Chapter 4:
Fourier Transform for Continuous-time Signals

In this chapter we continue our study of continuous-time signals and introduce the Fourier transform. This generalizes the concept of Fourier series to include functions that are not periodic. The frequency spectrum of a non-periodic signal is defined for all real values of the frequency variable.

Definition (Fourier Transform)

For a continuous-time signal x, we define its **Fourier Transform** to be

$$X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} \, dt$$

- The variable $\omega \in \mathbf{R}$ is called the **frequency variable.**
- We use lower case letters like x and f to denote continuous-time signals, and capital letters X and F to denote their Fourier transform.
- The Fourier transform is complex-valued, and to plot the transform requires separate graphs for the amplitude $|X(\omega)|$ and phase $\angle X(\omega)$.
- The amplitude and phase spectra are generalizations of the spectra for periodic signals.

Definition (Inverse Fourier Transform)

For a continuous-time signal x with Fourier Transform $X(\omega)$, the **Inverse Fourier Transform** of $X(\omega)$ is given by

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) e^{j\omega t} \, d\omega$$

- We say that x and X are Fourier Transform pairs, and write

$$x(t) \longleftrightarrow X(\omega)$$

- Recalling that the complex Fourier series for a periodic signal is

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t}$$

  we see that $X(\omega)$ is a generalization of the Fourier complex coefficients c_k. Thus a non-periodic function $x(t)$ can also be represented in terms of its frequency spectrum.

Theorem (Existence of the Fourier Transform)

A continuous-time signal x has a Fourier Transform if it satisfies the following conditions:

1. *x is* **absolutely integrable** *over* **R**, *i.e.*

$$\int_{-\infty}^{\infty} |x(t)| \, dt < \infty$$

2. *x has only finitely many maxima, minima and points of discontinuity on any interval of finite length.*

- Functions that satisfy these conditions are said to be **well-behaved.** Kamen and Heck argue that all signals that can be physically generated are well-behaved.
- The Fourier transform generalizes the concepts of amplitude and phase spectra to accommodate non-periodic signals. If $|X(\omega)|$ is large for certain values of ω, it means that those frequencies make up a large component of the signal $x(t)$.

Example

Consider the function

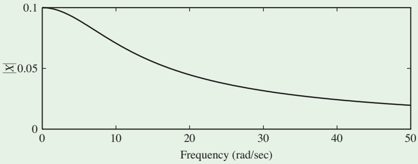
$$x(t) = e^{-bt}u(t)$$

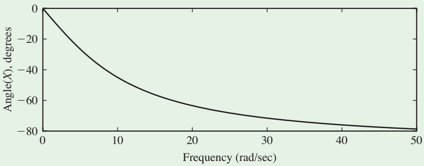
where $b > 0$ and $u(t)$ is the unit step function. It can be shown that

$$X(\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}\, dt = \frac{1}{b + j\omega}$$

so the amplitude and phase spectra are

$$|X(\omega)| = \frac{1}{\sqrt{b^2 + \omega^2}}, \quad \angle X(\omega) = -\tan^{-1}\left(\frac{\omega}{b}\right)$$





Using Euler's formulae $e^{j\theta} = \cos\theta + j\sin(\theta)$, we can write

$$X(\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}\, dt = \int_{-\infty}^{\infty} x(t)\cos(\omega t)\, dt - j\int_{-\infty}^{\infty} x(t)\sin(\omega t)\, dt$$

Definition (Rectangular form of the Fourier Transform)

Let

$$\begin{aligned} R(\omega) &= \int_{-\infty}^{\infty} x(t)\cos(\omega t)\, dt, \\ I(\omega) &= -\int_{-\infty}^{\infty} x(t)\sin(\omega t)\, dt \end{aligned}$$

Then

$$X(\omega) = R(\omega) + jI(\omega)$$

This the **Rectangular form** of the Fourier transform $X(\omega)$.

Definition (Polar form of the Fourier Transform)

Let

$$
\begin{aligned}
|X(\omega)| &= \sqrt{R^2(\omega) + I^2(\omega)} \\
\angle X(\omega) &= \begin{cases} \tan^{-1}\left(\frac{I(\omega)}{R(\omega)}\right), & R(\omega) \geq 0 \\ \pi + \tan^{-1}\left(\frac{I(\omega)}{R(\omega)}\right), & R(\omega) < 0 \end{cases}
\end{aligned}
$$

Then the **Polar form** of the Fourier transform $X(\omega)$ is

$$X(\omega) = |X(\omega)| \exp(j \angle X(\omega))$$

Theorem

For any continuous-time signal x with Fourier Transform X,

1. $|X(-\omega)| = |X(\omega)|$
2. $\angle X(-\omega) = -\angle X(\omega)$

Thus $|X(\omega)|$ is an even function, and $\angle X(\omega)$ is an odd function of ω.

