COMP90038 Algorithms and Complexity

NP-Completeness

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Lecture 22

Semester 1, 2021

Concrete Complexity

We have been concerned with the analysis of algorithms' running times (best, average, worst cases).

Our approach has been to give a bound for the asymptotic behaviour of running time, as a function of input size.

For example, the quicksort algorithm is $O(n^2)$ in the worst case, whereas mergesort is $O(n \log n)$.

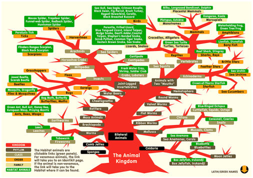
Abstract Complexity

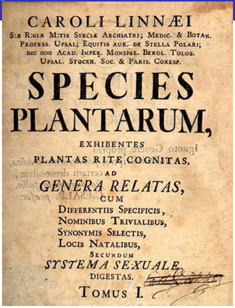
Complexity theory instead asks:

"What is the inherent difficulty of the problem?"

How do we know when we have come up with an algorithm which is optimal (in the asymptotic sense).

Taxonomy in the Sciences





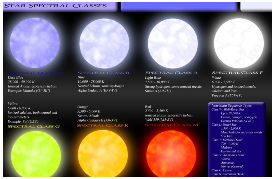
CAROLI LINNÆI

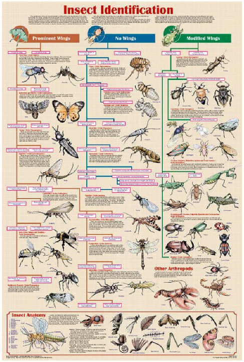
S:æ R:giæ M:tis Sveciæ Archiatri; Medic. & Botan. Profess. Upsal; Equitis aur. de Stella Polari; nec non Acad. Imper. Monspel. Berol. Tolos. Upsal. Stockh. Soc. & Paris. Coresp.

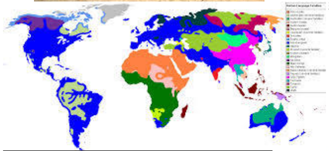
SPECIES PLANTARUM,

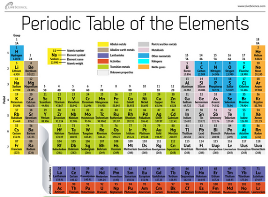
EXHIBENTES PLANTAS RITE COGNITAS, AD GENERA RELATAS, CUM DIFFERENTIIS SPECIFICIS, NOMINIBUS TRIVIALIBUS, SYNONYMIS SELECTIS, LOCIS NATALIBUS, SECUNDUM SYSTEMA SEXUALE DIGESTAS.

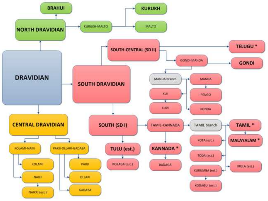
TOMUS I.

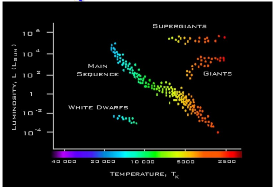






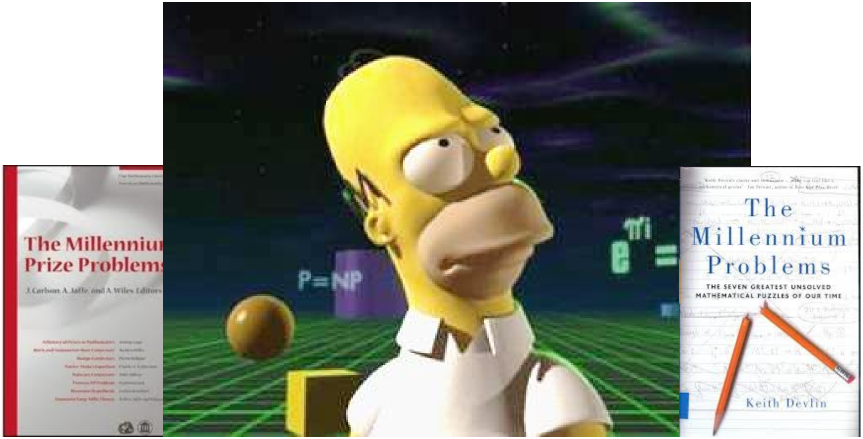

Periodic Table of the Elements





A Million Dollar Question: Is P = NP?

