HYPERKÄHLER MANIFOLDS

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Abstract. These are personal notes for the course on hyperkähler manifolds taught by Alessio Bottini at Universität Bonn in the Winter 2025-2026 semester. Please email me at s37mbrai@uni-bonn.de if you notice any typo.

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We first review some complex geometry.

Definition 1.0.1. A **complex manifold** is a locally ringed space $(X, \mathcal{O}_X)$ such that

- X is Hausdorff and second countable (this part is to ensure we actually have a topological manifold);
- $(X, \mathcal{O}_X)$ is locally isomorphic to $(\Delta, \mathcal{O}_\Delta)$, where $\Delta \subset \mathbb{C}^n$ is the polydisc.

Example 1.0.2. Let $f_i, \ldots, f_d \in \mathbb{C}[z_1, \ldots, z_n]$ be complex polynomials such that the Jacobian of

$$f = (f_1, \ldots, f_d) : \mathbb{C}^n \to \mathbb{C}^d$$

has everywhere full rank on the vanishing set $V = V(f) \subset \mathbb{C}^n$. By the holomorphic implicit function theorem (regular value theorem), V is a complex manifold.

Example 1.0.3. If X is a smooth algebraic variety over $\mathbb{C}$, we may cover it by affines V_i which are of the same form as in Example 1.0.2. We may consider the analytic topology X^{an} obtained by gluing the different charts V_i. Similarly, we may define the sheaf $\mathcal{O}_X^{an}$ on X^{an} by considering the sheaf of holomorphic functions on each V_i (the transitions $V_i \to V_j$ are regular algebraic, hence holomorphic, so that this gluing makes sense). Then, $(X^{an}, \mathcal{O}_X^{an})$ is a complex manifold.

Note that in Example 1.0.3, we obtain a natural map of ringed spaces

$$\alpha : (X^{an}, \mathcal{O}_X^{an}) \to (X, \mathcal{O}_X)$$

since the analytic topology is finer than the Zariski topology, and regular functions are holomorphic. In particular, we obtain a functor between abelian categories:

$$\alpha^* : \mathcal{O}_X\text{-mod} \to \mathcal{O}_X^{an}\text{-mod}$$

restricting to

$$\alpha^* : \mathrm{Coh}(X) \to \mathrm{Coh}(X^{an}).$$

Theorem 1.0.4 (Géométrie algébrique géométrie analytique; [Ser56])**.** *If X is smooth*[1] *and proper functor $\alpha^* : Coh(X) \to Coh(X^{an})$ is an equivalence, which moreover induces an isomorphism of sheaf cohomology.*

1.1. **Almost complex structures.**

A complex manifold $(X, \mathcal{O}_X)$ has an underlying smooth manifold (X, C_X^∞), where C_X^∞ denotes the sheaf of smooth functions on X; indeed, if X has complex charts $U_i \subset \mathbb{C}^n$, the transitions are holomorphic, hence C^∞.

Notation 1.1.1. Since the indices i will be ubiquitous, ι shall denote the root of -1 for these notes (this spares the cumbersome $\sqrt{-1}$ alternative).

On each chart U_i, we have multiplication by ι, but this does not globalise, as ι does not commute with holomorphic functions: in the Taylor expansion, we have terms which are of degree m where $m \neq 1 \mod 4$. However, the differential of ι may be globalised, as we get rid of the higher order terms. In a local chart $U_i \subset \mathbb{C}^n$, the (real) tangent bundle has a local frame

$$T_\mathbb{R} U_i = \langle \partial_{x_j}, \partial_{y_j} : 1 \leq j \leq n \rangle,$$

on which $I := d\iota$ acts by

$$\begin{cases} \partial_{x_j} \mapsto \partial_{y_j} \\ \partial_{y_j} \mapsto -\partial_{x_j}. \end{cases}$$

Definition 1.1.2. An **almost complex structure** on a smooth manifold X is an endomorphism $I \in \mathrm{End}(T_\mathbb{R} X)$ such that $I^2 = -1$. We say that I is integrable if X is a complex manifold and I is obtained by locally differentiating ι.

Question 1.1.3. Given I an almost complex structure, when is it integrable?

Let us first set up some tools in order to address this question appropriately. Assume only for now that X is a smooth manifold and I is an almost complex structure. We can consider the complexified tangent bundle

$$T_\mathbb{C} X := T_\mathbb{R} X \otimes_\mathbb{R} \mathbb{C},$$

[1] This can be dropped by considering complex analytic spaces (rather than manifolds).

to which we can extend the action of I. Since $I^2 = -1$, the minimal polynomial of I is x^2+1, which is separable over $\mathbb{C}$, meaning that I is diagonalisable, with eigenvalues $\pm\iota$. The eigenspaces must have the same dimension as I acts on the *real* tangent space. We thus obtain a decomposition

$$T_{\mathbb{C}}X = T^{1,0}X \oplus T^{0,1}X = T^{1,0}X \oplus \overline{T^{1,0}X}.$$

Note that we have

$$T^{1,0}X = \{(v - iIv) : v \in T_{\mathbb{C}}X\} \qquad T^{0,1}X = \{(v + iIv) : v \in T_{\mathbb{C}}X\}.$$

Notation 1.1.4. We will use the following notation

- $\mathcal{A}^0(X) := C^\infty_X$;
- $\mathcal{A}^k(X)$ denotes the sheaf of (smooth) degree k real forms;
- $\mathcal{A}^k(X,\mathbb{C}) = \oplus_{p+q=k}\mathcal{A}^{p,q}(X)$ denotes the sheaf of sections of $\bigwedge^k T^*_{\mathbb{C}}X$ (i.e. smooth complex degree k forms) and $\mathcal{A}^{p,q}(X)$ denotes the sheaf of sections of $\bigwedge^p T^{1,0}X \otimes \bigwedge^q T^{0,1}X$;
- $d : \mathcal{A}^k(X,\mathbb{C}) \to \mathcal{A}^{k+1}(X,\mathbb{C})$ denotes the complexification of the usual exterior derivative, and can be decomposed by types as $d = \partial + \bar\partial$, where ∂ denotes the part corresponding to the differentiation in holomorphic coordinates, and similarly $\bar\partial$ for anti-holomorphic coordinates.
- $A^k(X)$, $A^k(X,\mathbb{C})$, and $A^{p,q}(X)$ denotes the global sections of $\mathcal{A}^k(X), \mathcal{A}^k(X,\mathbb{C})$, and $\mathcal{A}^{p,q}(X)$ respectively.
- $\mathcal{T}_X$ denotes the sheaf of homolorphic vector fields, i.e. $\mathcal{T}_X := \mathcal{D}er(\mathcal{O}_X,\mathcal{O}_X)$;
- $\Omega_X := \mathcal{T}^*_X$ denotes the cotangent sheaf.

The following theorem answers Question 1.1.3.

Theorem 1.1.5 (Newlander-Niremberg)**.** *I is integrable if and only if $\bar\partial^2 = 0$.*

Note that this is equivalent to $T^{1,0}X$ being closed under the (complex) Lie bracket (this is much related to the Frobenius theorem of differential geometry), and also equivalent of the vanishing of a certain tensor N_I called the *Nijenhuis* tensor.

1.2. **Metrics.** Let E be a real vector bundle on (X, C^∞_X). A **Riemannian metric** g on E is a section of $\mathrm{Sym}^2 E^\vee$ such that for all $p \in X$, g_p is positive definite. If E is a complex bundle, a **Hermitian metric** h is a map of sheaves $E \otimes \overline{E} \to C^\infty(X,\mathbb{C})$ such that each h_p is Hermitian, i.e. $h_p(e,f) = \overline{h_p(f,e)}$ and $h_p(e,e) > 0$ for all $e, f \in E_p$. When $E = T_{\mathbb{R}}X$, we say that g (resp. h) is a **Riemannian** (resp. **Hermitiam**) **metric on** X (here, the almost complex structure I is used to put a $\mathbb{C}$-structure on $T_{\mathbb{R}}X$).

If X is a complex manifold and h is a Hermitian metric, then we can write

$$h = g - i\omega$$

where $g = \mathfrak{Re}(h)$ and $\omega = -\mathfrak{Im}(h)$. We obtain that g is a Riemannian metric, and ω is skew-symmetric since

$$\omega(X,Y) = \frac{\iota}{2}(h - \overline{h})$$

and h is conjugate skew-symmetric. Thus, $\omega \in A^2(X)$.

Definition 1.2.1. (X,h) is **Kähler** if $d\omega = 0$.

That h is linear in the first variable and anti-linear in the second ensures that $h(I-,I-) = h(-,-)$, implying that $g(I-,I-) = g(-,-)$, a property that is sometimes called **compatibility** of the metric with I. We have

$$\omega(-,-) = \frac{\iota}{2}(h(-,-) - \overline{h}(-,-)) = \frac{1}{2}(h(I-,-) + \overline{h}(I-,-)) = g(I-,-),$$

which also implies

$$\omega(-,I-) = g(-,-).$$

Definition 1.2.2. A form $\omega \in A^2(X)$ is called **positive** if $\omega(u,Iu) > 0$ for all $u \in T_{\mathbb{R}}X$. We see that a de Rham cohomology class in $H^2(X,\mathbb{C})$ is **positive** if it can be represented by a positive form. If moreover ω is I-invariant (or equivalently, of type $(1,1)$ after embedding $A^2(X) \subset A(X,\mathbb{C})$), we say ω is **Kähler**.

If ω is Kähler, we may define the hermitian metric $h_\omega = \omega(-,I-) - i\omega$, and we have that ω is Kähler if and only if (X,h_ω) is Kähler.

Example 1.2.3. Let $X = \mathbb{P}^n$, with projective coordinates $Z_0, \ldots, Z_n$. Let U_i be the $Z_i \neq 0$ chart, and define $z_j = \frac{Z_j}{Z_i}$. We may define on U_i the metric

$$\omega_{FS} = \omega = i\partial\overline{\partial} \log \left(1 + \sum_j z_j \overline{z}_j \right),$$

and one checks that these glue to a global form, which we call the **Fubini-Study metric**. Written as a Kähler potential this way shows that it is a Kähler metric.

Note that if (X, ω) is Kähler, restricting the metric to a complex submanifold Y preserves all properties of Definition 1.2.2, and so (Y, ω_Y) is Kähler. Thus, any projective manifold is Kähler.

1.3. **Connections.** Let E be a complex (the real case is identical) vector bundle on (X, C_X^∞). A **complex connection** in E is a $\mathbb{C}$-linear map

$$\nabla : \mathcal{A}^0(E, \mathbb{C}) \to \mathcal{A}^1(E, \mathbb{C}),$$

(here $\mathcal{A}^i(E, \mathbb{C}) = \mathcal{A}^i(X, \mathbb{C}) \otimes \Gamma(E)$) such that

$$\nabla(f \cdot s) = df \otimes s + f \cdot \nabla s$$

for all section s of E and $f \in C_X^\infty$.

If E is a holomorphic bundle on a complex manifold, we can define the operator

$$\overline{\partial} : \mathcal{A}^0(E) \to \mathcal{A}^{0,1}(E)$$

as follows: if σ_i is a local frame, and $s = s^i \sigma_i$ a section, we let

$$\overline{\partial}(s^i \sigma_i) := (\overline{\partial} s^i) \otimes \sigma_i. \tag{1.1}$$

Indeed, given another frame τ_j related by $\sigma_i = g_{ij} \tau_j$, we have

$$\overline{\partial}(s^i) \otimes \sigma_i = \overline{\partial}(s_i) \otimes g_{ij} \tau_j = \overline{\partial}(g_{ij} s^i) \otimes \tau_j$$

since the transitions g_{ij} are holomorphic by assumption.

