Chapter 5:
Fourier Transform for Discrete–time Signals

In this chapter we develop the discrete-time counterpart to the study of continuous time signals given in the previous chapter. We introduce the discrete-time Fourier transform (DTFT). The frequency spectrum of a non-periodic signal is defined for all real values of the frequency variable. We also consider the discrete Fourier transform (DFT) which involves representing a signal in terms of finitely many frequencies.

Definition (Discrete-time Fourier Transform)

For a discrete-time signal x, we define its **discrete-time Fourier Transform (DTFT)** to be

$$X(\Omega) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\Omega n}$$

- The variable $\Omega \in \mathbf{R}$ is called the **frequency variable**. We use lower case letters like x and f to denote discrete-time signals, and capital letters X and F to denote their Fourier transform.
- We may compare it with the continuous-time Fourier transform (CTFT)

$$X(\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}\, dt$$

  Since n is an integer, we naturally replace the integral (dt) in the CTFT with a sum in the DTFT. We also use Ω and ω to distinguish the discrete-time and continuous-time frequency variables.

Definition (Inverse DTFT)

For a discrete-time signal x with DTFT $X(\Omega)$, the **Inverse DTFT** of $X(\Omega)$ is given by

$$x[n] = \frac{1}{2\pi}\int_0^{2\pi} X(\Omega)e^{j\Omega n}\, d\Omega$$

We say that x and X are DTFT pairs, and write

$$x[n] \longleftrightarrow X(\Omega)$$

Theorem

For any discrete-time signal x, its DTFT X is a periodic function with period 2π:

$$X(\Omega + 2\pi) = X(\Omega), \quad \text{for all } \Omega \in \mathbf{R}$$

Since $X(\Omega)$ and $e^{j\Omega n}$ are both periodic functions of Ω with period 2π, we can integrate over any interval of length 2π:

$$x[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(\Omega) e^{j\Omega n} \, d\Omega$$

Theorem (Existence of the DTFT)

A discrete-time signal x has a DTFT if it is **absolutely summable** *over* **Z***, i.e.*

$$\sum_{n=-\infty}^{\infty} |x[n]| < \infty$$

- Functions that satisfy this condition are said to be have a DTFT **in the ordinary sense.**
- Any time-limited signal (a signal that has finite support) will have a DTFT in the ordinary sense.

Example

Consider the discrete-time signal x with

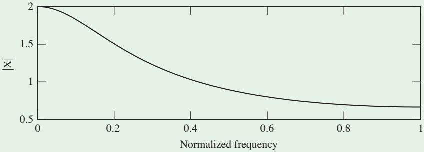
$$x[n] = a^n u[n]$$

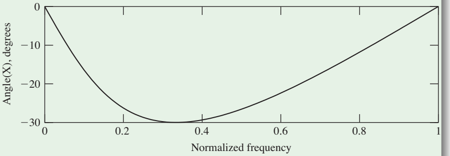
where $|a| < 1$, with $a \neq 0$. It can be shown that

$$X(\Omega) = \frac{1}{1 - ae^{-j\Omega}}$$

so the amplitude and phase spectra are

$$|X(\Omega)| = \frac{1}{\sqrt{1 - 2a\cos(\Omega) + a^2}}, \quad \angle X(\Omega) = -\underline{/1 - ae^{-j\Omega}}$$





Example

$$\delta[n] \longleftrightarrow 1$$

because, by the Sifting Theorem,

$$\sum_{n=-\infty}^{\infty} \delta[n] e^{-j\Omega n} = e^0 = 1$$

Definition (Rectangular form of the Fourier Transform)

$$\begin{aligned} \text{Let} \quad R(\Omega) &= \sum_{n=-\infty}^{\infty} x[n] \cos(\Omega n) \\ I(\Omega) &= -\sum_{n=-\infty}^{\infty} x[n] \sin(\Omega n) \\ \text{then} \quad X(\Omega) &= R(\Omega) + jI(\Omega) \end{aligned}$$

This the **Rectangular form** of the Fourier transform $X(\Omega)$.

