rangle \\ &= \langle A_i^{\ k} \otimes t_k, t^j \rangle + \langle s_i, B^j_{\ k} \otimes s^k \rangle \\ &= A_i^{\ j} + B^j_{\ i}, \end{aligned}$$
where $B = (B^j_{\ i})$ is the connection form of $\nabla^\vee$. And so we have $B = -A^t$ as sections of $\mathcal{A}^2(End(E), \mathbb{C}) = \mathcal{A}^2(End(E^\vee), \mathbb{C})$. Using Cartan's formula for the curvature of a connection, we conclude
$$F_{\nabla^\vee} = d(-A^t) + (-A^t) \wedge (-A^t) = -(dA + A \wedge A)^t = -F_\nabla^t.$$
- connections pull back, that is if $f : Y \to X$ is a smooth map and $E$ is a bundle on $X$ with connection $\nabla_E$, we may define the connection $\nabla_{f^*E}$ by *locally* demanding
$$\nabla_{f^*E}(f^*s) = f^*\nabla s.$$

**Corollary 2.1.2.** *Let $E_1, E_2$ be complex vector bundles on $X$. The following hold:*

1. $ch(E_1 \oplus E_2)) = ch(E_1) + ch(E_2)$ *and* $c(E_1 \oplus E_2) = c(E_1) \cup c(E_2)$;
2. $ch(E_1 \otimes E_2) = ch(E_1) \cup ch(E_2)$;
3. $c_k(E^\vee) = (-1)^k c_k(E)$
4. $ch_k(f^*E) = f^*(ch_k(E))$ *and and* $c_k(f^*E) = f^*(c_k(E))$.

Note that $c_k(E) \in H^{2k}(X,\mathbb{C})$ is in fact real: indeed, conjugation acts via $\overline{F_\nabla} = F_{\overline{\nabla}}$, and up to choosing a hermitian metric, we have $E^\vee \simeq \overline{E}$ and $F_{\overline{\nabla}} = -F_\nabla^t$, and in a Chern splitting, the eigenvalues $\omega_1,\dots,\omega_r$ are purely imaginary. We obtain

$$c_k(E) = P_k(\frac{\iota}{2\pi}F_\nabla) = \left[\frac{\iota^k}{(2\pi)^k}P_k(F_\nabla)\right] = \left[\frac{\iota^k}{(2\pi)^k}\sigma_k(\omega_1,\dots,\omega_r)\right]$$

while

$$\overline{c_k(E)} = \left[\frac{(-1)^k\iota^k}{(2\pi)^k}\sigma_k(-\omega_1,\dots,-\omega_r)\right] = \left[\frac{(-1)^{2k}\iota^k}{2\pi}\sigma_k(\omega_1,\dots,\omega_r)\right] = c_k(E)$$

where $\sigma_k$ denotes the $k$th standard symmetric polynomial, so that $c_k(E) \in H^{2k}(X,\mathbb{R})$. The $k$th Chern class is also $(1,1)$. Indeed, consider the Chern connection $\nabla = \nabla^{1,0} + \nabla^{0,1} = \nabla^{1,0} + \overline{\partial}$. From this decomposition, we see that the $(0,2)$-part of the curvature is $\overline{\partial}^2 = 0$. Similarly, one can use the fact that the hermitian metric is parallel to show that the $(2,0)$ part vanishes so that all $\omega_i$ are of type $(1,1)$, from which one obtains that $c_k(E)$ is of type $(k,k)$.

Let $D \subset X$ be a divisor. It is a fact that the fundamental class of $D$ is the Chern class of $\mathcal{O}(D)$, i.e.

$$[D] = c_1(\mathcal{O}(D)) \in H^2(X,\mathbb{C}),$$

showing that the first — and therefore any — Chern class is integral.

An important theorem relating to chern classes is Kodaira's embedding theorem.

**Theorem 2.1.3** ([Kod54]). *A (holomorphic) line bundle $L$ on a complex manifold $X$ is ample (i.e. induces an embedding in projective space) if and only if it admits a metric $h$ such that $\frac{\iota}{2\pi}F_{D_h}$ is a positive form.*

In particular, let $h_0$ be any hermitian metric on $L$, and let $\omega_0 = \frac{\iota}{2\pi}F_{D_{h_0}}$. Assuming $X$ is Kähler, if $c_1(L) = [\omega_0]$ is positive, i.e. if it has a positive form $\omega$ as representative of the cohomology class, then we can write $\omega = \omega_0 + \frac{\iota}{2\pi}\partial\overline{\partial}\phi$ for some function $\phi$ by the $\partial\overline{\partial}$-lemma. Then, one may compute that the metric $h := e^{-\phi}\cdot h_0$ satisfies $\frac{\iota}{2\pi}F_{D_h} = \omega$; indeed the $(1,0)$-part of the Chern of the connection is

$$h^{-1}\partial h = \partial\log h = \partial\log(e^{-\phi}h_0) = \partial\log h_0 + \partial(-\phi)$$

so that the curvature is given by

$$F_{D_{h_0}} + \overline{\partial}\partial(-\phi) = F_{D_{h_0}} + \partial\overline{\partial}\phi.$$

In particular, a line bundle $L$ is ample if and only if $c_1(L)$ is positive, i.e. is represented by a Kähler form. Now, on a compact manifold, slightly perturbing a Kähler form inside $H^{1,1}(X,\mathbb{R})$ still yields a Kähler form, since it preserves the positivity criterion. Thus, Kähler forms form an open positive cone $\mathcal{K}_X$ inside of $H^{1,1}(X,\mathbb{R})$ (scaling by a positive real preserves Kählerness). Moreover, by the Lefschetz theorem on $(1,1)$ classes, the Chern map $\mathrm{Pic}(X) \to H^{1,1}(X,\mathbb{Z})$ is surjective. Thus, we conclude that a compact complex manifold is projective if and only if $\mathcal{K}_X$ intersects with $H^{1,1}(X,\mathbb{Z})$ (or equivalently $H^{1,1}(X,\mathbb{Q})$) inside of $H^{1,1}(X,\mathbb{R})$.