A (complex) connection being valued in $\mathcal{A}^1(E, \mathbb{C}) = \mathcal{A}^{1,0}(E) \oplus \mathcal{A}^{0,1}(E)$, we may split $\nabla = \nabla^{1,0} + \nabla^{0,1}$.

Definition 1.3.1. The complex connection ∇ in E is said to be **compatible** with the holomorphic structure if $\nabla^{0,1} = \overline{\partial}$. Suppose E has a hermitian metric h. We say ∇ is **compatible** with h if for any sections e, f, we have equality of forms

$$d(h(e, f)) = h(\nabla e, f) + h(e, \nabla f).$$

More geometrically, this says that h is parallel to the connection, i.e. constant along parallel transport, i.e. the connection has $U(n)$-holonomy. We say ∇ is a **Chern connection** if it is both compatible with the holomorphic structure and the hermitian metric.

Theorem 1.3.2 (Chern). *There exists a unique Chern connection.*

When $E = T_{\mathbb{R}}X$, the Chern connection ought to be regarded as the complex geometric analogue of the Levi-Civita connection from Riemannian geometry. In fact this is more than an analogy. If h is a hermitian metric, the Levi-Civita connection of $g = \mathfrak{Re}(h)$ can be complexified to a complex connection. It is a theorem that the Levi-Civita connection is the Chern connection if and only if (X, h) is Kähler.

We can extend the connection $\nabla : \mathcal{A}^0(E, \mathbb{C}) \to \mathcal{A}^1(E, \mathbb{C})$ to a connection

$$\nabla : \mathcal{A}^p(E, \mathbb{C}) \to \mathcal{A}^{p+1}(E, \mathbb{C})$$

for all positive p via

$$\nabla(\omega \otimes s) = d\omega \otimes s + (-1)^p \omega \wedge \nabla s,$$

where ω is a p-form and s is a section of E.

Remark 1.3.3. Note that this is different from the usual extension of a connection to tensors since we are dealing with skew-symmetric forms. In particular, this satisfies a different Leibniz rule:

$$\nabla(fs) = df \wedge s \otimes d\nabla s.$$

Definition 1.3.4. We define the **curvature** of ∇ to be the composition $\nabla^2 = \nabla \circ \nabla = F_\nabla$.

Note that

$$\begin{aligned}\nabla(\nabla f s) = \nabla(df \otimes s + f\nabla s) &= ddf - df \wedge \nabla s + \nabla(f\nabla s)\\ &= -df \wedge \nabla s + df \wedge \nabla s + f\nabla^2 s = f\nabla^2 s\end{aligned}$$

so that F_∇ is C^∞_X-linear, that is a section of $\mathcal{A}^2(End(E), \mathbb{C})$.

We may also define

$$F_\nabla^k := \underbrace{F_\nabla \circ \cdots \circ F_\nabla}_{k} \in \mathcal{A}^{2k}(End(E), \mathbb{C}).$$

We define the k**th Chern character** of ∇ to be

$$\mathrm{ch}_k(E, \nabla) := \mathrm{Tr}\left(\frac{1}{k!}\left(\frac{\iota}{2\pi}F_\nabla^k\right)\right) \in A^{2k}(X, \mathbb{C}).$$

Theorem 1.3.5 (Chern-Weil)**.** *The following is true about the Chern character.*

(1) *$ch_k(E, \nabla)$ is closed;*
(2) *The cohomology class $ch_k(E) := [ch_k(E, \nabla)] \in H^{2k}_{dR}(X, \mathbb{C})$ is independent of ∇;*
(3) *$ch_k(E)$ is real, i.e. in $H^{2k}_{dR}(X, \mathbb{R})$ (in fact, it is integral);*
(4) *The total Chern character $\sum_k ch_k(E)$ is equal to the cohomology class of $Tr(\exp(\frac{\iota}{2\pi}F_\nabla))$ (this one directly follows from developing the exponential).*

2.1. **Chern classes.** Let V be a vector space over $\mathbb{C}$ of dimension r. Let $P \in \mathbb{C}[End(V)]$ be a homogeneous polynomial of degree k. Assume moreover P is $GL(V)$ invariant, that is $P(A^{-1}BA) = P(B)$ for any $A \in GL(V)$.

Let now E be a complex vector bundle and ∇ a connection. By $GL(V)$ invariance, $P(\frac{\iota}{2\pi}F_\nabla)$ is well-defined, and lives in $A^{2k}(X, \mathbb{C})$.

Fact 2.1.1 (Chern-Weil)**.** $P(\frac{\iota}{2\pi}F_\nabla)$ is closed, and the class $[P(\frac{\iota}{2\pi}F_\nabla)] \in H^{2k}(X, \mathbb{C})$ is independent of ∇.

Consider now the $GL(V)$-invariant homogeneous polynomials P_k returning the coefficients of the characteristic polynomials (i.e. P_k kth elementary symmetric polynomial on the eigenvalues). We can explicitly define P_k by the formula:

$$\det(I + tB) = \sum_k P_k(B)t^k.$$

We define the k**th Chern class** of E to be the cohomology class of $P_k(\frac{i}{2\pi}F_\nabla)$, and the **total Chern class** of E to be $c(E) := \sum_{i=0}^k c_k(E)$.

The Chern classes and characters satisfy certain properties:

- $c_0(E) = 1$ and $ch_0 = r$;
- $c_d = 0$ if $d > r$. In particular, if L is a line bundle, $c(L) = 1 + c_1(L)$;
- $ch(L) = \exp(c_1(L)) := 1 + c_1(L) + \frac{c_1(L)^2}{2!} + \cdots \in H^{2\bullet}(X, \mathbb{C})$.

To derive further elementary properties of the Chern characters and classes, let us first observe how we can assemble connections into new connections

Let E_1 and E_2 be vector bundles with respective (complex) connections ∇_1 and ∇_2. Then,

- $\nabla_{E_1 \oplus E_2} := \nabla_1 \oplus \nabla_2$ is a connection on $E_1 \oplus E_2$, and $F_{\nabla_1 \oplus \nabla_2} = F_{\nabla_1} \oplus F_{\nabla_2}$, where this is seen as a block matrix
$$\begin{pmatrix} F_{\nabla_1} & \\ & F_{\nabla_2} \end{pmatrix};$$
- $\nabla_{E_1 \otimes E_2} := \nabla_1 \otimes \mathrm{id}_{E_2} + \mathrm{id}_{E_1} \otimes \nabla_2$ is a connection on $E_1 \otimes E_2$;
- the assignment
$$(\nabla^\vee \phi)(s) := d(\phi(s)) - \phi(\nabla(s))$$
where $s \in E$ and $\phi \in E^\vee$ defines a connection on $E^\vee$ (note that this is the usual way to extend connections on tensors). In other words, if $\langle -, - \rangle$ denotes the natural pairing on $E \otimes E^\vee$, the dual connection is defined by
$$d\langle s, \phi \rangle = \langle \nabla s, \phi \rangle + \langle s, \nabla^\vee \phi \rangle.$$
Let us try to compare the two curvatures. Given s_i and t_j frames of E, we consider the connection form $A = (A_i^{\ j})$ satisfying $\nabla s_i = A_i^{\ j} \otimes t_j$. Let s^i and t^j be the dual frames. We obtain
$$\begin{aligned} d\langle s_i, t^j \rangle = 0 &= \langle \nabla s_i, t^j \rangle + \langle s_i, \nabla^\vee t^j \rangle \\ &= \langle A_i^{\ k} \otimes t_k, t^j \rangle + \langle s_i, B^j_{\ k} \otimes s^k \rangle \\ &= A_i^{\ j} + B^j_{\ i}, \end{aligned}$$
where $B = (B^j_{\ i})$ is the connection form of $\nabla^\vee$. And so we have $B = -A^t$ as sections of $\mathcal{A}^2(End(E), \mathbb{C}) = \mathcal{A}^2(End(E^\vee), \mathbb{C})$. Using Cartan's formula for the curvature of a connection, we conclude
$$F_{\nabla^\vee} = d(-A^t) + (-A^t) \wedge (-A^t) = -(dA + A \wedge A)^t = -F_\nabla^t.$$
- connections pull back, that is if $f : Y \to X$ is a smooth map and E is a bundle on X with connection ∇_E, we may define the connection ∇_{f^*E} by *locally* demanding
$$\nabla_{f^*E}(f^*s) = f^*\nabla s.$$

Corollary 2.1.2. *Let E_1, E_2 be complex vector bundles on X. The following hold:*

1. $ch(E_1 \oplus E_2)) = ch(E_1) + ch(E_2)$ *and* $c(E_1 \oplus E_2) = c(E_1) \cup c(E_2)$*;*
2. $ch(E_1 \otimes E_2) = ch(E_1) \cup ch(E_2)$*;*
3. $c_k(E^\vee) = (-1)^k c_k(E)$
4. $ch_k(f^*E) = f^*(ch_k(E))$ *and and* $c_k(f^*E) = f^*(c_k(E))$*.*

Note that $c_k(E) \in H^{2k}(X, \mathbb{C})$ is in fact real: indeed, conjugation acts via $\overline{F_\nabla} = F_{\overline{\nabla}}$, and up to choosing a hermitian metric, we have $E^\vee \simeq \overline{E}$ and $F_{\overline{\nabla}} = -F_\nabla^t$, and in a Chern splitting, the eigenvalues $\omega_1, \dots, \omega_r$ are purely imaginary. We obtain

$$c_k(E) = P_k(\frac{\iota}{2\pi} F_\nabla) = \left[\frac{\iota^k}{(2\pi)^k} P_k(F_\nabla)\right] = \left[\frac{\iota^k}{(2\pi)^k} \sigma_k(\omega_1, \dots, \omega_r)\right]$$

while

$$\overline{c_k(E)} = \left[\frac{(-1)^k \iota^k}{(2\pi)^k} \sigma_k(-\omega_1, \dots, -\omega_r)\right] = \left[\frac{(-1)^{2k} \iota^k}{2\pi} \sigma_k(\omega_1, \dots, \omega_r)\right] = c_k(E)$$

where σ_k denotes the kth standard symmetric polynomial, so that $c_k(E) \in H^{2k}(X, \mathbb{R})$. The kth Chern class is also $(1,1)$. Indeed, consider the Chern connection $\nabla = \nabla^{1,0} + \nabla^{0,1} = \nabla^{1,0} + \overline{\partial}$. From this decomposition, we see that the $(0,2)$-part of the curvature is $\overline{\partial}^2 = 0$. Similarly, one can use the fact that the hermitian metric is parallel to show that the $(2,0)$ part vanishes so that all ω_i are of type $(1,1)$, from which one obtains that $c_k(E)$ is of type (k,k).

Let $D \subset X$ be a divisor. It is a fact that the fundamental class of D is the Chern class of $\mathcal{O}(D)$, i.e.

$$[D] = c_1(\mathcal{O}(D)) \in H^2(X, \mathbb{C}),$$

showing that the first — and therefore any — Chern class is integral.

An important theorem relating to chern classes is Kodaira's embedding theorem.

