ELEN 3002 MID-TEST 2019

1. $y(t) = \int_0^t (t-s)\,x(s)\,ds$

(a) System is time varying.
Let $x(t) = t.$
Then
$$y(t) = \int_0^t (t-s)\,s\,ds$$
$$= \int_0^t ts - s^2\,ds$$
$$= \left[\tfrac{1}{2}ts^2 - \tfrac{1}{3}s^3\right]_0^t$$
$$= \tfrac{1}{2}t^3 - \tfrac{1}{3}t^3 \quad (2)$$
$$= \tfrac{1}{6}t^3$$
Next let
$$x_1(t) = x(t-1)$$
$$= t-1.$$
Then let $y_1(t)$ = output from $x_1(t)$ (1)
$$= \int_0^t (t-s)(s-1)\,ds$$
$$= \int_0^t ts - s^2 - t + s\,ds$$
$$= \left[\tfrac{1}{2}ts^2 - \tfrac{1}{3}s^3 - ts + \tfrac{1}{2}s^2\right]_0^t$$
$$= \tfrac{1}{2}t^3 - \tfrac{1}{3}t^3 - t^2 + \tfrac{1}{2}t^2$$
$$= \tfrac{1}{6}t^3 - \tfrac{1}{2}t^2 \quad (3)$$
Also
$$y(t-1) = \tfrac{1}{6}(t-1)^3$$
$$= \tfrac{1}{6}\left[t^3 - 3t^2 - 3t + 1\right]$$
$$\neq y_1(t). \quad (2)$$
Hence not time invariant

Total: 8 marks

(2) System is defined by

(a) $y[n] - (1+i)\, y[n] = -x[n]$
$n \geq 1.$

Hence

$$y[1] = (1+i)\, y[0] - x[n]$$
$$= (1+i)\, y[0] - c \quad (1)$$

and

$$y[2] = (1+i)\, y[1] - c$$
$$= (1+i)\left[(1+i)\, y[0] - c\right] - c$$
$$= (1+i)^2 y[0] - c\left[(1+i) + 1\right] \quad (1)$$

So in general

$$y[n] = (1+i)^n y[0] - \quad (2)$$
$$- c\left[(1+i)^{n-1} + \dots + (1+i) + 1\right]$$

Using the sum of a geometric series we obtain

$$y[n] = (1+i)^n y[0]$$
$$- c\, \frac{1 - (1+i)^n}{1 - (1+i)} \quad (2)$$
$$= (1+i)^n y[0] + \frac{c}{i}\left[1 - (1+i)^n\right]$$

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(b) The natural response is
(1) $(1+i)^n y[0]$, due to the initial condition $y[0]$ (1)

The step response is
(1) $\frac{c}{i}\left[1 - (1+i)^n\right]$, due to the step input $x[n] = c$. (1)

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(2) (c) For the loan to be paid after N months, we need

