THE UNIVERSITY OF MELBOURNE

Semester 2, 2019

Department of Electrical and Electronic Engineering

ELEN30012 SIGNALS AND SYSTEMS

Time allowed: 180 minutes

Reading time: 15 minutes

This paper has 5 pages

Authorised Materials:

Only Casio FX82 and Casio FX100 calculators may be used.

Students may bring **TWO** sheets of A4 paper containing their own notes into the examination room. Students may write on both sides of these sheets of paper.

Instruction to Invigilators:

Students are to be provided with one script book. Additional script books are to be provided on request.

All script books, the examination paper and any personal notes brought into the examination room by students are to be collected after the examination.

Instruction to Students:

Attempt **ALL** questions. You may attempt the questions in any order. The marks given for each question are shown in brackets after the question numbers. The examination paper has a total of 100 marks.

You must show your work in order to receive credit.

Ensure your student number is written on all your script books. No annotating of script books is allowed during reading time or after the end of writing time.

Answer all questions on the right-hand lined pages of the script book. The left-hand page is for rough working only and will not be marked.

Any personal notes brought into the examination room must be left in the room after the examination.

Question 1 (16 marks)

Consider the continuous-time system with input-output trajectories (v, y) described by the integral equation

$$y(t) = \int_0^t (t - \lambda)v(\lambda)\, d\lambda$$

with $t \geq 0$.

(a) **[3 marks]** Is this a linear system? Justify your answer with a proof or counterexample.

(b) **[4 marks]** Is this a time-invariant system? Justify your answer with a proof or counterexample.

(c) **[1 mark]** Obtain h, the impulse response of this system.

(d) **[2 marks]** Obtain the response of this system from the ramp input $r(t) = tu(t)$.

(e) **[6 marks]** Let (v, y) be a trajectory of the system, where v is such that

$$v(t) = p_2(t - 2)$$

and p_2 is the unit pulse of width $\tau = 2$.

Sketch v for $t \geq 0$, and hence find y for $t \geq 0$.

Question 2 (15 marks)

Consider the continuous-time signals

$$f(t) = e^{-t}u(t) \qquad \text{and } f_1(t) = f(t-1)$$

Let F and F_1 be the continuous-time Fourier transforms (CTFT) of f and f_1.

(a) **[2 marks]** Use the definition of the CTFT to find F.

(b) **[3 marks]** Obtain expressions for $|F|$ and $\angle F$, the amplitude and phase spectra of f.

(c) **[4 mark]** On separate graphs, plot $|F|$ and $\angle F$, for $-\infty \leq \omega \leq \infty$.

(d) **[2 marks]** Obtain expressions for $|F_1|$ and $\angle F_1$, in terms of $|F|$ and $\angle F$.

(e) **[4 marks]** On separate graphs, plot $|F_1|$ and $\angle F_1$ for $-\infty \leq \omega \leq \infty$.

Question 3 (12 marks)

Consider the discrete-time signal x defined by $x[n] = \cos(\frac{\pi n}{2})$, for all $n \in \mathbf{Z}$.

(a) **[4 marks]** Write down an expression for X, the generalized discrete-time Fourier transform (DTFT) of x, and plot its graph for $-4\pi \leq \Omega \leq 4\pi$.

(b) **[4 marks]** The signal x is applied as an input to an ideal filter whose frequency response is such that

$$H(\Omega) = \begin{cases} 1, & \text{if } 2k\pi \leq \Omega < (2k+1)\pi \\ 0, & \text{if } (2k+1)\pi \leq \Omega < (2k+2)\pi \end{cases}$$

for all $k \in \mathbf{Z}$. Let y be the output of x from this filter, and let Y be its DTFT. Plot the graph of $Y(\Omega)$ for $-4\pi \leq \Omega \leq 4\pi$.

(c) **[4 marks]** Obtain an expression for Y and hence find $y[n]$ for all $n \in \mathbf{Z}$.

Question 4 (12 marks)

(a) **[2 marks]** Let $x[n] \longleftrightarrow X(z)$ be a z-Transform pair. Show that for any $a \in \mathbf{R}$,

$$x[n]a^n \longleftrightarrow X\left(\frac{z}{a}\right)$$

is a z-Transform pair.

(b) **[4 marks]** Let $x[n] \longleftrightarrow X(z)$ be a z-Transform pair. Show that

$$x[n+1] \longleftrightarrow zX(z) - x[0]z$$

is a z-Transform pair.

(c) A discrete-time signal x has z-Transform

$$X(z) = \frac{1}{8z^2 - 2z - 1}$$

Find z-Transforms for each of the following signals:

(i) **[2 marks]** $v_1[n] = x[n-4]u[n-4]$

(ii) **[2 marks]** $v_2[n] = (x \star x)[n]$

(iii) **[2 marks]** $v_3[n] = nx[n]$

Simplify your answers as much as possible.