## 2.2. Hirzebruch-Riemann-Roch.

**Definition 2.2.1.** Let $X$ be a compact complex manifold. Let $\nabla$ be a connection in the tangent bundle. We define the **Todd class** of $X$ to be

$$td(X) := \left[\det\left(\frac{\frac{\iota}{2\pi}F_\nabla}{1-\exp(\frac{-\iota}{2\pi}F_\nabla)}\right)\right] \in H^{2\bullet}(X,\mathbb{C})$$

In terms of Chern roots $\omega_1,\dots,\omega_r$, we have that

$$td(X) = \prod_{i=1}^r \frac{\omega_i}{1-e^{-\omega_i}}$$

It can be computed that we have

$$td_0(X) = 1;\quad td_1(X) = \frac{c_1}{2};\quad td_2(X) = \frac{1}{12}(c_1^2 + c_2);\quad td_3(X) = \frac{c_1c_2}{24}\quad td_4(X); = \frac{-c_1^4 + 4c_2 + c_1c_3 + 3c_2^2 - c_4}{720};\quad \cdots$$

where $td_k(X)$ denotes the $k$th homogeneous component of $td(X)$ and $c_i = c_i(X) := c_i(\mathcal{T}_X)$.

**Theorem 2.2.2** (Hirzebruch-Rieman-Roch)**.** *Let $E$ be a holomorphic vector bundle on a compact complex manifold $X$. Then, we have equality*

$$\chi(X,E) := \sum_k (-1)^k h^i(X,E) = \int_X ch(E) \cup td(X). \tag{2.1}$$

Note that since we are integrating over $X$, we only need to consider the top degree parts.

**Example 2.2.3.** Let $X = C$ be a compact Riemann surface and $L$ be a line bundle on $X$. we have $ch(E) = 1 + c_1(L)$ and $td(1) = 1 + \frac{c_1(X)}{2}$. thus, we have

$$\chi(X,L) = \int_X 1 + c_1(L) + \frac{c_1(X)}{2} + \frac{c_1(L)c_1(X)}{2} = \int_X c_1(L) + \frac{c_1(X)}{2} = \deg(L) + \frac{\deg(\mathcal{T}_C)}{2}.$$

What is remarkable about this theorem is that the left-hand side of (2.1) is purely holomorphic (or algebraic) whilst the right-hand side is purely topological. Another similar theorem is the algebro-geometric Gauss-Bonnet theorem.

**Theorem 2.2.4.** *Let $X$ be a compact complex dimension of dimension $n$. Then,*

$$\chi_{top}(X) := \sum_i (-1)^i b_i(X) = \int_X c_n(X).$$

Recall the classical relation between the Euler characteristic $\chi_{top}$ and the genus $g$ of a topological surface: $\chi_{top} = 2 - 2g$. In particular, this implies for a compact Riemann surface $C$ as above, that

$$\int_X c_1(X) = \deg(\mathcal{T}_C) = 2 - 2g.$$

In particular, in light of what we found in Example 2.2.3, we recover the classical Riemann-Roch theorem:

$$\chi(X,L) = \deg(L) - g + 1.$$

## 2.3. **Kähler-Einstein manifolds.**

**Question 2.3.1.** When does a smooth projective variety over $\mathbb{C}$ admit a "canonical" metric?

**Definition 2.3.2.** Let $(X,\omega)$ be a compact Kähler manifold, and let $D_\omega$ be the corresponding Chern connection. We define the **Ricci form** $\mathrm{Ric}(\omega)$ of $\omega$ to be

$$\mathrm{Ric}(\omega) = i\mathrm{Tr}(F_{D_\omega}) \in A^2(X,\mathbb{C}).$$

We say that $(X,\omega)$ is **Kähler-Einstein** if $\mathrm{Ric}(\omega) = \lambda\omega$ for some constant $\lambda \in \mathbb{R}$.

**Remark 2.3.3.** We make the following comments.

(1) Recall that we argued earlier that all the Chern roots of $F_{D_\omega}$ were pure imaginary of type $(1,1)$ so that $\mathrm{Ric}(\omega)$ is real of type $(1,1)$.
(2) Note also that $D_\omega$ is invariant under rescaling $\omega$ by some $\lambda > 0$ (indeed, parallelness of $h$ is unaffected so we get the same connection). Thus, we may always assume that $\lambda = -1, 0, 1$.
(3) since $c_1(X) = [\frac{\iota}{2\pi}\mathrm{Tr}F_{D_\omega}]$ by definition, we have $[\mathrm{Ric}(\omega)] = 2\pi c_1(X) \in H^2(X,\mathbb{R})$.
(4) $\lambda$ is proportional to the scalar curvature, and so $(X,\omega)$ being Kähler-Einstein implies that the scalar curvature with respect to $g_\omega = \omega(I-,-)$ is constant.
(5) If $X$ is Kähler-Einstein, then we have

$$c_1(X) = \begin{cases} 0 \\ \pm \text{ positive form.} \end{cases}$$

**Definition 2.3.4.** We say that a complex manifold $X$ is **Calabi-Yau** if $c_1(X) = 0$, **Fano** if $c_1(X)$ is positive, of **general type** (or **canonically polarised**) if $-c_1(X)$ is positive.

Note that by Kodaira's embedding theorem, Fano and general type manifolds are projective.

