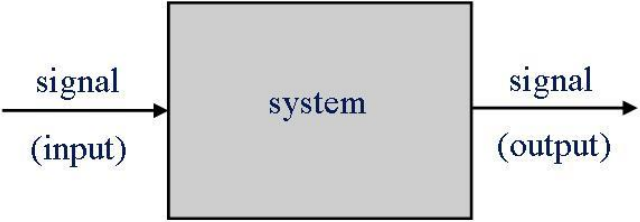
Chapter 1:
Fundamental Concepts of Signals and Systems

In this Chapter we introduce and motivate the mathematical study of signals and systems for electrical engineering. We discuss some their fundamantal properties, and we also meet some important examples signals and systems taken from real-world applications.

1. Signals contain information eg. sounds, images, voltage, motion, video, text.
2. Systems transform signals eg. digital encoder, audio equalizer, modems, control systems, digital circuits, analog circuits



Examples of signals:

- Position as trajectory in time of moving object
- Voltage signal
- Ultrasound
- Semaphore
- Scoreboard
- ECG signal

Examples of systems:

- Violin
- Mobile phone
- Robot
- Car
- Bionic ear
- Bridge
- Wine Glass
- Baby

We denote some important sets as follows:

Definition

- $\mathbf{Z}$ is the set of integers: $\{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$.
- $\mathbf{Z}_0+$ is the set of positive integers with zero: $\{0, 1, 2, 3, \ldots\}$.
- $\mathbf{R}$ is the set of real numbers.
- $\mathbf{C}$ is the set of complex numbers.
- $\mathbf{R}^n$ is the set of real vectors of length n, e.g. $(1, \sqrt{3}, e, -.5) \in \mathbf{R}^4$.

We distinguish between two types of signals:

Definition

- Functions with domain $\mathbf{R}$ are called **continuous-time** signals. We write $x : \mathbf{R} \mapsto \mathbf{R}$ and denote the output of the signal as $x(t)$.
- Functions with domain $\mathbf{Z}$ (or $\mathbf{Z}_0^+$) are called **discrete-time** signals. We write $x : \mathbf{Z} \mapsto \mathbf{R}$ and denote the output of the signal as $x[n]$.

Example (Speech waveforms)

- Figure 1.1 shows a continuous-time signal; it is a function that maps real numbers (time) to real numbers (air pressure).
- Figure 1.2 shows a discrete-time signal; it is a function that maps integers (time) to real numbers (air pressure).

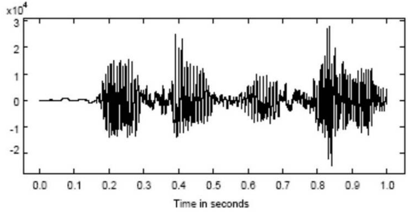


Figure 1.1: Waveform of a speech fragment.

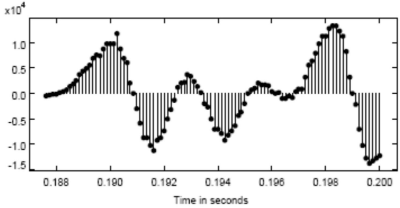


Figure 1.2: Discrete-time representation of a speech fragment.

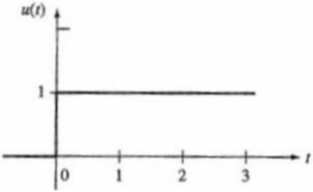
Definition (Continuous-time unit step and unit ramp signals)

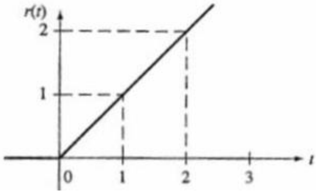
- The **unit step** function $u : \mathbf{R} \mapsto \mathbf{R}$ is defined as

$$u(t) = \begin{cases} 1 & \text{if } t \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

- The **unit ramp** function $r : \mathbf{R} \mapsto \mathbf{R}$ is defined as

$$r(t) = \begin{cases} t & \text{if } t \geq 0 \\ 0 & \text{otherwise} \end{cases}$$





Definition (Impulse signal)