Theorem 2.1.3 ([Kod54]). *A (holomorphic) line bundle L on a complex manifold X is ample (i.e. induces an embedding in projective space) if and only if it admits a metric h such that $\frac{\iota}{2\pi} F_{D_h}$ is a positive form.*

In particular, let h_0 be any hermitian metric on L, and let $\omega_0 = \frac{\iota}{2\pi} F_{D_{h_0}}$. Assuming X is Kähler, if $c_1(L) = [\omega_0]$ is positive, i.e. if it has a positive form ω as representative of the cohomology class, then we can write $\omega = \omega_0 + \frac{\iota}{2\pi} \partial\overline{\partial}\phi$ for some function ϕ by the $\partial\overline{\partial}$-lemma. Then, one may compute that the metric $h := e^{-\phi} \cdot h_0$ satisfies $\frac{\iota}{2\pi} F_{D_h} = \omega$; indeed the $(1,0)$-part of the Chern of the connection is

$$h^{-1}\partial h = \partial \log h = \partial \log(e^{-\phi} h_0) = \partial \log h_0 + \partial(-\phi)$$

so that the curvature is given by

$$F_{D_{h_0}} + \overline{\partial}\partial(-\phi) = F_{D_{h_0}} + \partial\overline{\partial}\phi.$$

In particular, a line bundle L is ample if and only if $c_1(L)$ is positive, i.e. is represented by a Kähler form. Now, on a compact manifold, slightly perturbing a Kähler form inside $H^{1,1}(X, \mathbb{R})$ still yields a Kähler form, since it preserves the positivity criterion. Thus, Kähler forms form an open positive cone $\mathcal{K}_X$ inside of $H^{1,1}(X, \mathbb{R})$ (scaling by a positive real preserves Kählerness). Moreover, by the Lefschetz theorem on $(1,1)$ classes, the Chern map $\operatorname{Pic}(X) \to H^{1,1}(X, \mathbb{Z})$ is surjective. Thus, we conclude that a compact complex manifold is projective if and only if $\mathcal{K}_X$ intersects with $H^{1,1}(X, \mathbb{Z})$ (or equivalently $H^{1,1}(X, \mathbb{Q})$) inside of $H^{1,1}(X, \mathbb{R})$.

2.2. Hirzebruch-Riemann-Roch.

Definition 2.2.1. Let X be a compact complex manifold. Let ∇ be a connection in the tangent bundle. We define the **Todd class** of X to be

$$td(X) := \left[\det\left(\frac{\frac{\iota}{2\pi} F_\nabla}{1 - \exp(\frac{-\iota}{2\pi} F_\nabla)}\right)\right] \in H^{2\bullet}(X, \mathbb{C})$$

In terms of Chern roots $\omega_1, \dots, \omega_r$, we have that

$$td(X) = \prod_{i=1}^r \frac{\omega_i}{1 - e^{-\omega_i}}$$

It can be computed that we have

$$td_0(X) = 1; \quad td_1(X) = \frac{c_1}{2}; \quad td_2(X) = \frac{1}{12}(c_1^2 + c_2); \quad td_3(X) = \frac{c_1 c_2}{24} \quad td_4(X); = \frac{-c_1^4 + 4c_2 + c_1 c_3 + 3c_2^2 - c_4}{720}; \quad \cdots$$

where $td_k(X)$ denotes the kth homogeneous component of $td(X)$ and $c_i = c_i(X) := c_i(\mathcal{T}_X)$.

Theorem 2.2.2 (Hirzebruch-Rieman-Roch). *Let E be a holomorphic vector bundle on a compact complex manifold X. Then, we have equality*

$$\chi(X,E) := \sum_k (-1)^k h^i(X,E) = \int_X ch(E) \cup td(X). \tag{2.1}$$

Note that since we are integrating over X, we only need to consider the top degree parts.

Example 2.2.3. Let $X = C$ be a compact Riemann surface and L be a line bundle on X. we have $ch(E) = 1 + c_1(L)$ and $td(1) = 1 + \frac{c_1(X)}{2}$. thus, we have

$$\chi(X,L) = \int_X 1 + c_1(L) + \frac{c_1(X)}{2} + \frac{c_1(L)c_1(X)}{2} = \int_X c_1(L) + \frac{c_1(X)}{2} = \deg(L) + \frac{\deg(\mathcal{T}_C)}{2}.$$

What is remarkable about this theorem is that the left-hand side of (2.1) is purely holomorphic (or algebraic) whilst the right-hand side is purely topological. Another similar theorem is the algebro-geometric Gauss-Bonnet theorem.

Theorem 2.2.4. *Let X be a compact complex dimension of dimension n. Then,*

$$\chi_{top}(X) := \sum_i (-1)^i b_i(X) = \int_X c_n(X).$$

Recall the classical relation between the Euler characteristic χ_{top} and the genus g of a topological surface: $\chi_{top} = 2 - 2g$. In particular, this implies for a compact Riemann surface C as above, that

$$\int_X c_1(X) = \deg(\mathcal{T}_C) = 2 - 2g.$$

In particular, in light of what we found in Example 2.2.3, we recover the classical Riemann-Roch theorem:

$$\chi(X,L) = \deg(L) - g + 1.$$

2.3. Kähler-Einstein manifolds.

Question 2.3.1. When does a smooth projective variety over $\mathbb{C}$ admit a "canonical" metric?

Definition 2.3.2. Let (X,ω) be a compact Kähler manifold, and let D_ω be the corresponding Chern connection. We define the **Ricci form** $\mathrm{Ric}(\omega)$ of ω to be

$$\mathrm{Ric}(\omega) = i\mathrm{Tr}(F_{D_\omega}) \in A^2(X,\mathbb{C}).$$

We say that (X,ω) is **Kähler-Einstein** if $\mathrm{Ric}(\omega) = \lambda\omega$ for some constant $\lambda \in \mathbb{R}$.

Remark 2.3.3. We make the following comments.

(1) Recall that we argued earlier that all the Chern roots of F_{D_ω} were pure imaginary of type $(1,1)$ so that $\mathrm{Ric}(\omega)$ is real of type $(1,1)$.
(2) Note also that D_ω is invariant under rescaling ω by some $\lambda > 0$ (indeed, parallelness of h is unaffected so we get the same connection). Thus, we may always assume that $\lambda = -1, 0, 1$.
(3) since $c_1(X) = [\frac{\iota}{2\pi}\mathrm{Tr}F_{D_\omega}]$ by definition, we have $[\mathrm{Ric}(\omega)] = 2\pi c_1(X) \in H^2(X,\mathbb{R})$.
(4) λ is proportional to the scalar curvature, and so (X,ω) being Kähler-Einstein implies that the scalar curvature with respect to $g_\omega = \omega(I-,-)$ is constant.
(5) If X is Kähler-Einstein, then we have

$$c_1(X) = \begin{cases} 0 \\ \pm \text{ positive form.} \end{cases}$$

Definition 2.3.4. We say that a complex manifold X is **Calabi-Yau** if $c_1(X) = 0$, **Fano** if $c_1(X)$ is positive, of **general type** (or **canonically polarised**) if $-c_1(X)$ is positive.

Note that by Kodaira's embedding theorem, Fano and general type manifolds are projective.

Caution 2.3.5. It is not because a manifold fits in this trichotomy that it admits a Kähler-Einstein metric. In fact, there exist Fano varieties with no Kähler-Einstein metric. Whether a Fano variety admits such metric is equivalent to K-stability, a purely algebro-geometric notion. Nonetheless, Yau (cf. [Yau78]) proved that any Calabi-Yau manifold admits a Kähler-Einstein metric, and Aubin–Yau (cf. [Aub76; Yau78]) proved the same for general type manifolds.

Note also that not all manifolds fit in this trichotomy.

Example 2.3.6 (Curves)**.** Let us see how these categories apply to curves.

- $g = 0$ gives only $\mathbb{P}^1$. Since it is diffeomorphic to a sphere, we have positive scalar curvature. And indeed, the Fubini-Study metric is Kähler-Einstein with $\lambda = 1$. Note also that $\mathbb{P}^1$ is Fano.
- The $g = 1$ case corresponds to elliptic curves. These are Ricci-flat and Calabi-Yau.
- The case $g > 1$ are of general type, and there exists a Kähler-Einstein metric with negative scalar curvature.

For Fano manifolds, here is a summary of the known classifications:

(1) In dimension 1 there is only the projective line.
(2) In dimension 2, they are called *del Pezzo* surfaces. There are 10 different deformation families. First, $\mathbb{P}^2$ and $\mathbb{P}^1 \times \mathbb{P}^1$ which are isolates. The other 8 families are obtained by blowing up $\mathbb{P}^2$ at d points in general position, where $1 \leq d \leq 8$.
(3) In dimension 3, Mukai proved there are 105 families.
(4) In dimension 4, we know there are finitely many families but it remains open to know how many.

For Calabi-Yau manifolds, there is the following structural theorem.

Theorem 2.3.7 (Beauville–Bogomolov; Bog74 and Bea83)**.** *Let X be Kähler and Calabi-Yau. Then, there exists an étale cover $\tilde{X} \to X$ such that*

$$\tilde{X} = T \times \prod_j X_j \times \prod_i V_i$$

*where T is a torus, X_j is **hyperkähler** for all j, and V_i are **strict Calabi-Yau** for all i.*

We now define the terms.

Definition 2.3.8. A compact Kähler manifold V is called **strict Calabi-Yau** if

- $K_V \simeq \mathcal{O}_V$ is trivial, where K_V denotes the canonical bundle;
- V is simply connected;
- $H^i(V, \mathcal{O}_V) = 0$ for all $0 < i < \dim V$.

A complex manifold X is **hyperkähler** if

- it is simply connected;
- $H^0(X, \Omega_X^2) \simeq \mathbb{C}\sigma$ where σ is holomorphic symplectic (in particular, it induces an isomorphism $\mathcal{T}_X \simeq \Omega_X$).

Remark 2.3.9. If V is a strict Calabi-Yau of dimension greater than two, then $h^{2,0} = h^{0,2} = 0$. In particular, $H^{1,1}(X, \mathbb{C}) = H^2(X, \mathbb{C})$, and so $H^{1,1}(X, \mathbb{R}) = H^2(X, \mathbb{R}) = H^{1,1}(X, \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{R}$. Since the Kähler cone is not empty by assumption, we conclude that it intersects $H^{1,1}(X, \mathbb{Z})$, so that V is projective by our discussion on Kodaira's embedding theorem. In dimension 2, non-projective K3 surfaces yield an example of non-projective strict Calabi-Yau manifolds.

Remark 2.3.10. Theorem 2.3.7 has an important corollarly. Namely, if X is a Kähler, we have a degree n étale map $f : \tilde{X} \to X$ with the canonical bunlde $K_{\tilde{X}}$ of $\tilde{X}$ trivial. Since f is étale, $f^*K_X = K_{\tilde{X}}$ is trivial, and so by the projection formula,

$$f_*\mathcal{O}_{\tilde{X}} = f_*f^*K_X = K_X \otimes f_*\mathcal{O}_{\tilde{X}}.$$

Taking determinants, we have

$$\det(f_*\mathcal{O}_{\tilde{X}}) = K_X^n \otimes \det(f_*\mathcal{O}_{\tilde{X}}),$$

so that $K_X^n = \mathcal{O}_X$. Hence, the power of the canonical bundle of a Kähler Calabi-Yau is always trivial.

3. Third lecture: Hodge theory

In this lecture, we recollect Hodge theory.

3.1. **Linear algebra.**

Let us first explore the constructions of Hodge theory in the setting of linear algebra, our toy model.

Let V be a real vector space of dimension n and $\langle -, - \rangle$ be an inner product on V. The scalar product induces a scalar product on $\bigwedge^k V$ via declaring

$$\langle v_1 \wedge \cdots \wedge v_k, u_1 \wedge \cdots \wedge u_k \rangle := \det(\langle u_i, v_j \rangle)_{ij}.$$

Moreover, if $e_1, \ldots, e_n$ is an *ordered* orthonormal basis of V, the vectors $e_{i_1} \wedge \cdots \wedge e_{i_k}$ for $i_1 < \cdots < i_k$ form an orthonormal basis of $\bigwedge^k V$.