Definition (Polar form of the Fourier Transform)

The **Polar form** of the Fourier transform $X(\Omega)$ is

$$X(\Omega) = |X(\Omega)| \exp(j\angle X(\Omega))$$

where

$$\begin{aligned} |X(\Omega)| &= \sqrt{R^2(\Omega) + I^2(\Omega)} \\ \angle X(\Omega) &= \begin{cases} \tan^{-1}\left(\frac{I(\Omega)}{R(\Omega)}\right), & R(\Omega) \geq 0 \\ \pi + \tan^{-1}\left(\frac{I(\Omega)}{R(\Omega)}\right), & R(\Omega) < 0 \end{cases} \end{aligned}$$

Theorem

For any discrete-time signal x with Fourier Transform X,

1. $X(-\Omega) = \overline{X(\Omega)}$,
2. $|X(-\Omega)| = |X(\Omega)|$, *so* $|X(\Omega)|$ *is an even function.*
3. $\angle X(-\Omega) = -\angle X(\Omega)$, *so* $\angle X(\Omega)$ *is an odd function.*

Theorem (Even and Odd signals)

1. *For any discrete-time even signal x with DTFT X,*

$$\begin{aligned} R(\Omega) &= x[0] + 2\sum_{n=1}^{\infty} x[n]\cos(\Omega n) \\ I(\Omega) &= 0 \\ \textit{and so} \quad X(\Omega) &= x[0] + 2\sum_{n=1}^{\infty} x[n]\cos(\Omega n) \end{aligned}$$

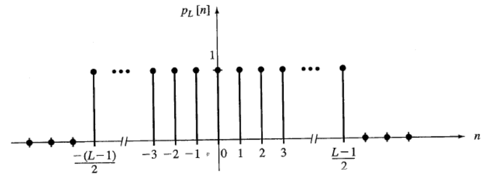
2. *For any discrete-time odd signal x with DTFT X,*

$$\begin{aligned} R(\Omega) &= x[0] \\ I(\Omega) &= -2\sum_{n=1}^{\infty} x[n]\sin(\Omega n) \\ \textit{and so} \quad X(\Omega) &= x[0] - j2\sum_{n=1}^{\infty} x[n]\sin(\Omega n) \end{aligned}$$

Example (Rectangular Pulse)

The discrete-time rectangular pulse function of width L, for any odd integer $L > 0$, is defined as

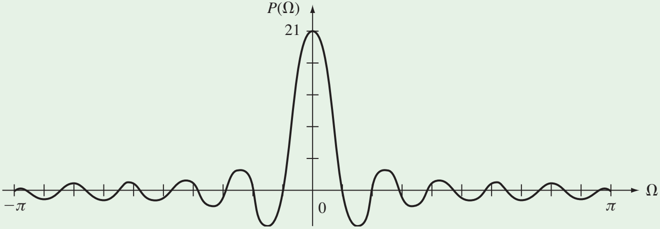
$$p_L[n] = \begin{cases} 1, & -(L-1)/2 \le n \le (L-1)/2 \\ 0, & \text{otherwise} \end{cases}$$



If we introduce $q = (L-1)/2$, then it can be shown that

$$P_L(\Omega) = \frac{\sin((q+1/2)\Omega)}{\sin(\Omega/2)}$$

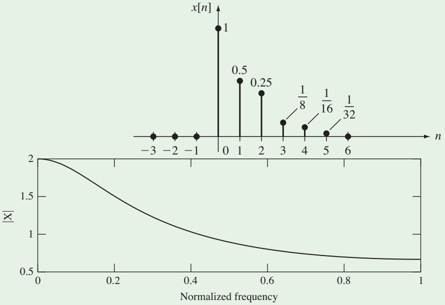
Example (Rectangular Pulse II)

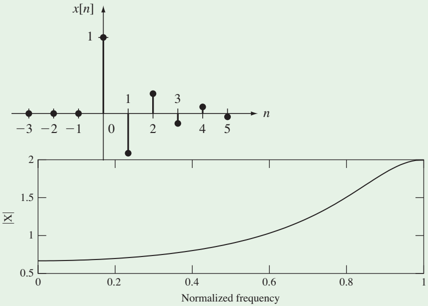


For large values of L, the spectrum of $P_L(\Omega)$ resembles a sinc function on the interval $-\pi \leq \Omega \leq \pi$.

