(82) A continuous-time signal $x(t)$ has LT

$$X(s) = \frac{s+1}{s^2+5s+7}$$

Find the LT of

(a) $v(t) = x(3t-4)\,u(3t-4)$.

Solution: Firstly we introduce

$$\boxed{w(t) = x(3t)}$$

Then by Time Scaling

$$W(s) = \frac{1}{3} X\left(\frac{s}{3}\right)$$

$$= \frac{1}{3}\left[\frac{s/3+1}{(s/3)^2+5(s/3)+7}\right]\left(\frac{9}{9}\right)$$

$$= \frac{s+3}{s^2+15s+63}$$

Then

$$v(t) = x(3t-4)\,u(3t-4)$$

$$= x(3(t-4/3))\,u(t-4/3)$$

because $u(\alpha s) = u(s)$ for $\alpha > 0$

$$= w(t-4/3)\,u(t-4/3)$$

So by Time Shifting

$$V(s) = W(s)\,e^{-4/3\, s}$$

$$= \frac{(s+3)\,e^{-4s/3}}{s^2+15s+63}.$$

82 (c) Assume $x(t)$ has LT

$$X(s) = \frac{s+1}{s^2+5s+7}$$

Let $v(t) = \frac{d^2x}{dt^2}$

and assume $x(0^-) = 1$, $\dot{x}(0^-) = -4$

Then $v(t)$ has LT

$$V(s) = s^2 X(s) - s\,x(0^-) - \dot{x}(0^-)$$

$$= \frac{s^2(s+1)}{s^2+5s+7} - s + 4$$

$$= \frac{s^2(s+1) - (s-4)(s^2+5s+7)}{s^2+5s+7}$$

$$= \frac{s^3 + s^2 - (s^3+5s^2+7s) + (4s^2+20s+28)}{s^2+5s+7}$$

$$= \frac{13s+28}{s^2+5s+7}.$$

(83)

Let $p = \sigma + j\omega \in \mathbb{C}$

Then for any $c \in \mathbb{C}$,

$$\boxed{c e^{pt} + \bar{c} e^{\bar{p} t} = 2|c| e^{\sigma t} \cos(\omega t + \angle c)}$$

Proof: Since $c \in \mathbb{C}$, we have

$$c = |c| e^{j \angle c}, \quad \bar{c} = |c| e^{-j \angle c}$$

Hence

$$\begin{aligned} \text{LHS} &= |c| e^{(\sigma + j\omega) t} \cdot e^{j \angle c} \\ &\quad + |c| e^{(\sigma - j\omega) t} e^{-j \angle c} \\ &= |c| e^{\sigma t} e^{j(\omega t + \angle c)} \\ &\quad + |c| e^{\sigma t} e^{-j(\omega t + \angle c)} \\ &= 2|c| e^{\sigma t} \left(\frac{1}{2}\right) \left[e^{j(\omega t + \angle c)} + e^{-j(\omega t + \angle c)} \right] \\ &= 2|c| e^{\sigma t} \cos(\omega t + \angle c) \end{aligned}$$

by Euler's Identity.

Prove the Time Shift Property:

(84) If $x(t) \longleftrightarrow X(s)$

then

$$x(t-c)\,u(t-c) \longleftrightarrow e^{-cs} X(s)$$

for any $c > 0$.

Proof: We know that

$$X(s) = \int_0^\infty x(t)\, e^{-st}\, dt$$

Let $f(t) = x(t-c)\,u(t-c)$.

Then
$$F(s) = \int_0^\infty f(t)\, e^{-st}\, dt$$
$$= \int_0^\infty x(t-c)\,u(t-c)\,e^{-st}\, dt$$
$$= \int_c^\infty x(t-c)\, e^{-st}\, dt \quad \text{because } c > 0$$

Let $\boxed{p = t - c} \Rightarrow t = p + c$, $dt = dp$

and $t = c \Rightarrow p = 0$

$t = \infty \Rightarrow p = \infty$.

$$F(s) = \int_0^\infty x(p)\, e^{-s(p+c)}\, dp$$
$$= e^{-sc} \int_0^\infty x(p)\, e^{-sp}\, dp$$
$$= e^{-sc} \int_0^\infty x(t)\, e^{-st}\, dt$$
$$= e^{-sc} X(s).$$

as required.

