THE UNIVERSITY OF MELBOURNE

November 2017 Assessment
Department of Electrical and Electronic Engineering
ELEN 30012 SIGNALS AND SYSTEMS

Time allowed: 180 minutes
Reading time: 15 minutes

This paper has 8 pages including the 2-page Formulae Sheet

Authorised materials:

Only the following calculators may be used:
Casio FX82
Casio FX100

Instructions to invigilators:
The examination paper IS TO REMAIN in the examination room.

Instruction to students:
Attempt **ALL** questions. You may attempt the questions in any order.
The marks given for each question are shown in brackets after the question numbers. The examination paper has a total of 100 marks.

You must show your work in order to receive credit.

Ensure your student number is written on all script books and answer sheets during writing time.
No annotating is allowed in reading time or after the end of writing time.

Answer all questions and show all work in the script book.

Question 1 (13 marks)

(a) [**4 marks**] Consider the discrete-time signal x given by

$$x[n] = 1 - \sum_{k=2}^{\infty} \delta[n+1-k]$$

(i) Plot $x[n]$ for $n = \ -1,\ 0,\ 1,\ 2.$

(ii) Find the values of the integers M and n_0 such that

$$x[n] = u[Mn - n_0]$$

(b) [**3 marks**] Find a continuous-time signal $x : \mathbf{R} \to \mathbf{R}$ that satisfies the equation

$$x(t-2)u(t-2) = (t^2+1)u(t-2)$$

(c) [**3 marks**] Give an example of a discrete-time signal that is periodic with fundamental period $L = 3$. Explain why the signal has period $L = 3$.

(d) [**3 marks**] The 3-point Moving Average filter is given by the input-output relationship

$$y[n] = \frac{1}{3}\left(x[n] + x[n-1] + x[n-2]\right)$$

Assume the system has zero initial conditions, i.e. $y[n] = 0$ for $n < 0$. Find $y[n]$ for all $n \geq 0$ when $x[n] = u[n]$.

Question 2 (13 marks)

A Linear Time-Invariant (LTI) discrete-time system with input-output trajectories (v, y) is defined by the difference equation

$$y[n+2] - 3y[n+1] - 4y[n] = 2v[n+1] + v[n]$$

(a) [**2 marks**] Obtain the input-output transfer function H for this system.

(b) [**2 marks**] Is the system stable? Justify your answer.

(c) [**5 marks**] Introduce suitable state variables to obtain a state representation

$$\begin{aligned} x[n+1] &= Ax[n] + Bv[n] \\ y[n] &= Cx[n] \end{aligned}$$

for this system in controller canonical form.

(d) [**4 marks**] Use z-transforms to show that

$$Y(z) = C(zI - A)^{-1}BV(z)$$

and use this to verify the transfer function H you obtained in part (a).

Question 3 (16 marks)

An LTI continuous-time system has impulse response

$$h(t) = \delta(t-2) + \delta(t-1)$$

(a) [**1 mark**] Is this system causal? Give a reason for your answer.

(b) [**1 mark**] Show that h is absolutely integrable.

(c) [**2 marks**] Let x be an input signal for this system. Obtain an expression for the system output y in terms of x.

(d) [**4 marks**] Suppose the input signal x is given by

$$x(t) = \begin{cases} t+1, & t \geq 0 \\ 0, & \text{elsewhere} \end{cases}$$

Use your answer to part (c) to find $y(t)$.

(e) [**4 marks**] Find the system frequency response H and use it to find the system output y when the input signal is $v(t) = \cos(\frac{\pi t}{2})$.

(f) [**4 marks**] Show that the system frequency response has linear phase for $-\pi \leq \omega \leq \pi$.