- The **Dirac delta** function $\delta : \mathbf{R} \mapsto \mathbf{R} \cup \{\infty\}$ is defined as the function with the properties
  1. For all $t \in \mathbf{R}$, if $t \neq 0$, then $\delta(t) = 0$
  2. For all $\varepsilon > 0$

$$\int_{-\varepsilon}^{\varepsilon} \delta(\lambda)\, d\lambda = 1$$

The δ function is an example of a **generalized function** and is known as the **Impulse signal**. It is not a function in normal sense. It may be thought of as the limit of the sequence of step functions of unit area: For each $k \in \mathbf{Z}^{+}$, define the functions $f_k : \mathbf{R} \mapsto \mathbf{R}$ with

$$f_k(t) = \begin{cases} k & \text{if } \frac{-1}{2k} \leq t \leq \frac{1}{2k} \\ 0 & \text{otherwise} \end{cases}$$

Then

$$\delta = \lim_{k \to \infty} f_k$$

Definition (Impulse signal)

Figure 1.3 gives a sketch of the function sequence $f_k(t)$.
Figure 1.4 shows how a graphical representation of $K\delta$ for any constant $K > 0$.

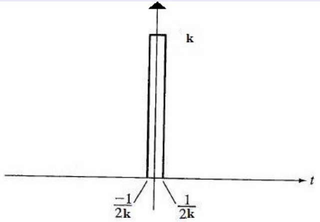


FIGURE 1.3

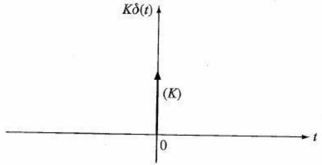


FIGURE 1.4

The unit step and impulse signals are closely related:

$$u(t) = \int_{-\infty}^{t} \delta(\lambda)\, d\lambda, \quad \text{for all } t \neq 0$$

Theorem (Sifting Property of the Impulse function)

Let $f : \mathbf{R} \mapsto \mathbf{R}$ be continuous at any point $t_0 \in \mathbf{R}$. Then

$$\int_{-\infty}^{t} f(\lambda)\delta(\lambda - t_0)\, d\lambda = f(t_0)u(t - t_0)$$

and

$$\int_{-\infty}^{\infty} f(\lambda)\delta(\lambda - t_0)\, d\lambda = f(t_0)$$

The Sifting Theorem is actually an alternative way of defining the impulse function. Together with convolution operation, it will allow us to express the outputs of any system in terms of its inputs and the impulse function.

Definition (Time shift signals)

Figure 1.6(a) gives a sketch of $u(t-2)$, the unit step function with a right shift of two units.
Figure 1.6(b) gives a sketch of $u(t+2)$, the unit step function with a left shift of two units.

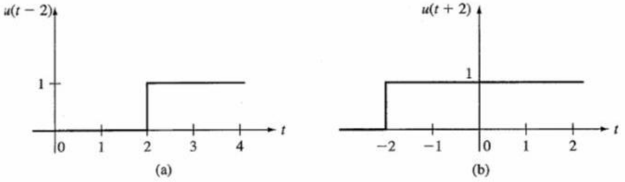


FIGURE 1.6
Two-second shifts of $u(t)$: (a) right shift; (b) left shift.

In general, for any function $f(t)$ and for any constant $a > 0$,

1. $f(t-a)$ is right shifted, compared to $f(t)$.
2. $f(t+a)$ is left shifted, compared to $f(t)$.

Definition (Finite duration signals)

A signal $x : \mathbf{R} \mapsto \mathbf{R}$ is of **finite duration** if there exist constants a and b such that $x(t) = 0$ for all $t < a$ and all $t > b$. If a and b are respectively the largest and smallest such numbers, then we say that $[a, b]$ is the **support** of the function x, and its **duration** d is given by $d = b - a$.