Question 5 (11 marks)

(a) [**6 marks**] Let X_k be the N-point Discrete Fourier Transform (DFT) of a signal x with record length $L \leq N$. Show that

$$X_{N-k} = \bar{X}_k \qquad \text{for all } k = 0, 1, \ ..., N-1$$

(b) A discrete-time signal x has record length $L = 4$, and its 4-point DFT is such that

$$X_0 = 2, \ X_2 = -2 \text{ and } \ X_3 = -2 - j2$$

(i) [**1 mark**] Find X_1.

(ii) [**4 marks**] Find $x[0]$ and $x[1]$.

Question 6 (14 marks)

A linear time-invariant discrete-time system has state representation

$$\begin{aligned} \begin{bmatrix} x_1[n+1] \\ x_2[n+1] \end{bmatrix} &= \begin{bmatrix} -1 & 1 \\ -1 & -2 \end{bmatrix} \begin{bmatrix} x_1[n] \\ x_2[n] \end{bmatrix} + \begin{bmatrix} 0.5 & 1 \\ -1 & -0.5 \end{bmatrix} \begin{bmatrix} v_1[n] \\ v_2[n] \end{bmatrix} \\ \begin{bmatrix} y_1[n] \\ y_2[n] \end{bmatrix} &= \begin{bmatrix} 2 & 1 \\ -1 & -2 \end{bmatrix} \begin{bmatrix} x_1[n] \\ x_2[n] \end{bmatrix} \end{aligned}$$

(a) [**1 mark**] Is this a scalar-input scalar-output system (SISO) or a multiple-input multiple-output (MIMO) system? Give a reason for your answer (write about 10 words).

(b) [**2 marks**] Is this a bounded-input bounded output (BIBO) stable system? Justify your answer.

(c) [**2 marks**] Is there an invertible matrix P such that the coordinate transformation $\bar{x} = Px$ will convert the system to an equivalent diagonal state representation? Give a reason for your answer (write about 10 words).

Note: You are NOT not required to find P, only to explain whether it exists.

(d) [**4 marks**] Draw the unit delay realization for this system. Ensure that the input signals v_1 and v_2 are drawn on the left-hand side of your diagram, and output signals y_1 and y_2 are on the right-hand side of your diagram.

(e) [**2 marks**] Compute $x[1]$ and $x[2]$ when $x[0] = [1 \ 1]^T$ and $v[n] = [n \ n]^T$.

(f) [**3 marks**] Suppose $x[0] = [0 \ 0]^T$. Find an input sequence $v[0]$, $v[1]$ that will drive the system state to $x[2] = [-1 \ 2]^T$.

Question 7 (12 marks)

The equation for a mass-spring-damper system shown in Figure 1 may be written as

$$M\ddot{y}(t) + D\dot{y}(t) + Ky(t) = v(t).$$

Here y is the output, which is the displacement of the mass M, and v is the input force. Assume that $M = 1$, $D = 0$ and $K = 9$.

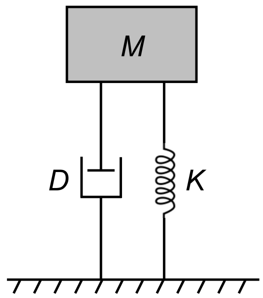

Figure 1: Mass-spring-damper system.

(a) [**4 marks**] Find the damping ratio and natural frequency for this system.

(b) [**1 mark**] Is this a stable system? Give a reason for your answer (write about 10 words).

(c) [**1 mark**] Does the system have a dominant real pole? Give a reason for your answer (write about 10 words).

(d) [**1 mark**] Is it overdamped, critically damped, underdamped or undamped? Give a reason for your answer (write about 10 words).

(e) [**5 marks**] A step input of $v(t) = u(t)$ is applied to the system. Find the output y and sketch its graph for $0 \le t \le 2\pi$. Assume $y(0) = \dot{y}(0) = 0$.

Question 8 (8 marks)

Let $x(t)$ be a continuous-time function with Laplace transform $X(s)$, and assume that $x(t)$ is continuous at $t = 0$. For any integer $n \ge 1$, let $x^{(n)}(t)$ denote the n-th derivative of $x(t)$. Assume further that $x^{(n)}(t)$ is continuous at $t = 0$, for all integers $n \ge 1$. Use the definition of Laplace transform and Mathematical Induction to show that

$$x^{(n)}(t) \longleftrightarrow s^n X(s) - \sum_{i=1}^{n} s^{i-1} x^{(n-i)}(0)$$

for all integers $n \ge 1$.

END OF EXAMINATION