Example (Comparison of frequency spectra)

We saw that $a^n u[n] \longleftrightarrow \frac{1}{1-ae^{-j\Omega}}$, for $|a| < 1$. We compare $a = 0.5$ with $a = -0.5$:





Since $a = -0.5$ has greater time variations, its DTFT has more high-frequency components.

The DTFT has very similar properties to the CTFT that can be used to compute transforms of signals without using the definition of the transform. Some of these are

Theorem (Linearity)

If $x_1[n] \longleftrightarrow X_1(\Omega)$ *and* $x_2[n] \longleftrightarrow X_2(\Omega)$*, and a and b are any two scalars, then*

$$ax_1[n] + bx_2[n] \longleftrightarrow aX_1(\Omega) + bX_2(\Omega)$$

Theorem (Time shifting)

If $x[n] \longleftrightarrow X(\Omega)$ *and* $q \in \mathbf{Z}$*, then*

$$x[n-q] \longleftrightarrow X(\Omega)e^{-j\Omega q}$$

Theorem (Time reversal or Flipping)

If $x[n] \longleftrightarrow X(\Omega)$ *then*

$$x[-n] \longleftrightarrow X(-\Omega)$$

Theorem (Modulation)

If $x[n] \longleftrightarrow X(\Omega)$ *and* $\Omega_0 \in \mathbf{R}$, *then*

$$\begin{aligned} x[n]e^{j\Omega_0 n} &\longleftrightarrow X(\Omega - \Omega_0) \\ x[n]\cos(\Omega_0 n) &\longleftrightarrow \frac{1}{2}[X(\Omega + \Omega_0) + X(\Omega - \Omega_0)] \\ x[n]\sin(\Omega_0 n) &\longleftrightarrow \frac{j}{2}[X(\Omega + \Omega_0) - X(\Omega - \Omega_0)] \end{aligned}$$

Theorem (Convolution)

If $x[n] \longleftrightarrow X(\Omega)$ *and* $v[n] \longleftrightarrow V(\Omega)$, *then*

$$(x \star v)[n] \longleftrightarrow X(\Omega)V(\Omega)$$

Theorem

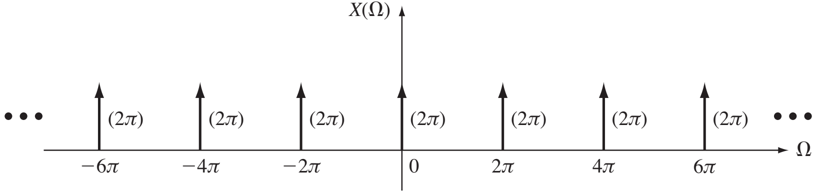
$$1 \longleftrightarrow 2\pi \sum_{k=-\infty}^{\infty} \delta(\Omega - 2\pi k)$$

Proof: Let $X(\Omega) = 2\pi \sum_{k=-\infty}^{\infty} \delta(\Omega - 2\pi k)$ and take its inverse DTFT:

$$\begin{aligned} x[n] &= \frac{1}{2\pi}\int_{-\pi}^{\pi} X(\Omega)e^{j\Omega n}\, d\Omega \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} 2\pi \sum_{k=-\infty}^{\infty} \delta(\Omega - 2\pi k)e^{j\Omega n}\, d\Omega \\ &= \int_{-\pi}^{\pi} \delta(\Omega)e^{j\Omega n}\, d\Omega \\ &= e^{0} \\ &= 1, \quad \text{for all } n \end{aligned}$$

by the Sifting theorem. So the constant function 1 and $X(\Omega)$ are a DTFT pair.

The impulse train $X(\Omega) = 2\pi \sum_{k=-\infty}^{\infty} \delta(\Omega - 2\pi k)$ may be plotted as



Note that $X(\Omega)$ is periodic with period 2π, as expected for the DTFT.

Remark

The discrete-time signals $\cos[\Omega n]$ *and* $\sin[\Omega n]$, *do not have DTFT transforms in the ordinary sense because they are not absolutely summable , i.e.*

$$\sum_{n=-\infty}^{\infty} |\sin(\Omega n)| = \infty$$

However we can obtain **generalized Fourier transforms** *for them using the Dirac function* $\delta(t)$.