Definition 3.1.1. The **volume form** of V (with respect to the chosen ordered basis) is

$$\mathrm{vol}_V = \mathrm{vol} := e_1 \wedge \cdots \wedge e_n.$$

For any $k \leq n$, we define the **Hodge operator** to be the map

$$\begin{aligned} * : \bigwedge^k V &\to \bigwedge^{n-k} V \\ e_{i_1} \wedge \cdots \wedge e_{i_k} &\mapsto \varepsilon e_{j_1} \wedge \cdots \wedge e_{j_{n-k}}, \end{aligned}$$

where $\{e_{j_1}, \ldots, e_{n-k}\}$ is the complement of $\{e_{i_1}, \ldots, e_{i_k}\}$ in the full basis $\{e_1, \ldots, e_n\}$ and this map is well-defined because of the permutation index $\varepsilon = \mathrm{sgn}(i_1, \ldots, i_k, j_1, \ldots, j_{n-k})$.

Let $\alpha = e_{i_1} \wedge \cdots \wedge e_{i_k}$ and $\beta = e_{j_1} \wedge \cdots \wedge e_{j_k}$ be elements of the basis of $\bigwedge^k V$. We have that

$$\alpha \wedge *\beta = \begin{cases} 0 & \text{if } \{i_1, \ldots, i_k\} \neq \{j_1, \ldots, j_k\} \\ \mathrm{vol} & \text{otherwise.} \end{cases}$$

In any case, we have that

$$\alpha \wedge *\beta = \langle \alpha, \beta \rangle \cdot \mathrm{vol}, \tag{3.1}$$

and by bilinearity of the left and right hand side, we see that (3.1) holds for any $\alpha, \beta \in \bigwedge^k V$.

Let I be a complex structure on V, i.e. $I^2 = -\mathrm{id}_V$. Recall that it forces the dimension of V to be even. Suppose furthermore that the inner product is compatible with I, i.e. $\langle I-, I- \rangle = \langle -, - \rangle$. Let $\langle -, - \rangle$ also denote the $\mathbb{C}$-sesquilinear extension of the inner product to a hermitian product on the complexified space $V_{\mathbb{C}}$.

The decomposition $V = V^{1,0} \oplus V^{0,1}$ into $\pm\iota$ eigenspaces is orthodonal for $\langle -, - \rangle$, indeed, for $v \in V^{1,0}$ and $u \in V^{0,1}$, we have

$$\langle v, u \rangle = \langle Iv, Iu \rangle = \langle \iota v, \iota u \rangle = -\langle v, u \rangle.$$

By extension, this shows that the decomposition

$$\bigwedge^k V_{\mathbb{C}} = \bigoplus_{p+q=k} \bigwedge^{p,q} V \qquad \text{where } \bigwedge^{p,q} V := \bigwedge^p V^{1,0} \otimes \bigwedge^q V^{0,1} \tag{3.2}$$

is orthogonal. We may extend the hodge star operator $\mathbb{C}$-linearly to

$$* : \bigwedge^k V_{\mathbb{C}} \to \bigwedge^{n-k} V_{\mathbb{C}},$$

and by looking at an orthonormal basis for (3.2), we see that the operator restricts to

$$* : \bigwedge^{p,q} V \to \bigwedge^{n-q,n-p} V.$$

Note that the equality (3.1) now becomes

$$\alpha \wedge *\overline{\beta} = \langle \alpha, \beta \rangle \mathrm{vol}_{V_{\mathbb{C}}}.$$

3.2. **Harmonic forms.** From now on, (X, h) is a Kähler manifold and $g = \mathfrak{Re}(h)$ is the compatible Riemannian metric associated to h. We will write $\langle -, - \rangle$ for the hermitian metric induced by h on $T^*_{\mathbb{R}}X$. We have the real volume form

$$\mathrm{vol} \in A^{2n}(X).$$

The decomposition

$$\mathcal{A}^k(X, \mathbb{C}) = \bigoplus_{p+q=k} \mathcal{A}^{p,q}(X)$$

is orthogonal with respect to $\langle -, - \rangle$. As before, we have the hodge operator

$$* : \mathcal{A}^{p,q}(X) \to \mathcal{A}^{n-q,n-p}.$$

Recall also our three different exterior derivatives:

$$\begin{aligned} d &: \mathcal{A}^k(X) \to \mathcal{A}^{k+1}(X) \\ \partial &: \mathcal{A}^{p,q}(X) \to \mathcal{A}^{p+1,q}(X) \\ \overline{\partial} &: \mathcal{A}^{p,q}(X) \to \mathcal{A}^{p,q+1}(X). \end{aligned}$$

We use the hodge operator to define other operators:

$$\begin{aligned} d^* &:= (-1)^k *^{-1} \circ d \circ * : \mathcal{A}^k(X) \to \mathcal{A}^{k-1}(X) \\ \partial^* &:= - * \circ \overline{\partial} \circ * : \mathcal{A}^{p,q}(X) \to \mathcal{A}^{p-1,q}(X) \\ \overline{\partial}^* &:= - * \circ \partial \circ * : \mathcal{A}^{p,q}(X) \to \mathcal{A}^{p,q-1}(X). \end{aligned}$$

The mismatch in the sign is a convention, and we note that $*^2 = (-1)^{k(n-k)}$. Extending d^* $\mathbb{C}$-linearly, these operators satisfy

$$d^* = \partial^* + \overline{\partial}^*$$

We use these operators to define three different Laplacians:

$$\begin{aligned} \Delta_d &:= dd^* + d^*d : \mathcal{A}^k(X) \to \mathcal{A}^k(X) \\ \Delta_\partial &:= \partial\partial^* + \partial^*\partial : \mathcal{A}^{p,q}(X) \to \mathcal{A}^{p,q}(X) \\ \Delta_{\overline{\partial}} &:= \overline{\partial}\,\overline{\partial}^* + \overline{\partial}^*\overline{\partial} : \mathcal{A}^{p,q}(X) \to \mathcal{A}^{p,q}(X). \end{aligned}$$

A k-form α is **d-harmonic** if $\Delta_d\alpha = 0$, and a (p,q)-form β is **$\overline{\partial}$-harmonic** if $\Delta_{\overline{\partial}}\beta = 0$. We will write $\mathcal{H}^k(X) := \ker \Delta_d$ for the space of d-harmonic k-forms (note, we can do this either over $\mathbb{R}$ or $\mathbb{C}$ and do not specify), and $\mathcal{H}^{p,q}(X) := \ker \Delta_{\overline{\partial}}$ for the space of $\overline{\partial}$-harmonic (p,q)-forms.

We have an L^2 inner product

$$\begin{aligned} A^k(X) \otimes A^k(X) &\to \mathbb{R} \\ (\alpha, \beta) &\mapsto \int_X \langle \alpha, \beta \rangle \mathrm{vol} = \int_X \alpha \wedge *\beta. \end{aligned}$$

From the right hand side, we compute that for $\alpha \in A^k(X)$ and $\beta \in A^{k+1}(X)$,

$$(d\alpha, \beta) = \int_X d\alpha \wedge *\beta = \int_X \alpha \wedge (-1)^{k+1} d(*\beta) = \int_X \alpha \wedge *(-1)^{k+1} *^{-1} d(*\beta) = \int_X \alpha \wedge *d^*\beta = (\alpha, d^*\beta),$$

so that d and d^* are formal adjoints with respect to $(-,-)$. From this, we see that

$$(\alpha, \Delta_d\alpha) = (\alpha, dd^*\alpha + d^*d\alpha) = (d^*\alpha, d^*\alpha) + (d\alpha, d\alpha),$$

and so by positive definiteness, $\Delta\alpha = 0$ if and only if $d\alpha = d^*\alpha = 0$. This yields a map

$$\mathcal{H}^k(X, \mathbb{R}) \to H^k(X, \mathbb{R}).$$

Theorem 3.2.1 (Hod41)**.** *This map is an isomorphism.*

Remark 3.2.2. Note that this works equally well over $\mathbb{C}$ witht the hermitian product

$$(\alpha, \beta) \mapsto \int_X \langle \alpha, \beta \rangle \mathrm{vol} = \int_X \alpha \wedge *\overline{\beta}$$

3.3. **The Dolbeault side.** We can get a similar theorem by taking into account the (p,q) type of forms. First note that for any p, we have a complex

$$0 \to \Omega_X^p \to \mathcal{A}^{p,0}(X) \to \mathcal{A}^{p,1}(X) \to \cdots \to \mathcal{A}^{p,n}(X) \to 0.$$

The $\overline{\partial}$-Poincaré lemma (also called the Dolbeault–Grothendieck lemma) says that this complex is exact, i.e. that analytically locally, a $\overline{\partial}$-closed form is $\overline{\partial}$-exact. Moreover, since each $\mathcal{A}^{p,q}(X)$ admits partitions of unity (these are smooth sections), this is an acyclic resolution. As corollary, we obtain an isomorphism

$$H^q(X, \Omega_X^p) = \mathbb{H}^q(\mathcal{A}^{p,0}(X) \to \cdots \to \mathcal{A}^{p,n}(X) \to 0) = \frac{\ker\left(\overline{\partial} : A^{p,q}(X) \to A^{p,q+1}(X)\right)}{\operatorname{im}\left(\overline{\partial} : A^{p-1,q}(X) \to A^{p,q}(X)\right)} =: H^{p,q}(X), \tag{3.3}$$

and we call the right-hand side the (p,q) Dolbeault cohomology of X.

As before, we have an inner product on (p,q)-forms:

$$(\alpha, \beta) := \int_X \alpha \wedge *\overline{\beta},$$

and the same computation shows that ∂^* and $\overline{\partial}^*$ are formal adjoints. Therefore, $\Delta_{\overline{\partial}}\alpha = 0$ if and only if $\overline{\partial}\alpha = \overline{\partial}^*\alpha = 0$. This yields a map

$$\mathcal{H}^{p,q}(X) \to H^{p,q}(X).$$

Theorem 3.3.1. *This map is an isomorphism.*

It is worth noting that Theorem 3.3.1 does not require X to be Kähler.

Proposition 3.3.2. *If X is compact Kähler, then*

$$\Delta_d = 2\Delta_{\overline{\partial}} = 2\Delta_{\partial}.$$

Corollary 3.3.3 (Hodge decomposition)**.** *Let X be a compact Kähler manifold. We have a decomposition*

$$H^k(X, \mathbb{C}) = \oplus_{p+q=k} H^{p,q}(X)$$

such that $H^{p,q}(X) = \overline{H^{q,p}(X)}$ and $H^{p,q}(X) \simeq H^q(X, \Omega_X^p)$.