Definition (Rectangular pulse signal)

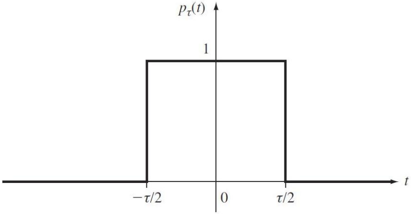
For any $\tau > 0$ we define the **rectangular pulse of width** τ as

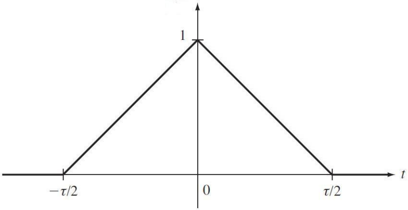
$$p_\tau(t) = \begin{cases} 1, & \text{if } \frac{-\tau}{2} \leq t < \frac{\tau}{2} \\ 0, & \text{otherwise} \end{cases}$$

Definition (Triangular pulse signal)

For any $\tau > 0$ we define the **triangular pulse of width** τ as

$$\Lambda_\tau(t) = \begin{cases} 1 - \frac{2|t|}{\tau}, & \text{if } \frac{-\tau}{2} \leq t < \frac{\tau}{2} \\ 0, & \text{otherwise} \end{cases}$$





- The rectangular pulse signal can be expressed in terms of step functions:

$$p_\tau(t) = u\left(t + \frac{\tau}{2}\right) - u\left(t - \frac{\tau}{2}\right)$$

- The triangular and rectangular pulse signals are related by

$$\Lambda_\tau(t) = \left(1 - \frac{2|t|}{\tau}\right) p_\tau(t)$$

Definition (Periodic signals)

A signal $x : \mathbf{R} \mapsto \mathbf{R}$ is **periodic** if there exists a constant $T > 0$ such that for all $t \in \mathbf{R}$,

$$x(t) = x(t + T)$$

If T is the smallest number with this property, then T is the **fundamental period** of x.

Theorem

Let $x_1 : \mathbf{R} \mapsto \mathbf{R}$ and $x_2 : \mathbf{R} \mapsto \mathbf{R}$ be periodic signals with fundamental periods T_1 and T_2 respectively. Then the sum of the signals x_1 and x_2 is a periodic function if and only if T_1/T_2 is a rational number, i.e. if their exist positive coprime integers m and n such that

$$\frac{T_1}{T_2} = \frac{n}{m}$$

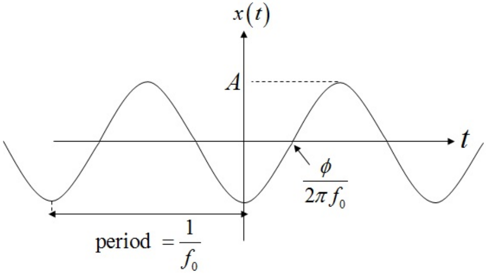
If m and n exist, then $x_1 + x_2$ has period $T_0 = mT_1 = nT_2$.

Example (sinusoidal signals)

For the sinusoidal signal $x : \mathbf{R} \mapsto \mathbf{R}$

$$x(t) = A\sin(\omega t + \phi)$$

- A is the **amplitude.**
- ω is **angular frequency** in rad/s, and ϕ is the **phase** in radians.
- $f_0 = \omega/2\pi$ and $T = 1/f_0$ are the **fundamental frequency** and **fundamental period**, respectively.



Definition (Obtaining Periodic signals from finite duration signals)

Let $x : \mathbf{R} \mapsto \mathbf{R}$ be a finite-duration signal with support $[a, b]$. Let $d \geq b - a$ and define the signal $y : \mathbf{R} \mapsto \mathbf{R}$ with

$$y(t) = \sum_{k=-\infty}^{\infty} x(t - kd) \quad \text{for all } t \in \mathbf{R}$$

Then y is called the **shift-and-add summation** signal of x.

Theorem

The signal y defined above is periodic with fundamental period $T_0 = d$.