(85) (b) Consider

$$X(s) = \frac{s+1}{s(s+2.5-j\frac{\sqrt{3}}{2})(s+2.5+j\frac{\sqrt{3}}{2})}$$

The poles of X are

$$p_1 = 0, \quad p_2 = -2.5 + j\sqrt{3}/2$$
$$p_3 = -2.5 - j\sqrt{3}/2.$$

To express X(s) in partial fractions we compute residues:

$$c_1 = [(s-p_1)X(s)]_{s=p_1}$$
$$= \left.\frac{s+1}{(s+2.5-j\sqrt{3}/2)(s+2.5+j\sqrt{3}/2)}\right|_{s=0}$$
$$= \frac{1}{(2.5-j\sqrt{3}/2)(2.5+j\sqrt{3}/2)}$$
$$= \frac{1}{(2.5)^2 + \frac{3}{4}}$$
$$= \frac{1}{7}$$

$$c_2 = [(s-p_2)X(s)]_{s=p_2}$$
$$= \left.\frac{s+1}{s(s+2.5+j\sqrt{3}/2)}\right|_{s=p_2}$$
$$= \frac{-2.5+j\sqrt{3}/2+1}{(-2.5+j\sqrt{3}/2)(-2.5+j\sqrt{3}/2+2.5+j\sqrt{3}/2)}$$
$$= \frac{-1.5+j\sqrt{3}/2}{(-2.5+j\sqrt{3}/2)(j\sqrt{3})}$$
$$= \frac{-1.5+j\sqrt{3}/2}{-3/2-2.5\sqrt{3}\,j}$$

Q5 (b) $c_2 = \frac{(-1.5 + j\sqrt{3}/2)}{(-3/2 - 2.5\sqrt{3}j)} \left(\frac{-\frac{3}{2} + 2.5\sqrt{3}j}{-\frac{3}{2} + 2.5\sqrt{3}j} \right)$

$= -0.0714 - 0.3712j$

Hence $c_3 = \bar{c}_2$
$= -0.0714 + 0.3712j$

So $X(s) = \frac{c_1}{s} + \frac{c_2}{s + 2.5 - j\frac{\sqrt{3}}{2}} + \frac{\bar{c}_2}{s + 2.5 + j\frac{\sqrt{3}}{2}}$

We note that

$|c_2| = 0.378$

$\angle c_2 = -1.761^c$ ⟵ radians

So $x(t) = c_1 e^{p_1 t} + c_2 e^{p_2 t} + \bar{c}_2 e^{\bar{p}_2 t}$

$= \left[\frac{1}{7} + 2|c_2| e^{-2.5t} \cos\left(\frac{\sqrt{3}}{2}t + \angle c_2\right)\right] u(t)$

$= \left[\frac{1}{7} + 0.7559 e^{-2.5t} \cos\left(\frac{\sqrt{3}}{2}t - 1.761\right)\right] u(t).$

NOTE: $\boxed{\begin{aligned} p_2 &= -2.5 + j\sqrt{3}/2 \\ &= \sigma + j\omega \end{aligned}}$

$$\boxed{e^{-bt} u(t) \longleftrightarrow \frac{1}{s+b}}$$

89

Show that if $x[n] \longleftrightarrow X(z)$

then

$$n\,x[n] \longleftrightarrow -z\frac{dX}{dz}.$$

Proof : By assumption $X(z) = \sum_{n=0}^{+\infty} x[n]z^{-n}$ $\boxed{\text{KNOW}}$

Let $f[n] = n\,x[n]$.

We want to show that

$$F(z) = -z\frac{dX}{dz}.$$ $\boxed{\text{WANT}}$

Consider $\frac{dX}{dz} = \frac{d}{dz}\left(\sum_{n=0}^{+\infty} x[n]z^{-n}\right)$

$$= \sum_{n=0}^{+\infty} -n\,x[n]\,z^{-n-1}$$

Hence

$$-z\frac{dX}{dz} = \sum_{n=0}^{\infty} \underbrace{n\,x[n]}\,z^{-n}$$

$$= \sum_{n=0}^{\infty} f[n]z^{-n}$$

$$= F(z)$$

as required.