Proof. Note that $\Delta_{\overline{\partial}}$ preserves the (p,q)-type, and thus so does Δ_d by Proposition 3.3.2. Therefore, we have

$$\bigoplus_{p+q=k} H^{p,q}(X) \simeq \bigoplus_{p+q=k} \mathcal{H}^{p,q}(X) = \mathcal{H}^k(X, \mathbb{C}) \xrightarrow{\sim} H^k(X, \mathbb{C}). \tag{3.4}$$

It remains to show that $H^{p,q}(X) = \overline{H^{q,p}(X)}$, and this follows from the fact that $\Delta_d(\overline{\alpha}) = \overline{\Delta_d(\alpha)}$, as Δ_d is a real operator. ☐

Remark 3.3.4. It is worthwile to observe that along the isomorphism (3.4), $H^{p,q}(X)$ is identified with the subset $\{[\alpha] : d\alpha = 0 \text{ and } \alpha \in A^{p,q}\} \subset H^k(X, \mathbb{C})$. Note moreover that the Hodge decomposition is compatible witht the wedge product (that is, wedging (p,q) $\overline{\partial}$-closed forms yields the same algebra structure as that on $H^\bullet(X, \mathbb{C})$ along (3.4)). However, there is no clear algebra structure on

$$\bigoplus_{p,q \geq 0} \mathcal{H}^{p,q}(X),$$

as wedging two harmonic forms need not yield a harmonic form (this comes from the fact that d^* does not satisfy a Leibniz rule).

Corollary 3.3.5. *Let X be a compact Kähler manifold. If k is odd, then the kth Betti number $b_k(X)$ is even.*

Remark 3.3.6. For any Kähler form ω,

$$\int_X \omega^{\dim X} = n! \cdot \operatorname{vol}(X) > 0,$$

implying that ω is not d-exact (assuming X compact), i.e. that $b_k > 0$ for k odd.

Remark 3.3.7 (Hodge diamond)**.** We can make the Hodge numbers fit in what is called the Hodge diamond:

$$
\begin{array}{ccccccccc}
 & & & & h^{0,0} & & & & \\
 & & & h^{1,0} & & h^{0,1} & & & \\
 & & h^{2,0} & & h^{1,1} & & h^{0,2} & & \\
 & \iddots & & & & & & \ddots & \\
h^{n,0} & & \cdots & & \circlearrowleft & & \cdots & & h^{0,n} \\
 & \ddots & & & & & & \iddots & \\
 & & h^{n,n-2} & & h^{n-1,n-1} & & h^{n-2,n} & & \\
 & & & h^{n,n-1} & & h^{n-1,n} & & & \\
 & & & & h^{n,n} & & & & \\
 & & & & \longleftrightarrow & & & &
\end{array}
$$

where $n = \dim_{\mathbb{C}} X$, and these are the only non-zero Hodge numbers. Moreover, there are some symmetries. We already saw Hodge symmetry, which implies $h^{p,q} = h^{q,p}$, which is represented by the arrow in the bottom.

Recalll that some version of Poincaré duality says that there is a non-degenerate pairing

$$
\begin{aligned}
H^k(X,\mathbb{C}) \times H^{2n-k}(X,\mathbb{C}) &\to \mathbb{C} \\
(\alpha,\beta) &\mapsto \int_X \alpha \wedge \beta,
\end{aligned}
\tag{3.5}
$$

yielding an isomorphism $H^k(X,\mathbb{C})^\vee = H_k(X,\mathbb{C}) \simeq H^{2n-k}(X,\mathbb{C})$. This pairing restricts to a non-degenerate pairing

$$H^{p,q}(X) \times H^{n-p,n-q}(X) \to \mathbb{C};$$

indeed, if a d-closed form of type (p,q) is non-zero, we know by (3.5) that we may pair it with a form β with $\alpha \wedge \beta \neq 0$, but this forces the type of β to be $(n-p, n-q)$. Therefore, we get an isomorphism

$$H^{p,q}(X) = H^{n-p,n-q}(X)^\vee. \tag{3.6}$$

In particular, our diamond has the symmetry $h^{p,q} = h^{n-p,n-q}$, which is represented by the central circling arrow in the Hodge diamond. Note that (3.6) can also be seen using Serre duality:

$$H^{p,q}(X) = H^q(X,\Omega_X^p) = H^{n-q}(X,(\Omega_X^p)^\vee \otimes K_X)^\vee = H^{n-q}(X,\Omega_X^{n-p})^\vee = H^{n-p,n-q}(X)^\vee,$$

where the isomorphism $(\Omega_X^p)^\vee \otimes K_X = \bigwedge^p \mathcal{T}_X \otimes K_X \simeq \Omega^{n-p}$ comes from contraction of vector fields:

$$
\begin{aligned}
\bigwedge^p \mathcal{T}_X \otimes K_X &\xrightarrow{\sim} \Omega^{n-p} \\
X_1 \wedge \cdots \wedge X_p \otimes \alpha &\mapsto \alpha(X_1,\dots,X_p,-,\dots,-).
\end{aligned}
$$

Note that there is also a way to see this duality with the Hodge star operator.

The Hodge diamond also satisfies a unimodal condition. Namely, in each row (hence each column by Poincaré/Serre-duality), the Hodge numbers increase before reaching half, then decrease (the latter follows from the former by Hodge symmetry).

3.4. **Lefschetz theorems.** Let X be compact Kähler manifold with Kähler form ω. As ω is a $(1,1)$ real form, we obtain an operator

$$
\begin{aligned}
L_\omega : A^k(X) &\to A^{k+2}(X) \\
\alpha &\mapsto \omega \wedge \alpha.
\end{aligned}
$$

In the complexification, this restrict to

$$L_\omega : A^{p,q}(X) \to A^{p+1,q+1}(X),$$

and these operators descend to cohomology by definition. We define the operator

$$\Lambda_\omega := *^{-1} \circ L_\omega \circ *,$$

and the degree operator

$$h : H^*(X) \to H^*(X)$$

where $h|_{H^k(X)} = (k-n)\mathrm{id}|_{H^k(X)}$. Here, we do not specify the coefficients, as we want to work over either $\mathbb{R}$ or $\mathbb{C}$.

Theorem 3.4.1. *On cohomology, we have $[L_\omega, \Lambda_\omega] = h$, $[h, L_\omega] = 2L_\omega$ and $[2, \Lambda_\omega] = -2\Lambda_\omega$, that is, L_ω, Λ_ω and h form an $\mathfrak{sl}_2$-triple. Moreover, we have $[L_\omega, \Delta_d] = 0 = [\Lambda_\omega, \Delta_d]$.*

From this $\mathfrak{sl}_2$-representation, one can deduce the following.

Theorem 3.4.2 (Hard Lefschetz)**.** *Let $k \leq n$.*

(1) *the map*
$$L_\omega^{n-k} : H^k(X) \to H^{2n-k}(X)$$
is an isomorphism.

(2) *Let $H^k(X)_{prim} \subset H^k(X)$ be the kernel of L_ω^{n-k+1}. We have a **Lefschetz decomposition***
$$H^k(X) = \bigoplus_{2r \leq kr} L_\omega^r H^{k-2r}(X)_{prim}.$$
Moreover, over $\mathbb{C}$, this decomposition is compatible with the Hodge decomposition; i.e, if we define
$$H^{p,q}(X)_{prim} := \ker\left(L_\omega^{2n-p-q+1} : H^{p,q}(X) \to H^{n-p+1,n-q+1}(X)\right),$$
we have
$$H^{p,q}(X) = \bigoplus_{2r \leq p+q} L_\omega^r H^{p-r,q-r}(X)_{prim},$$
and
$$H^k(X, \mathbb{C})_{prim} = \bigoplus_{p+q=k} H^{p,q}(X)_{prim}.$$

Remark 3.4.3. Note that when X is projective, we may choose ω to be integral and the Lefschetz decomposition also holds over $\mathbb{Q}$.

Example 3.4.4. Let us study the consequences of these theorems on the cohomology of a surface X with Kähler form ω. We have the diagram

$$\begin{tikzcd}
H^0(X) \arrow[r, equal] \arrow[dddd, bend right=60, leftrightarrow, "L_\omega^2"'] & H^0(X)_{\mathrm{prim}} \\
H^1(X) \arrow[r, equal] \arrow[dd, bend right=60, leftrightarrow, "L_\omega"'] & H^1(X)_{\mathrm{prim}} \\
H^2(X) = H^2(X)_{\mathrm{prim}} \oplus LH^0(X) & \\
H^3(X) & \\
H^4(X), &
\end{tikzcd}$$

where $H^0(X)_{\mathrm{prim}} = H^0(X)$ and $H^1(X)_{\mathrm{prim}} = H^1(X)$ since the primitive parts are defined as the kernel of maps to a cohomology groups that vanish for dimension reasons. Since $H^0(X)$ is generated by the identity element in the cohomology ring, we obtain (over say $\mathbb{R}$ coefficients) $H^2(X, \mathbb{R}) = \mathbb{R}[\omega] \oplus H^2(X, \mathbb{R})_{\mathrm{prim}}$. As we soon shall see, this decomposition is orthogonal with respect to a certain intersection pairing, so that we may write $H^2(X, \mathbb{R}) = \mathbb{R}[\omega] \oplus [\omega]^\perp$.

3.5. **Hodge index theorem.** Let X be compact and ω be a Kähler form. Consider the complex Poincaré pairing
$$H^k(X, \mathbb{C}) \times H^{2n-k}(X, \mathbb{C}) \to \mathbb{C}$$
$$(\alpha, \beta) \mapsto \int_X \alpha \wedge \beta.$$
This pairing is skew symmetric for k odd and symmetric for k even. We use the polarisation ω to turn this into a pairing on $H^k(X, \mathbb{C})$ for $k \leq n$:
$$Q_k : H^k(X, \mathbb{C}) \times H^k(X, \mathbb{C}) \to \mathbb{C}$$
$$(\alpha, \beta) \mapsto \int_X \alpha \wedge L_\omega^{n-k}(\beta).$$
We now define the hermitian pairing $H_k(\alpha, \beta) := \iota^k Q_k(\alpha, \overline{\beta})$ called the **Hodge–Riemann bilinear form**.

Theorem 3.5.1 (Hodge–Riemann bilinear relations)**.** *The following hold true*

(1) *The Hodge decomposition is orthogonal with respect to* H_k*;*
(2) *The Lefschetz decomposition is orthogonal with respect to* H_k*, and for* $\alpha, \beta \in H^{k-2r}(X, \mathbb{C})_{prim}$*, we have*

$$H_k(L^r(\alpha), L^r(\beta)) = (-1)^{k+r} H_{k-2r}(\alpha, \beta);$$

(3) *The form*

$$(-1)^{\frac{k(k-1)}{2}} \iota^{p-q-k} H_k$$

is positive definite on $H^{p,q}(X)_{prim}$*.*

Proof. (1) Let $\alpha \in H^{p,q}(X)$ and $\beta \in H^{p',q'}(X)$ with $p+q = p'+q'$. By definition, we have

$$H_k(\alpha, \beta) = \iota^k \int_X \alpha \wedge \omega^{n-k} \wedge \overline{\beta},$$

but the form $\alpha\omega^{n-k} \wedge \wedge \overline{\beta}$ is of degree $2n$ but not of type (n,n). Hence it vanishes, implying that the integral vanishes.

(2) Let $\alpha' = L_\omega^r(\alpha)$ for $\alpha \in H^{k-2r}(X)_{\text{prim}}$ and $\beta' = L_\omega^s(\beta)$ for $\beta \in H^{k-2s}(X)_{\text{prim}}$. Without loss of generality, assume $r > s$. We have that

$$H_k(\alpha', \beta') = \iota^k \int_X \alpha' \wedge L_\omega^{n-k+s}(\overline{\beta'}) = \iota^k \int_X \omega^r \alpha \wedge \omega^{n-k+s} \wedge \overline{\beta} = (-1)^k \iota^k \int_X \alpha \wedge L_\omega^{n-k+r+s}(\overline{\beta}) = 0$$

since $n-k+r+s > n-k+2s$, so that primitiveness of β ensures it is in the kernel of $L_\omega^{n-k+r+s}$.