**Caution 2.3.5.** It is not because a manifold fits in this trichotomy that it admits a Kähler-Einstein metric. In fact, there exist Fano varieties with no Kähler-Einstein metric. Whether a Fano variety admits such metric is equivalent to $K$-stability, a purely algebro-geometric notion. Nonetheless, Yau (cf. [Yau78]) proved that any Calabi-Yau manifold admits a Kähler-Einstein metric, and Aubin–Yau (cf. [Aub76; Yau78]) proved the same for general type manifolds.

Note also that not all manifolds fit in this trichotomy.

**Example 2.3.6** (Curves)**.** Let us see how these categories apply to curves.

- $g = 0$ gives only $\mathbb{P}^1$. Since it is diffeomorphic to a sphere, we have positive scalar curvature. And indeed, the Fubini-Study metric is Kähler-Einstein with $\lambda = 1$. Note also that $\mathbb{P}^1$ is Fano.
- The $g = 1$ case corresponds to elliptic curves. These are Ricci-flat and Calabi-Yau.
- The case $g > 1$ are of general type, and there exists a Kähler-Einstein metric with negative scalar curvature.

For Fano manifolds, here is a summary of the known classifications:

(1) In dimension 1 there is only the projective line.
(2) In dimension 2, they are called *del Pezzo* surfaces. There are 10 different deformation families. First, $\mathbb{P}^2$ and $\mathbb{P}^1 \times \mathbb{P}^1$ which are isolates. The other 8 families are obtained by blowing up $\mathbb{P}^2$ at d points in general position, where $1 \leq d \leq 8$.
(3) In dimension 3, Mukai proved there are 105 families.
(4) In dimension 4, we know there are finitely many families but it remains open to know how many.

For Calabi-Yau manifolds, there is the following structural theorem.

**Theorem 2.3.7** (Beauville–Bogomolov; Bog74 and Bea83)**.** *Let $X$ be Kähler and Calabi-Yau. Then, there exists an étale cover $\tilde{X} \to X$ such that*

$$\tilde{X} = T \times \prod_j X_j \times \prod_i V_i$$

*where $T$ is a torus, $X_j$ is* ***hyperkähler*** *for all $j$, and $V_i$ are* ***strict Calabi-Yau*** *for all $i$.*

We now define the terms.

**Definition 2.3.8.** A compact Kähler manifold $V$ is called **strict Calabi-Yau** if

- $K_V \simeq \mathcal{O}_V$ is trivial, where $K_V$ denotes the canonical bundle;
- $V$ is simply connected;
- $H^i(V, \mathcal{O}_V) = 0$ for all $0 < i < \dim V$.

A complex manifold $X$ is **hyperkähler** if

- it is simply connected;
- $H^0(X, \Omega_X^2) \simeq \mathbb{C}\sigma$ where $\sigma$ is holomorphic symplectic (in particular, it induces an isomorphism $\mathcal{T}_X \simeq \Omega_X$).

**Remark 2.3.9.** If $V$ is a strict Calabi-Yau of dimension greater than two, then $h^{2,0} = h^{0,2} = 0$. In particular, $H^{1,1}(X, \mathbb{C}) = H^2(X, \mathbb{C})$, and so $H^{1,1}(X, \mathbb{R}) = H^2(X, \mathbb{R}) = H^{1,1}(X, \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{R}$. Since the Kähler cone is not empty by assumption, we conclude that it intersects $H^{1,1}(X, \mathbb{Z})$, so that $V$ is projective by our discussion on Kodaira's embedding theorem. In dimension 2, non-projective K3 surfaces yield an example of non-projective strict Calabi-Yau manifolds.

**Remark 2.3.10.** Theorem 2.3.7 has an important corollarly. Namely, if $X$ is a Kähler, we have a degree $n$ étale map $f : \tilde{X} \to X$ with the canonical bunlde $K_{\tilde{X}}$ of $\tilde{X}$ trivial. Since $f$ is étale, $f^*K_X = K_{\tilde{X}}$, which implies that $nK_X = 0$ (this follows from existence of a norm map $\mathcal{O}_{\tilde{X}} \to \mathcal{O}_X$ mapping $f^*g \mapsto g^n$; see [Stacks, Tag 0BD3 and Tag 0BCY] for the algebraic case). Thus, for Kähler manifolds, being Calabi-Yau and that $nK_X = 0$ for some $n \geq 0$ are equivalent conditions.

In this lecture, we recollect Hodge theory.

3.1. **Linear algebra.** Let us first explore the constructions of Hodge theory in the setting of linear algebra, our toy model.

Let $V$ be a real vector space of dimension $n$ and $\langle -,-\rangle$ be an inner product on $V$. The scalar product induces a scalar product on $\bigwedge^k V$ via declaring

$$\langle v_1 \wedge \cdots \wedge v_k, u_1 \wedge \cdots \wedge u_k\rangle := \det(\langle u_i, v_j\rangle)_{ij}.$$

Moreover, if $e_1, \ldots, e_n$ is an *ordered* orthonormal basis of $V$, the vectors $e_{i_1} \wedge \cdots \wedge e_{i_k}$ for $i_1 < \cdots < i_k$ form an orthonormal basis of $\bigwedge^k V$.

**Definition 3.1.1.** The **volume form** of $V$ (with respect to the chosen ordered basis) is

$$\mathrm{vol}_V = \mathrm{vol} := e_1 \wedge \cdots \wedge e_n.$$

For any $k \leq n$, we define the **Hodge operator** to be the map

$$\begin{aligned} * : \bigwedge^k V &\to \bigwedge^{n-k} V \\ e_{i_1} \wedge \cdots \wedge e_{i_k} &\mapsto \varepsilon e_{j_1} \wedge \cdots \wedge e_{j_{n-k}}, \end{aligned}$$

where $\{e_{j_1}, \ldots, e_{n-k}\}$ is the complement of $\{e_{i_1}, \ldots, e_{i_k}\}$ in the full basis $\{e_1, \ldots, e_n\}$ and this map is well-defined because of the permutation index $\varepsilon = \mathrm{sgn}(i_1, \ldots, i_k, j_1, \ldots, j_{n-k})$.