$$y[N] = 0 \quad (2)$$
$$= (1+i)^N y[0] + \frac{c}{i}\left[1-(1+i)^N\right]$$

So $\frac{c}{i}\left[1-(1+i)^N\right] = -(1+i)^N y[0]$

$$\Rightarrow c = \frac{i(1+i)^N y[0]}{(1+i)^N - 1}.$$

If $N=24$, $i=1\%$, $y[0]=10{,}000$ then the monthly payments need to

$$c = \frac{(0.01)(1.01)^{24}(10{,}000)}{(1.01)^{24}-1}$$

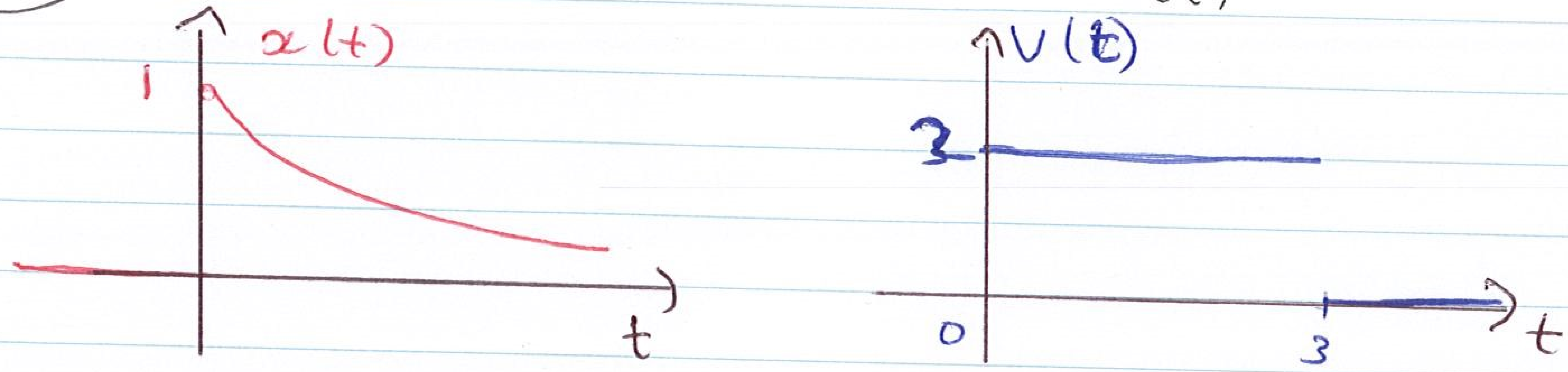
$$= \$470.73 \quad (2)$$

Total: 14 marks $\boxed{4}$

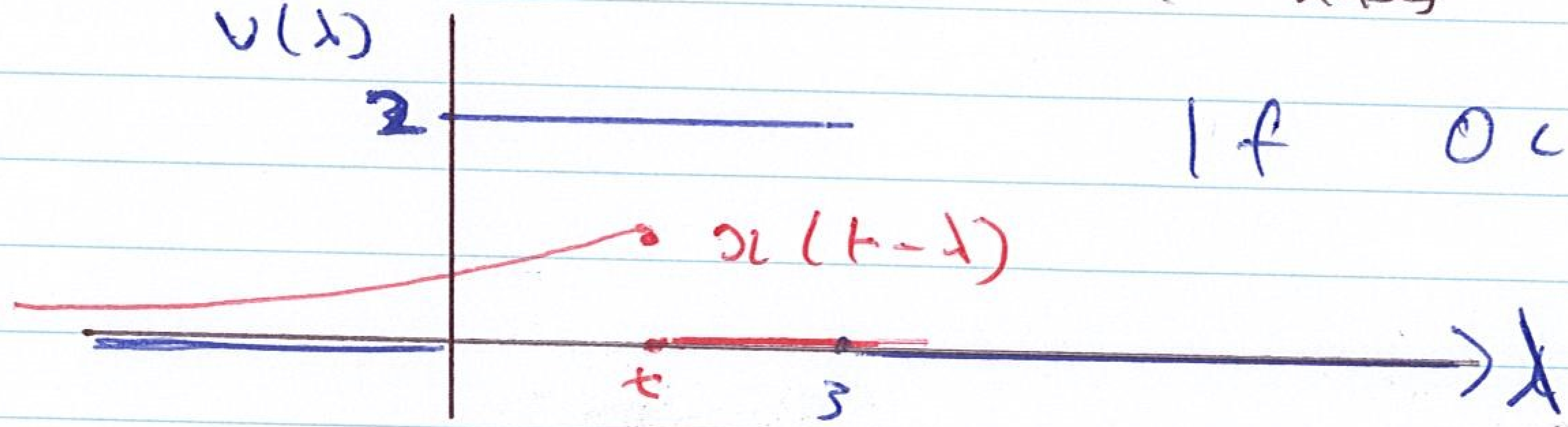
(3) (a) $h(t) = \begin{cases} e^{-t}, & t \geq 0 \\ 0, & \text{otherwise} \end{cases}$

The system is causal because the impulse response $h(t) = 0$ for $t < 0$. (1)

