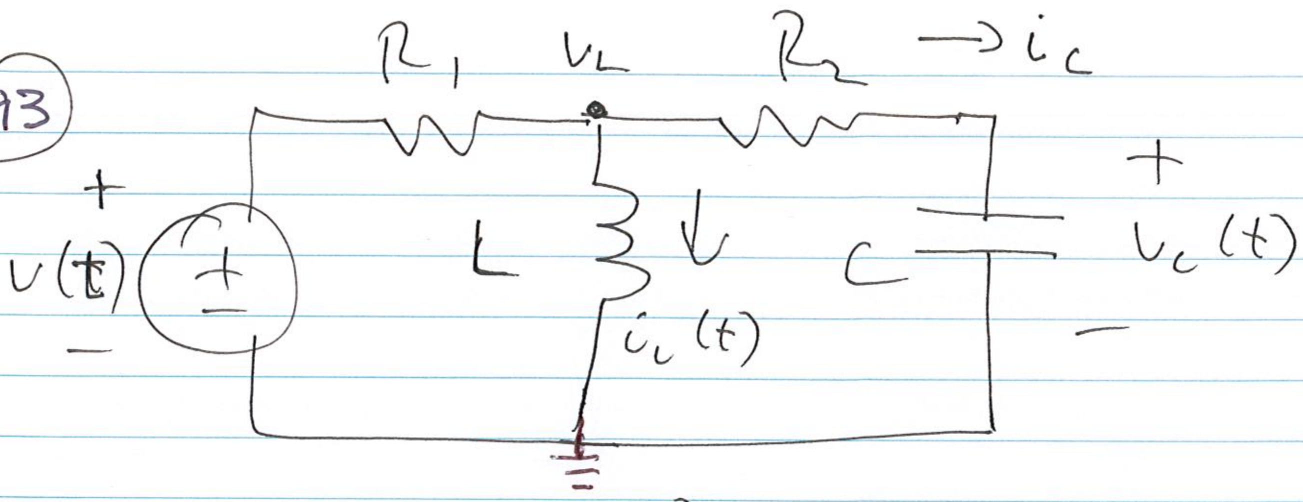
93



Here $i_L(t) = x_1(t)$ and $v_c(t) = x_2(t)$ are the state variables, and the voltage source $v(t)$ is the forcing term or input.

Let v_L = voltage across the inductor

$$= L \frac{di_L}{dt}$$

Let i_c = current through capacitor

$$= C \frac{dv_c}{dt}$$

So by Kirchoff Current Law at node v_L.

$$\frac{v - v_L}{R_1} = i_L + i_c. \quad — (1)$$

By OHM'S LAW

$$v_L - v_c = R_2 i_c. \quad — (2)$$

We obtain

$$\frac{v}{R_1} - \frac{L}{R_1}\dot{x}_1 = x_1 + C\dot{x}_2 \quad — (3)$$

$$L\dot{x}_1 - x_2 = R_2 C \dot{x}_2 \quad — (4)$$

So

$$\cancel{v\left(\frac{R_2}{R_1}\right) - L\left(\frac{R_2}{R_1}\right)\dot{x}_1 = R_2 x_1 + RC\dot{x}_2} \quad — (5)$$

(93) In matrix form this is

$$\begin{bmatrix} -\dfrac{L}{R_1} & -C \\ L & -R_2 C \end{bmatrix} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} - \begin{bmatrix} \dfrac{V}{R_1} \\ 0 \end{bmatrix}$$

Multiplying through by the inverse of $\begin{bmatrix} -L/R_1 & -C \\ L & -R_2 C \end{bmatrix}$ gives the state space representation

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} -\dfrac{R_1 R_2}{L(R_1+R_2)} & \dfrac{R_1}{L(R_1+R_2)} \\ -\dfrac{R_1}{L(R_1+R_2)} & -\dfrac{1}{C(R_1+R_2)} \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

$$+ \begin{bmatrix} \dfrac{R_2}{L(R_1+R_2)} \\ \dfrac{1}{C(R_1+R_2)} \end{bmatrix} V$$

$$y(t) = i_L(t) + V_C(t)$$

$$= \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

4 (a) $\ddot{y}(t) + 6\dot{y}(t) + 9y(t) = -v(t)$

We introduce $x_1(t) = -y(t)$

$x_2(t) = -\dot{y}(t) = \dot{x}_1(t)$

(Equation is second-order so we introduce two state variables).