Let $\alpha = e_{i_1} \wedge \cdots \wedge e_{i_k}$ and $\beta = e_{j_1} \wedge \cdots \wedge e_{j_k}$ be elements of the basis of $\bigwedge^k V$. We have that

$$\alpha \wedge *\beta = \begin{cases} 0 & \text{if } \{i_1, \ldots, i_k\} \neq \{j_1, \ldots, j_k\} \\ \mathrm{vol} & \text{otherwise.} \end{cases}$$

In any case, we have that

$$\alpha \wedge *\beta = \langle \alpha, \beta\rangle \cdot \mathrm{vol}, \tag{3.1}$$

and by bilinearity of the left and right hand side, we see that (3.1) holds for any $\alpha, \beta \in \bigwedge^k V$.

Let $I$ be a complex structure on $V$, i.e. $I^2 = -\mathrm{id}_V$. Recall that it forces the dimension of $V$ to be even. Suppose furthermore that the inner product is compatible with $I$, i.e. $\langle I-, I-\rangle = \langle -,-\rangle$. Let $\langle -,-\rangle$ also denote the $\mathbb{C}$-sesquilinear extension of the inner product to a hermitian product on the complexified space $V_{\mathbb{C}}$.

The decomposition $V = V^{1,0} \oplus V^{0,1}$ into $\pm\iota$ eigenspaces is orthodonal for $\langle -,-\rangle$, indeed, for $v \in V^{1,0}$ and $u \in V^{0,1}$, we have

$$\langle v, u\rangle = \langle Iv, Iu\rangle = \langle \iota v, \iota u\rangle = -\langle v, u\rangle.$$

By extension, this shows that the decomposition

$$\bigwedge^k V_{\mathbb{C}} = \bigoplus_{p+q=k} \bigwedge^{p,q} V \qquad \text{where } \bigwedge^{p,q} V := \bigwedge^p V^{1,0} \otimes \bigwedge^q V^{0,1} \tag{3.2}$$

is orthogonal. We may extend the hodge star operator $\mathbb{C}$-linearly to

$$* : \bigwedge^k V_{\mathbb{C}} \to \bigwedge^{n-k} V_{\mathbb{C}},$$

and by looking at an orthonormal basis for (3.2), we see that the operator restricts to

$$* : \bigwedge^{p,q} V \to \bigwedge^{n-q,n-p} V.$$

Note that the equality (3.1) now becomes

$$\alpha \wedge *\overline{\beta} = \langle \alpha, \beta\rangle \mathrm{vol}_{V_{\mathbb{C}}},$$

where $\mathrm{vol}_{V_{\mathbb{C}}} := (e_1 + \iota e_1) \wedge (e_1 - \iota e_1) \wedge \cdots \wedge (e_n + \iota e_n) \wedge (e_n - \iota e_n)$.

3.2. **Harmonic forms.** From now on, $(X,h)$ is a Kähler manifold and $g = \mathfrak{Re}(h)$ is the compatible Riemannian metric associated to $h$. We will write $\langle -,- \rangle$ for the hermitian metric induced by $h$ on $T^*_{\mathbb{R}}X$. We have the real volume form

$$\mathrm{vol} \in A^{2n}(X).$$

The decomposition

$$\mathcal{A}^k(X,\mathbb{C}) = \bigoplus_{p+q=k} \mathcal{A}^{p,q}(X)$$

is orthogonal with respect to $\langle -,- \rangle$. As before, we have the hodge operator

$$* : \mathcal{A}^{p,q}(X) \to \mathcal{A}^{n-q,n-p}.$$

Recall also our three different exterior derivatives:

$$\begin{aligned} d &: \mathcal{A}^k(X) \to \mathcal{A}^{k+1}(X) \\ \partial &: \mathcal{A}^{p,q}(X) \to \mathcal{A}^{p+1,q}(X) \\ \overline{\partial} &: \mathcal{A}^{p,q}(X) \to \mathcal{A}^{p,q+1}(X). \end{aligned}$$

We use the hodge operator to define other operators:

$$\begin{aligned} d^* &:= (-1)^k *^{-1} \circ d \circ * : \mathcal{A}^k(X) \to \mathcal{A}^{k-1}(X) \\ \partial^* &:= - * \circ \overline{\partial} \circ * : \mathcal{A}^{p,q}(X) \to \mathcal{A}^{p-1,q}(X) \\ \overline{\partial}^* &:= - * \circ \partial \circ * : \mathcal{A}^{p,q}(X) \to \mathcal{A}^{p,q-1}(X). \end{aligned}$$

The mismatch in the sign is a convention, and we note that $*^2 = (-1)^{k(n-k)}$. Extending $d^*$ $\mathbb{C}$-linearly, these operators satisfy

$$d^* = \partial^* + \overline{\partial}^*$$

We use these operators to define three different Laplacians:

$$\begin{aligned} \Delta_d &:= dd^* + d^*d : \mathcal{A}^k(X) \to \mathcal{A}^k(X) \\ \Delta_\partial &:= \partial\partial^* + \partial^*\partial : \mathcal{A}^{p,q}(X) \to \mathcal{A}^{p,q}(X) \\ \Delta_{\overline{\partial}} &:= \overline{\partial}\,\overline{\partial}^* + \overline{\partial}^*\overline{\partial} : \mathcal{A}^{p,q}(X) \to \mathcal{A}^{p,q}(X). \end{aligned}$$

A $k$-form $\alpha$ is $d$**-harmonic** if $\Delta_d\alpha = 0$, and a $(p,q)$-form $\beta$ is $\overline{\partial}$**-harmonic** if $\Delta_{\overline{\partial}}\beta = 0$. We will write $\mathcal{H}^k(X) := \ker \Delta_d$ for the space of $d$-harmonic $k$-forms (note, we can do this either over $\mathbb{R}$ or $\mathbb{C}$ and do not specify), and $\mathcal{H}^{p,q}(X) := \ker \Delta_{\overline{\partial}}$ for the space of $\overline{\partial}$-harmonic $(p,q)$-forms.