91 Let $p \in \mathbb{C}$ be such that
$|p| = \sigma$, $\angle p = \Omega$
for some $\sigma, \Omega \in \mathbb{R}$.
Let $c \in \mathbb{C}$ be arbitrary. Then
for any integer $n > 1$

$$\boxed{c p^n + \bar{c} \bar{p}^n = 2|c| \sigma^n \cos(\Omega n + \angle c)}$$

Proof: We know that

$$p = \sigma e^{j\Omega}, \quad \bar{p} = \sigma e^{-j\Omega}$$

$$c = |c| e^{j\angle c}, \quad \bar{c} = |c| e^{-j\angle c}$$

Then $c p^n + \bar{c} \bar{p}^n$

$$= |c| e^{j\angle c} (\sigma e^{j\Omega})^n + |c| e^{-j\angle c} (\sigma e^{-j\Omega})^n$$

$$= |c| e^{j\angle c} \sigma^n e^{j\Omega n} + |c| e^{-j\angle c} \sigma^n e^{-j\Omega n}$$

$$= |c| \sigma^n \left[e^{j(\Omega n + \angle c)} + e^{-j(\Omega n + \angle c)} \right]$$

$$= 2|c| \left(\frac{\sigma^n}{2}\right) \left[e^{j(\Omega n + \angle c)} + e^{-j(\Omega n + \angle c)} \right]$$

$$= 2|c| \sigma^n \cos(\Omega n + \angle c)$$

as required.

Q2 (d) $\frac{X(z)}{z} = \frac{2z+1}{z^2(10z^2-z-2)}$

Poles are
$p_1 = 0,\ r = 2$
$p_3 = -0.4$
$p_4 = \frac{1}{2}$

$$= \frac{2z+1}{z^2(5z+2)(2z-1)}$$

$$= \frac{c_0}{z} + \frac{c_1}{z^2} + \frac{c_2}{z+2/5} + \frac{c_3}{z-1/2}$$

$$c_1 = \left[z^2 \frac{X(z)}{z}\right]_{z=0}$$
$$= \frac{2z+1}{10z^2-z-2}\Bigg|_{z=0}$$
$$= -\frac{1}{2}$$

$$c_0 = \left[\frac{d}{dz}\left(z^2 \frac{X(z)}{z}\right)\right]_{z=0}$$
$$= \left[\frac{d}{dz}\left(\frac{2z+1}{10z^2-z-2}\right)\right]_{z=0}$$
$$= \frac{-(20z^2-20z-3)}{-10z^2+z+2}\Bigg|_{z=0}$$
$$= -\frac{3}{4}$$

OK to use MATLAB here

$$c_2 = \left[\left(z+\frac{2}{5}\right)\frac{X(z)}{z}\right]_{z=-0.4}$$
$$= \frac{2z+1}{5z^2(2z-1)}\Bigg|_{z=0.4}$$
$$= -0.1389$$

$$c_3 = \left[\left(z-\frac{1}{2}\right)\frac{X(z)}{z}\right]_{z=0.5}$$
$$= \left[\frac{2z+1}{2z^2(5z+2)}\right]_{z=0.5}$$
$$= 0.8889.$$

Q2 (d)

Hence $\frac{X(z)}{z} = \frac{-3}{4z} - \frac{1}{2z^2} - \frac{0.1389}{z+2/5} + \frac{0.8889}{z-1/2}$

$\Rightarrow X(z) = \frac{-3}{4} - \frac{1}{2z} - \frac{0.1389z}{z+2/5} + \frac{0.8889z}{z-1/2}.$

We use the ZT pairs

$$\delta[n] \longleftrightarrow 1$$

$$\delta[n-q] \longleftrightarrow z^{-q}$$

$$a^n u[n] \longleftrightarrow \frac{z}{z-a}$$

to obtain

$$x[n] = \frac{-3\delta[n]}{4} - \frac{1}{2}\delta[n-1] - 0.1389(-0.4)^n u[n] + \frac{8}{9}(0.5)^n u[n]$$

MATLAB check

```
>> B = [2 1]            ≡ 2z+1
>> A = [10 -1 -2 0 0]   ≡ 10z^4-z^3-2z^2
>> [R, P] = residue(B, A)
```

Obtain

R = 0.8889 −0.1389 −0.75 −0.5

P = 0.5 −0.4 0 0.