Then $\dot{x}_1(t) = x_2(t)$

$$\begin{aligned}\dot{x}_2(t) &= -\ddot{y}(t)\\ &= 6\dot{y}(t) + 9y(t) + v(t)\\ &= -6x_2(t) - 9x_1(t) + v(t)\end{aligned}$$

So
$$\begin{bmatrix}\dot{x}_1(t)\\ \dot{x}_2(t)\end{bmatrix} = \begin{bmatrix}0 & 1\\ -9 & -6\end{bmatrix}\begin{bmatrix}x_1(t)\\ x_2(t)\end{bmatrix} + \begin{bmatrix}0\\ 1\end{bmatrix}v(t)$$

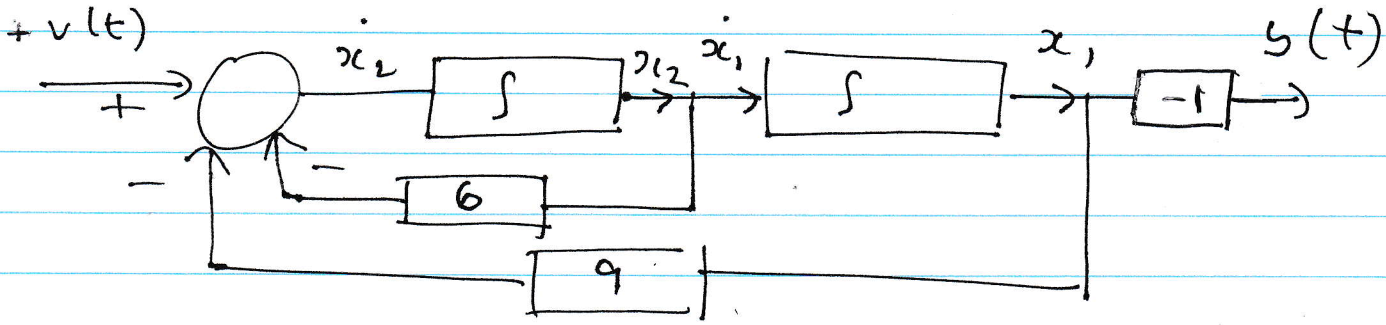
$$y(t) = \begin{bmatrix}-1 & 0\end{bmatrix}\begin{bmatrix}x_1(t)\\ x_2(t)\end{bmatrix}$$

Hence the state matrices are

$$A = \begin{bmatrix}0 & 1\\ -9 & -6\end{bmatrix}, \quad B = \begin{bmatrix}0\\ 1\end{bmatrix}$$