Example

Consider the finite duration signal x with

$$x(t) = \begin{cases} 1, & \text{if } 0 \leq t < 1 \\ -1, & \text{if } -1 \leq t < 0 \\ 0, & \text{otherwise} \end{cases}$$

Example

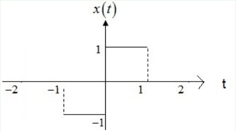
x has support $[-1,1]$. Shift x to the left and right by $d=2$ time units:

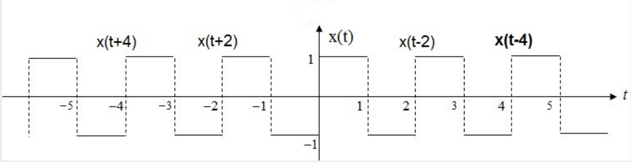
$$x(t-2)=\begin{cases} 1, & \text{if } 2\le t<3 \\ -1, & \text{if } 1\le t<2 \\ 0, & \text{otherwise} \end{cases} \quad \text{and} \quad x(t+2)=\begin{cases} 1, & \text{if } -2\le t<-1 \\ -1, & \text{if } -3\le t<-2 \\ 0, & \text{otherwise} \end{cases}$$

If we add up all possible $2k$ time unit shifts of x, we obtain the signal

$$y(t)=\sum_{k=-\infty}^{\infty} x(t-2k)$$

with period $T=2$.





Definition (Constructing finite duration signals from periodic signals)

Let $x : \mathbf{R} \mapsto \mathbf{R}$ be a periodic signal with period $T > 0$. Define the signal $y : \mathbf{R} \mapsto \mathbf{R}$ with

$$y(t) = x(t)p_T(t)$$

where $p_T(t)$ is the rectangular pulse function of width T. Then

- y has finite duration with support $[\frac{-T}{2}, \frac{T}{2}]$.
- It is called a **rectangular window** signal of x.

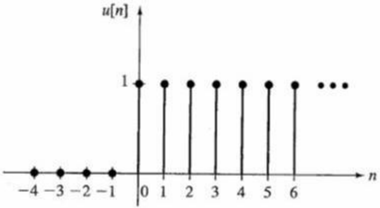
Definition (Discrete-time unit step and unit ramp signals)

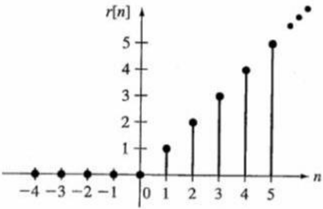
- The **unit step** function $u : \mathbf{Z} \mapsto \mathbf{R}$ is defined as

$$u[n] = \begin{cases} 1 & \text{if } n \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

- The **unit ramp** function $r : \mathbf{Z} \mapsto \mathbf{R}$ is defined as

$$r[n] = \begin{cases} n & \text{if } n \geq 0 \\ 0 & \text{otherwise} \end{cases}$$





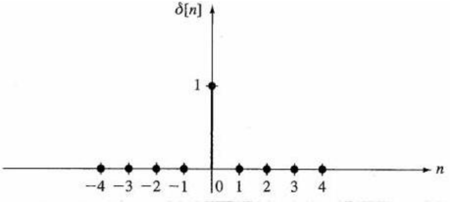
Definition (Discrete-time unit pulse and rectangular pulse)

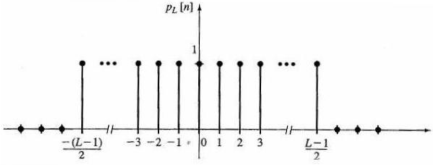
The **unit pulse function** function $\delta : \mathbf{Z} \mapsto \mathbf{R}$ is defined as

$$\delta[n] = \begin{cases} 1, & \text{if } n = 0 \\ 0, & \text{otherwise} \end{cases}$$

It is also known as the **Kronecker** delta function. For any positive odd integer $L > 0$, the discrete-time **rectangular pulse of width** L is

$$p_L[n] = \begin{cases} 1, & \text{if } \frac{-(L-1)}{2} \leq n \leq \frac{L-1}{2} \\ 0, & \text{otherwise} \end{cases}$$





Theorem (Sifting Property of the Pulse function)

Let $x : \mathbf{Z} \mapsto \mathbf{R}$*. Then for integer* n*,*

$$\sum_{i=-\infty}^{\infty} x[i]\delta[n-i] = x[n]$$

The Sifting Theorem for discrete-time systems, together with convolution operation, will allow us to express the outputs of any discrete-time system in terms of its inputs and the pulse function.

Definition (Periodic signals)

A signal $x : \mathbf{Z} \mapsto \mathbf{R}$ is **periodic** if there exists a positive integer $L > 0$ such that for all $n \in \mathbf{Z}$,

$$x[n] = x[n+L]$$

If L is the least integer number with this property, then L is the **fundamental period** of x.