Question 4 (10 marks)

Let x and y be discrete-time signals, and let X and Y be their z-transforms. Use the definition of z-transform to prove the following transform pairs:

(a) [**3 marks**] (Linearity) For any two real numbers a and b

$$ax[n] + by[n] \longleftrightarrow aX(z) + bY(z)$$

(b) [**3 marks**] (Right Time Shift Theorem)

$$x[n-1]u[n-1] \longleftrightarrow X(z)z^{-1}$$

(c) [**4 marks**] (Left Time Shift Theorem)

$$x[n+1] \longleftrightarrow zX(z) - x[0]z$$

Question 5 (14 marks)

Consider the continuous-time periodic signal

$$v(t) = 2 + 3\cos(2t) + 4\sin(3t)$$

(a) [**3 marks**] Is v an even function, odd function, or neither? Justify your answer with a proof or counterexample.

(b) [**3 marks**] Find the fundamental period T_0 and angular frequency ω_0 of v.

(c) [**3 marks**] Find the coefficients c_k^v of the complex exponential Fourier series for v:

$$v(t) = \sum_{k=-\infty}^{\infty} c_k^v e^{jk\omega_0 t}$$

(d) [**5 marks**] The signal v is applied as an input to an LTI continuous-time system with frequency response function

$$H(\omega) = \begin{cases} 10e^{-j5\omega}, & |\omega| > \frac{\pi}{T_0} \\ 0 & \text{otherwise} \end{cases}$$

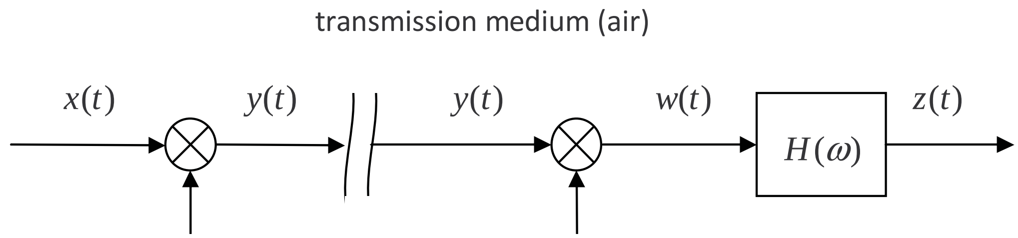
The resulting output y has complex exponential Fourier series

$$y(t) = \sum_{k=-\infty}^{\infty} c_k^y e^{jk\omega_0 t}$$

Find the coefficients c_k^y. You may express your answers in polar form $re^{j\theta}$.

Question 6 (13 marks)

An Amplitude Modulation communication system is depicted below:



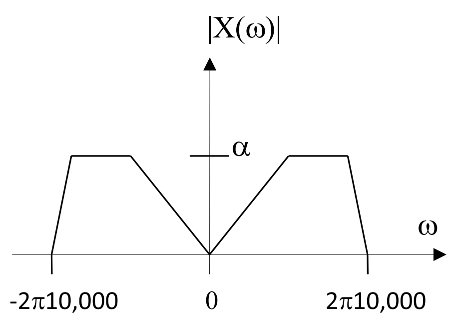
The circular components multiply their input signals. The transmission medium (air, for radio signals) is assumed to pass the signal y unaltered. An AM radio station transmit signals at a **carrier frequency** of $\omega_c = 2\pi \times 774$ kHz. The transmitted AM signal y is

$$y(t) = x(t)\cos(\omega_c t) \tag{1}$$

(a) [**4 marks**] Let X be the Fourier Transform of x. Show that Y, the Fouruier transform of y, is given by

$$Y(\omega) = \frac{1}{2}[X(\omega + \omega_c) + X(\omega - \omega_c)]$$

(b) [**2 marks**] Assume the Fourier amplitude spectrum $|X(\omega)|$ of the audio signal x is



Sketch $|Y(\omega)|$, clearly indicating all relevant values.

(c) [**3 marks**] The transmitted signal y can be demodulated at the receiver by multiplying by a sinusoidal wave at the carrier frequency to obtain the signal w, where

$$w(t) = y(t)\cos(\omega_c t). \tag{2}$$

(i) Express W, the Fourier transform of w, in terms of X.

(ii) Sketch $|W(\omega)|$, noting the important magnitudes and frequencies.

