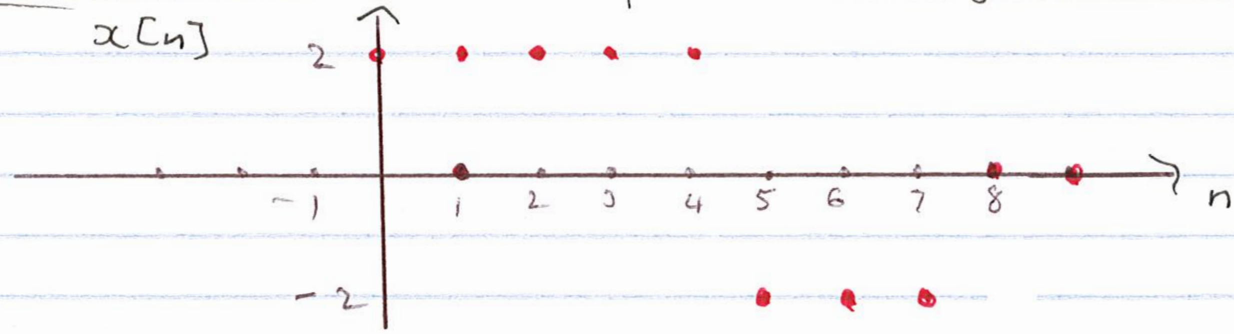
58: Let

$$x[n] = \begin{cases} 2, & n = 0, 1, 2, 3, 4 \\ -2, & n = 5, 6, 7 \\ 0, & \text{otherwise.} \end{cases}$$

Find $X(\Omega)$, the DTFT of x.

Solution: First we plot $x[n]$:



Consider the rectangular pulse functions

$$p_5[n] = \begin{cases} 1, & -2 \le n \le 2 \\ 0, & \text{otherwise} \end{cases}$$

$$p_3[n] = \begin{cases} 1, & -1 \le n \le 1 \\ 0, & \text{otherwise} \end{cases}$$

Then

$$x[n] = 2p_5[n-2] - 2p_3[n-6]$$

We know that for an odd integer L

$$p_L[n] \longleftrightarrow \frac{\sin[(q+\frac{1}{2})\Omega]}{\sin[\Omega/2)}$$

where $q = (L-1)/2$.

So by time shifting

$$X(\Omega) = \frac{2\sin(\frac{5}{2}\Omega)\, e^{-j2\Omega}}{\sin(\Omega/2)} - \frac{2\sin(\frac{3}{2}\Omega)\, e^{-j6\Omega}}{\sin(\Omega/2)}.$$

61 (a) Use the definition of DTFT to prove Time shifting:

$$x[n-q] \leftrightarrow X(\Omega)e^{-j\Omega q}$$

$q \in \mathbb{Z}$.

Proof: We know that $x[n] \leftrightarrow X(\Omega)$

So $$X(\Omega) = \sum_{n=-\infty}^{+\infty} x[n]e^{-j\Omega n}$$

Let $y[n] = x[n-q]$

Then $$Y(\Omega) = \sum_{n=-\infty}^{+\infty} y[n]e^{-j\Omega n}$$

$$= \sum_{n=-\infty}^{+\infty} x[n-q]e^{-j\Omega n}$$

Introduce $\boxed{m = n - q \Rightarrow n = m + q}$

and $\boxed{m(\infty) = \infty, \quad m(-\infty) = -\infty.}$

So we can write

$$Y(\Omega) = \sum_{m=-\infty}^{+\infty} x[m]e^{-j\Omega(m+q)}$$

$$= e^{-j\Omega q}\sum_{m=-\infty}^{+\infty} x[m]e^{-j\Omega m}$$

$$= e^{-j\Omega q}\sum_{n=-\infty}^{+\infty} x[n]e^{-j\Omega n}$$

$$= e^{-j\Omega q}X(\Omega)$$

as required.

(61) (b) Prove the Flipping Theorem for the DTFT:

$$x[-n] \Leftrightarrow X(-\Omega).$$

Proof: Let $x[n] \Leftrightarrow X(\Omega)$ be a DTFT pair. Then

$$X(\Omega) = \sum_{n=-\infty}^{+\infty} x[n]\, e^{-j\Omega n}.$$

Let $f[n] = x[-n]$. Then

$$F(\Omega) = \sum_{n=-\infty}^{+\infty} f[n]\, e^{-j\Omega n}$$

$$= \sum_{n=-\infty}^{+\infty} x[-n]\, e^{-j\Omega n}$$

Introduce $m = -n$, so $m(\infty) = -\infty$, $m(-\infty) = \infty$.

$$= \sum_{m=\infty}^{-\infty} x[m]\, e^{j\Omega m}$$

$$= \sum_{m=-\infty}^{+\infty} x[m]\, e^{j\Omega m}$$

(The order of summation does not affect the sum of an absolutely convergent series)

$$= X(-\Omega)$$

Hence $x[-n] \Leftrightarrow X(-\Omega)$

as required.