Theorem

Let $x_1 : \mathbf{Z} \mapsto \mathbf{R}$ and $x_2 : \mathbf{Z} \mapsto \mathbf{R}$ be periodic signals with fundamental periods L_1 and L_2 respectively. Then the sum of the signals x_1 and x_2 is a periodic function with period

$$L_0 = lcm(L_1, L_2)$$

where lcm denotes the least common multiple of L_1 and L_2.

Theorem (Discrete-time sinusoidal signals)

The signal $x : \mathbf{Z} \mapsto \mathbf{R}$ *with*

$$x[n] = A\cos[\Omega n + \phi]$$

is periodic if and only if there exist integers q *and* r *such that*

$$\frac{\Omega}{2\pi} = \frac{q}{r}$$

If q *and* r *are coprime, then the fundamental period of* x *is*

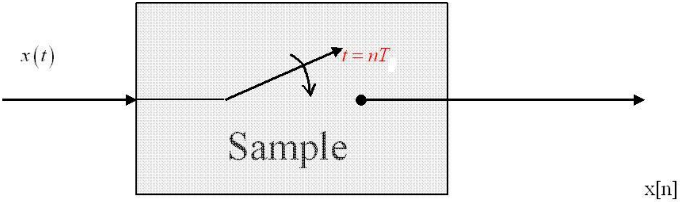
$$r = \frac{2\pi q}{\Omega}$$

Many discrete-time signals are generated by sampling a continuous-time signals. The sampling may be done by closing a switch for a very brief time interval.

Definition (Uniform sampling signal)

Given a continuous-time signal x and a **sampling interval** $T > 0$, we define the **sampled signal** by

$$x[n] = x(nT)$$



Example (Sampling sinusoids)

Let $x : \mathbf{R} \mapsto \mathbf{R}$ be the sinusoidal signal

$$x(t) = \sin(2\pi f_0 t)$$

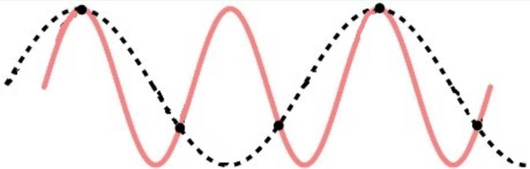
with fundamental frequency f_0. Suppose we sample x with period T_s, so the sampling frequency is $f_s = \frac{1}{T_s}$. Then the sampled signal is

$$x[n] = x(nT_s) = \sin[2\pi f n], \quad \text{where } f = \frac{f_0}{f_s}$$

Then $x[n]$ is periodic (as a discrete-time signal) only if f is a rational number.

Example (Sampling sinusoids: Aliasing)

Sampling two different continuous-time sinusoids can give the same discrete-time signal, if the sampling frequency is small enough:



Here the black and pink sinusoidal curves yield the same discrete signal (black dots). This is called **aliasing**.

Theorem (Shannon-Nyquist Sampling Theorem)

To uniquely sample a continuous-time sinusoidal signal (and hence avoid aliasing), the sampling frequency f_s must be at least twice f_0, the fundamental frequency of the sinusoid.

Definition (Systems)

Systems convert input signals into output signals:

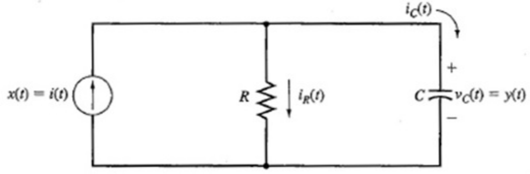
- The **system behaviour** consists of all input-output pairs (x, y), called **trajectories**, that satisfy the laws of the system.
- A **continuous-time system** operates on continuous-time signals.
- A **discrete-time system** operates on discrete-time signals.
- The system is usually described by a mathematical model that determines how the outputs are obtained from the inputs, e.g.
  - difference equations or differential equations;
  - convolution formula;
  - La place or Fourier Transfer function.

