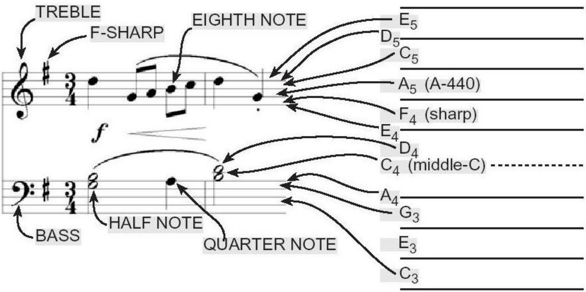
Chapter 3: Fourier Series Representations of Signals

In this chapter we study periodic continuous-time signals and introduce Fourier series methods for expressing them as the sum of sinusoidal functions. This leads to the notion of the frequency spectrum of a signal, which describes how a signal may be constructed in terms of its sinusoidal frequency components.

Music consists of sinusoidal signals (notes) played simultaneously.



Some of these notes are

- C4 (also known as middle C) is 262 Hz.
- A4 (also known A-440) is 440 Hz.
- F4 $\sharp$ is 370 Hz.

The note A4 is the sinusoidal function

$$A \sin(880\pi t)$$

Musicologists describe three properties of a musical note:

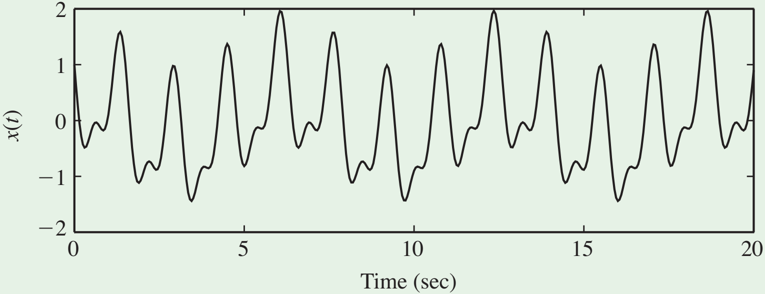
- **Loudness**, which is the amplitude A.
- **Pitch**, which is the frequency 440 Hz.
- **Timbre**, which describes everything about the note that is not loudness or pitch!

When the note A4 is played on a violin and on a piano, the pitch (and perhaps loudness) are the same. Timbre describes the unique sound qualities of a particular instrument - these cannot be precisely described mathematically! So timbre is often described in subjective terms, like colour and warmth.

Example (Sums of sinusoids I)

Consider the signal

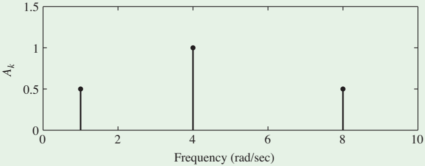
$$x(t) = 0.5\cos t + \cos(4t + \pi/3) + 0.5\cos(8t + \pi/2)$$

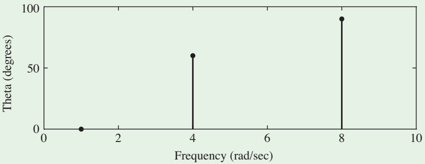


The three sinusoidal signals making up $x(t)$ have frequencies $\omega_1 = 1$, $\omega_2 = 4$, and $\omega_3 = 8$, with corresponding periods $T_1 = 2\pi \approx 6.3$, $T_2 = \frac{\pi}{2} \approx 1.6$ and $T_3 = \frac{\pi}{4} \approx 0.8$.

Example (Sums of sinusoids II)

- The amplitudes and phase angles of the three sinusoids are $A_1 = 0.5$, $A_2 = 1$ and $A_3 = 0.5$, and $\theta_1 = 0$, $\theta_2 = \pi/3 = 60^\circ$ and $\theta_3 = \pi/2 = 90^\circ$.
- If we plot these as functions of the frequencies ω_1, ω_2, and ω_3, we obtain the **amplitude spectrum** and **phase spectrum** of $x(t)$.





- The amplitude spectrum shows the magnitudes of the three frequency components of $x(t)$.
- The frequency spectrum shows which frequencies are represented in the signal.