To do this we use the modulation theorem

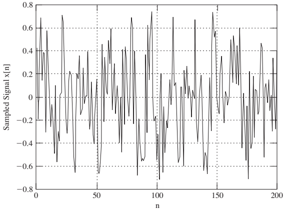
$$
\begin{aligned}
x[n]e^{j\Omega_0 n} &\longleftrightarrow X(\Omega-\Omega_0) \\
x[n]\cos(\Omega_0 n) &\longleftrightarrow \frac{1}{2}[X(\Omega+\Omega_0)+X(\Omega-\Omega_0)] \\
x[n]\sin(\Omega_0 n) &\longleftrightarrow \frac{j}{2}[X(\Omega+\Omega_0)-X(\Omega-\Omega_0)]
\end{aligned}
$$

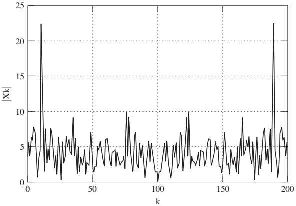
to obtain the generalized DTFT pairs

Theorem

$$
\begin{aligned}
e^{j\Omega_0 n} &\longleftrightarrow 2\pi\sum_{k=-\infty}^{\infty}\delta(\Omega-\Omega_0-2\pi k) \\
\cos(\Omega_0 n) &\longleftrightarrow \pi\sum_{k=-\infty}^{\infty}[\delta(\Omega+\Omega_0-2\pi k)+\delta(\Omega-\Omega_0-2\pi k)] \\
\sin(\Omega_0 n) &\longleftrightarrow j\pi\sum_{k=-\infty}^{\infty}[\delta(\Omega+\Omega_0-2\pi k)-\delta(\Omega-\Omega_0-2\pi k)]
\end{aligned}
$$

Up until now we have assumed that the signals we want analyse via Fourier methods were available in closed form, meaning that we have a mathematical formula for $x(t)$ or $x[n]$ that defines the signal. This is seldom the case with real-world data:





- On the left we see the plot of a time-domain signal obtained from 200 measurements, taken over a certain time-interval.
- The right graph shows its amplitude spectrum, revealing a dominant frequency.
- How do we obtain the frequency spectrum from the measurements?

Definition (Discrete Fourier Transform)

Let x be a finite duration discrete-time signal with support $[0, L-1]$, for some integer $L > 1$. (Recall this means that $x[n] = 0$ for $n < 0$, and $n \geq L$.) The integer L is called the **record length.** For any integer $N \geq L$, we define the ***N*-point discrete Fourier transform (DFT)** X_k of x to be

$$X_k = \sum_{n=0}^{L-1} x[n] e^{-j2\pi kn/N}, \quad \text{where } k = 0, 1, \ldots, N-1$$

- Since x has finite duration, the DFT X_k always exists.
- X_k is a function of the discrete variable k, and takes N complex values $X_0, \ldots, X_{N-1}$.
- The DFT is a periodic discrete function with period N.

Example

Find the 4-point DFT of the signal $x[n]$ given by

$$x[0]=1,\ x[1]=2,\ x[2]=2,\ x[3]=1, \quad x[n]=0 \text{ otherwise}$$

Then x has support $[0,3]$ so $L=4$, and the 4-point DFT is

$$\begin{aligned} X_k &= \sum_{n=0}^{3} x[n]e^{-j\pi kn/2}, \\ &= x[0]+x[1]e^{-j\pi k/2}+x[2]e^{-j\pi k}+x[3]e^{-j\pi 3k/2} \end{aligned}$$

Hence

$$\begin{aligned} X_0 &= 1+2e^0+2e^0+e^0=6 \\ X_1 &= 1+2e^{-j\pi/2}+2e^{-j\pi}+e^{-j3\pi/2}=\sqrt{2}e^{j5\pi/4} \\ X_2 &= 1+2e^{-j\pi}+2e^{-j2\pi}+e^{-j3\pi}=0 \\ X_3 &= 1+2e^{-j3\pi/2}+2e^{-j3\pi}+e^{-j9\pi/2}=\sqrt{2}e^{j3\pi/4} \end{aligned}$$

Theorem (Symmetry)

If X_k is the N-point DFT of a signal x with record length $L \leq N$, then

$$X_{N-k} = \bar{X}_k \quad \textit{for all } k = 0, 1, \ldots, N-1$$

A finite duration signal x can be recovered from X_k via the inverse DFT:

Definition (Inverse Discrete Fourier Transform)

The **inverse discrete Fourier transform** of an N-point DFT X_k is defined to be

$$x[n] = \frac{1}{N} \sum_{k=0}^{N-1} X_k e^{j2\pi kn/N}, \quad \text{where } n = 0, 1, \ldots, L-1$$

where $L \leq N$ is the record length of x.