Definition (Systems)

- If the input $x = 0$, the output y is called the **zero-input response**. It is also known as the **natural response** or the **unforced response**. It depends only upon the initial condition of the system.
- If the input $x = u$, the step signal, and the initial condition is zero, then y is called the **step response**.
- If the input $x = \delta$, the impulse (or pulse) signal, and the initial condition is zero, then y is called the **impulse (or pulse) response**.
- The input is sometimes called the **forcing term** or **driving function**, since it is usually the input that causes the system to produce an output.

Example (RC circuit)

The RC (resistive-capacitive) electric circuit is an example of a continuous-time system



- The input $x(t)$ is the current source $i(t)$, and the output $y(t)$ is the capacitor voltage $v_C(t)$.
- Using Kirchoff's Laws and Ohm's Law, we can derive the differential equation for the system:

$$C\frac{dy}{dt} + \frac{1}{R}y(t) = x(t)$$

- To solve for $y(t)$, we need to know $x(t)$ and $y(0)$.

Example (RC circuit)

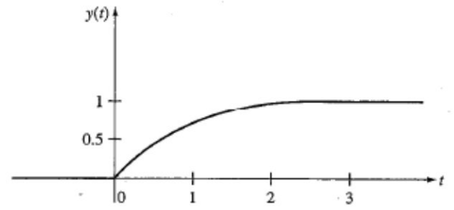
To find the unit step response we must solve

$$C\frac{dy}{dt} + \frac{1}{R}y(t) = u(t)$$

with $y(0) = 0$. The solution is found to be

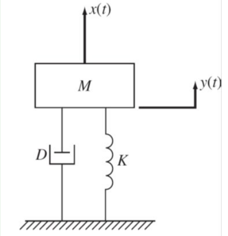
$$y(t) = R[1 - e^{-t/RC}], \quad \text{for } t \geq 0$$

The step response is the voltage on the capacitor as it charges up under a dc voltage source. Suppose $R = 1\ \Omega$ and $C = 1\ F$:



Example (Mass-spring damper)

The mass-spring damper is an example of a vibratory system:



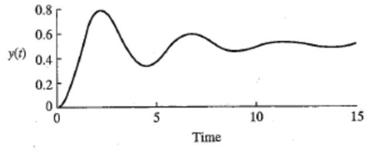
- The input $x(t)$ is the force applied to the mass, and the output $y(t)$ is the vertical displacement of the mass. The forces provided by the spring and damper are proportional to y and $\dot{y}$, respectively.
- Newton's Law leads to the differential equation:

$$M\frac{d^2y}{dt^2} + D\frac{dy}{dt} + Ky(t) = x(t)$$

- The system is comparable to an RLC electric circuit.

Example (Mass-spring damper)

If we assume $M = 1\ kg$, $D = 1 Ns/m$, and $K = N/m$, then the unit step response is

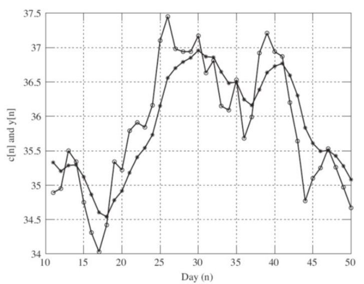


- The oscillations occur due to transfers between the kinetic energy of the mass and the potential energy of the spring and damper.
- The oscillations eventually decay as the applied force is balanced by the force of the spring.
- In very special cases **resonance** occurs, in which the oscillations grow with time, and can damage or destroy the system.

Example (Moving-average Filter)

For any positive integer N, the N**-point moving average filter** is a discrete-time system governed by the difference equation

$$y[n] = \frac{1}{N}\left[x[n] + x[n-1] + x[n-2] + \cdots + x[n-N+1]\right]$$



- The filter gives the average value of the N input values $x[n]$, $x[n-1]$, $x[n-2]$, $\ldots$, $x[n-N+1]$.
- The filter can be used to 'smooth' data to remove noise (random variations) and reveal underlying trends.

Definition (Linear Systems)

- A system is **additive** if for any two trajectories (x_1, y_1) and (x_2, y_2), the input-output pair $(x_1 + x_2, y_1 + y_2)$ is also a trajectory of the system.
- A system is **homogenous** if for any trajectory (x, y) and any real number a, the input-output pair (ax, ay) is also a trajectory of the system.
- A system is **linear** if it is both additive and homogenous.