This is one of the seven "millennium problems": The Clay Institute's seven most important unsolved mathematical problems.


The Millennium Prize Problems
J. Carlson, A. Jaffe, and A. Wiles Editors
P=NP
The Millennium Problems
THE SEVEN GREATEST UNSOLVED MATHEMATICAL PUZZLES OF OUR TIME
Keith Devlin

Who Wants to Be a Millionaire?

The "P versus NP" problem comes from computational complexity theory.

If you could use USD 1,000,000, solve this—and you might well get a bonus Turing Award.

Or solve any of the other millennium problems—all but one remain unsolved.

Algorithmic Problems

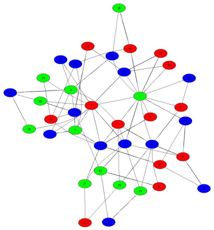
When we talk about a problem in computer science we almost always mean a family of instances of a general problem.

An algorithm for the problem has to work for all possible instances (input).

Example: The sorting problem—an instance is a sequence of items.

Example: The graph k-colouring problem—an instance is a graph.

Example: Equation solving problems—an instance is a set of, say, linear equations.



Easy and Hard Problems

A path in a graph G is simple if it visits each node of G at most once.

Consider this problem for undirected graphs G:

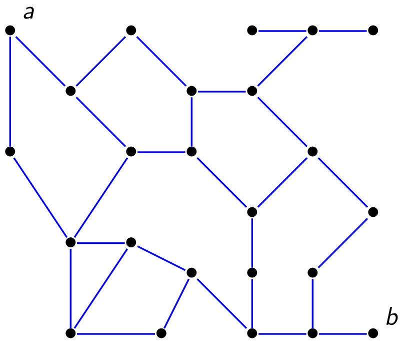
> SPATH: Given G and two nodes a and b in G,
> is there a simple path from a to b of length at most k?

And this problem:

> LPATH: Given G and two nodes a and b in G,
> is there a simple path from a to b of length at least k?

If you had a large graph G, which of the two problems would you rather have to solve?

Easy and Hard Problems



There are fast algorithms to solve SPATH.

Nobody knows of a fast algorithm for LPATH.

It is likely that the LPATH problem cannot be solved in polynomial time.

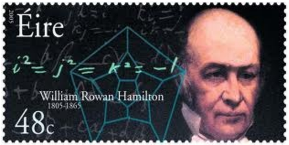
(But we don't know for sure.)

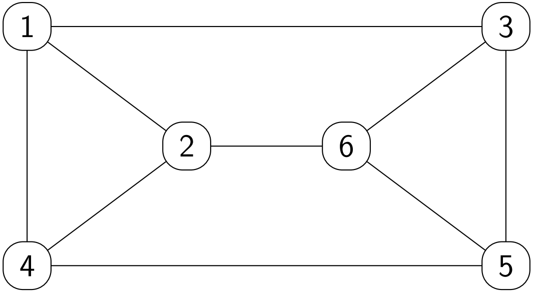
Hamiltonian Tours

The Hamiltonian tour problem (Lecture 5) again:

HAM: In a given graph, is there a path which visits each node of the graph once, returning to the origin?

Is there a Hamiltonian tour of this graph?



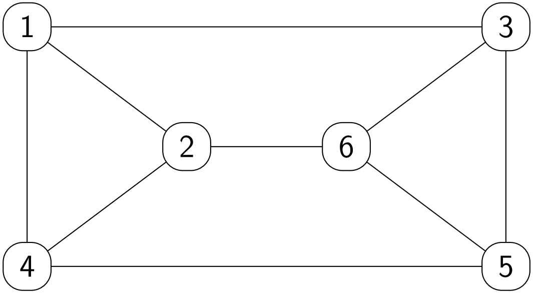


Eulerian Tours

The Eulerian tour problem is this:

> EUL: In a given graph, is there a path which visits each edge of the graph once, returning to the origin?