Definition (Trigonometric Fourier series I)

Let x be a periodic continuous-time signal with fundamental period T, and fundamental frequency ω_0. Then the **trigonometric Fourier series** for x is given by

$$x(t) = a_0 + \sum_{k=1}^{\infty} \left(a_k \cos(k\omega_0 t) + b_k \sin(k\omega_0 t) \right)$$

The coefficients a_0, a_k and b_k are known as the **Fourier coefficients** and may be calculated from **Euler's formulae**:

$$\begin{aligned}
a_0 &= \frac{1}{T}\int_0^T x(t)\,dt \\
a_k &= \frac{2}{T}\int_0^T x(t)\cos(k\omega_0 t)\,dt, \quad k = 1, 2, \ldots \\
b_k &= \frac{2}{T}\int_0^T x(t)\sin(k\omega_0 t)\,dt, \quad k = 1, 2, \ldots
\end{aligned}$$

Definition (Trigonometric Fourier series II)

The Fourier coefficients can be computed by integration over any interval of length T:

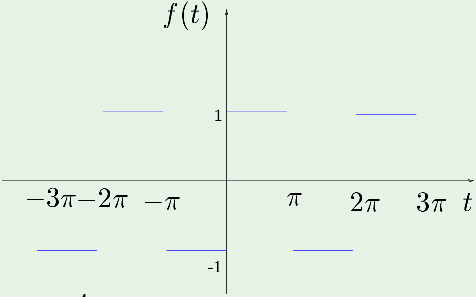
$$\begin{aligned}
a_0 &= \frac{1}{T}\int_{-T/2}^{T/2} x(t)\,dt \\
a_k &= \frac{2}{T}\int_{-T/2}^{T/2} x(t)\cos(k\omega_0 t)\,dt, \quad k = 1, 2, \ldots \\
b_k &= \frac{2}{T}\int_{-T/2}^{T/2} x(t)\sin(k\omega_0 t)\,dt, \quad k = 1, 2, \ldots
\end{aligned}$$

Example

Find the Fourier series for the square wave defined by

$$x(t) = \begin{cases} -1, & -\pi \le t < 0 \\ 1, & 0 \le t < \pi \end{cases}$$

with $x(t) = x(t+2\pi)$. First we sketch $x(t)$:



x has $T = 2\pi$, so $\omega_0 = 1$.

Example

We calculate the Fourier coefficients using Euler's formulae.

$$
\begin{aligned}
a_0 &= \frac{1}{T}\int_{-T/2}^{T/2} x(t)\,dt \\
&= \frac{1}{2\pi}\int_{-\pi}^{0} -1\,dt + \frac{1}{2\pi}\int_{0}^{\pi} 1\,dt \\
&= 0 \\
a_k &= \frac{2}{T}\int_{-T/2}^{T/2} x(t)\cos(k\omega_0 t)\,dt, \qquad \text{for } k \geq 1 \\
&= \frac{1}{\pi}\int_{-\pi}^{0} -\cos(kt)\,dt + \frac{1}{\pi}\int_{0}^{\pi} \cos(kt)\,dt \\
&= \frac{1}{\pi}\left[\frac{-\sin(kt)}{k}\right]_{-\pi}^{0} + \frac{1}{\pi}\left[\frac{\sin(kt)}{k}\right]_{0}^{\pi} \\
&= 0
\end{aligned}
$$

Example

$$
\begin{aligned}
b_k &= \frac{2}{T}\int_{-T/2}^{T/2} x(t)\sin(k\omega_0 t)\,dt, \quad \text{for } k \geq 1 \\
&= \frac{1}{\pi}\int_{-\pi}^{0} -\sin(kt)\,dt + \frac{1}{\pi}\int_{0}^{\pi} \sin(kt)\,dt \\
&= \frac{1}{\pi}\left[\frac{\cos(nt)}{k}\right]_{-\pi}^{0} + \frac{1}{\pi}\left[\frac{-\cos(nt)}{k}\right]_{0}^{\pi} \\
&= \frac{2(1-\cos(k\pi))}{k\pi} \\
&= \frac{2(1-(-1)^k)}{k\pi}
\end{aligned}
$$

noting that $\cos(k\pi) = (-1)^k$. The first few Fourier coefficients are

$$a_0 = a_k = 0,\; b_1 = \frac{4}{\pi}, b_2 = 0, b_3 = \frac{4}{3\pi}, b_4 = 0$$

Theorem (Fourier series Representation)

