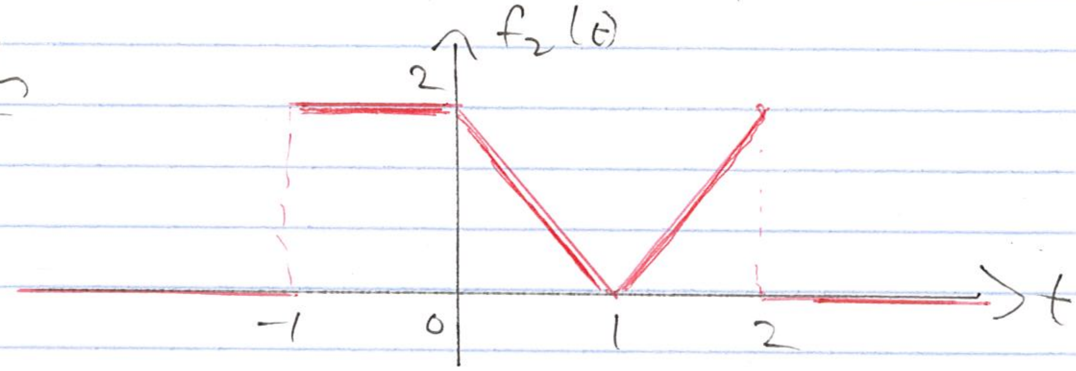
48 (b) $f_2(t) = \begin{cases} 0 & t < -1 \text{ and } t > 2 \\ 2, & -1 < t < 0 \\ 2(1-t), & 0 < t < 1 \\ 2(t-1), & 1 < t < 2 \end{cases}$

Sketch

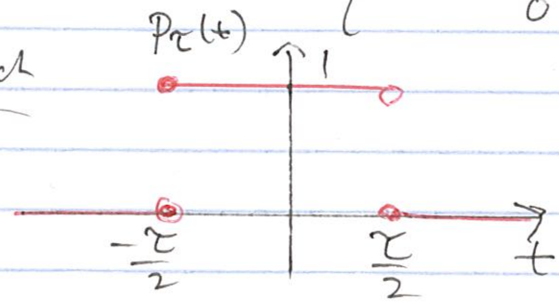


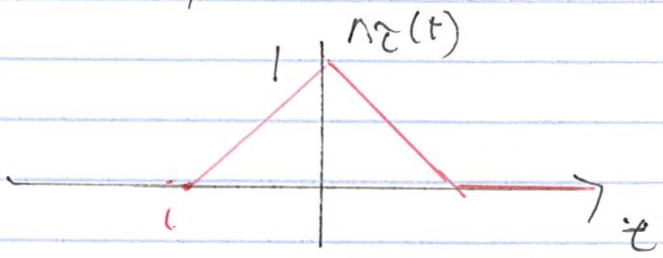
Recall the functions

$$p_\tau(t) = \begin{cases} 1, & -\tau/2 \le t < \tau/2 \\ 0, & \text{elsewhere} \end{cases}$$

$$\Lambda_\tau(t) = \begin{cases} 1 - 2|t|/\tau, & -\frac{\tau}{2} \le t < \frac{\tau}{2} \\ 0, & \text{elsewhere} \end{cases}$$

Sketch





So we can express

$$f_2(t) = 2p_3\left(t - \tfrac{1}{2}\right) - 2\Lambda_2(t-1)$$

and $p_\tau(t) \longleftrightarrow \tau \operatorname{sinc}\left(\frac{\tau w}{2\pi}\right)$

$$\Lambda_\tau(t) \longleftrightarrow \frac{\tau}{2} \operatorname{sinc}^2\left(\frac{\tau w}{4\pi}\right)$$

$$x(t-c) \longleftrightarrow X(w)e^{-jwc}$$

are ~~B~~ CTFT pairs.

48 (b) So

$$F_2(w) = 6\,\text{sinc}\left(\frac{3w}{2\pi}\right)e^{-\frac{jw}{2}}$$

$$- 2\,\text{sinc}^2\left(\frac{w}{2\pi}\right)e^{-jw}$$

(c) $f_3(t) = \begin{cases} 0 & , |t| > 0.5 \\ \cos(\pi t) & , |t| < 0.5 \end{cases}$