We have an $L^2$ inner product

$$\begin{aligned} A^k(X) \otimes A^k(X) &\to \mathbb{R} \\ (\alpha,\beta) &\mapsto \int_X \langle \alpha,\beta \rangle \mathrm{vol} = \int_X \alpha \wedge *\beta. \end{aligned}$$

From the right hand side, we compute that for $\alpha \in A^k(X)$ and $\beta \in A^{k+1}(X)$,

$$(d\alpha,\beta) = \int_X d\alpha \wedge *\beta = \int_X \alpha \wedge (-1)^{k+1} d(*\beta) = \int_X \alpha \wedge *(-1)^{k+1} *^{-1} d(*\beta) = \int_X \alpha \wedge *d^*\beta = (\alpha, d^*\beta),$$

so that $d$ and $d^*$ are formal adjoints with respect to $(-,-)$. From this, we see that

$$(\alpha, \Delta_d\alpha) = (\alpha, dd^*\alpha + d^*d\alpha) = (d^*\alpha, d^*\alpha) + (d\alpha, d\alpha),$$

and so by positive definiteness, $\Delta\alpha = 0$ if and only if $d\alpha = d^*\alpha = 0$. This yields a map

$$\mathcal{H}^k(X,\mathbb{R}) \to H^k(X,\mathbb{R}).$$

**Theorem 3.2.1** (Hod41)**.** *This map is an isomorphism.*

**Remark 3.2.2.** Note that this works equally well over $\mathbb{C}$ witht the hermitian product

$$(\alpha,\beta) \mapsto \int_X \langle \alpha,\beta \rangle \mathrm{vol} = \int_X \alpha \wedge *\overline{\beta}$$

3.3. **The Dolbeault side.** We can get a similar theorem by taking into account the $(p, q)$ type of forms. First note that for any $p$, we have a complex

$$0 \to \Omega_X^p \to \mathcal{A}^{p,0}(X) \to \mathcal{A}^{p,1}(X) \to \cdots \to \mathcal{A}^{p,n}(X) \to 0.$$

The $\overline{\partial}$-Poincaré lemma (also called the Dolbeault–Grothendieck lemma) says that this complex is exact, i.e. that analytically locally, a $\overline{\partial}$-closed form is $\overline{\partial}$-exact. Moreover, since each $\mathcal{A}^{p,q}(X)$ admits partitions of unity (these are smooth sections), this is an acyclic resolution. As corollary, we obtain an isomorphism

$$H^q(X, \Omega_X^p) = \mathbb{H}^q(\mathcal{A}^{p,0}(X) \to \cdots \to \mathcal{A}^{p,n}(X) \to 0) = \frac{\ker\left(\overline{\partial} : A^{p,q}(X) \to A^{p,q+1}(X)\right)}{\operatorname{im}\left(\overline{\partial} : A^{p-1,q}(X) \to A^{p,q}(X)\right)} =: H^{p,q}(X), \tag{3.3}$$

and we call the right-hand side the $(p, q)$ Dolbeault cohomology of $X$.

As before, we have an inner product on $(p, q)$-forms:

$$(\alpha, \beta) := \int_X \alpha \wedge *\overline{\beta},$$

and the same computation shows that $\partial^*$ and $\overline{\partial}^*$ are formal adjoints. Therefore, $\Delta_{\overline{\partial}}\alpha = 0$ if and only if $\overline{\partial}\alpha = \overline{\partial}^*\alpha = 0$. This yields a map

$$\mathcal{H}^{p,q}(X) \to H^{p,q}(X).$$

**Theorem 3.3.1.** *This map is an isomorphism.*

It is worth noting that Theorem 3.3.1 does not require $X$ to be Kähler.

**Proposition 3.3.2.** *If $X$ is compact Kähler, then*

$$\Delta_d = 2\Delta_{\overline{\partial}} = 2\Delta_{\partial}.$$

**Corollary 3.3.3** (Hodge decomposition)**.** *Let $X$ be a compact Kähler manifold. We have a decomposition*

$$H^k(X, \mathbb{C}) = \oplus_{p+q=k} H^{p,q}(X)$$

*such that $H^{p,q}(X) = \overline{H^{q,p}(X)}$ and $H^{p,q}(X) \simeq H^q(X, \Omega_X^p)$.*

*Proof.* Note that $\Delta_{\overline{\partial}}$ preserves the $(p, q)$-type, and thus so does $\Delta_d$ by Proposition 3.3.2. Therefore, we have

$$\bigoplus_{p+q=k} H^{p,q}(X) \simeq \bigoplus_{p+q=k} \mathcal{H}^{p,q}(X) = \mathcal{H}^k(X, \mathbb{C}) \xrightarrow{\sim} H^k(X, \mathbb{C}). \tag{3.4}$$

It remains to show that $H^{p,q}(X) = \overline{H^{q,p}(X)}$, and this follows from the fact that $\Delta_d(\overline{\alpha}) = \overline{\Delta_d(\alpha)}$, as $\Delta_d$ is a real operator. ☐

**Remark 3.3.4.** It is worthwile to observe that along the isomorphism (3.4), $H^{p,q}(X)$ is identified with the subset $\{[\alpha] : d\alpha = 0 \text{ and } \alpha \in A^{p,q}\} \subset H^k(X, \mathbb{C})$. Note moreover that the Hodge decomposition is compatible witht the wedge product (that is, wedging $(p, q)$ $\overline{\partial}$-closed forms yields the same algebra structure as that on $H^\bullet(X, \mathbb{C})$ along (3.4)). However, there is no clear algebra structure on

$$\bigoplus_{p,q \geq 0} \mathcal{H}^{p,q}(X),$$

as wedging two harmonic forms need not yield a harmonic form (this comes from the fact that $d^*$ does not satisfy a Leibniz rule).