3 Sketch $x(t)$ and $v(t)$

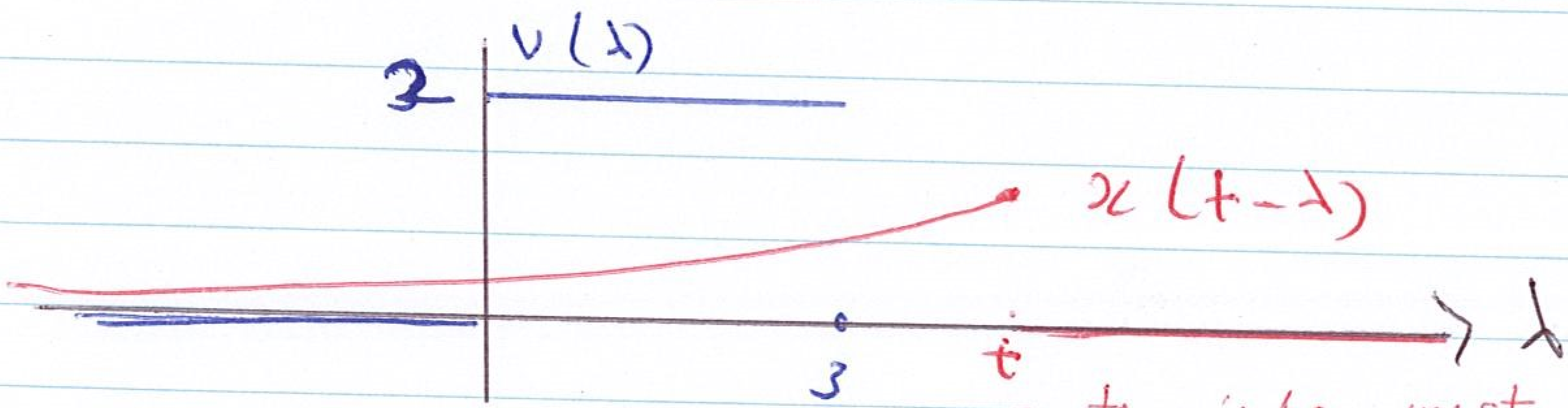


Hence we need to plot $v(\lambda)$ and $x(t-\lambda)$ on the same axes



If $0 < t \le 3$

(2)



(2)

terminals must be correct

So if $0 < t \le 3$: $(x*v)(t) = \int_0^t 2e^{-(t-\lambda)}\,d\lambda$ (2)

$$= \left[2e^{-(t-\lambda)}\right]_0^t$$

$$= 2(1 - e^{-t})$$ (2)

and if $t \geq 3$: $(x*v)(t) = \int_0^3 2e^{-(t-\lambda)}\,d\lambda$ (2)

terminals must be correct

$$= \left[2e^{-(t-\lambda)}\right]_0^3$$

$$= 2\left[e^{-(t-3)} - e^{-t}\right]$$ (2)

Total: 13 marks

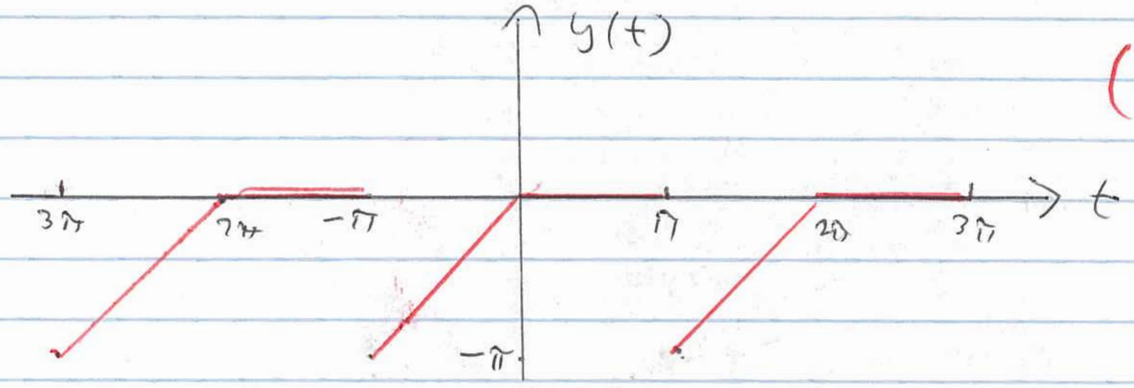
$\boxed{12}$

(4) $f(t) = \begin{cases} t, & -\pi < t < 0 \\ 0, & \text{otherwise} \end{cases}$

and

$$y(t) = \sum_{k=-\infty}^{+\infty} f(t - 2\pi k).$$

(a)



(3)

(b) Yes, y is periodic with period $T = 2\pi$

$$y(t + 2\pi) = \sum_{k=-\infty}^{+\infty} f(t + 2\pi - 2\pi k) \quad (1)$$

$$= \sum_{k=-\infty}^{+\infty} f(t - 2\pi(k-1))$$

(1)

Let $m = k-1$, $k = 1+m$, $m(\infty)=\infty$, $m(-\infty)=-\infty$

$$= \sum_{m=-\infty}^{+\infty} f(t - 2\pi m) \quad (1)$$

[4]

$$= y(t). \quad (1)$$

Total: $\boxed{7}$

(5) We know that $x(t) \leftrightarrow X(w)$

Let $f(t) = x(-t)$.

Then $F(w) = \int_{-\infty}^{\infty} f(t) e^{-jwt} dt$

$$= \int_{-\infty}^{\infty} x(-t) e^{-jwt} dt. \quad (2)$$

Let $s = -t$, $ds = -dt$,

$x(\infty) = -\infty$, $x(-\infty) = \infty$ (2)

So $F(w) = \int_{\infty}^{-\infty} -x(s) e^{+jws} ds$ (2)

$$= \int_{-\infty}^{\infty} x(s) e^{jws} ds$$

$$= X(-w). \quad (2)$$

So $x(t) \leftrightarrow X(-w)$.

$\boxed{8}$