Then

$$f_3(t) = \cos(\pi t)\, p_1(t)$$

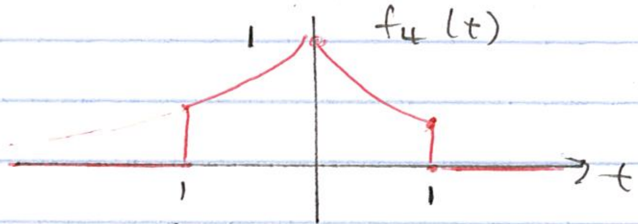
and so by modulation

$$F_3(w) = \frac{1}{2}\left[P_1(w+\pi) + P_1(w-\pi)\right]$$

$$= \frac{1}{2}\left[\text{sinc}\left(\frac{w+\pi}{2\pi}\right) + \text{sinc}\left(\frac{w-\pi}{2\pi}\right)\right]$$

(d) $f_4(t) = \begin{cases} 0 & , |t| > 1 \\ e^{-|t|} & , |t| \le 1 \end{cases}$

Sketch



Hint: Write $f_4(t) = v(t) + v(-t)$

Introduce $v(t) = e^{-t}(u(t) - u(t-1))$

Recall

$$e^{-bt}u(t) \longleftrightarrow \frac{1}{b+jw}$$

So $v(t) \longleftrightarrow \dfrac{1}{1+jw} - \dfrac{1}{1+jw}e^{-jw}$

Also $v(-t) \longleftrightarrow V(-w)$.

$$= \frac{1}{1-jw} - \frac{1}{1-jw}e^{jw}$$

(48) d) Then by definition

$$V(\omega) = \int_{-\infty}^{\infty} v(t) e^{-j\omega t} dt$$

$$= \int_0^1 e^{-t} e^{-j\omega t} dt$$

$$= \int_0^1 e^{-(1+j\omega)t} dt$$

$$= \frac{-1}{1+j\omega} \left[e^{-(1+j\omega)t} \right]_0^1$$

$$= \frac{1}{1+j\omega} \left[1 - e^{-(1+j\omega)} \right]$$

So $F_4(\omega) = V(\omega) + V(-\omega)$

by Flipping theorem

$$= \frac{1}{1+j\omega} \left[1 - e^{-(1+j\omega)} \right] + \frac{1}{1-j\omega} \left[1 - e^{-(1-j\omega)} \right]$$

$$= \frac{1}{1+j\omega} + \frac{1}{1-j\omega} - e^{-1} \left[\frac{e^{-j\omega}}{1+j\omega} + \frac{e^{j\omega}}{1-j\omega} \right]$$

$$= \frac{(1-j\omega) + (1+j\omega)}{1+\omega^2} - e^{-1} \left[\frac{(1-j\omega)e^{-j\omega} + (1+j\omega)e^{j\omega}}{1+\omega^2} \right]$$

$$= \frac{2}{1+\omega^2} - \frac{e^{-1}}{1+\omega^2} \left[e^{-j\omega} + e^{j\omega} + j\omega \left(e^{j\omega} - e^{-j\omega} \right) \right]$$

$$= \frac{2}{1+\omega^2} - \frac{e^{-1}}{1+\omega^2} \left[2\cos(\omega) - 2\omega \sin(\omega) \right]$$

$$= \frac{2 - 2e^{-1} \left[\cos(\omega) - \omega \sin(\omega) \right]}{1+\omega^2}$$

Conside the function,

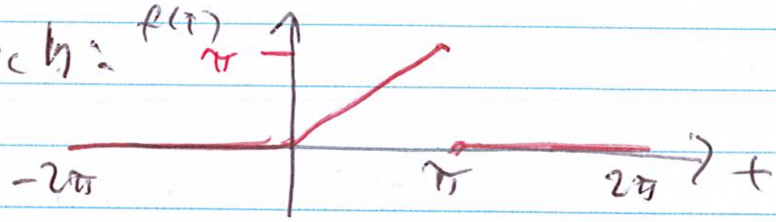
$$f(t) = t\,(u(t) - u(t-\pi)).$$

49

$$= \begin{cases} t, & 0 \le t \le \pi \\ 0, & \text{elsewhere} \end{cases}$$

Find the Fourier transform of $f(t)$

Solution: Sketch:



$$F(w) = \int_{-\infty}^{\infty} f(t)\, e^{-iwt}\, dt$$

$$= \int_0^{\pi} t\, e^{-iwt}\, dt.$$

Let $u(t) = t$, $\quad v'(t) = e^{-iwt}$

$u'(t) = 1$, $\quad v(t) = \frac{-1}{iw} e^{-iwt}$

So

$$F(w) = \left[\frac{-t}{iw} e^{-iwt}\right]_0^{\pi} + \int_0^{\pi} \frac{1}{iw} e^{-iwt}\, dt$$