Let x be a periodic continuous-time signal with fundamental period T. Then x can be represented by its Fourier series if it satisfies the **Dirichlet conditions***:*

1. *x is* **absolutely integrable** *over one period, i.e.*

$$\int_0^T |x(t)|\, dt < \infty$$

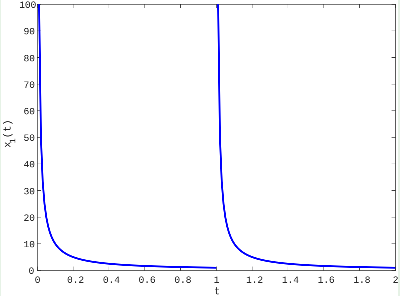
2. *x has only finitely many maxima and minima over one period.*
3. *x has only finitely many points of discontinuity over one period.*

It is generally believed that 'virtually all' periodic signals of interest to engineers satisfy the Dirichlet conditions. From now on we always assume that all periodic signals under discussion satisfy the Dirichlet conditions and hence have a Fourier series representation.

Example (A function that fails the first Dirichlet condition)

Let $x_1(t) = \frac{1}{t}$ for $0 \leq t < 1$, and $x_1(t) = x_1(t+1)$. Then

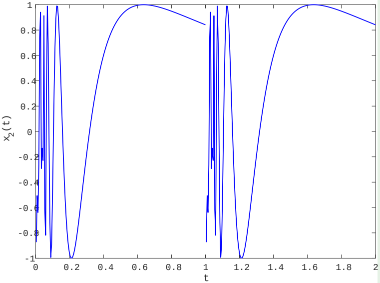
$$\int_0^1 |x_1(t)|\, dt = \int_0^1 \frac{dt}{t} = \infty$$



Other examples of functions that fail the first Dirichlet condition include $x(t) = t^{-\alpha}$, for any $\alpha > 1$.

Example (A function that does not meet the second Dirichlet condition)

Let $x_2(t) = \sin\left(\frac{1}{t}\right)$ for $0 \leq t < 1$, and $x_2(t) = x_2(t+1)$.



$\sin(t)$ has local maxima for t_k such that

$$t_k = 2\pi k + \frac{\pi}{2}$$

for any positive integer k. Hence f_2 has maxima for t_k such that

$$\frac{1}{t_k} = 2\pi k + \frac{\pi}{2} \Longleftrightarrow t_k = \frac{1}{2\pi k + \frac{\pi}{2}}$$

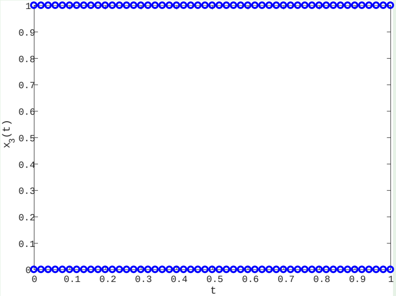
Thus x_2 has infinitely many distinct local maxima on the interval $[0, 1]$. It also has infinitely many distinct local minima on the interval $[0, 1]$, at times $t_k = \frac{1}{2\pi k - \frac{\pi}{2}}$.

Example (A function that does not meet the third Dirichlet condition)

Let x_3 be defined as

$$x_3(t) = \begin{cases} 1, & \text{if } t \text{ is an irrational number} \\ 0, & \text{if } t \text{ is a rational number} \end{cases}$$

and let $x_3(t) = x_3(t+1)$. Then x_3 is discontinuous everywhere.



Definition (Finite Fourier series)

Let x be a periodic continuous-time signal with fundamental period T. For any integer N, we define the **finite Fourier series** x_N of x as

$$x_N(t) = a_0 + \sum_{k=1}^{N} \left(a_k \cos(k\omega_0 t) + b_k \sin(k\omega_0 t)\right)$$

Theorem (Convergence of Fourier series I)

If x satisfies the Dirichlet conditions, then

$$\lim_{N\to\infty} \frac{1}{T} \int_0^T |x(t) - x_N(t)|^2 \, dt = 0$$

where x_N denotes the finite Fourier series for x.

We say that x_N **converges to x in the L_2 norm**, because the mean-squared error converges to zero.