Now if α', β' are chosen as above but $r = s$, we have

$$\begin{aligned} H_k(\alpha', \beta') &= \iota^k \int_X \alpha \wedge L^{n-k+r}(\overline{\beta'}) = \iota^k \int_X \omega^r \wedge \alpha \wedge \omega^{n-k+r} \wedge \overline{\beta} \\ &= (-1)^k \iota^{2r} \iota^{k-2r} \int_X \alpha \wedge \omega^{n-k+2r} \wedge \overline{\beta} = (-1)^{k+r} H_{k-2r}(\alpha, \beta). \end{aligned}$$

(3) For $\alpha \in H^{p,q}(X)_{\text{prim}}$. For such form, on Kähler manifolds, we have

$$*\alpha = \iota^{p-q} (-1)^{\frac{k(k-1)}{2}} \frac{\omega^{n-k}}{(n-k)!} \wedge \alpha;$$

see [Voi98, Proposition 6.29]. Therefore, we obtain that

$$\begin{aligned} H_k(\alpha, \alpha) &= \iota^k \int_X \alpha \wedge \omega^{n-k} \wedge \overline{\alpha} = (n-k)!(-1)^{\frac{k(k-1)}{2}} \iota^{k-p+q} \int_X \alpha \wedge *\overline{\alpha} \\ &= (n-k)!(-1)^{\frac{k(k-1)}{2}} \iota^{k-p+q} \int_X \langle \alpha, \alpha \rangle \mathrm{vol}_{\mathbb{C}} \end{aligned}$$

so that

$$(-1)^{\frac{k(k-1)}{2}} \iota^{p-q+k} H_k$$

is positive-definite. □

Corollary 3.5.2 (Hodge index theorem)**.** *Let* X *be a compact Kähler surface. Then, the signature of the Poincaré intersection pairing* Q_2 *on* $H^2(X, \mathbb{R})$ *is*

$$(2h^{2,0} + 1, h^{1,1} - 1).$$

Proof. Let $\alpha \in H^2(X, \mathbb{R})$. We may decompose into types: $\alpha = \alpha^{2,0} + \alpha^{1,1} + \alpha^{0,2}$. The fact that $\alpha = \overline{\alpha}$ forces $\alpha^{1,1} \in H^{1,1}(X, \mathbb{R})$, and $\alpha^{2,0} = \overline{\alpha^{2,0}}$. We thus have the decomposition

$$H^2(X, \mathbb{R}) = ((H^{2,0}(X) \oplus H^{0,2}) \cap H^2(X, \mathbb{R})) \oplus H^{1,1}(X, \mathbb{R}),$$

which we know to be orthogonal with respect to the Poincaré pairing. Any $\alpha \in (H^{2,0}(X) \oplus H^{0,2}(X)) \cap H^2(X, \mathbb{R})$ is primitive for degree reasons: taking the cup product with ω yields types $(3,1)$ or $(1,3)$. Thus, we have

$$Q_2(\alpha, \alpha) = \int_X \alpha \wedge \alpha = \int_X \alpha^{2,0} \wedge \overline{\alpha^{2,0}},$$

which is positive by Theorem 3.5.1(3). Now, we have the Lefschetz decomposition

$$H^{1,1}(X, \mathbb{R}) = H^{1,1}(X, \mathbb{R})_{\text{prim}} \oplus L_\omega H^0(X, \mathbb{R}) = H^{1,1}(X, \mathbb{R})_{\text{prim}} \oplus \mathbb{R}[\omega].$$

This decomposition is orthogonal: if $\alpha \in H^{1,1}(X,\mathbb{R})_{\text{prim}}$, then

$$Q_2(\omega, \alpha) = \int_X \omega \wedge \alpha = 0$$

as $\omega \wedge \alpha = 0$ by definition of primitive cohomology. We have

$$Q_2(\omega, \omega) = \int_X \omega^2 = 2 \cdot \text{vol}(X) > 0.$$

There remains to compute Q_2 on real $(1,1)$ primitive classes. But by Theorem 3.5.1,

$$\int_X \alpha^2 < 0,$$

and so we obtain the right count for the index of the pairing. □

3.6. **(1,1) classes.** Consider now the exponential exact sequence

$$0 \to \mathbb{Z} \to \mathcal{O}_X \xrightarrow{exp(2\pi i -)} \mathcal{O}_X^* \to 0.$$

It is a fact that the induced connecting homomorphism δ in the long exact cohomological sequence

$$\cdots \to H^1(X,\mathbb{Z}) \to H^1(X,\mathcal{O}_X) \to \text{Pic}(X) \xrightarrow{\delta} H^2(X,\mathbb{Z}) \to H^2(X,\mathcal{O}_X) \to \cdots$$

is related to the Chern map $c_1 : \text{Pic}(X) \to H^2(X,\mathbb{C})$: after composing δ with the map $H^2(X,\mathbb{Z}) \to H^2(X,\mathbb{C})$ (which kills torsion), we have $\delta(\mathcal{L}) = -c_1(\mathcal{L})$, where $\mathcal{L} \in \text{Pic}(X)$.

Theorem 3.6.1 (Lefschetz theorem on $(1,1)$ classes)**.** *If X is compact Kähler, then the first chern map c_1 composed with $H^2(X,\mathbb{Z}) \to H^2(X,\mathbb{C})$ is surjective onto $H^{1,1}(X,\mathbb{Z})$.*

Proof. Note that the compositio is indeed valued in $H^{1,1}(X,\mathbb{Z})$ by definition of the Chern class (indeed, the $(2,0)$ and $(0,2)$ part of the curvature of the Chern connection vanish). The map $H^2(X,\mathbb{Z}) \to H^2(X,\mathcal{O}_X)$ factors as

$$H^2(X,\mathbb{Z}) \to H^2(X,\mathbb{C}) \to H^{0,2}(X,\mathbb{C}) \simeq H^2(X,\mathcal{O}_X),$$

where the middle arrow is the projection onto the $(2,0)$ part. This can be seen as follows: the map $\mathbb{Z} \to \mathcal{O}_X$ factors as $\mathbb{Z} \to \mathbb{C} \to \mathcal{O}_X$, and the map in cohomology $H^2(\mathbb{C}, X) \to H^2(X,\mathcal{O}_X)$ may be computed by considering the map between the resolutions

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbb{C} & \longrightarrow & \mathcal{A}^1(\mathbb{C}) & \xrightarrow{d} & \mathcal{A}^2(\mathbb{C}) & \xrightarrow{d} & \cdots \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathcal{O}_X & \longrightarrow & \mathcal{A}^{0,1} & \xrightarrow{\bar\partial} & \mathcal{A}^{0,2} & \xrightarrow{\bar\partial} & \cdots, \end{array}$$

where the vertical arrows are projections. Thus, the map $H^2(X,\mathbb{Z}) \to H^2(X,\mathcal{O}_X)$ vanishes on $(1,1)$ classes, so that $(1,1)$ classes are in the kernel of this map, and equivalently in the image of $c_1 = -\delta$. □

Definition 3.6.2. We define the **Néron-Severi** group of X, denoted $\text{NS}(X)$, to be the image of c_1 in $H^2(X,\mathbb{Z})$. We note that $\text{NS}(X)/\text{torsion} = H^{1,1}(X,\mathbb{Z})$, as $H^{1,1}(X,\mathbb{Z})$ is defined to be the intersection of $H^{1,1}(X)$ with the image of $H^2(X,\mathbb{Z})$ in $H^2(X,\mathbb{C})$.

We define the rank $\rho(X)$ of $\text{NS}(X)$ (or equivalently of $H^{1,1}(X,\mathbb{Z})$) to be the **Picard number**. To motivate this terminology, note that there is an exact sequence

$$0 \to \text{Pic}^0(X) \to \text{Pic}(X) \to \text{NS}(X) \to 0,$$

where $\text{Pic}^0(X)$ is the connected component of the identity.

Remark 3.6.3. The integral Hodge conjecture, which has been proven to be false, states that the map

$$\begin{aligned} \{\text{algebraic } \mathbb{Z}\text{-cycles of dimension } k\} &\to H^{(k,k)}(X,\mathbb{Z}) \\ Z &\mapsto [Z] \end{aligned}$$

is surjective. Nevertheless, the Lefschetz theorem on $(1,1)$ classes shows that it is true for $k = 1$. The rational Hodge conjecture, notoriously unsolved, asks whether taking this map over $\mathbb{Q}$-coefficients is surjective.

4. Fourth Lecture: K3 surfaces

4.1. **K3 Surfaces.** We now focus on K3 surfaces, which are the fundamental examples in dimension 2 arising from the Beauville-Bogomolov decomposition theorem.

Definition 4.1.1 (Strong definition)**.** A **K3 surface** is a compact connected Kähler surface S such that

(1) The canonical bundle is trivial, $K_S \simeq \mathcal{O}_S$;
(2) S is simply connected, i.e. $\pi_1(S) = \{e\}$.

K3 surfaces are exactly strict Calabi-Yau manifolds of dimension 2. To show this, we introduce a second definition of K3 surfaces, which we will show are equivalent.

Definition 4.1.2 (Weak definition)**.** A K3 surface is a compact, connected Kähler surface S such that

(1) $K_S \simeq \mathcal{O}_S$;
(2) $H^1(S, \mathcal{O}_S) = 0$.

Proposition 4.1.3. *The two definitions (Definition 4.1.1 and Definition 4.1.2) are equivalent. In particular, K3 surfaces are exactly strict Calabi-Yau manifolds surfaces.*

Proof. (4.1.1 $\implies$ 4.1.2): Assume S satisfies the strong definition. We need to show $H^1(S, \mathcal{O}_S) = 0$. Since S is simply connected, $H_1(S, \mathbb{C})$ and hence $H^1(S, \mathbb{C}) = H_1(S, \mathbb{C})^\vee$ vanish. Since S is Kähler, the Hodge decomposition gives $H^1(S, \mathbb{C}) = H^{1,0}(S) \oplus H^{0,1}(S)$. Thus $H^{0,1}(S) \simeq H^1(S, \mathcal{O}_S) = 0$.

(4.1.2 $\implies$ 4.1.1): Assume S satisfies the weak definition. A deep theorem proved by Siu in [Siu83] that any such surface is Kähler. We need to prove that S is simply connected.

First, we compute the holomorphic Euler characteristic:

$$\begin{aligned}\chi(S, \mathcal{O}_S) &= h^0(S, \mathcal{O}_S) - h^1(S, \mathcal{O}_S) + h^2(S, \mathcal{O}_S) \\ &= 1 - 0 + h^0(S, K_S) \quad \text{(by Serre duality)} \\ &= 1 - 0 + 1 = 2.\end{aligned}$$

We apply the Beauville-Bogomolov decomposition theorem (Theorem 2.3.7) to conclude that there exists an étale cover $\pi : \tilde{S} \to S$ where $\tilde{S}$ is either a complex torus, a Hyperkähler surface, a strict Calabi-Yau surface, or a product of two elliptic curves. Suppose first that $\tilde{S}$ is either a product of two elliptic curves or of a complex torus. Then,

$$\chi(\tilde{S}, \mathcal{O}_{\tilde{S}}) = \int_{\tilde{S}} ch(\mathcal{O}_S) \cup td(S) = \int_{\tilde{S}} \frac{c_1^2(S) + c_2(S)}{12} = 0$$

since $\tilde{S}$ is flat. But by Hirzebruch-Riemann-Roch again, we have $\chi(\tilde{S}, \mathcal{O}_{\tilde{S}}) = \deg(\pi)\chi(S, \mathcal{O}_S) = 2\deg(\pi)$, which forces $\deg(\pi) = 0$, contradicting that π is a covering.

