34 (b) Let

$$x(t) = \sin\left(2t - \frac{\pi}{4}\right) + 2\cos\left(2t - \frac{\pi}{3}\right)$$

$$= \sin(2t)\cdot\cos\left(\frac{\pi}{4}\right) - \cos(2t)\sin\left(\frac{\pi}{4}\right)$$

$$+ 2\cos(2t)\cos\left(\frac{\pi}{3}\right)$$

$$+ 2\sin(2t)\sin\left(\frac{\pi}{3}\right)$$

$$= \left(-\sin\left(\frac{\pi}{4}\right) + 2\cos\left(\frac{\pi}{3}\right)\right)\cos(2t)$$

$$+ \left(\cos\left(\frac{\pi}{4}\right) + 2\sin\left(\frac{\pi}{3}\right)\right)\sin(2t)$$

$$= 0.293\cos(2t) + 2.439\sin(2t)$$

35 (b) Express $x(t)$ in cosine-with-phase form.

We have $a_2 = 0.293$, $b_2 = 2.439$.

So $A_2 = \sqrt{a^2 + b^2}$

$$= 2.457$$

$$\theta_2 = \tan^{-1}\left(\frac{-b_2}{a_2}\right), \text{ as } a_2 > 0$$

$$= \tan^{-1}\left(\frac{-2.439}{0.293}\right)$$

$$= -1.451^c$$

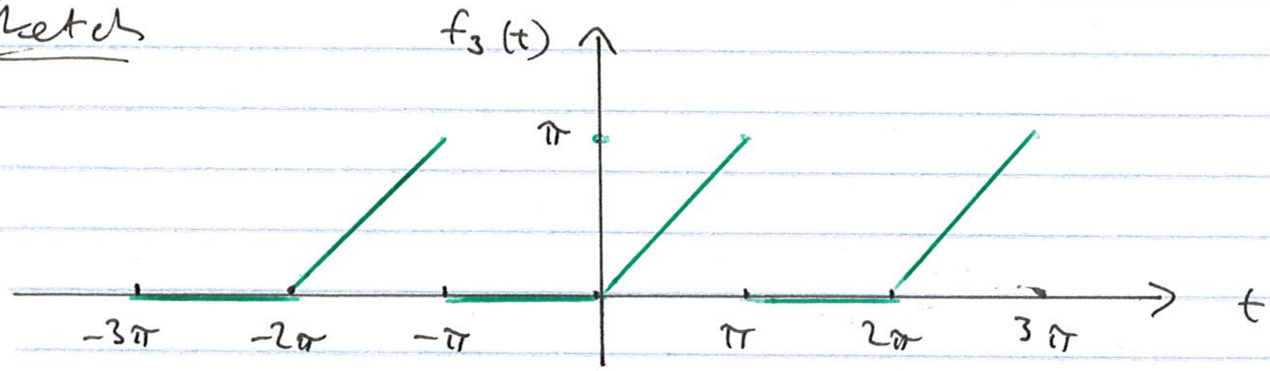
So $x(t) = 2.457\cos(2t - 1.451)$

38 $f_3(t) = \begin{cases} 0, & -\pi < t < 0 \\ t, & 0 < t < \pi \end{cases}$

and $f_3(t) = f_3(t + 2\pi)$.

(a) f_3 is periodic with period $T = 2\pi$.

Sketch



(b) Condition (1): On the interval $[0, 2\pi]$ $|f_3(t)| \leq \pi$, hence f_3 is bounded which means it is absolutely integrable.

Condition (2): On the interval $[0, 2\pi]$ f_3 reaches a maximum at $t = \pi$ and minimum at $t = 0$. Hence it has only two maxima / minima.

Condition (3): f_3 is discontinuous at the points $t = \pm\pi$. Hence it has only ~~one~~ two points of discontinuity on the interval $[0, 2\pi]$.

So f_3 meets all three Dirichlet conditions, and hence it can be represented by its Fourier series.