Theorem (Linearity theorem)

Let (x_1, y_1) and (x_2, y_2) be trajectories of a system, and let a and b be arbitrary real numbers. Then the system is linear if and only if the input-output pair $(ax_1 + bx_2, ay_1 + by_2)$ is also a trajectory of the system.

Example (Linear System)

The mass-spring damper is an example of a linear system:

$$M\frac{d^2y}{dt^2} + D\frac{dy}{dt} + Ky(t) = x(t)$$

To see this we let (x_1, y_1) and (x_2, y_2) be trajectories of the system. Then

$$\begin{aligned} M\frac{d^2y_1}{dt^2} + D\frac{dy_1}{dt} + Ky_1(t) &= x_1(t) \\ M\frac{d^2y_2}{dt^2} + D\frac{dy_2}{dt} + Ky_2(t) &= x_2(t) \end{aligned}$$

Next we let a and b be arbitrary real numbers, and consider the signals

$$x_3(t) = ax_1 + bx_2, \quad y_3 = ay_1 + by_2$$

Example

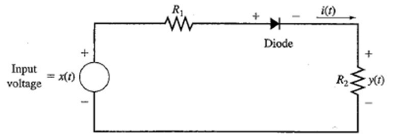
Then

$$
\begin{aligned}
& M\frac{d^2y_3}{dt^2} + D\frac{dy_3}{dt} + Ky_3(t) \\
= \; & M\frac{d^2(ay_1+by_2)}{dt^2} + D\frac{d(ay_1+by_2)}{dt} + K(ay_1(t)+by_2(t)) \\
= \; & aM\frac{d^2y_1}{dt^2} + aD\frac{dy_1}{dt} + aKy_1(t) + bM\frac{d^2y_2}{dt^2} + bD\frac{dy_2}{dt} + bKy_2(t) \\
= \; & ax_1(t) + bx_2(t) \\
= \; & x_3(t)
\end{aligned}
$$

Hence (x_3, y_3) is also a trajectory of the system, and we conclude by the Linearity theorem that the mass-spring damper is a linear system.

Example (Nonlinear System)

Any electric circuit containing a diode is a nonlinear system.



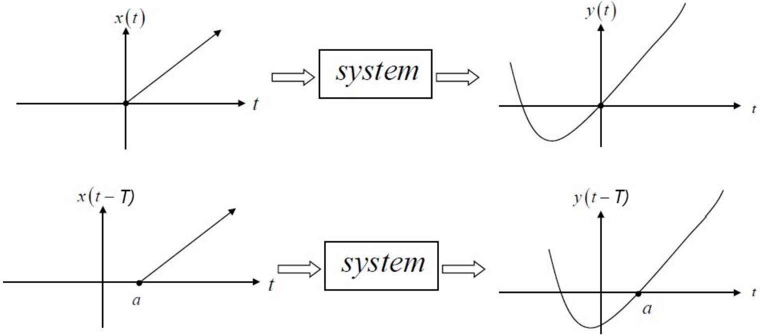
- The input-output relationship can be shown to be

$$y(t) = \begin{cases} \frac{R_2 x(t)}{R_1 + R_2} & \text{if } x(t) \geq 0 \\ 0 & \text{if } x(t) < 0 \end{cases}$$

- Suppose $x(t) = 1$ for all t; then $x(t) \geq 0$ and the output is $y(t) = \frac{R_2}{R_1+R_2}$. So $(1, \frac{R_2}{R_1+R_2})$ is a trajectory of the system. Now consider the scalar $a = -1$ and the input $ax(t) = -1 < 0$. Hence the output is 0, and $(-1, 0)$ is a trajectory of the system. This contradicts the homogeneity requirement that $(-1, \frac{-R_2}{R_1+R_2})$ be a trajectory of the system. So the system is nonlinear.

Definition (Time-invariant Systems)

A system is **time-invariant** if, for any trajectory $(x(t), y(t))$ and any constant $T \in \mathbf{R}$, the input-output pair $(x(t-T), y(t-T))$ is also a trajectory of the system.