Recall that the DTFT of a discrete signal x of record length L is

$$X(\Omega) = \sum_{n=0}^{L-1} x[n]e^{-j\Omega n}$$

and the N-point DFT of x is

$$X_k = \sum_{n=0}^{L-1} x[n]e^{-j2\pi kn/N}, \quad \text{where } k = 0, 1, \ldots, N-1$$

Hence the connection between the DTFT and DFT is given by

Theorem

Let x be a finite duration discrete-time signal with support $[0, L-1]$, let $X(\Omega)$ be the DTFT of x, and let X_k be its N-point DFT, where $N \geq L$. Then

$$X_k = X\left(\frac{2\pi k}{N}\right) \quad \textit{for all } k = 0, 1, \ldots, N-1$$

Thus the DFT is equivalent to sampling the DTFT at the N frequency values $\Omega = 0, \frac{2\pi}{N}, \frac{4\pi}{N}, \ldots, \frac{2(N-1)\pi}{N}$.

Example

Let $x[n] = p_{11}[n-5]$, where p_{11} is the rectangular discrete pulse of width $L = 11$. Then

$$x[n] = \begin{cases} 1, & \text{if } 0 \leq n \leq 10 \\ 0, & \text{otherwise} \end{cases}$$

so x has support $[0, 10]$. We saw that before that the DTFT of the rectangular pulse p_{11} is

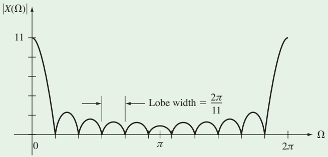
$$P_{11}(\Omega) = \frac{\sin(5.5\Omega)}{\sin(\Omega/2)}$$

Since $x[n] = p_{11}[n-5]$, by the time-shifting theorem, the DTFT of x is

$$X(\Omega) = P_{11}(\Omega)e^{-j5\Omega} = \frac{\sin(5.5\Omega)}{\sin(\Omega/2)}e^{-j5\Omega}$$

Example

Hence the amplitude spectrum of the DTFT of x is $|X(\Omega)| = \frac{|\sin(11\Omega/2)|}{|\sin(\Omega/2)|}$:



Choosing $N = L = 11$ and obtaining the 11-point DFT of x yields

$$
\begin{aligned}
|X_k| &= \left| X\left(\frac{2\pi k}{11}\right) \right| \\
&= \begin{cases} 11, & \text{if } k = 0 \\ 0, & \text{otherwise} \end{cases}
\end{aligned}
$$

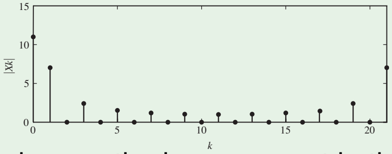
BUT this $|X_k|$ gives a poor approximation to $|X(\Omega)|$: agreement only at $k = 0, 1, 2, \ldots, 10$ which correspond to $\Omega = 0, \frac{2\pi}{11}, \ldots \frac{20\pi}{11}$.

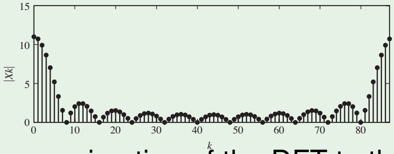
Example

To improve the approximation, we increase the number of samples N in the DFT so the sampling frequencies $\frac{2\pi k}{N}$ are closer together. If $N = 22$ then the DFT is given by

$$X_k = \sum_{n=0}^{10} x[n]e^{-j2\pi kn/22} = X\left(\frac{\pi k}{11}\right) \text{ for all } k = 0, 1, \ldots, 21$$

The plots show the DFT for $N = 22$ and $N = 88$:





We observe the improvement in the approximation of the DFT to the DTFT as the sample size N increases.