Theorem (Even and Odd signals)

1. *For any continuous-time even signal x with Fourier Transform X,*

$$\begin{aligned} R(\omega) &= 2\int_0^{\infty} x(t)\cos(\omega t)\, dt, \\ I(\omega) &= 0 \end{aligned}$$

and so $\quad X(\omega) = 2\int_0^{\infty} x(t)\cos(\omega t)\, dt$

2. *For any continuous-time odd signal x with Fourier Transform X,*

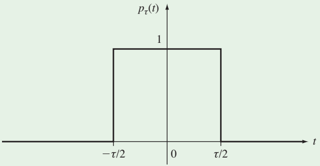
$$\begin{aligned} R(\omega) &= 0 \\ I(\omega) &= -2\int_0^{\infty} x(t)\sin(\omega t)\, dt \end{aligned}$$

and so $\quad X(\omega) = -j2\int_0^{\infty} x(t)\sin(\omega t)\, dt$

Example (Rectangular Pulse)

Consider the rectangular pulse function of width τ

$$p_\tau(t) = \begin{cases} 1, & -\frac{\tau}{2} \leq t < \frac{\tau}{2} \\ 0, & \text{otherwise} \end{cases}$$



Then p_τ is an even function and hence we can use

$$\begin{aligned} P_\tau(\omega) &= 2\int_0^\infty x(t)\cos(\omega t)\, dt \\ &= 2\int_0^{\tau/2} \cos(\omega t)\, dt \\ &= \frac{2}{\omega}\sin\left(\frac{\omega\tau}{2}\right) \end{aligned}$$

Example (Rectangular Pulse II)

The transform of the rectangular pulse can be expressed in terms of the sinc function, which is defined as

$$\mathrm{sinc}(a\omega) = \frac{\sin(a\pi\omega)}{a\pi\omega}, \quad \text{for any } a \in \mathbf{R}$$

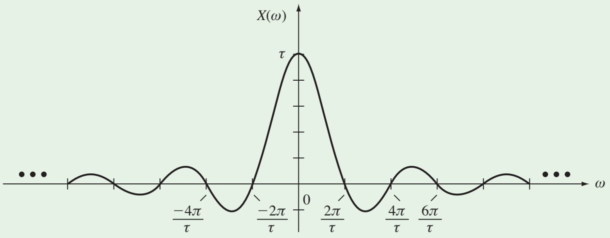
We saw that

$$\begin{aligned} P_\tau(\omega) &= \frac{2}{\omega}\sin\left(\frac{\omega\tau}{2}\right) \\ &= \frac{2}{\omega}\left[\frac{\sin\left(\frac{\pi\omega\tau}{2\pi}\right)}{\left(\frac{\pi\omega\tau}{2\pi}\right)}\left(\frac{\pi\omega\tau}{2\pi}\right)\right] \\ &= \tau\,\mathrm{sinc}\left(\frac{\tau\omega}{2\pi}\right) \end{aligned}$$

using $a = \frac{\tau}{2\pi}$.

Example (Rectangular Pulse III)

Since $P_\tau(\omega)$ is real-valued, we can plot it as a function of ω:

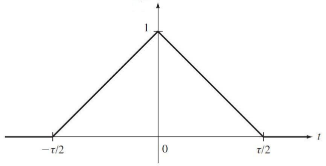


- For small values of the pulse width τ, the spectrum spreads out across a larger frequency band.
- For large values of τ, the spectrum becomes narrow and taller (more like a spike).

Example (Triangular Pulse)

Consider the triangular pulse function of width τ:

$$\Lambda_\tau(t) = \begin{cases} 1 - \frac{2|t|}{\tau}, & -\frac{\tau}{2} \le t < \frac{\tau}{2} \\ 0, & \text{otherwise} \end{cases}$$



Then Λ_τ is an even function and hence we can use

$$\begin{aligned} L_\tau(\omega) &= 2\int_0^\infty \Lambda_\tau(t)\cos(\omega t)\,dt \\ &= 2\int_0^{\tau/2} \left(1 - \frac{2t}{\tau}\right)\cos(\omega t)\,dt \\ &= \frac{8}{\tau\omega^2}\left(\sin^2\left(\frac{\tau\omega}{4}\right)\right) \end{aligned}$$

Example (Triangular Pulse II)

Recall that

$$\mathrm{sinc}(a\omega) = \frac{\sin(a\pi\omega)}{a\pi\omega}, \quad \text{for any } a \in \mathbf{R}$$