$\boxed{38}$ (c) Since $T = 2\pi$, $\omega_0 = 1$

Then $a_0 = \frac{1}{T}\int_0^T f_3(t)\,dt$

$= \frac{1}{2\pi}\int_0^\pi t\,dt + \frac{1}{2\pi}\int_\pi^{2\pi} 0\,dt.$

$= \frac{\pi}{4}$

$a_k = \frac{2}{T}\int_0^T f_3(t)\cos(k\omega_0 t)\,dt$

$= \frac{1}{\pi}\int_0^\pi t\cos(kt)\,dt$

$= \frac{1}{\pi}\left[\frac{t\sin(kt)}{k} + \frac{\cos(kt)}{k^2}\right]_0^\pi$

using Integration by Parts

$= \frac{(-1)^k - 1}{k^2\pi}$

$b_k = \frac{2}{T}\int_0^{2\pi} f_3(t)\,\sin(k\omega_0 t)\,dt$

$= \frac{1}{\pi}\int_0^\pi t\sin(kt)\,dt$

$= \frac{1}{\pi}\left[-\frac{t\cos(kt)}{k} + \frac{\sin(kt)}{k^2}\right]_0^\pi$

$= \frac{(-1)^{k+1}}{k}$, (By Parts)

Evaluating the first few coefficients gives

$a_0 = \frac{\pi}{4}$, $a_1 = \frac{-2}{\pi}$, $a_2 = 0$, $a_3 = \frac{-2}{9\pi}$

$b_1 = 1$, $b_2 = \frac{-1}{2}$, $b_3 = \frac{1}{3}$

$\boxed{40}$ (b) The product of two odd functions is an odd function.

This statement says that if $f(t)$ and $g(t)$ are odd functions, then $f(t)\,g(t)$ is an odd function, i.e.

if $f(-t) = -f(t)$

$g(-t) = -g(t)$.

Then let $h(t) = g(t)\,f(t)$

we have $h(-t) = -h(t)$.

The statement is <u>FALSE</u>. So we find a counterexample.

Consider $f(t) = t$

$g(t) = t^3$.

Then $f(-t) = -t = -f(t)$

$g(-t) = (-t)^3$

$= -t^3 = -g(t)$.

So $f(t)$ and $g(t)$ are both odd functions. Let

$$h(t) = f(t)\,g(t) = t\,t^3 = t^4.$$

Then $h(-t) = (-t)^4$

$= t^4$

$= h(t)$.

So h is an <u>even</u> function, not an odd function.

4 I) Let $a > 0$ and let x and v be even functions. Show that

(a) $$\int_{-a}^{a} x(t)\,v(t)\,dt = 2\int_0^a x(t)\,v(t)\,dt$$

Proof: $$\int_{-a}^{a} x(t)\,v(t)\,dt =$$
$$= \int_{-a}^{0} x(t)\,v(t)\,dt + \int_0^a x(t)\,v(t)\,dt.$$

Introduce the change of variables $s = -t$. Then $t = -s$, $dt = -ds$ and $s(0) = 0$, $s(-a) = a$.

Hence by the Substitution method of Integration

$$\int_{-a}^{0} x(t)\,v(t)\,dt = \int_{s(-a)}^{s(0)} -x(-s)\,v(-s)\,ds$$
$$= \int_{a}^{0} -x(s)\,v(s)\,ds$$

as x and v are even functions

$$= \int_0^a x(s)\,v(s)\,ds$$
$$= \int_0^a x(t)\,v(t)\,dt \qquad \boxed{s = t}$$

Hence

$$\int_{-a}^{a} x(t)\,v(t)\,dt = \int_0^a x(s)\,v(s)\,ds + \int_0^a x(t)\,v(t)\,dt$$
$$= 2\int_0^a x(t)\,v(t)\,dt$$

$\boxed{44}$

Given the saw tooth wave

$$f(t) = \begin{cases} 0, & -\pi < t < 0 \\ t, & 0 < t < \pi \end{cases}$$

and $f(t) = f(t + 2\pi)$.