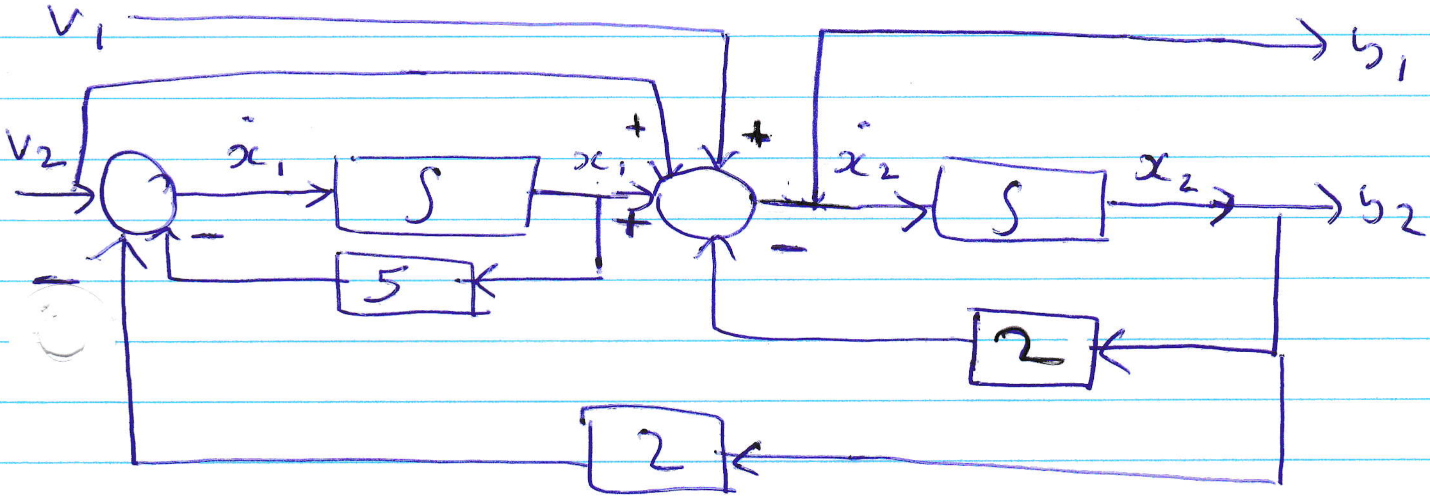
$$C = \begin{bmatrix}-1 & 0\end{bmatrix}, \quad D = 0.$$

(b)



(96) System has two inputs v_1, v_2
two outputs y_1, y_2

(a) We need to introduce two state variables x_1, x_2.



At the summer for $\dot{x}_1$ we have

$$\dot{x}_1(t) = -5x_1(t) - 2x_2(t) + v_2(t)$$

At the summer for $\dot{x}_2$ we have

$$\dot{x}_2(t) = x_1(t) - 2x_2(t) + v_1(t) + v_2(t)$$

The outputs are

$$y_1(t) = \dot{x}_2(t)$$
$$y_2(t) = x_2(t).$$

So
$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} -5 & -2 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} v_1(t) \\ v_2(t) \end{bmatrix}$$

$$\begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1(t) \\ v_2(t) \end{bmatrix}$$

98 $\begin{bmatrix} x_1[n+1] \\ x_2[n+1] \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ -1 & -2 \end{bmatrix} \begin{bmatrix} x_1[n] \\ x_2[n] \end{bmatrix} + \begin{bmatrix} 0.5 & 1 \\ -1 & -0.5 \end{bmatrix} \begin{bmatrix} v_1[n] \\ v_2[n] \end{bmatrix}$

$$\begin{bmatrix} y_1[n] \\ y_2[n] \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ -1 & -2 \end{bmatrix} \begin{bmatrix} x_1[n] \\ x_2[n] \end{bmatrix}$$

(a) If $x(0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $v[n] = \begin{bmatrix} n \\ n \end{bmatrix}$

then

$$x[1] = \begin{bmatrix} -1 & 1 \\ -1 & -2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 0.5 & 1 \\ -1 & -0.5 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ -3 \end{bmatrix}$$

$$x[2] = \begin{bmatrix} -1 & 1 \\ -1 & -2 \end{bmatrix} \begin{bmatrix} 0 \\ -3 \end{bmatrix} + \begin{bmatrix} 0.5 & 1 \\ -1 & -0.5 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} -3 \\ 6 \end{bmatrix} + \begin{bmatrix} 1.5 \\ -1.5 \end{bmatrix}$$

$$= \begin{bmatrix} -1.5 \\ 4.5 \end{bmatrix}$$

$$x[3] = \begin{bmatrix} -1 & 1 \\ -1 & -2 \end{bmatrix} \begin{bmatrix} -1.5 \\ 4.5 \end{bmatrix} + \begin{bmatrix} 0.5 & 1 \\ -1 & -0.5 \end{bmatrix} \begin{bmatrix} 2 \\ 2 \end{bmatrix}$$

$$= \begin{bmatrix} 6 \\ -7.5 \end{bmatrix} + \begin{bmatrix} 3 \\ -3 \end{bmatrix}$$

$$= \begin{bmatrix} 9 \\ -10.5 \end{bmatrix}$$

(b) We want to find $v[0]$ and $v[1]$ such that $x[2] = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$