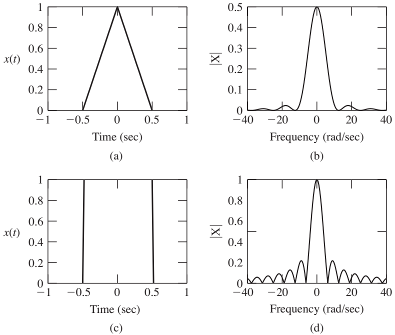
Hence

$$\begin{aligned} L_\tau(\omega) &= \frac{8}{\tau\omega^2}\left(\sin^2\left(\frac{\tau\omega}{4}\right)\right) \\ &= \frac{8}{\tau\omega^2}\left[\left(\frac{\sin\left(\frac{\tau\pi\omega}{4\pi}\right)}{\frac{\tau\pi\omega}{4\pi}}\right)\left(\frac{\tau\pi\omega}{4\pi}\right)\right]^2 \\ &= \frac{\tau}{2}\,\mathrm{sinc}^2\left(\frac{\tau\omega}{4\pi}\right) \end{aligned}$$

using $a = \frac{\tau}{4\pi}$ and

$$\mathrm{sinc}(a\omega) = \frac{\sin(a\pi\omega)}{a\pi\omega}$$

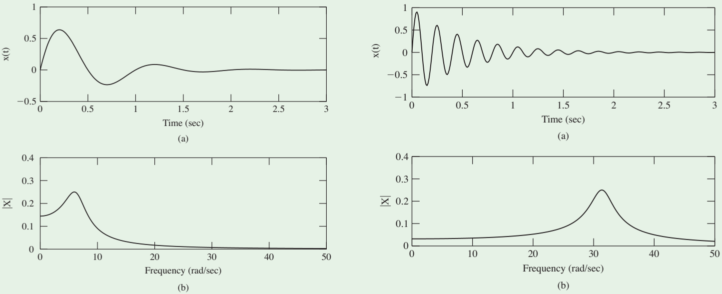
The magnitude spectra of the rectangular and triangular pulse signals are



- Rapid changes in time-domain (e.g. discontinuities) leads to more high frequency content in the spectrum. Thus the side lobes of the rectangular pulse are larger.
- The main lobe of the triangular pulse is wider, showing more low frequency content.

Example (Decaying sinusoid)

$$x(t) = e^{-at}\sin(b\pi t)u(t) \longleftrightarrow X(\omega) = \frac{2\pi}{4 - \omega^2 + 4\pi^2 + j4\omega}$$



- When $b = 2$ we have a dominant frequency at $\omega = 2\pi$.
- When $b = 10$ the time-domain signal has more rapid fluctuations, leading to more high frequency content and a dominant frequency at $\omega = 10\pi$.

The Fourier Transform has many properties that allow us to compute transforms of signals without using the definition of the transform. Some of these are

Theorem (Linearity)

The Fourier Transform is linear: if $x_1(t) \longleftrightarrow X_1(\omega)$ *and* $x_2(t) \longleftrightarrow X_2(\omega)$*, and a and b are any two scalars, then*

$$ax_1(t) + bx_2(t) \longleftrightarrow aX_1(\omega) + bX_2(\omega)$$

Theorem (Time shifting)

If $x(t) \longleftrightarrow X(\omega)$ *and* $c \in \mathbf{R}$*, then*

$$x(t-c) \longleftrightarrow X(\omega)e^{-j\omega c}$$

Theorem (Time scaling)

If $x(t) \longleftrightarrow X(\omega)$ *and* $a > 0$, *then*

$$x(at) \longleftrightarrow \frac{1}{a} X\left(\frac{\omega}{a}\right)$$

Theorem (Time reversal or Flipping)

If $x(t) \longleftrightarrow X(\omega)$ *then*

$$x(-t) \longleftrightarrow X(-\omega)$$

Corollary

If $x(t) \longleftrightarrow X(\omega)$ *and* $a \in \mathbf{R}$ *with* $a \neq 0$, *then*

$$x(at) \longleftrightarrow \frac{1}{|a|} X\left(\frac{\omega}{a}\right)$$

Theorem (Modulation)

If $x(t) \longleftrightarrow X(\omega)$ *and* $\omega_0 \in \mathbf{R}$*, then*

$$\begin{aligned} x(t)e^{j\omega_0 t} &\longleftrightarrow X(\omega-\omega_0) \\ x(t)\cos(\omega_0 t) &\longleftrightarrow \frac{1}{2}[X(\omega+\omega_0)+X(\omega-\omega_0)] \\ x(t)\sin(\omega_0 t) &\longleftrightarrow \frac{j}{2}[X(\omega+\omega_0)-X(\omega-\omega_0)] \end{aligned}$$

Theorem (Convolution)

If $x(t) \longleftrightarrow X(\omega)$ *and* $v(t) \longleftrightarrow V(\omega)$*, then*

$$(x \star v)(t) \longleftrightarrow X(\omega)V(\omega)$$

Theorem (Duality)

If $x(t) \longleftrightarrow X(\omega)$*, then*

$$X(t) \longleftrightarrow 2\pi x(-\omega)$$

Some important signals like $\cos(\omega t)$ and $\sin(\omega t)$ do not have Fourier transforms in the ordinary sense because they are not absolutely integrable, i.e.

$$\int_{-\infty}^{\infty} |\sin(\omega t)| \, dt = \infty$$

Nonetheless these functions have Fourier series representations - so they *should* have Fourier transforms.