**Corollary 3.3.5.** *Let $X$ be a compact Kähler manifold. If $k$ is odd, then the $k$th Betti number $b_k(X)$ is even.*

**Remark 3.3.6.** For any Kähler form $\omega$,

$$\int_X \omega^{\dim X} = n! \cdot \operatorname{vol}(X) > 0,$$

implying that $\omega$ is not $d$-exact (assuming $X$ compact), i.e. that $b_k > 0$ for $k$ odd.

**Remark 3.3.7** (Hodge diamond)**.** We can make the Hodge numbers fit in what is called the Hodge diamond:

$$
\begin{array}{ccccccccc}
 & & & & h^{0,0} & & & & \\
 & & & h^{1,0} & & h^{0,1} & & & \\
 & & h^{2,0} & & h^{1,1} & & h^{0,2} & & \\
 & \iddots & & & & & & \ddots & \\
h^{n,0} & & \cdots & & \circlearrowleft & & \cdots & & h^{0,n} \\
 & \ddots & & & & & & \iddots & \\
 & & h^{n,n-2} & & h^{n-1,n-1} & & h^{n-2,n} & & \\
 & & & h^{n,n-1} & & h^{n-1,n} & & & \\
 & & & & h^{n,n} & & & & \\
 & & & & \longleftrightarrow & & & &
\end{array}
$$

where $n = \dim_{\mathbb{C}} X$, and these are the only non-zero Hodge numbers. Moreover, there are some symmetries. We already saw Hodge symmetry, which implies $h^{p,q} = h^{q,p}$, which is represented by the arrow in the bottom.

Recalll that some version of Poincaré duality says that there is a non-degenerate pairing

$$
\begin{aligned}
H^k(X, \mathbb{C}) \times H^{2n-k}(X, \mathbb{C}) &\to \mathbb{C} \\
(\alpha, \beta) &\mapsto \int_X \alpha \wedge \beta,
\end{aligned}
\tag{3.5}
$$

yielding an isomorphism $H^k(X, \mathbb{C})^\vee = H_k(X, \mathbb{C}) \simeq H^{2n-k}(X, \mathbb{C})$. This pairing restricts to a non-degenerate pairing

$$H^{p,q}(X) \times H^{n-p,n-q}(X) \to \mathbb{C};$$

indeed, if a $d$-closed form of type $(p, q)$ is non-zero, we know by (3.5) that we may pair it with a form $\beta$ with $\alpha \wedge \beta \neq 0$, but this forces the type of $\beta$ to be $(n - p, n - q)$. Therefore, we get an isomorphism

$$H^{p,q}(X) = H^{n-p,n-q}(X)^\vee. \tag{3.6}$$

In particular, our diamond has the symmetry $h^{p,q} = h^{n-p,n-q}$, which is represented by the central circling arrow in the Hodge diamond. Note that (3.6) can also be seen using Serre duality:

$$H^{p,q}(X) = H^q(X, \Omega^p_X) = H^{n-q}(X, (\Omega^p_X)^\vee \otimes K_X)^\vee = H^{n-q}(X, \Omega^{n-p}_X)^\vee = H^{n-p,n-q}(X)^\vee,$$

where the isomorphism $(\Omega^p_X)^\vee \otimes K_X = \bigwedge^p \mathcal{T}_X \otimes K_X \simeq \Omega^{n-p}$ comes from contraction of vector fields:

$$
\begin{aligned}
\bigwedge^p \mathcal{T}_X \otimes K_X &\xrightarrow{\sim} \Omega^{n-p} \\
X_1 \wedge \cdots \wedge X_p \otimes \alpha &\mapsto \alpha(X_1, \ldots, X_p, -, \ldots, -).
\end{aligned}
$$

Note that there is also a way to see this duality with the Hodge star operator.

The Hodge diamond also satisfies a unimodal condition. Namely, in each row (hence each column by Poincaré/Serre-duality), the Hodge numbers increase before reaching half, then decrease (the latter follows from the former by Hodge symmetry).

3.4. **Lefschetz theorems.** Let $X$ be compact Kähler manifold with Kähler form $\omega$. As $\omega$ is a $(1, 1)$ real form, we obtain an operator

$$
\begin{aligned}
L_\omega : A^k(X) &\to A^{k+2}(X) \\
\alpha &\mapsto \omega \wedge \alpha.
\end{aligned}
$$

In the complexification, this restrict to

$$L_\omega : A^{p,q}(X) \to A^{p+1,q+1}(X),$$

and these operators descend to cohomology by definition. We define the operator

$$\Lambda_\omega := *^{-1} \circ L_\omega \circ *,$$

and the degree operator

$$h : H^*(X) \to H^*(X)$$

where $h|_{H^k(X)} = (k - n)\mathrm{id}|_{H^k(X)}$. Here, we do not specify the coefficients, as we want to work over either $\mathbb{R}$ or $\mathbb{C}$.

**Theorem 3.4.1.** *On cohomology, we have $[L_\omega, \Lambda_\omega] = h$, $[h, L_\omega] = 2L_\omega$ and $[2, \Lambda_\omega] = -2\Lambda_\omega$, that is, $L_\omega, \Lambda_\omega$ and $h$ form an $\mathfrak{sl}_2$-triple. Moreover, we have $[L_\omega, \Delta_d] = 0 = [\Lambda_\omega, \Delta_d]$.*

From this $\mathfrak{sl}_2$-representation, one can deduce the following.