We are told that $x[0] = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

98 (b) Then

$$x[1] = A\,x[0] + B\,v[0]$$
$$= B\,v[0]$$

as $x[0] = [0\ 0]^T$.

and

$$x[2] = A\,x[1] + B\,v[1]$$
$$= A\,B\,v[0] + B\,v[1]$$
$$= \begin{bmatrix} -1 \\ 2 \end{bmatrix}.$$

Solve ~~for~~ $v[0]$ and $v[1]$.

Note that $B = \begin{bmatrix} 0.5 & 1 \\ -1 & -0.5 \end{bmatrix}$

is invertible with

$$B^{-1} = \begin{bmatrix} -2/3 & -4/3 \\ 4/3 & 2/3 \end{bmatrix}$$

We can solve

$$v[1] = B^{-1}\left(x[2] - A\,B\,v[0]\right)$$

with $v[0]$ = arbitrary choice.

If we choose $= [0\ 0]$

then $v[1] = \begin{bmatrix} -2/3 & -4/3 \\ 4/3 & 2/3 \end{bmatrix}\begin{bmatrix} -1 \\ 2 \end{bmatrix}$

$$= \begin{bmatrix} -2 \\ 0 \end{bmatrix}$$

(c) We have $x[0] = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$ and want $x[2] = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

Then $x[2] = A\,x[1] + B\,v[1]$

$$= A^2 x[0] + AB\,v[0] + B\,v[1]$$

If we choose $v[0] = [0\ 0]^T$

Then we solve

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} = A^2 x[0] + B\,v[1] \Rightarrow v[1] = -B^{-1}A^2 x[0]$$
$$= \begin{bmatrix} 0 \\ -6 \end{bmatrix}.$$

(101) System has state matrices (A, B, C) with

$$A = \begin{bmatrix} 0 & 2 \\ 0 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 \\ 1 & -1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 3 \end{bmatrix}$$

(a) Find e^{At}.

First we find the eigenvalues of A. Note that A is an upper triangular matrix. Hence the eigenvalues appear on the main diagonal.

We have $\lambda_1 = 0$, $\lambda_2 = -1$.

Next we find the eigenvectors.

λ_1 : Solve $\left[A - \lambda_1 I \mid 0 \right]$

$$\Rightarrow \left[\begin{array}{cc|c} 0 & 2 & 0 \\ 0 & -1 & 0 \end{array}\right]$$

$$\Rightarrow 0x + 2y = 0$$

$$\Rightarrow y = 0, \text{ choose } x = 1$$

So $v_1 = (1, 0)^T$.

λ_2: Solve $\left[A - \lambda_2 I \mid 0 \right]$

$$\Rightarrow \left[\begin{array}{cc|c} 1 & 2 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$$\Rightarrow x + 2y = 0$$

Choose $x = 1 \Rightarrow y = \frac{-1}{2}$.

So $v_2 = (1, -\frac{1}{2})^T$.

Let $D = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix}$

and $P = \left[v_1 \; v_2 \right]^{-1}$

$$= \begin{bmatrix} 1 & 1 \\ 0 & -\frac{1}{2} \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 2 \\ 0 & -2 \end{bmatrix}$$

10.1 (a) and $P^{-1} = \begin{bmatrix} 1 & 1 \\ 0 & -\frac{1}{2} \end{bmatrix}$.