Is there a Eulerian tour of this graph?



More Problems

Try to rank these problems according to inherent difficulty:

1. SAT: Given a propositional formula φ, is φ satisfiable?
2. SUBSET-SUM: Given a set S of positive integers and a positive integer t, is there a subset of S that adds up to t?
3. 3COL: Given a graph G, is it possible to colour the nodes of G using only three colours, so that no edge connects two nodes of the same colour?

Polynomial-Time Verifiability

SAT, SUBSET-SUM, 3COL, LPATH, and HAM share an interesting property.

If somebody claims to have a solution (a "yes instance") then we can quickly test their claim.

In other words, these problems seem hard to solve, but solutions allow for efficient verification. They are polynomial-time verifiable.

That property is shared by a very large number of interesting problems in planning, scheduling, design, information retrieval, networks, games, ...

Turing Machines

A Turing machine has an infinite tape through which it takes its input and performs its computations.

It can

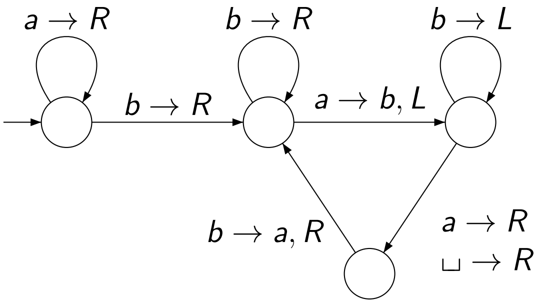
- both read from and write to the tape,
- move either left or right over the tape.

The tape is unbounded in both directions.

A Turing machine may fail to halt.

Turing Machine Example: Sorting

This (halting) Turing machine sorts a sequence of *a*s and *b*s.



The tape alphabet is $\{\sqcup, a, b\}$.

Turing Machines and Computability

Surprisingly, Turing machines appear to have the same "computational power" as any other sensible computing device.

By this we mean that any function that can be implemented in C, Java, Haskell, ... can be implemented on a Turing machine.

The Turing machine has a certain universality property: There is a Turing machine which is able to simulate any other Turing machine.

Here we shall assume that Turing machines are used to implement decision procedures: algorithms with a yes/no answer.

Non-Deterministic Turing Machines

One variant of the Turing machine has a powerful guessing capability: If different moves are available, the machine will favour a move that ultimately leads to a ‘yes’ answer.

Adding this kind of non-determinism to the capabilities of Turing machines does not change what Turing machines can compute.

However, it may have an impact on efficiency.

What a non-deterministic Turing machine can compute in polynomial time corresponds exactly to the class of polynomial-time verifiable problems.

Is there anything it can do in polynomial time that a deterministic machine cannot?

P and NP

P is the class of problems solvable in polynomial time by a deterministic Turing machine.

NP is the class of problems solvable in polynomial time by a non-deterministic Turing machine.

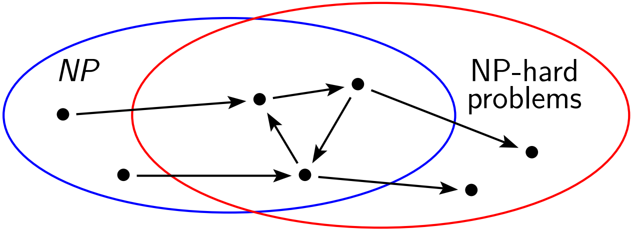
Clearly $P \subseteq NP$. $\boxed{\text{Is } P = NP\text{?}}$

Most computer scientists consider this computer science's

most important open question

NP-Completeness and NP-Hardness

NP-complete problems are the NP-hard problems in *NP*.



Here arrows indicate polynomial time reduction (a transitive relation).

Note: All problems in *NPC* stand or fall together!

NP-Complete Problems

SAT, SUBSET-SUM, 3COL, LPATH, and HAM, as well as thousands of other interesting and practically relevant problems have been shown to be NP-complete.