Therefore, $\tilde{S}$ is either a strict Calabi-Yau or a Hyperkähler manifold (we will see *a posteriori* that these two conditions coincide for surfaces), and in particular is simply connected. Thus, $\deg(\pi) = 1$ and $S = \tilde{S}$ is simply connected. ☐

Remark 4.1.4. Importantly, it is also true that K3 surfaces are exactly Hyperkähler manifolds of dimension 2, but we shall see this later.

4.2. **Cohomology and Picard group of K3 surfaces.**

Theorem 4.2.1. *Let S be a K3 surface. Then:*

(1) $H^0(S, \mathbb{Z}) \simeq H^4(S, \mathbb{Z}) \simeq \mathbb{Z}$.
(2) $H^1(S, \mathbb{Z}) = H^3(S, \mathbb{Z}) = 0$.
(3) $H^2(S, \mathbb{Z}) \simeq \mathbb{Z}^{22}$ *and is torsion-free.*
(4) *The intersection pairing (cup product)*

$$(-,-) : H^2(S, \mathbb{Z}) \times H^2(S, \mathbb{Z}) \to \mathbb{Z}, \quad (\alpha, \beta) \mapsto \int_S \alpha \cup \beta$$

is symmetric, bilinear, and unimodular (i.e. induces an isomorphism $H^2(S, \mathbb{Z}) \to H^2(S, \mathbb{Z})^\vee$).

(5) *The signature of the pairing is* $(3, 19)$.
(6) *The pairing is even, i.e.,* $\alpha^2 := (\alpha, \alpha) \equiv 0 \pmod 2$ *for all* $\alpha \in H^2(S, \mathbb{Z})$.

Proof. We use the strong definition.

(1): S is compact and oriented.

(2): Since S is simply connected, $H_1(S,\mathbb{Z})=0$. By the universal coefficient theorem, $H^1(S,\mathbb{Z})=0$ (there is no torsion in $H_0(S,\mathbb{Z})$). By Poincaré duality, $H^3(S,\mathbb{Z})\simeq H_1(S,\mathbb{Z})^\vee=0$.

(3) By the universal coefficient theorem, the torsion subgroup of $H^2(S,\mathbb{Z})$ comes from the torsion of $H_1(S,\mathbb{Z})=0$. So $H^2(S,\mathbb{Z})$ is torsion-free.

To compute the rank $b_2(S)$, we use the topological Euler characteristic $e(S)$.

$$\chi_{top}(S)=\sum(-1)^i b_i(S)=1-0+b_2(S)-0+1=2+b_2(S).$$

By the Gauss-Bonnet theorem, we have

$$\chi_{top}(S)=\int_S c_2(S).$$

However, Hirzebruch-Riemann-Roch theorem gives us

$$\chi(S,\mathcal{O}_S)=\int_S \frac{c_1(S)^2+c_2(S)}{12}=\int_S \frac{c_2(S)}{12},$$

since S is Ricci-flat by definition. Since $\chi(S,\mathcal{O}_S)=2$, we conclude that $b_2(S)=2\cdot 12-2=22$.

(4) Unimodularity follows from Poincaré duality over $\mathbb{Z}$. Symmetry just follows from the degree in cohomology.

(5) This is a direct consequence of the Hodge index theorem (Corollary 3.5.2) and the fact that $h^{1,1}(S)=20$ since $h^{2,0}=h^0(S,\mathcal{O}_S)=1$.

(6) For algebraic classes $\alpha=c_1(L)\in NS(S)$, we use Hirzebruch-Riemann-Roch:

$$\chi(S,L)=\frac{c_1(L)^2}{2}+\chi(S,\mathcal{O}_S)=\frac{\alpha^2}{2}+2.$$

Since $\chi(S,L)\in\mathbb{Z}$, α^2 must be even.

For the general $\alpha\in H^2(S,\mathbb{Z})$, this follows from Wu's formula. □

Remark 4.2.2 (Hodge Diamond)**.** The Hodge diamond of a K3 surface is the following

$$\begin{array}{ccccc} & & 1 & & \\ & 0 & & 0 & \\ 1 & & 20 & & 1 \\ & 0 & & 0 & \\ & & 1 & & \end{array}$$

Proposition 4.2.3. *For a K3 surface S, the Picard group is isomorphic to the Néron-Severi group.*

Proof. Consider the exponential sequence:

$$\cdots\to H^1(S,\mathcal{O}_S)\to \operatorname{Pic}(S)\xrightarrow{-c_1} H^2(S,\mathbb{Z})\to\cdots$$

Since $H^1(S,\mathcal{O}_S)=0$, the map c_1 is injective. Thus $\operatorname{Pic}(S)\simeq \operatorname{im}(c_1)=\operatorname{NS}(S)$. □

Remark 4.2.4. This implies that a line bundle on a K3 surface is determined by its first Chern class.

The Picard number satisfies $\rho(S)\le h^{1,1}(S)=20$. If S is algebraic, $\rho(S)\ge 1$ by Kodaira's embedding theorem. For the very general[2] K3 surface, $\rho(S)=0$.

4.3. **Examples of K3 surfaces.** We now discuss examples of how K3 surfaces may be constructed.

Example 4.3.1 (Quartic surface in $\mathbb{P}^3$)**.** Let $S=\{f=0\}\subset\mathbb{P}^3$ be a smooth hypersurface defined by a homogeneous polynomial f of degree 4. We verify that S is a K3 surface.

By adjunction formula, we have

$$K_S=(K_{\mathbb{P}^3}\otimes\mathcal{O}_{\mathbb{P}^3}(4))|_S=(\mathcal{O}_{\mathbb{P}^3}(-4)\otimes\mathcal{O}_{\mathbb{P}^3}(4))|_S=\mathcal{O}_S.$$

There remains to show $H^1(S,\mathcal{O}_S)=0$: Consider the short exact sequence defining S:

$$0\to\mathcal{O}_{\mathbb{P}^3}(-4)\xrightarrow{\cdot f}\mathcal{O}_{\mathbb{P}^3}\to\mathcal{O}_S\to 0.$$

[2]The notion of "very general" is to be distinguished from "generic": a property holds for the very general K3 surface if, in the appropriate moduli space, it holds outside of a *countable* union of (analytic) Zariski-closed sets.

The long exact sequence in cohomology gives:

$$\cdots \to H^1(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}) \to H^1(S, \mathcal{O}_S) \to H^2(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(-4)) \to \cdots.$$

Since $H^1(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}) = H^2(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(-4)) = 0$, we conclude $H^1(X, \mathcal{O}_X) = 0$.

To construct further examples, we review the cyclic covering trick, used to construct branched covers. If we are given a covering $f : X \to Y$, the ramification divisor R is that where the rank of the differential drops, i.e. the zero locus of

$$\det(df) : f^*K_Y \to K_X.$$

This shows that $R \in |f^*K_Y^\vee \otimes K_X|$. Quite tautologically, the Hurwitz formula follows:

$$K_X = f^*K_Y \otimes \mathcal{O}(R).$$

We say that f is ramified over $f(R)_{\text{red}} = B$. Conversely, given the choice of an effective divisor $B \subset Y$, we want to describe ways to construct finite coverings of Y that are ramified over B.

Construction 4.3.2 (Cyclic covering trick)**.** Assume for simplicity that Y is algebraic, the analytic case shall be discussed in Remark 4.3.6. Let $B \subset Y$ be an effective reduced divisor, and suppose that $\mathcal{O}(B)$ is a mth power for some $m \geq 2$, that is, there exists a line bundle $\mathcal{L} \in \operatorname{Pic}(Y)$ with $\mathcal{L}^m = \mathcal{O}(B)$, and let $s \in \mathcal{O}(B)$ be a defining section for D.

Let $\mathbb{V}(\mathcal{L})$ be the total space of $\mathcal{L}$. We have

$$\mathbb{V}(\mathcal{L}) = \operatorname{Spec}(\operatorname{Sym}^\bullet \mathcal{L}^\vee).$$

Let $\pi : \mathbb{V}(\mathcal{L}) \to Y$ be the projection. We have a tautological section

$$\tau \in H^0(\mathbb{V}(\mathcal{L}), \pi^*\mathcal{L}) = \mathcal{L} \otimes \bigoplus_{i \leq 0} \mathcal{L}^i$$

given by the identity element of $\mathcal{L}^\vee \otimes \mathcal{L}$ and whose zero locus coincides with that of the zero section $Y \subset \mathbb{V}(\mathcal{L})$. Consider the variety X defined as the zero locus of the section $\tau^m - \pi^*s \in \pi^*\mathcal{L}^m$, i.e.

$$X := Z(\tau^m - \pi^*s) \subset \mathbb{V}(\mathcal{L}).$$

the map $f : X \hookrightarrow \mathbb{V}(\mathcal{L}) \xrightarrow{\pi} Y$ is finite, and it is ramified over B

Remark 4.3.3. By using the Jacobian criterion on the local equations for X, it is obvious that X is smooth if and only if B is.

Lemma 4.3.4. *Let $\pi : X \to Y$ be the m-cyclic cover defined by B as above.*

(1) *The pushforward of the structure sheaf is $f_*\mathcal{O}_X \simeq \bigoplus_{j=0}^{m-1} \mathcal{L}^{-j}$.*
(2) *The canonical bundle is $K_X \simeq f^*(K_Y \otimes \mathcal{L}^{m-1})$.*

Proof. (1): Consider the short exact sequence defining X in $\mathbb{V}(\mathcal{L})$:

$$0 \to \mathcal{O}_{\mathbb{V}(\mathcal{L})}(-X) \xrightarrow{\tau^m - \pi^*s} \mathcal{O}_{\mathbb{V}(\mathcal{L})} \to \mathcal{O}_X \to 0.$$

We have by definition $\mathcal{O}_{\mathbb{V}(\mathcal{L})}(-X) \simeq \pi^*(\mathcal{L}^{-m})$. We push-forward via π, which preserves exactness since π is affine:

$$0 \to \pi_*(\pi^*L^{-m}) = \mathcal{L}^{-m} \otimes \operatorname{Sym}^\bullet \mathcal{L}^\vee \xrightarrow{-s} \operatorname{Sym}^\bullet \mathcal{L}^\vee \to f_*\mathcal{O}_X \to 0, \tag{4.1}$$

where the equality is obtained from the projection formula. Since s has degree m, this multiplication map identifies a section in $\operatorname{Sym}^\bullet \mathcal{L}^\vee$ of degree $a + mb$ (where $a, b \leq 0$, $a > -m$) with a section of degree a. Therefore, as $\mathcal{O}_Y$-modules, we have

$$f_*\mathcal{O}_X \simeq \bigoplus_{j=0}^{m-1} \mathcal{L}^{-j}.$$

(2): By the adjunction formula, we have

$$K_X \simeq \left(K_{\mathbb{V}(\mathcal{L})} \otimes \mathcal{O}_{\mathbb{V}(\mathcal{L})}(X)\right)|_X = \left(\pi^*(K_Y \otimes \mathcal{L}^{-1}) \otimes \pi^*\mathcal{L}^m\right)|_X = f^*(K_Y \otimes \mathcal{L}^{m-1}).$$

Alternatively, looking at the defining equation, the ramification divisor of $X \to Y$ is $(m-1)Z(\tau) \subset X$, and so we have by the Hurwitz formula

$$K_X = f^*(K_Y \otimes \mathcal{L}^{m-1}).$$

□

Remark 4.3.5. Such a cyclic cover has a μ_m-action, acting transitively on the fibers, which can be seen from the μ_m-graded structure on $f_*\mathcal{O}_X$. Conversely, it can be shown that any cover that has such a μ_m-action arises as a cyclic covering as constructed above.