Theorem (Convergence of Fourier series II)

Let x be a periodic continuous-time signal, and assume x satisfies the Dirichlet conditions. Let $t_0 \in \mathbf{R}$. Then

1. *If x is continuous at t_0, then*

$$\lim_{N\to\infty} x_N(t_0) = x(t_0)$$

2. *If x is discontinuous at t_0, then*

$$\lim_{N\to\infty} x_N(t_0) = \frac{1}{2}(x(t_0^-) + x(t_0^+))$$

   where $x(t_0^-)$ and $x(t_0^+)$ are the left and right limits of x at t_0.

If x is continuous at t_0, then $x_N(t_0)$ converges pointwise to $x(t_0)$.
If x is discontinuous at t_0, then $x_N(t_0)$ converges to the average of the left and right limits of $x(t_0)$.

Definition (Even and Odd functions)

1. A signal x is an **even function** if

$$x(t) = x(-t) \quad \text{for all } t \in \mathbf{R}$$

   The graph of an even functions is symmetric about a reflection in the y-axis. Examples of even functions are: any constant function, $\cos(\omega t)$, and t^2.

2. A signal x is an **odd function** if

$$x(t) = -x(-t) \quad \text{for all } t \in \mathbf{R}$$

   The graph of an odd function is symmetric about a reflection in the y-axis, followed by a reflection in the x-axis. Examples of odd functions are: $\sin(\omega t)$, t and t^3.

Theorem (Fourier cosine series)

Suppose x is an even periodic function with period $T = 2L$. Then x may be represented by a **Fourier cosine series**

$$x(t) = a_0 + \sum_{k=1}^{\infty} a_k \cos(k\omega_0 t)$$

$$\text{where } a_0 = \frac{1}{L}\int_0^L x(t)\,dt \quad \text{and} \quad a_k = \frac{2}{L}\int_0^L x(t)\cos(k\omega_0 t)\,dt$$

Theorem (Fourier sine series)

Suppose x is an odd periodic function with period $T = 2L$. Then x may be represented by a **Fourier sine series**

$$x(t) = \sum_{k=1}^{\infty} b_k \sin(k\omega_0 t)$$

$$\text{where} \quad b_k = \frac{2}{L}\int_0^L x(t)\sin(k\omega_0 t)\,dt$$

Definition (Cosine-with-phase Fourier series)

A trigonometric Fourier series for a periodic signal x

$$x(t) = a_0 + \sum_{k=1}^{\infty} \left(a_k \cos(k\omega_0 t) + b_k \sin(k\omega_0 t)\right)$$

can be expressed in the **cosine-with-phase form**

$$x(t) = a_0 + \sum_{k=1}^{\infty} A_k \cos(k\omega_0 t + \theta_k)$$

where

$$\begin{aligned} A_k &= \sqrt{a_k^2 + b_k^2} \\ \theta_k &= \begin{cases} \tan^{-1}\left(\frac{-b_k}{a_k}\right), & k = 1, 2, \ldots, \text{when } a_k \geq 0 \\ \pi + \tan^{-1}\left(\frac{-b_k}{a_k}\right), & k = 1, 2, \ldots, \text{when } a_k < 0 \end{cases} \end{aligned}$$

Definition (Complex Fourier series I)

Let x be a periodic continuous-time signal with fundamental period T, and frequency ω_0. The **complex Fourier series** for x is given by

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t}$$

The **complex Fourier coefficients** c_k may be calculated from

$$c_k = \frac{1}{T} \int_0^T x(t) e^{-jk\omega_0 t} \, dt, \quad k = 0, \pm 1, \pm 2, \ldots$$

Alternatively, we may integrate over any interval of length T, e.g.

$$c_k = \frac{1}{T} \int_{-T/2}^{T/2} x(t) e^{-jk\omega_0 t} \, dt, \quad k = 0, \pm 1, \pm 2, \ldots$$

Note that $c_k = \bar{c}_{-k}, \qquad k = 1, 2, \ldots.$

Theorem (Complex Fourier series)