Next we see how to define the Fourier transform of these functions using the generalized function $\delta(t)$. Recall the Sifting Theorem

$$\int_{-\infty}^{\infty} f(\lambda)\delta(\lambda - t_0) \, d\lambda = f(t_0)$$

Hence

$$\int_{-\infty}^{\infty} \delta(t)e^{-j\omega t} \, dt = e^0 = 1$$

so $\delta(t)$ and 1 are a **generalized Fourier transform** pair:

$$\delta(t) \longleftrightarrow 1$$

Hence by duality we have

$$1 \longleftrightarrow 2\pi\delta(\omega)$$

as $\delta(-\omega) = \delta(\omega)$. We recall the modulation theorem

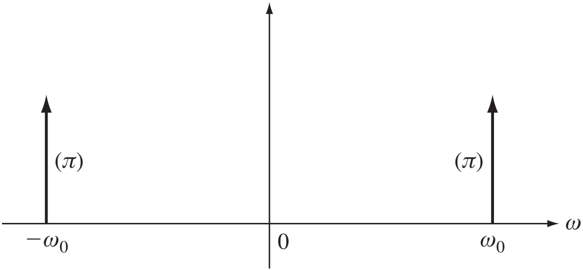
$$\begin{aligned}
x(t)e^{j\omega_0 t} &\longleftrightarrow X(\omega - \omega_0) \\
x(t)\cos(\omega_0 t) &\longleftrightarrow \frac{1}{2}[X(\omega + \omega_0) + X(\omega - \omega_0)] \\
x(t)\sin(\omega_0 t) &\longleftrightarrow \frac{j}{2}[X(\omega + \omega_0) - X(\omega - \omega_0)]
\end{aligned}$$

to obtain the generalized Fourier transform pairs

Theorem

$$\begin{aligned}
e^{j\omega_0 t} &\longleftrightarrow 2\pi\delta(\omega - \omega_0) \\
\cos(\omega_0 t) &\longleftrightarrow \pi[\delta(\omega + \omega_0) + \delta(\omega - \omega_0)] \\
\sin(\omega_0 t) &\longleftrightarrow j\pi[\delta(\omega + \omega_0) - \delta(\omega - \omega_0)]
\end{aligned}$$

A frequency plot of the transform of $\cos(\omega_0 t)$ looks like



For a periodic signal with complex Fourier series

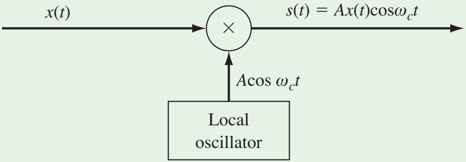
$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t}$$

the Fourier transform is a train of impulse signals

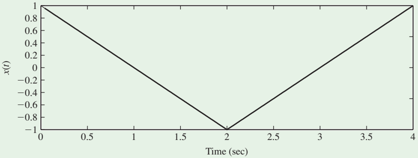
$$X(\omega) = \sum_{k=-\infty}^{\infty} 2\pi c_k \delta(\omega - k\omega_0)$$

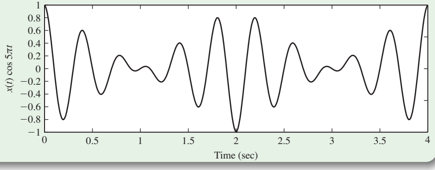
Example (Analog signal modulation)

Transmission of signal over a channel may be done by modulating the signal with a carrier signal.



The signal $x(t)$ is modulated by the carrier signal $A\cos(\omega_c t)$



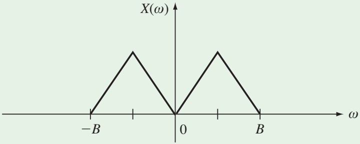


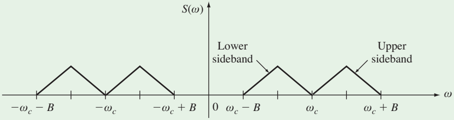
Example (Analog signal modulation II)

The transmitted signal is $s(t) = Ax(t)\cos(\omega t)$ with transform

$$S(\omega) = \frac{A}{2}[X(\omega + \omega_c) + X(\omega - \omega_c)]$$

The transforms of the original **baseband** signal and the modulated **passband** signal are





Usually the frequency ω_c of the carrier signal is much higher than the frequencies of the base signal, so that the spectrum of the passband signal is in a range that is suitable for transmission through cables or free space.