Example (Time-invariant Systems)

The Mass-spring damper is a time-invariant system:

$$\frac{d^2y}{dt^2} + \frac{dy}{dt} + 2y(t) = x(t)$$

To show this we let (x, y) be a trajectory of the system, and introduce the signals

$$\begin{aligned} x_1(t) &= x(t - T) \\ y_1(t) &= y(t - T) \end{aligned}$$

and show that (x_1, y_1) is also a trajectory of the system.

Example

$$
\begin{aligned}
\frac{d^2 y_1(t)}{dt^2} + \frac{dy_1(t)}{dt} + y_1(t) &= \frac{d^2 y(t-T)}{dt^2} + \frac{dy(t-T)}{dt} + 2y(t-T) \\
&= \frac{d^2 y(s)}{ds^2}\left(\frac{ds}{dt}\right)^2 + \frac{dy(s)}{ds}\left(\frac{ds}{dt}\right) + 2y(s) \\
&\quad \text{using the change of variables} \\
&\qquad s = t - T; \text{ then } ds = dt \\
&= \frac{d^2 y(s)}{ds^2} + 2\frac{dy(s)}{ds} + 2y(s) \\
&= x(s), \quad \text{as } (x, y) \text{ is a trajectory} \\
&= x(t-T) \\
&= x_1(t)
\end{aligned}
$$

Thus (x_1, y_1) is also a trajectory of the system, and the system is time-invariant.

Example (Time-varying Systems)

More examples of time-invariant systems

- $y[n] = \frac{1}{N}\left[x[n] + x[n-1] + \cdots + x[n-N+1]\right]$ (Moving average filter)
- $y(t) = \frac{x^2(t) - x(t-3) + x^3(t+1)}{1 + x^4(t)}$

Some examples of time-varying systems

- $C\frac{dy}{dt} + \frac{1}{R(t)}y(t) = u(t)$, where $R(t) = 2 + \sin(t)$ (RC circuit with variable resistor).
- $y(t) = tx(t)$ (Amplifier with a time-varying gain)

Definition (Causal Systems)

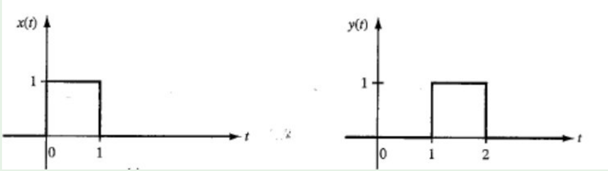
A system is **causal** if, for any time t_1, the output response $y(t_1)$ resulting from the input $x(t)$ does not depends on values of the input $x(t)$ for $t > t_1$.

Example (Ideal time delay system)

The ideal time delay system

$$y(t) = x(t-1)$$

is causal because the output at time t depends upon the input at time $t-1$.

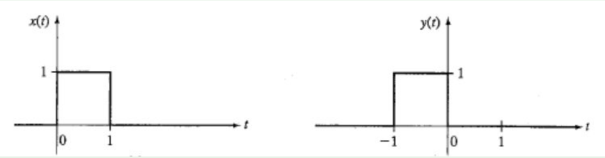


Example (Ideal predictor system)

The ideal predictor system

$$y(t) = x(t+1)$$

is non-causal because the output at time t depends upon the input at time $t+1$.



- Non-causal systems (with time as the independent variable) are impossible to build.
- Non-causal systems arise in image processing, where the independent variable(s) represent spatial coordinates rather than time.

Definition (Memoryless Systems)

A system is **memoryless** if, for any time t_1, the output response at time t_1 depends only upon the input at time t_1.

Definition (Systems with Memory)

A causal system has **memory** if, for some time t_1, the output response at time t_1 depends upon the input at time $t < t_1$.

Example

- The (output) voltage $v_R(t)$ across a resistor with (input) current $i(t)$ is a memoryless system.

$$v_R(t) = Ri(t) \quad \text{(Ohm's Law)}$$

- The capacitor voltage $v_C(t)$ from a current $i(t)$ has memory:

$$v_C(t) = \frac{1}{C}\int_{t_0}^{t} i(s)ds + v_C(t_0), \quad \text{for } t \geq t_0$$