For any periodic continuous-time signal x with fundamental period T, and fundamental frequency ω_0, the trigonometric and complex Fourier coefficients are related by

$$\begin{aligned}
c_0 &= a_0 \\
c_k &= \frac{1}{2}(a_k - jb_k), \quad k = 1,2,\ldots \\
c_{-k} &= \frac{1}{2}(a_k + jb_k), \quad k = 1,2,\ldots
\end{aligned}$$

or equivalently,

$$\begin{aligned}
a_0 &= c_0 \\
a_k &= c_k + c_{-k}, \quad k = 1,2,\ldots \\
b_k &= j(c_k - c_{-k}), \quad k = 1,2,\ldots
\end{aligned}$$

The cosine-with-phase Fourier series of a periodic signal x is

$$x(t) = a_0 + \sum_{k=1}^{\infty} A_k \cos(k\omega_0 t + \theta_k)$$

Thus the component of x at each frequency $k\omega_0$ has both an amplitude (A_k) and a phase (θ_k) component. This information is also contained in the complex Fourier coefficient c_k.

Definition (Amplitude and phase spectra)

- The **amplitude spectrum** of x is $|c_k|$.
- The **phase spectrum** of x is $\angle c_k$.

Theorem (Amplitude and phase spectra)

- $|c_k| = |c_{-k}|$ *for all* $k = 1, 2, \ldots$, *so the amplitude spectrum is an even function of* k.
- $\angle c_{-k} = -\angle c_k$, *for all* $k = 1, 2, \ldots$, *so the phase spectrum is an odd function of* k.

Theorem (Amplitude and phase spectra II)

- *If x is a periodic signal with trigonometric Fourier coefficients a_k and b_k, then the amplitude and phase spectrum are given by*

$$
\begin{aligned}
|c_k| &= \frac{1}{2}\sqrt{a_k^2 + b_k^2}, \quad k = 1, 2, \ldots \\
\angle c_k &= \begin{cases} \tan^{-1}\left(\frac{-b_k}{a_k}\right), & k = 1, 2, \ldots \text{ when } a_k \geq 0 \\ \pi + \tan^{-1}\left(\frac{-b_k}{a_k}\right), & k = 1, 2, \ldots \text{ when } a_k < 0 \end{cases}
\end{aligned}
$$

- *If x is a periodic signal with cosine-with-phase coefficients A_k and θ_k, then the amplitude and phase spectrum are given by*

$$
\begin{aligned}
|c_k| &= \frac{1}{2} A_k, \quad k = 1, 2, \ldots \\
\angle c_k &= \theta_k, \quad k = 1, 2, \ldots
\end{aligned}
$$

Example (Example)

Consider the signal

$$x(t) = \cos(t) + 0.5\cos(4t + \pi/3) + \cos(8t + \pi/2)$$

The signal is in cosine-with-phase form with coefficients

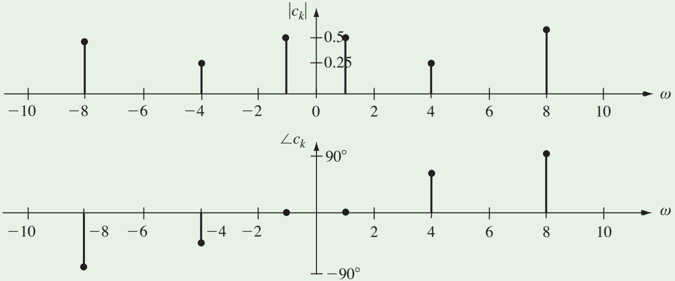
$$A_1 = 1, \theta_1 = 0; \quad A_4 = 0.5, \theta_4 = \pi/3; \quad A_8 = 1, \theta_8 = \pi/2$$

and hence the complex Fourier coefficients are

$$\begin{aligned} c_1 &= \frac{1}{2}, \quad c_4 = \frac{0.5}{2} e^{j\pi/3} = 0.25\underline{/60^\circ}, c_8 = \frac{1}{2} e^{j\pi/2} = 0.5\underline{/90^\circ} \\ c_{-1} &= \frac{1}{2}, \quad c_{-4} = \bar{c}_4 = 0.25\underline{/-60^\circ}, c_{-8} = \bar{c}_8 = 0.5\underline{/-90^\circ} \end{aligned}$$

Example (Example II)

The amplitude and phase spectra of x are shown below:



As expected, we see that $|c_k|$ is an even function and $\underline{/c_k}$ is an odd function.