Question 6 (continued)

(d) [**4 marks**] After performing the demodulation in (**??**), an AM receiver filters the received signal w through a filter H with frequency response such that $H(\omega) = 1$ for $|\omega| \leq 2\pi \times 10,000$ and $H(\omega) = 0$ for $|\omega| > 2\pi \times 20,000$. Let z be the filtered signal.

(i) Obtain an equation relating $Z(\omega)$ to $X(\omega)$. Explain your answer using about 20 to 30 words.

(ii) Sketch $|Z(\omega)|$, noting the important magnitudes and frequencies.

Question 7 (13 marks)

Let x be the discrete-time signal given by

$$x[n] = \delta[n] - \delta[n-1]$$

(a) [**3 marks**] Let X be the discrete-time Fourier transform (DTFT) of x. Compute $|X(\Omega)|$, the amplitude spectrum of x. Simplify your expression as much as possible.

(b) [**2 marks**] Let $N \geq 2$ be an integer, and let X_k be the N-point Discrete Fourier Transform (DFT) of x. Express X_k as a function of k and N, for $k = 0, 1, \ldots, N-1$.

(c) [**2 marks**] Assume $N = 4$. Evaluate $|X_k|$ for $k = 0, 1, 2, 3$.

(d) [**3 marks**] Plot $|X(\Omega)|$ for $0 \leq \Omega \leq 2\pi$, noting the important magnitudes and frequencies. Using the same axes, plot $|X_k|$ for $k = 0, 1, 2, 3$.

(e) [**3 marks**] By referring to your graph in part (d), explain how $|X_k|$ is related to $|X(\Omega)|$, and so explain what happens to $|X_k|$ as $N \to \infty$. Explain your answer using about 20 to 30 words.

Question 8 (8 marks)

Let x be a discrete-time signal and let X denote its DTFT. Let f be a continuous-time signal defined by

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) p_{2\pi}(\omega) e^{j\omega t} \, d\omega$$

where $p_{2\pi}$ is the continuous-time rectangular pulse signal of width 2π. Show that for all integer n,

$$f(n) = x[n]$$

Formulae for the Exam

1. **Trigonometric formulae**

$$
\begin{aligned}
\sin(\theta \pm \alpha) &= \sin(\theta)\cos(\alpha) \pm \cos(\theta)\sin(\alpha) \\
\cos(\theta \pm \alpha) &= \cos(\theta)\cos(\alpha) \mp \sin(\theta)\sin(\alpha) \\
e^{j\theta} &= \cos(\theta) + j\sin(\theta) \\
\cos(\theta) &= \frac{1}{2}\left(e^{j\theta} + e^{-j\theta}\right) \\
\sin(\theta) &= \frac{1}{2j}\left(e^{j\theta} - e^{-j\theta}\right) \\
cp^n + \bar{c}\bar{p}^n &= 2|c|\sigma^n \cos(\Omega n + \angle c) \\
ce^{pt} + \bar{c}e^{\bar{p}t} &= 2|c|e^{\sigma t}\cos(\omega t + \angle c)
\end{aligned}
$$

2. **Convolution**

$$(x \star v)[n] = \sum_{i=-\infty}^{\infty} x[i]v[n-i], \qquad (x \star v)(t) = \int_{-\infty}^{\infty} x(\lambda)v(t-\lambda)\, d\lambda$$

3. **Integration Methods**

$$\int_a^b v(x)u'(x)\,dx = [u(x)v(x)]_a^b - \int_a^b v'(x)u(x)\,dx, \qquad \int_a^b f(u(x))u'(x)\,dx = \int_{u(a)}^{u(b)} f(u)\,du$$