**Theorem 3.4.2** (Hard Lefschetz)**.** *Let $k \leq n$.*

(1) *the map*

$$L_\omega^{n-k} : H^k(X) \to H^{2n-k}(X)$$

*is an isomorphism.*

(2) *Let $H^k(X)_{prim} \subset H^k(X)$ be the kernel of $L_\omega^{n-k+1}$. We have a* ***Lefschetz decomposition***

$$H^k(X) = \bigoplus_{2r \leq kr} L_\omega^r H^{k-2r}(X)_{prim}.$$

*Moreover, over $\mathbb{C}$, this decomposition is compatible with the Hodge decomposition; i.e, if we define*

$$H^{p,q}(X)_{prim} := \ker\left(L_\omega^{2n-p-q+1} : H^{p,q}(X) \to H^{n-p+1,n-q+1}(X)\right),$$

*we have*

$$H^{p,q}(X) = \bigoplus_{2r \leq p+q} L_\omega^r H^{p-r,q-r}(X)_{prim},$$

*and*

$$H^k(X, \mathbb{C})_{prim} = \bigoplus_{p+q=k} H^{p,q}(X)_{prim}.$$

**Remark 3.4.3.** Note that when $X$ is projective, we may choose $\omega$ to be integral and the Lefschetz decomposition also holds over $\mathbb{Q}$.

**Example 3.4.4.** Let us study the consequences of these theorems on the cohomology of a surface $X$ with Kähler form $\omega$. We have the diagram

$$\begin{tikzcd}
H^0(X) \arrow[dddd, bend right=60, "L_\omega^2"'] \arrow[r, equal] & H^0(X)_{\mathrm{prim}} \\
H^1(X) \arrow[dd, bend right, "L_\omega"'] \arrow[r, equal] & H^1(X)_{\mathrm{prim}} \\
H^2(X) = H^2(X)_{\mathrm{prim}} \oplus LH^0(X) & \\
H^3(X) & \\
H^4(X), &
\end{tikzcd}$$

where $H^0(X)_{\mathrm{prim}} = H^0(X)$ and $H^1(X)_{\mathrm{prim}} = H^1(X)$ since the primitive parts are defined as the kernel of maps to a cohomology groups that vanish for dimension reasons. Since $H^0(X)$ is generated by the identity element in the cohomology ring, we obtain (over say $\mathbb{R}$ coefficients) $H^2(X, \mathbb{R}) = \mathbb{R}[\omega] \oplus H^2(X, \mathbb{R})_{\mathrm{prim}}$. As we soon shall see, this decomposition is orthogonal with respect to a certain intersection pairing, so that we may write $H^2(X, \mathbb{R}) = \mathbb{R}[\omega] \oplus [\omega]^\perp$.

3.5. **Hodge index theorem.** Let $X$ be compact and $\omega$ be a Kähler form. Consider the complex Poincaré pairing

$$\begin{aligned} H^k(X, \mathbb{C}) \times H^{2n-k}(X, \mathbb{C}) &\to \mathbb{C} \\ (\alpha, \beta) &\mapsto \int_X \alpha \wedge \beta. \end{aligned}$$

This pairing is skew symmetric for $k$ odd and symmetric for $k$ even. We use the polarisation $\omega$ to turn this into a pairing on $H^k(X, \mathbb{C})$ for $k \leq n$:

$$\begin{aligned} Q_k : H^k(X, \mathbb{C}) \times H^k(X, \mathbb{C}) &\to \mathbb{C} \\ (\alpha, \beta) &\mapsto \int_X \alpha \wedge L_\omega^{n-k}(\beta). \end{aligned}$$

We now define the hermitian pairing $H_k(\alpha, \beta) := \iota^k Q_k(\alpha, \overline{\beta})$ called the **Hodge–Riemann bilinear form**.

**Theorem 3.5.1** (Hodge–Riemann bilinear relations)**.** *The following hold true*

(1) *The Hodge decomposition is orthogonal with respect to* $H_k$*;*
(2) *The Lefschetz decomposition is orthogonal with respect to* $H_k$*, and for* $\alpha, \beta \in H^{k-2r}(X, \mathbb{C})_{prim}$*, we have*

$$H_k(L^r(\alpha), L^r(\beta)) = (-1)^{k+r} H_{k-2r}(\alpha, \beta);$$

(3) *The form*

$$(-1)^{\frac{k(k-1)}{2}} \iota^{p-q-k} H_k$$

*is positive definite on* $H^{p,q}(X)_{prim}$*.*

*Proof.* (1) Let $\alpha \in H^{p,q}(X)$ and $\beta \in H^{p',q'}(X)$ with $p+q = p'+q'$. By definition, we have

$$H_k(\alpha, \beta) = \iota^k \int_X \alpha \wedge \omega^{n-k} \wedge \overline{\beta},$$

but the form $\alpha\omega^{n-k} \wedge \wedge \overline{\beta}$ is of degree $2n$ but not of type $(n,n)$. Hence it vanishes, implying that the integral vanishes.

(2) Let $\alpha' = L^r_\omega(\alpha)$ for $\alpha \in H^{k-2r}(X)_{\text{prim}}$ and $\beta' = L^s_\omega(\beta)$ for $\beta \in H^{k-2s}(X)_{\text{prim}}$. Without loss of generality, assume $r > s$. We have that

$$H_k(\alpha', \beta') = \iota^k \int_X \alpha' \wedge L_\omega^{n-k+s}(\overline{\beta'}) = \iota^k \int_X \omega^r \alpha \wedge \omega^{n-k+s} \wedge \overline{\beta} = (-1)^k \iota^k \int_X \alpha \wedge L_\omega^{n-k+r+s}(\overline{\beta}) = 0$$

since $n-k+r+s > n-k+2s$, so that primitiveness of $\beta$ ensures it is in the kernel of $L_\omega^{n-k+r+s}$.