This explains the continuing interest in NP-completeness and related concepts from complexity theory.

For such problems we do not know of solutions that are essentially better than exhaustive search.

There are many other interesting complexity classes, including space complexity classes and probabilistic classes.

Open Problems

So we don't know whether $P = NP$, and there are many other unsolved problems.

We know that $P \subset \text{EXPTIME}$, that is, there are problems that can be solved in exponential time but provably not in polynomial time.

We also know:

$$P \subseteq RP \subseteq NP \subseteq \text{PSPACE} = \text{NPSPACE} \subseteq \text{EXPTIME}$$

But which of these inclusions are strict?
(Some must be; all could be.)

Dealing with NP-Completeness

Pseudo-polynomial problems (SUBSET-SUM and KNAPSACK are in this class): Unless you have really large data, don't worry; for reasonably-sized sets and numbers, the bad behaviour will not have kicked in yet.

Clever engineering to push the boundary slowly: SAT solvers.

Approximation algorithms: Settle for less than perfection.

Live happily with intractability: Sometimes the bad instances never turn up in practice. Example: Type inference in Haskell and ML.

Approximation Algorithms

For intractable optimization problems, it makes sense to look for approximation algorithms that are fast and still find solutions that are reasonably close to the optimal.

Example: Bin Packing

Bin packing is closely related to the knapsack problem.

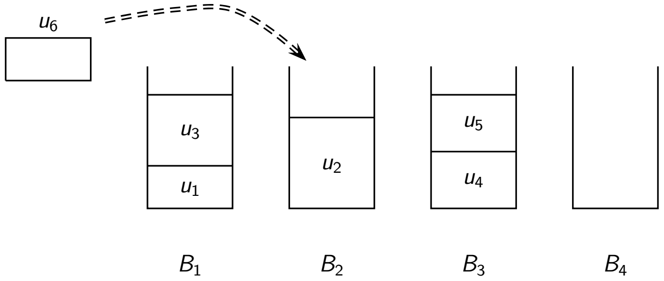
Given a finite set $U = \{u_1, u_2, \ldots, u_n\}$ of items and a rational size $s(u) \in [0, 1]$ for each item $u \in U$, partition U into disjoint subsets $U_1, U_2, \ldots, U_k$ such that

1. the sum of the sizes of items in U_i is at most 1; and
2. k is as small as possible.

The bin-packing problem is NP-hard.

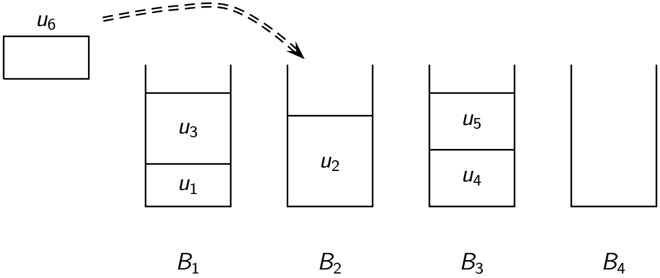
Bin Packing

Each subset U_i gives the set of items to be placed in a unit-sized "bin", with the objective of using as few bins as possible:



An example of using the (fast) "First Fit" heuristic: Use first bin that has the capacity.

Bin Packing: First Fit as Approximation



Invariant: At most one non-empty bin is half-or-less full. (If there were two such bins, their content would have been placed together.)

Hence the number of bins used with First Fit is never more than twice the minimal number required.

Bin Packing: "First Fit Decreasing"

First Fit behaves worst when we are left with many large items towards the end.

The variant in which the items are taken in order of decreasing size performs better.

The added cost (for sorting the items) is not large.

Detailed analysis shows that bin packing using the "First Fit Decreasing" heuristic guarantees that the number of bins used cannot exceed $\frac{11n}{9} + 4$, where n is the optimal solution.

Computability Theory

That completes our Cook's Tour of complexity theory.

Sometimes we run into more trouble than mere intractability!

There are practically important computational problems that have no algorithmic solution!

But that's a topic you'll have to discover in some other subject, on computability theory.