Hence $e^{At} = P^{-1} e^{Dt} P$

$$= \begin{bmatrix} 1 & 1 \\ 0 & -\frac{1}{2} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & e^{-t} \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & -2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 \\ 0 & -\frac{1}{2} \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & -2e^{-t} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 2 - 2e^{-t} \\ 0 & e^{-t} \end{bmatrix}$$

(b) If the input $v = 0$ and the initial state $x(0) = [1 \ -1]$ tha the zero-input response is

$$y_{zi}(t) = C e^{At} x(0)$$

$$= [1 \quad 3] \begin{bmatrix} 1 & 2-2e^{-t} \\ 0 & e^{-t} \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$= [1 \quad 3] \begin{bmatrix} 1 - (2 - 2e^{-t}) \\ -e^{-t} \end{bmatrix}$$

$$= 1 - (2 - 2e^{-t}) - 3e^{-t}$$

$$= -1 - e^{-t}$$

104 Let A and $\bar{A}$ be square matrices such that

$$A = P\bar{A}P^{-1} \qquad (*)$$

for some invertible matrix P.
Show that A and $\bar{A}$ have the same eigenvalues.

Proof : Let $\lambda \in \mathbb{C}$ be an eigenvalue of $\bar{A}$. We want to show that λ is also an eigenvalue of A.

Since λ is an eigenvalue of $\bar{A}$, there exists an eigenvector $\bar{x}$ of $\bar{A}$ such that

$$\bar{A}\bar{x} = \lambda \bar{x} \qquad (+)$$

Now consider the vector

$$x = P\bar{x}$$

~~Then~~ Then

$$\begin{aligned} Ax &= (P\bar{A}P^{-1})x \quad \text{by } (*) \\ &= P\bar{A}P^{-1}P\bar{x} \\ &= P\bar{A}\bar{x} \\ &= \lambda P\bar{x} \quad \text{by } (+) \\ &= \lambda x \end{aligned}$$

So x is an eigenvector of A ~~and~~ and λ is its eigenvalue.
An identical argument shows that every eigenvalue of A is an eigenvalue of $\bar{A}$. $\square$

(104) (b) Let

$$A = \begin{bmatrix} -2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \bar{A} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_0 & -a_1 & -a_2 \end{bmatrix}$$

Since we have P such that

$$A = P \bar{A} P^{-1}$$

we know that A and $\bar{A}$ have the same eigenvalues, by part (a).

Since A is a diagonal matrix, its eigenvalues are ~~on~~ the main diagonal

$$\lambda_1 = -2, \quad \lambda_2 = -1, \quad \lambda_3 = 1.$$

The characteristic polynomial of $\bar{A}$ is

$$|\bar{A} - \lambda I| = \begin{vmatrix} -\lambda & 1 & 0 \\ 0 & -\lambda & 1 \\ -a_0 & -a_1 & (-a_2 - \lambda) \end{vmatrix}$$

$$= -\lambda \begin{vmatrix} -\lambda & 1 \\ -a_1 & -a_2-\lambda \end{vmatrix} - \begin{vmatrix} 0 & 1 \\ -a_0 & -a_2-\lambda \end{vmatrix}$$

$$= -\lambda\left[(-\lambda)(-a_2-\lambda) + a_1\right] - \left[0 + a_0\right]$$

$$= -\lambda(\lambda^2 + a_2\lambda + a_1) - a_0$$

$$= -(\lambda^3 + a_2\lambda^2 + a_1\lambda + a_0) \leftarrow$$

The characteristic polynomial of A is

$$|A - \lambda I| = -(\lambda+2)(\lambda+1)(\lambda-1)$$

104 So

$$|A-\lambda I| = -(\lambda+2)\left[\lambda^2 - 1\right]$$

$$= -\left(\lambda^3 + 2\lambda^2 - \lambda - 2\right)$$

So we see that

$$\lambda^3 + 2\lambda^2 - \lambda - 2$$

$$= \lambda^3 + a_2\lambda^2 + a_1\lambda + a_0.$$

So $a_2 = 2$

$a_1 = -1$

$a_0 = -2$.

(c) We observe that the matrix P diagonalizes the matrix $\bar{A}$. By p 232 of the Lecture Notes,

$$P = \begin{bmatrix} v_1 & v_2 & v_3 \end{bmatrix}^{-1}$$ where $v_1, v_2\ v_3$ are the eigenvectors of $\bar{A}$

$$= \begin{bmatrix} -1 & -1 & -1 \\ 2 & 1 & -1 \\ -4 & -1 & -1 \end{bmatrix}^{-1}$$