Note that not every finite covering $f : X \to Y$ of degree m that is a cyclic covering. However, in characteristic coprime to m, we can say the following. The exact sequence

$$0 \to \mathcal{O}_Y \to f_*\mathcal{O}_X \to \mathrm{coker} \to 0$$

of $\mathcal{O}_Y$-modules splits as we have a retraction given by $\frac{1}{m}\mathrm{Tr}$, where $\mathrm{Tr} : \pi_*\mathcal{O}_X \to \mathcal{O}_Y$ is the trace, defined because we can view elements of $f_*\mathcal{O}_X$ as acting on $f_*\mathcal{O}_X$ by multiplication. Since $\pi_*\mathcal{O}_X$ is locally free of rank m, this implies that $\mathcal{T}^\vee := \mathrm{coker}$ is also locally free, of rank $m-1$. The splitting also ensures that we have a section $\mathcal{T}^\vee \to f_*\mathcal{O}_X$, and the universal property of the symmetric algebra yields a map of $\mathcal{O}_Y$-algebras

$$\mathrm{Sym}^\bullet \mathcal{T}^\vee \to f_*\mathcal{O}_X,$$

which is obviously surjective. Since f is affine, this map comes from a closed immersion

$$X \hookrightarrow \mathbb{V}(\mathcal{T})$$

over Y. $\mathcal{T}$ is called the *Tschirnhausen* bundle, and we have shown that any finite map (under characteristic assumptions) factors through its Tschirnhausen bundle. It is to be expected that this bundle would have rank $m-1$. Indeed, if we take m points in a very big vector space, the affine space they span has dimension $m-1$.

Remark 4.3.6. The only subtlety in the analytic case is that the total space of the line bundle $\mathcal{L}$ has more functions than $\mathrm{Sym}^\bullet\mathcal{L}^\vee$. But we can circumvent this e.g. by constructing directly the $\mathcal{O}_Y$-algebra structure on

$$\mathcal{O}_Y \oplus \mathcal{L}^{-1} \oplus \cdots \oplus \mathcal{L}^{m-1},$$

and showing afterwards it corresponds to $Z(\tau^m - \pi^* s) \subset \mathbb{V}(\mathcal{L})$. Alternatively, one may also show that the analytic counterpart of the exact sequence (4.1) yields the desired $\mathcal{O}_Y$-algebra structure in the same way, e.g. after locally injecting holomorphic functions into formal functions and taking Taylor expansions in local coordinates.

Example 4.3.7 (Double cover of $\mathbb{P}^2$ branched over a sextic)**.** Let $Y = \mathbb{P}^2$. Let $B \subset \mathbb{P}^2$ be a smooth curve of degree 6. We have $\mathcal{O}(B) = \mathcal{O}(6)$. We take $m = 2$ and $\mathcal{L} = \mathcal{O}(3)$. Let $f : X \to \mathbb{P}^2$ be the double cyclic cover branched along B. We check the K3 conditions.

1. Canonical bundle: Using Lemma 4.3.4 (2),

$$K_X = f^*(K_{\mathbb{P}^2} \otimes \mathcal{L}) = f^*(\mathcal{O}(-3) \otimes \mathcal{O}(3)) = \mathcal{O}_X.$$

2. Vanishing of $H^1(X, \mathcal{O}_X)$: Using Lemma 4.3.4 (1),

$$f_*\mathcal{O}_X = \mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{L}^{-1} = \mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}(-3).$$

Since f is finite,

$$H^1(X, \mathcal{O}_X) \simeq H^1(\mathbb{P}^2, f * \mathcal{O}_X) = H^1(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}) \oplus H^1(\mathbb{P}^2, \mathcal{O}(-3)) = 0.$$

Since B was chosen to be smooth, so is X, which is also connected since it is smooth and a ramified cover. Thus, X is a K3 surface.

Example 4.3.8 (Kummer surfaces)**.** Let A be a complex torus of dimension 2. Consider the involution $i : A \to A$, $x \mapsto -x$.

The fixed locus of i is the set of 2-torsion points $A[2]$, which consists of 16 points. These points induce 16 singularities in A/i. So we first blow up.

Let $p : \tilde{A} \to A$ be the blow-up of A at the 16 fixed points. The exceptional locus is a disjoint union of 16 smooth rational curves $\tilde{E}_i$. The involution i lifts to an involution $\tilde{i} : \tilde{A} \to \tilde{A}$ by the universal property of the blow-up: indeed, we have the commutative square

$$\begin{tikzcd} \tilde{A} \arrow[r, dotted, "\tilde{i}"] \arrow[d] \arrow[dr] & \tilde{A} \arrow[d] \\ A \arrow[r, "i"] & A, \end{tikzcd}$$

and so the diagonal arrow lifts to $\tilde{i}$ since the total transform of the 16 points under the diagonal arrow is a Cartier divisor. Incidentally, the fixed locus of $\tilde{i}$ is exactly the union of the exceptional divisors $\bigcup E_i$. This implies that the quotient $X = \tilde{A}/\tilde{i}$, which we define to be the **Kummer surface** of A, is smooth because the fixed locus has codimension 1. X is Kähler since it isthe quotient of a Kähler surface by a finite group. In order to check that it is a K3 surface, there are two conditions to check: that the canonical bundle is trivial, and that $H^1(X, \mathcal{O}_X) = 0$.

We first show that $H^1(X, \mathcal{O}_X) = 0$. Let $f : \tilde{A} \to \tilde{A}/\tilde{i}$ be the quotient map. Since f is finite, we have that the pullback $f^* : H^1(X, \mathbb{C}) \to H^1(\tilde{A}, \mathbb{C})$ is injective; indeed this is seen easily as for a given $\alpha \in H^1(X, \mathbb{C})$, the projection formula (singular cohomology version) gives

$$f_* f^* \alpha = \alpha \cup f_* 1_{\tilde{A}} = 2\alpha,$$

and so $f_* f^*$ is injective. Thus, we may identify

$$H^1(X, \mathbb{C}) = H^1(\tilde{A}, \mathbb{C})^{\mu_2} = H^1(A, \mathbb{C})^{\mu_2},$$

where the superscript indicates taking the invariants under i, and the last equality holds because blowing up a point does not affect singular cohomology[3]. But now, on $H^1(A, \mathbb{C}) = \mathbb{C}^4$, the involution acts as multiplication by -1; this can be seen directly by looking at how it acts on the generators $dz_1, dz_2, d\overline{z}_1$ and $d\overline{z}_2$. Therefore, $H^1(A, \mathbb{C})^{\mu_2} = H^1(X, \mathbb{C}) = 0$, implying that $H^1(X, \mathcal{O}_X) = 0$ by the Hodge decomposition.

We now show that $K_X = \mathcal{O}_X$. Let $b : \tilde{A} \to A$ be the blow-down. By the Hurwitz formula, we have

$$K_{\tilde{A}} \simeq f^* K_X \otimes \mathcal{O}\left(\sum \tilde{E}_i\right),$$

and similarly, by Hurwitz, we have

$$K_{\tilde{A}} \simeq b^* K_A \otimes \mathcal{O}\left(\sum \tilde{E}_i\right) = \mathcal{O}\left(\sum \tilde{E}_i\right),$$

so that $f^* K_X = \mathcal{O}_{\tilde{A}}$. Using the projection formula, we obtain

$$f_* \mathcal{O}_{\tilde{A}} \simeq f_* f^* K_X \simeq f_* \mathcal{O}_{\tilde{A}} \otimes K_X, \tag{4.2}$$

implying, after taking determinants, that K_X is 2-torsion. Now, f is a μ_2-covering, and so by our previous discussion on the cyclic covering trick, we have

$$f_* \mathcal{O}_{\tilde{A}} \simeq \mathcal{O}_X \oplus \mathcal{L}^\vee,$$

where $f^* \mathcal{L}^2 = \mathcal{O}\left(\sum \tilde{E}_i\right)$. Thus, we have

$$K_X \oplus \left(K_X \otimes \mathcal{L}^\vee\right) \simeq \mathcal{O}_X \oplus \mathcal{L}^\vee.$$

This isomorphism has to be given by a matrix of the form

$$\begin{pmatrix} \mathrm{Hom}(K_X, \mathcal{O}_X) & \mathrm{Hom}(K_X, \mathcal{L}^\vee) \\ \mathrm{Hom}(K_X \otimes \mathcal{L}^\vee, \mathcal{O}_X) & \mathrm{Hom}(K_X \otimes \mathcal{L}^\vee, \mathcal{L}^\vee) \end{pmatrix} = \begin{pmatrix} H^0(X, K_X^\vee) & H^0(X, (K_X \otimes \mathcal{L})^\vee) \\ H^0(X, K_X^\vee \otimes \mathcal{L}) & H^0(X, K_X^\vee). \end{pmatrix} \tag{4.3}$$

Suppose for the purpose of contradiction that $K_X \neq O_X$. Then, since $K_X = K_X^\vee$, it cannot have a non-zero global section, and so (4.3) is of the form

$$\begin{pmatrix} 0 & H^0(X, (K_X \otimes \mathcal{L})^\vee) \\ H^0(X, K_X^\vee \otimes \mathcal{L}) & 0, \end{pmatrix}$$

implying that we must have an isomorphism $K_X \simeq \mathcal{L}^\vee$, so that $\mathcal{L}$ is 2-torsion. But this is impossible, as $f^* \mathcal{L}^2 = \mathcal{O}\left(\sum \tilde{E}_i\right) \not\simeq \mathcal{O}_{\tilde{A}}$. Thus, $K_X \simeq \mathcal{O}_X$.

Remark 4.3.9. Note that the map $q : A \to A/i$ is flat. Let Z be the singular locus of A/i, comprising of 16 points. Recall that the blow-up (i.e. Rees) algebra of A at Z is

$$\mathcal{O}_{A/i}[\mathcal{I}_Z t],$$

where $\mathcal{I}_Z$ is the ideal at Z. We can equally recover the blow-up from the Rees algebra in the analytic setting, where there exists an analytic version of the projective spectrum; see [AHV18]. Now, since q is flat,

$$q^* \mathcal{O}_{A/i}[\mathcal{I}_Z t] = \mathcal{O}_A[(q^* \mathcal{I}_Z) t] = \mathcal{O}_A[\mathcal{I}_{Z'} t],$$

[3] In general, if $\tilde{Y} \to Y$ is the blow-up at a complex submanifold Z, we have the formula

$$H^\bullet(\tilde{Y}) = H^\bullet(Y) \oplus \bigoplus_{i=1}^{\operatorname{codim}(Z)-1} H^{\bullet - 2i}(Z).$$

where $Z' = q^{-1}(Z)$, that is, Z' consists of the 16 points fixed by i. The projective spectrum commutes with base change, and so if $\widetilde{A/i}$ denotes the blow up of A/i at Z, we have a cartesian diagram

$$\begin{tikzcd} \tilde{A} \arrow[r] \arrow[d] & \widetilde{A/i} \arrow[d] \\ A \arrow[r, "q"] & A/i \end{tikzcd}$$

now the quotient $q : A \to A/i$ may locally be defined as a GIT quotient, (we are quotienting by a finite group), and so it is universal under base change, hence $\tilde{A} \to \widetilde{A/i}$ is also the quotient of $\tilde{A}$ by the lifted action. But this implies that $\tilde{A}/\tilde{i} = \widetilde{A/i} = X$. That is, when constructing X, we may equally have taken the quotient first, and resolve the singularities afterwards.

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