Find the complex exponential form for the Fourier series of f. $T = 2\pi$

$\Rightarrow w_0 = 1$

Solution: We need to find

$$C_k = \frac{1}{T}\int_0^T f(t)\, e^{-jkt}\, dt.$$

Use Integration by Parts

$$= \frac{1}{2\pi}\int_0^\pi t e^{-jkt}\, dt$$

Then $u(t) = t$, $v'(t) = e^{-jkt}$

$u'(t) = 1$, $v(t) = \frac{j}{k} e^{-jkt}$

So

$k \neq 0$

$$C_k = \frac{1}{2\pi}\left[\frac{jt}{k} e^{-jkt}\right]_0^\pi - \int_0^\pi \frac{j}{k} e^{-jkt}\, dt$$

$$= \frac{1}{2\pi}\left[\frac{jt}{k} e^{-jkt} + \frac{1}{k^2} e^{-jkt}\right]_0^\pi$$

$$= \frac{1}{2k^2\pi}\left[jtk\, e^{-jkt} + e^{-jkt}\right]_0^\pi$$

$$= \frac{1}{2k^2\pi}\left[\left(jk\pi\, e^{-jk\pi} + e^{-jk\pi}\right) - (0 + 1)\right]$$

$$= \frac{1}{2k^2\pi}\left[e^{-jk\pi}(1 + jk\pi) - 1\right]$$

$$= \frac{1}{2k^2\pi}\left(e^{-jk\pi} - 1\right) + \frac{j e^{-jk\pi}}{2k}, \quad k \neq 0$$

$k = 0$:

$$c_0 = \frac{1}{2\pi}\int_0^\pi t\, dt$$

$$= \left(\frac{1}{2\pi}\right)\left(\frac{1}{2}\pi^2\right) = \frac{\pi}{4}.$$

Evaluate the first few coefficients

(44)

$$c_0 = \frac{\pi}{4}$$

$$c_1 = \frac{1}{2\pi}\left(e^{-j\pi} - 1\right) + j\left(\frac{1}{2}e^{-j\pi}\right)$$

$$= -\frac{1}{\pi} - \frac{1}{2}j$$

$$c_{-1} = \frac{1}{2\pi}\left(e^{j\pi} - 1\right) + j\left(\frac{-1}{2}e^{j\pi}\right)$$

$$= -\frac{1}{\pi} + \frac{1}{2}j$$

$$= \bar{c}_1$$

$$c_2 = \frac{1}{8\pi}\left(e^{-j2\pi} - 1\right) + j\left(\frac{1}{4}e^{-j2\pi}\right)$$

$$= \frac{1}{4}j$$

$$c_{-2} = \bar{c}_2$$

$$= -\frac{1}{4}j$$

Recall from Q (35) that the real Fourier coefficients are

$$a_k = \frac{(-1)^k - 1}{k^2\pi}, \quad b_k = \frac{(-1)^{k+1}}{k}$$

(b)

$$\text{So } c_k = \begin{cases} \frac{1}{2}(a_k - jb_k), & k \geq 1 \\ a_0 \\ \frac{1}{2}(a_k + jb_k), & k \leq -1 \end{cases}$$

So

$$c_1 = \frac{1}{2}\left(-\frac{2}{\pi} - j\right)$$

$$= -\frac{1}{\pi} - j/2 \quad \checkmark$$

$$c_{-1} = \frac{1}{2}\left(-2/\pi + j\right)$$

$$= -\frac{1}{\pi} + \frac{1}{2}j \quad \checkmark$$

$$c_2 = \frac{1}{2}\left(0 - \frac{1}{2}j\right)$$

$$= \frac{1}{4}j \quad \checkmark$$

$$c_{-2} = \frac{1}{2}\left(0 + \frac{1}{2}j\right)$$

$$= -\frac{1}{4}j \quad \checkmark.$$