Now if $\alpha', \beta'$ are chosen as above but $r = s$, we have

$$\begin{aligned} H_k(\alpha', \beta') &= \iota^k \int_X \alpha \wedge L^{n-k+r}(\overline{\beta'}) = \iota^k \int_X \omega^r \wedge \alpha \wedge \omega^{n-k+r} \wedge \overline{\beta} \\ &= (-1)^k \iota^{2r} \iota^{k-2r} \int_X \alpha \wedge \omega^{n-k+2r} \wedge \overline{\beta} = (-1)^{k+r} H_{k-2r}(\alpha, \beta). \end{aligned}$$

(3) For $\alpha \in H^{p,q}(X)_{\text{prim}}$. For such form, on Kähler manifolds, we have

$$*\alpha = \iota^{p-q} (-1)^{\frac{k(k-1)}{2}} \frac{\omega^{n-k}}{(n-k)!} \wedge \alpha;$$

see [Voi98, Proposition 6.29]. Therefore, we obtain that

$$\begin{aligned} H_k(\alpha, \alpha) &= \iota^k \int_X \alpha \wedge \omega^{n-k} \wedge \overline{\alpha} = (n-k)! (-1)^{\frac{k(k-1)}{2}} \iota^{k-p+q} \int_X \alpha \wedge * \overline{\alpha} \\ &= (n-k)! (-1)^{\frac{k(k-1)}{2}} \iota^{k-p+q} \int_X \langle \alpha, \alpha \rangle \mathrm{vol}_{\mathbb{C}} \end{aligned}$$

so that

$$(-1)^{\frac{k(k-1)}{2}} \iota^{p-q+k} H_k$$

is positive-definite. □

**Corollary 3.5.2** (Hodge index theorem)**.** *Let* $X$ *be a compact Kähler surface. Then, the signature of the Poincaré intersection pairing* $Q_2$ *on* $H^2(X, \mathbb{R})$ *is*

$$(2h^{2,0} + 1, h^{1,1} - 1).$$

*Proof.* Let $\alpha \in H^2(X, \mathbb{R})$. We may decompose into types: $\alpha = \alpha^{2,0} + \alpha^{1,1} + \alpha^{0,2}$. The fact that $\alpha = \overline{\alpha}$ forces $\alpha^{1,1} \in H^{1,1}(X, \mathbb{R})$, and $\alpha^{2,0} = \overline{\alpha^{2,0}}$. We thus have the decomposition

$$H^2(X, \mathbb{R}) = ((H^{2,0}(X) \oplus H^{0,2}) \cap H^2(X, \mathbb{R})) \oplus H^{1,1}(X, \mathbb{R}),$$

which we know to be orthogonal with respect to the Poincaré pairing. Any $\alpha \in (H^{2,0}(X) \oplus H^{0,2}(X)) \cap H^2(X, \mathbb{R})$ is primitive for degree reasons: taking the cup product with $\omega$ yields types $(3,1)$ or $(1,3)$. Thus, we have

$$Q_2(\alpha, \alpha) = \int_X \alpha \wedge \alpha = \int_X \alpha^{2,0} \wedge \overline{\alpha^{2,0}},$$

which is positive by Theorem 3.5.1(3). Now, we have the Lefschetz decomposition

$$H^{1,1}(X, \mathbb{R}) = H^{1,1}(X, \mathbb{R})_{\text{prim}} \oplus L_\omega H^0(X, \mathbb{R}) = H^{1,1}(X, \mathbb{R})_{\text{prim}} \oplus \mathbb{R}[\omega].$$

This decomposition is orthogonal: if $\alpha \in H^{1,1}(X,\mathbb{R})_{\text{prim}}$, then

$$Q_2(\omega,\alpha) = \int_X \omega \wedge \alpha = 0$$

as $\omega \wedge \alpha = 0$ by definition of primitive cohomology. We have

$$Q_2(\omega,\omega) = \int_X \omega^2 = 2 \cdot \text{vol}(X) > 0.$$

There remains to compute $Q_2$ on real $(1,1)$ primitive classes. But by Theorem 3.5.1,

$$\int_X \alpha^2 < 0,$$

and so we obtain the right count for the index of the pairing. $\square$

3.6. **(1,1) classes.** Consider now the exponential exact sequence

$$0 \to \mathbb{Z} \to \mathcal{O}_X \xrightarrow{exp(2\pi i -)} \mathcal{O}_X^* \to 0.$$

It is a fact that the induced connecting homomorphism $\delta$ in the long exact cohomological sequence

$$\cdots \to H^1(X,\mathbb{Z}) \to H^1(X,\mathcal{O}_X) \to \text{Pic}(X) \xrightarrow{\delta} H^2(X,\mathbb{Z}) \to H^2(X,\mathcal{O}_X) \to \cdots$$

is the opposite of the first chern map: $\delta = -c_1$.

**Theorem 3.6.1** (Lefschetz theorem on $(1,1)$ classes)**.** *If $X$ is compact Kähler, then the first chern map $c_1$ is surjective onto $H^{1,1}(X,\mathbb{Z})$.*

*Proof.* Note that the first chern map is indeed valued in $H^{1,1}(X,\mathbb{Z})$ by definition of the Chern class (indeed, the $(2,0)$ and $(0,2)$ part of the curvature of the Chern connection vanish). The map $H^2(X,\mathbb{Z}) \to H^2(X,\mathcal{O}_X)$ factors as

$$H^2(X,\mathbb{Z}) \hookrightarrow H^2(X,\mathbb{C}) \to H^{2,0}(X,\mathbb{C}) \simeq H^2(X,\mathcal{O}_X),$$

and thus vanishes on $(1,1)$ classes, so that $(1,1)$ classes are in the kernel of this map, and equivalently in the image of $c_1 = -\delta$. $\square$

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