$$= \left[\frac{-t}{iw} e^{-iwt} - \frac{1}{(iw)^2} e^{-iwt}\right]_0^{\pi}$$

$$= \left[\frac{-\pi}{iw} e^{-iw\pi} - \frac{1}{(iw)^2} e^{-iw\pi}\right] - \left[-\frac{1}{(iw)^2} e^{0}\right]$$

$$= \frac{-1}{w^2} + \frac{1}{w^2}\left[e^{-iw\pi} + i\pi w\, e^{-iw\pi}\right]$$

$$= \frac{1}{w^2}\left[(1 + i\pi w) e^{-iw\pi} - 1\right]$$

50 (a) Find the inverse FT of

$$X_1(w) = \cos(4w).$$

Let $x(t) = \cos(4t)$

Then $X(w) = \pi[\delta(w+4) + \delta(w-4)]$

By duality

$$\boxed{X(t) \leftrightarrow 2\pi\, x(-w)}$$

Hence

$$\pi[\delta(t+4) + \delta(t-4)] \leftrightarrow 2\pi \cos(-4w)$$

Hence

$$\frac{1}{2}[\delta(t+4) + \delta(t-4)] \leftrightarrow \cos(4w)$$

$$= X_1(w)$$

as cos is an even function.

So

$$x_1(t) = \frac{1}{2}[\delta(t+4) + \delta(t-4)]$$

(b) Find the inverse FT of

$$X_2(w) = \sin^2(3w).$$

From a similar argument to the above

$$\cancel{x_2(t) = \frac{1}{2}[\delta(t+4) - \delta(t-4)]}$$

50 (b) we obtain

$$j\pi[\delta(t+3)-\delta(t-3)] \longleftrightarrow 2\pi \sin(-3\omega)$$

So $\frac{-j}{2}[\delta(t+3)-\delta(t-3)] \longleftrightarrow \sin(3\omega)$

Let $y_2(t) = \frac{-j}{2}[\delta(t+3)-\delta(t-3)]$

Then by convolution

$$(y_2 * y_2)(t) \longleftrightarrow \sin^2(3\omega).$$

So
$$\begin{aligned} x_2(t) &= (y_2 * y_2)(t) \\ &= \int_{-\infty}^{\infty} y_2(t-\lambda)\, y_2(\lambda)\, d\lambda \\ &= -\frac{1}{4}\int_{-\infty}^{\infty} [\delta(t+3-\lambda)-\delta(t-3-\lambda)] \\ &\qquad [\delta(\lambda+3)-\delta(\lambda-3)]\, d\lambda \end{aligned}$$

Using ~~shift~~ sifting theorem gives

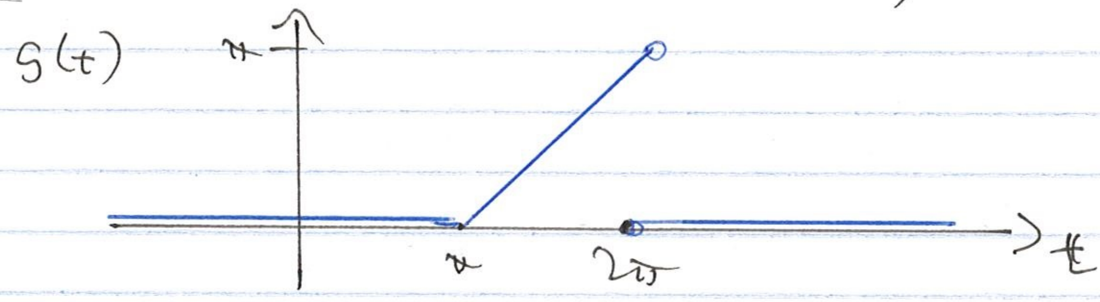
$$x_2(t) = \frac{-1}{4}\Big[(\delta(t+6)-\delta(t)) - (\delta(t)-\delta(t-6))\Big]$$

