1 (a) System is <u>nonlinear</u>.

Consider the input

$$v(t) = t \quad (1)$$

with output

$$y(t) = \int_0^t \lambda v^2(\lambda)\, d\lambda$$

$$= \int_0^t \lambda^3 \, d\lambda$$

$$= \frac{1}{4} t^4 \quad (1)$$

Next let $v_1(t) = 2t \overset{(1)}{=} 2v(t)$

Then

$$y_1(t) = \int_0^t \lambda v_1^2(\lambda)\, d\lambda$$

$$= \int_0^t 4\lambda^3 \, d\lambda$$

$$= t^4 \quad (1)$$

Hence $y_1(t) \neq 2y(t)$ (1)

and the homogeneity property fails. (1)

$\boxed{6}$

(b) Again let $v(t) = t$ (1) with output

$y(t) = \frac{1}{4} t^4$ (1)

Next let $v_1(t) = v(t-1) = (t-1)^2$ (1)

Then

$$y_1(t) = \int_0^t \lambda (\lambda - 1)^2 \, d\lambda$$

$$= \int_0^t \lambda^3 - 2\lambda^2 + \lambda \quad d\lambda$$

$$= \left[\frac{1}{4}\lambda^4 + \frac{2}{3}\lambda^3 + \frac{1}{2}\lambda^2 \right]_0^t$$

$$= \frac{1}{4} t^4 + \frac{2}{3} t^3 + \frac{1}{2} t^2 \quad (1)$$

Also

$$y(t-1) = \frac{1}{4}(t-1)^4$$

$$= \frac{1}{4}[t^4 - 3t^2 - 3t + 1]$$

$$\neq y_1(t) \quad (1)$$

So the time-invariance property (1) fails $\Rightarrow$ NOT TIME INVARIANT.

$\boxed{6}$

1 (c) The system is causal because, to compute $y(t_1)$ for any time t_1, we need to know the value of the input $v(t)$ for time $0 \le t \le t_1$ (1) and knowledge of $v(t)$ for time $t > t_1$ is not required. (1)

$\boxed{2}$

(d) The system has memory because the output at time t_1 depends on the input for $0 \le t \le t_1$. $\boxed{1}$

TOTAL QUESTION 1 : $\boxed{15}$

(2) (a) For $0 \le n < N_0$, the balance of the loan account is

$$y[1] = y_0(1+r) + 1 + (1+r) \quad (1)$$

$$y[2] = y_0(1+r)^2 + 1 + (1+r) + (1+r)^2 \quad (1)$$

$$\vdots$$

$$y[n] = y_0(1+r)^n + 1 + (1+r) + \dots + (1+r)^n \quad (1)$$

$$\text{So} \quad = y_0(1+r)^n + \frac{1-(1+r)^{n+1}}{1-(1+r)} \quad (1)$$

$$= y_0(1+r)^n + \frac{(1+r)^{n+1}-1}{r} \quad (2)$$

$\boxed{6}$

(b) At $y[N_0-1] = y_0(1+r)^{N_0-1} + \frac{(1+r)^{N_0}-1}{r}$ (3)

So for $n \geq N_0$,

$$y[n] = y[N_0-1](1+r)^{n-(N_0-1)} \quad (2)$$

$$= y_0(1+r)^n + \frac{[(1+r)^{N_0}-1](1+r)^{n-N_0+1}}{r} \quad (2)$$

$$= y_0(1+r)^n + \frac{1}{r}\left[(1+r)^{n+1} - (1+r)^{n-N_0+1}\right]$$

(2)

$\boxed{9}$

TOTAL QUESTION (2): $\boxed{15}$

Show that

(3) $$\sum_{n=0}^{N-1} \alpha^n = \frac{1-\alpha^N}{1-\alpha}.$$

$N=1$: $\text{LHS} = \sum_{n=0}^{0} \alpha^n = \alpha^0 = 1$

$$\text{RHS} = \frac{1-\alpha}{1-\alpha} = 1.$$

True for $N=1$. (2)

Assume true for $N = k \geq 1$:

$$\sum_{n=0}^{k-1} \alpha^n = \frac{1-\alpha^k}{1-\alpha}. \quad (1)$$

Then $$\sum_{n=0}^{k} \alpha^n = \sum_{n=0}^{k-1} \alpha^n + \alpha^k \quad (2)$$

$$= \frac{1-\alpha^k}{1-\alpha} + \alpha^k$$

$$= \frac{(1-\alpha^k) + (1-\alpha)\alpha^k}{1-\alpha}$$

$$= \frac{1-\alpha^k+\alpha^k-\alpha^{k+1}}{1-\alpha}$$

$$= \frac{1-\alpha^{k+1}}{1-\alpha} \quad (1)$$

TOTAL Q(3): $\boxed{8}$ $\boxed{8}$

Hence true for $N = k+1$. (1)

$\Rightarrow$ TRUE FOR ALL $N \geq 1$ by Induction (1)

(4) Using the first and third transforms we obtain

$$\cos(\omega_0 t) \leftrightarrow \pi[\delta(\omega+\omega_0)+\delta(\omega-\omega_0)]$$

Letting

$$x(t) = \cos(\omega_0 t)$$
$$X(\omega) = \pi[\delta(\omega+\omega_0)+\delta(\omega-\omega_0)] \quad (2)$$

and applying the duality theorem gives:

$$X(t) \leftrightarrow 2\pi x(-\omega)$$

ie.

$$\frac{1}{2}[\delta(t+\omega_0)+\delta(t-\omega_0)] \leftrightarrow \cos(-\omega_0 \omega) \quad (2)$$

Letting $\omega_0 = 4$, and noting that $\cos(\cdot)$ is an even function gives $(\frac{1}{2})$

$$\frac{1}{2}[\delta(t+4)+\delta[t-4]) \leftrightarrow \cos(4\omega) \quad (1\tfrac{1}{2})$$

So $y(t) = \frac{1}{2}[\delta(t+4)+\delta(t-4)]$ is the inverse FT of Y.

$\boxed{6}$

(b) Set $x(t) = \frac{1}{2}[\delta(t+2)+\delta(t-2)]$

Then $X(\omega) = \cos(2\omega)$. (1)

So the inverse of $G(\omega) = \cos(2\omega)\cos(4\omega)$ is $g(t) = (x * y)(t)$ (1)

$$= \int_{-\infty}^{\infty} \frac{1}{2}[\delta(\lambda+2)+\delta(\lambda-2)]\left(\frac{1}{2}\right)[\delta(t+4-\lambda)+\delta(t-4-\lambda)]\, d\lambda \quad (2)$$

$$= \underset{(\frac{1}{2})}{\tfrac{1}{4}\delta(t+6)} + \underset{(\frac{1}{2})}{\tfrac{1}{4}\delta(t-2)} + \underset{(\frac{1}{2})}{\tfrac{1}{4}\delta(t+2)} + \underset{(\frac{1}{2})}{\tfrac{1}{4}\delta(t-6)}$$

by the Sifting Theorem.

$\boxed{6}$

TOTAL QUESTION (4): $\boxed{12}$