4. **Fourier Series Formulae**

$$
\begin{aligned}
x(t) &= a_0 + \sum_{k=1}^{\infty}(a_k\cos(k\omega_0 t) + b_k\sin(k\omega_0 t)), \qquad \omega_0 = \frac{2\pi}{T} \\
& \quad a_0 = \frac{1}{T}\int_0^T x(t)\,dt,\ a_k = \frac{2}{T}\int_0^T x(t)\cos(k\omega_0 t)\,dt,\ b_k = \frac{2}{T}\int_0^T x(t)\sin(k\omega_0 t)\,dt \\
x(t) &= a_0 + \sum_{k=1}^{\infty} A_k\cos(k\omega_0 t + \theta_k), \qquad A_k = \sqrt{a_k^2 + b_k^2}, \qquad \theta_k = \begin{cases} \tan^{-1}\left(\frac{-b_k}{a_k}\right), & a_k \geq 0 \\ \pi + \tan^{-1}\left(\frac{-b_k}{a_k}\right), & a_k < 0 \end{cases} \\
x(t) &= \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t}, \qquad c_k = \frac{1}{T}\int_0^T x(t)e^{-jk\omega_0 t}\,dt, \qquad k = 0, \pm 1, \pm 2,\ \ldots
\end{aligned}
$$

$$
\begin{aligned}
c_0 &= a_0, \qquad c_k = \frac{1}{2}(a_k - jb_k), \qquad c_{-k} = \frac{1}{2}(a_k + jb_k),\ k = 1, 2,\ \ldots \\
a_0 &= c_0, \qquad a_k = c_k + c_{-k}, \qquad b_k = j(c_k - c_{-k}),\ k = 1, 2,\ \ldots
\end{aligned}
$$

$$|c_k| = \frac{1}{2}A_k, \qquad \angle c_k = \theta_k,\ k = 1, 2,\ \ldots$$

5. **Transforms**

$$\begin{aligned}
X(\omega) &= \int_{-\infty}^{\infty} x(t)e^{-j\omega t}\, dt, \\
x(t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)e^{j\omega t}\, d\omega \\
X(\Omega) &= \sum_{n=-\infty}^{\infty} x[n]e^{-j\Omega n} \\
x[n] &= \frac{1}{2\pi}\int_{0}^{2\pi} X(\Omega)e^{j\Omega n}\, d\Omega \\
X_k &= \sum_{n=0}^{L-1} x[n]e^{-j2\pi kn/N}, \\
x[n] &= \frac{1}{N}\sum_{k=0}^{N-1} X_k e^{j2\pi kn/N} \\
X(s) &= \int_{0}^{\infty} x(t)e^{-st}\, dt \\
X(z) &= \sum_{n=0}^{\infty} x[n]z^{-n}
\end{aligned}$$

6. **Residues**

$$\begin{aligned}
c_{r-i} &= \frac{1}{i!}\left[\frac{d^i}{ds^i}\left[(s-p_1)^r X(s)\right]\right]_{s=p_1}, \qquad i = 0, 1, 2,\ ..., r-1 \\
c_i &= \left[(s-p_i)X(s)\right]_{s=p_i}, \qquad i = r+1, r+2,\ ..., N \\
c_0 &= X(0) \\
c_{r-i} &= \frac{1}{i!}\left[\frac{d^i}{dz^i}\left[(z-p_1)^r \frac{X(z)}{z}\right]\right]_{z=p_1}, \qquad i = 0, 1, 2,\ ..., r-1 \\
c_i &= \left[(z-p_i)\frac{X(z)}{z}\right]_{z=p_i}, \qquad i = r+1, r+2,\ ..., N
\end{aligned}$$

7. **System Outputs**

$$\begin{aligned}
y(t) &= Ce^{At}x(0) + \int_0^t Ce^{A(t-\lambda)}Bv(\lambda)\, d\lambda + Dv(t), \qquad \text{for } t \geq 0 \\
y[n] &= CA^n x(0) + \sum_{i=0}^{n-1} CA^{n-i-1}Bv[i] + Dv[n], \qquad \text{for } n \geq 1 \\
y(t) &= |H(\omega_0)|\cos(\omega_0 t + \angle H(\omega_0)) \\
y[n] &= |H(\Omega_0)|\cos(\Omega_0 n + \angle H(\Omega_0))
\end{aligned}$$