(63) (a) Find the inverse DTFT of
$$X_1(\Omega) = \sin(\Omega)$$

Solution: We know that
$\delta[n] \leftrightarrow 1$ is a DTFT pair.
Hence by the time shifting theorem
$$\delta[n-q] = e^{-j\Omega q}$$
for any integer q.

Since
$$X_1(\Omega) = \frac{1}{2j}\left(e^{j\Omega} - e^{-j\Omega}\right)$$
(Euler's Theorem)
we can take inverse transform and obtain
$$x_1[n] = \frac{1}{2j}\left[\delta[n+1] - \delta[n-1]\right].$$

(b) Find the inverse DTFT of
$$X_2(\Omega) = \cos(\Omega).$$

Since $X_2(\Omega) = \frac{1}{2}\left(e^{j\Omega} + e^{-j\Omega}\right)$
using the Shifting Theorem we obtain
$$x_2[n] = \frac{1}{2}\left[\delta[n+1] + \delta[n-1]\right]$$

(c) Let $X_3(\Omega) = \cos^2(\Omega)$
$= X_2(\Omega)\, X_2(\Omega).$
Hence by Convolution Theorem for DTFT
$$x_3[n] = (x_2 * x_2)[n]$$
Use Array Method

		$x_2(-1)$	$x_2(0)$	$x_2(1)$
		~~$\frac{1}{2}$~~ 1	0	~~$\frac{1}{2}$~~ 1
$x_2(-1)$	1			
$x_2(0)$	0	0		$x_2(-1)$
$x_2(0)$	0	$\frac{1}{2}$		$x_2(0)$
$x_2(1)$	1			$x_2(1)$

	$x_2(-1)$	$x_2(0)$	$x_2(1)$
	$\frac{1}{2}$	0	$\frac{1}{2}$
$\frac{1}{2}$	1/4	0	1/4
0	0	0	0
$\frac{1}{2}$	$\frac{1}{4}$	0	$\frac{1}{4}$

Q3 (c) Since $x_2[n]$ has support $[-1, 1]$, we see that $x_2 * x_2$ has support $[-2, 2]$.

and $x_3[n] = \begin{cases} \frac{1}{4}, & n = -2 \text{ and } +2 \\ \frac{1}{2}, & n = 0 \\ 0, & \text{otherwise} \end{cases}$

$$= \frac{1}{4}\left[\delta[n+2] + 2\delta[n] + \delta[n-2]\right]$$

Q3 (d) Let $X_4(\Omega) = \sin(\Omega)\cos(\Omega)$

$$= X_1(\Omega)\, X_2(\Omega).$$

Then $x_4[n] = (x_1 * x_2)[n]$ by Convolution Theorem

Use Array method:

		$x_1(-1)$ $\frac{1}{2}j$	$x_1(0)$ 0	$x_1(1)$ $-\frac{1}{2}j$
$x_2(-1)$	$\frac{1}{2}$	$\frac{1}{4}j$	0	$-\frac{1}{4}j$
$x_2(0)$	0	0	0	0
$x_2(1)$	$\frac{1}{2}$	$\frac{1}{4}j$	0	$-\frac{1}{4}j$

So $x_4[n] = \begin{cases} \frac{1}{4}j, & n = -2 \\ -\frac{1}{4}j, & n = 2 \\ 0, & \text{otherwise.} \end{cases}$

$$= \frac{1}{4j}\left[\delta[n+2] - \delta[n-2]\right]$$

(66) Signal x has record length $L = 4$ and it 4-point DFT is
$X_0 = 2$, $X_1 = -2 + j2$, $X_2 = -2$, $X_3 = -2 - j2$
Find $x[n]$.

Solution The inverse DFT is

$$x[n] = \frac{1}{N} \sum_{k=0}^{N-1} X_k e^{j\frac{2\pi k n}{N}}$$

$n = 0, 1, \ldots, L-1$

So here $N = 4$, (4-point DFT)

$$x[0] = \frac{1}{4} \sum_{k=0}^{3} X_k e^{0}$$

$$= \frac{1}{4}\left[2 + (-2+j2) + (-2) + (-2-j2)\right]$$

$$= -1$$

$$x[1] = \frac{1}{4} \sum_{k=0}^{3} X_k e^{j\frac{2\pi k}{4}}$$

$$= \frac{1}{4}\left[2 + (-2+j2)e^{j\frac{\pi}{2}} + (-2)e^{j\pi} + (-2-j2)e^{j\frac{3\pi}{2}}\right]$$

$$= \frac{1}{4}\left[2 + (-2+j2)j + (-2)(-1) + (-2-j2)(-j)\right]$$

$$= 0$$

$$x[2] = \frac{1}{4} \sum_{k=0}^{3} X_k e^{j\pi k}$$

$$= \frac{1}{4}\left[2 + (-2+j2)(-1) + (-2)(1) + (-2-j2)(-1)\right]$$

$$= 1$$

$$x[3] = \frac{1}{4} \sum_{k=0}^{3} X_k e^{j\frac{3\pi k}{2}}$$

$$= \frac{1}{4}\left[2 + (-2+j2)(-j) + (-2)(-1) + (-2-j2)(j)\right]$$

$$= 2$$

$x[n] = 0$, otherwise.