$$= \frac{1}{4}[2\delta(t) - \delta(t+6) - \delta(t-6)]$$

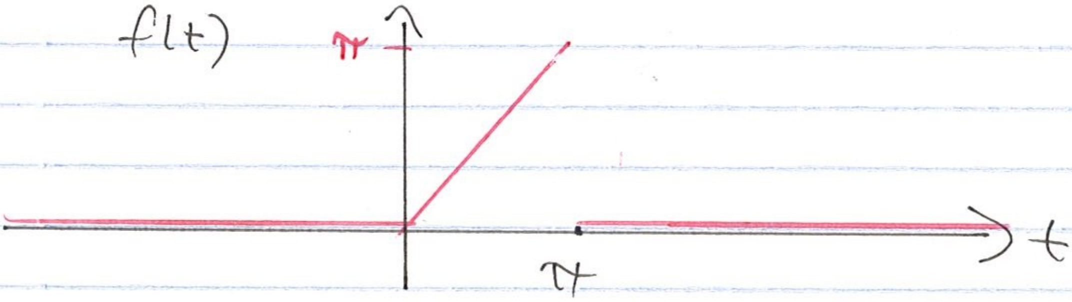
(51) Find the Fourier transform of

$$g(t) = (t-\pi)\left[u(t-\pi) - u(t-2\pi)\right]$$

<u>Solution</u> : First we sketch $g(t)$



we see that $g(t)$ is a delayed copy of $f(t) = t\,(u(t-\pi) - u(t))$



and Q(49) showed that

$$F(\omega) = \frac{1}{\omega^2}\left[(1+j\pi\omega)e^{-j\pi\omega} - 1\right]$$

Since $g(t) = f(t-\pi)$

we obtain by Shifting Theorem :

$$G(\omega) = F(\omega)\,e^{-j\pi\omega}$$

$$= \frac{1}{\omega^2}\left[(1+j\pi\omega)e^{-j\pi\omega} - e^{-j\pi\omega}\right]$$

(55) We are given that for $b > 0$

$$x(t) = e^{-bt} u(t)$$

have FT $$X(w) = \frac{1}{jw + b}.$$

(a) Let $x_1(t) = x(5t-4)$.

To find the FT of x_1, we introduce

$$x_0(t) = x(5t).$$

Then $$X_0(w) = \frac{1}{5} X\left(\frac{w}{5}\right) = \frac{1}{jw + 5b}.$$ by the Time Scaling theorem

Then $$x_1(t) = x(5t-4) = x_0\left(t - \frac{4}{5}\right)$$

So by the Time-Shifting theorem

$$X_1(w) = X_0(w) e^{-jw4/5} = \frac{e^{-jw4/5}}{jw + 5b}.$$

(b) Let $x_2(t) = e^{j2t} x(t)$

Then by Modulation

$$X_2(w) = X(w-2) = \frac{1}{j(w-2) + b}.$$

55 (c) Let $x_3(t) = (x * x)(t)$

Then $X_3(t) = X(w)\,X(w)$ } by the Convolution Theorem

$$= \frac{1}{(jw+b)^2}$$

(d) Let $x_4(t) = \dfrac{1}{jt - b}$

$$= \frac{-1}{-jt + b}.$$

Then $x_4(-w) = \dfrac{-1}{jw + b}.$

Hence by the Duality Theorem

$$X_4(t) = -2\pi e^{-bt}\, u(t)$$

and hence $X_4(w) = -2\pi e^{-bw}\, u(w).$

56 (a) Prove the Time Shift Theorem

Let $x(t)$ have FT $X(w)$.
Let $f(t) = x(t-c)$, $c \in \mathbb{R}$
Show that $F(w) = X(w)e^{-jwc}$.

Proof: By definition,

$$F(w) = \int_{-\infty}^{\infty} f(t) e^{-jwt} \, dt$$

$$= \int_{-\infty}^{\infty} x(t-c) e^{-jwt} \, dt$$

Let $\boxed{s = t - c}$, $ds = dt$, $t = s + c$.
$s(\infty) = \infty$, $s(-\infty) = -\infty$

Then

$$F(w) = \int_{-\infty}^{\infty} x(s) e^{-jw(s+c)} \, ds$$

$$= e^{-jwc} \int_{-\infty}^{\infty} x(s) e^{-jws} \, ds$$

$$= e^{-jwc} X(w)$$

So $x(t-c) \longleftrightarrow e^{-jwc} X(w)$

is a Fourier Transform Pair.

56 (c) Let $x(t) \leftrightarrow X(w)$ be an FT pair
Let $a > 0$. Then

$$x(at) \leftrightarrow \frac{1}{a} X\left(\frac{w}{a}\right)$$

Proof: Let $x_1(t) = x(at)$

Then $X_1(w) = \int_{-\infty}^{\infty} x(at)\, e^{-jwt}\, dt$

Let $s = at$, $s(\infty) = \infty$, $s(-\infty) = -\infty$, as $a > 0$.
$ds = a\, dt$.

So $X_1(w) = \int_{-\infty}^{\infty} \frac{1}{a} x(s)\, e^{-j\frac{ws}{a}}\, ds$

$$= \frac{1}{a} \int_{-\infty}^{\infty} x(s)\, e^{-jws/a}\, ds$$

$$= \frac{1}{a} X\left(\frac{w}{a}\right).$$

as required.