

# Lecture 3: K-Nearest Neighbors

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**COMP90049**

**Introduction to Machine Learning**

Semester 2, 2021

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Acknowledgement: Lea Frermann



## Last time... Machine Learning concepts

- data, features, classes
- models, training
- practical considerations

## Today... Our first machine learning algorithm

- K-nearest neighbors
- Application to classification
- Application to regression

Also: the topic of your **first assignment!**

- Released on Canvas on Wed 4th August at 7pm!
- Questions: assignment1 discussion board (don't share solutions!)



## Introduction

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## K-Nearest Neighbors: Example

Your 'photographic memory' of all handwritten digits you've ever seen:



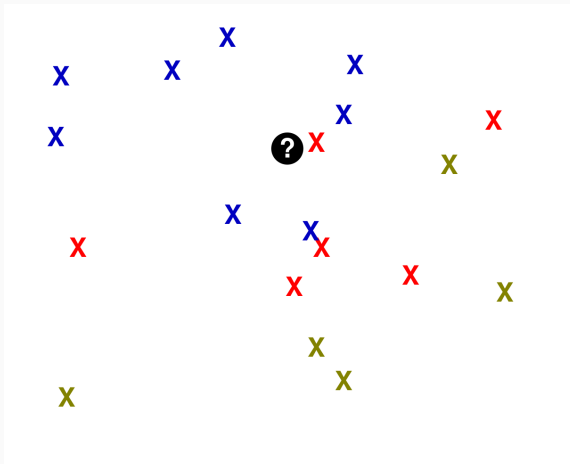
## K-Nearest Neighbors: Example

Your 'photographic memory' of all handwritten digits you've ever seen:

Given a new drawing, determine the digit by comparing it to all digits in your 'memory'.

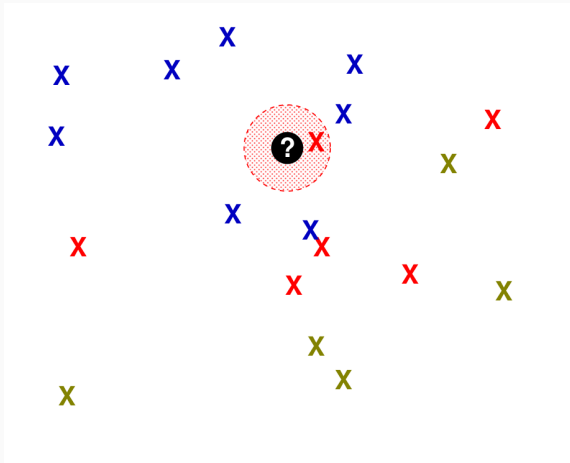


## K-Nearest Neighbors: Visualization



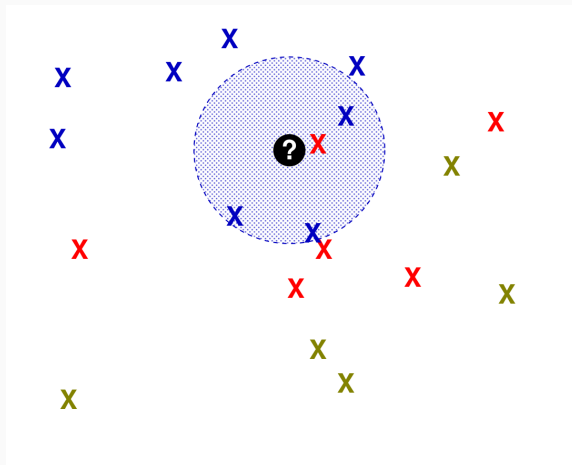
$K$  nearest neighbors =  $K$  closest stored data points

# K-Nearest Neighbors: Visualization



1 nearest neighbor = single closest stored data point

## K-Nearest Neighbors: Visualization



4 nearest neighbors = 4 closest stored data points



## Training

- Store all training examples

## Testing

- Compute **distance** of test instance to all training data points
- Find the K closest training data points (*nearest neighbors*)
- Compute **target concept** of the test instance based on labels of the training instances

## KNN Classification

- Return the most common class label among neighbors
- Example: cat vs dog images; text classification; ...

## KNN Regression

- Return the average value of among  $K$  nearest neighbors
- Example: housing price prediction;

## Four problems

1. **How to represent each data point?**
2. How to measure the distance between data points?
3. What if the neighbors disagree?
4. How to select  $K$ ?

# Feature Vectors

A data set of 6 instances (a...f) with 4 features and a label

	Outlook	Temperature	Humidity	Windy	Play
a	sunny	hot	high	FALSE	no
b	sunny	hot	high	TRUE	no
c	overcast	hot	high	FALSE	yes
d	rainy	mild	high	FALSE	yes
e	rainy	cool	normal	FALSE	yes
f	rainy	cool	normal	TRUE	no

# Feature Vectors

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We can represent each instance as a feature vector

$$\text{feature vector} = \begin{bmatrix} \text{Outlook} \\ \text{Temperature} \\ \text{Humidity} \\ \text{Windy} \end{bmatrix}$$



# Feature (or attribute) Types

## Recall, from last lecture?

### 1. Nominal

- set of values with no intrinsic ordering
- possibly *boolean*

### 2. Ordinal

- explicitly ordered

### 3. Numerical

- real-valued, often no upper bound, easily mathematical manipulatable
- vector valued



1. How to represent each data point?
2. **How to measure the distance between data points?**
3. What if the neighbors disagree?
4. How to select  $K$ ?

# Comparing Nominal Feature Vectors

First, we convert the nominal features into numeric features.

instance	features			
	color	shape	taste	size
apple	red	round	sweet	—
banana	yellow	curved	sweet	—
cherry	red	round	sweet	small

instance	features					
	red	yellow	round	sweet	curved	small
apple	1	0	1	1	0	?
banana	0	1	0	1	1	?
cherry	1	0	1	1	0	1





## Comparing Nominal Features: Hamming Distance

instance	features					
	red	yellow	round	sweet	curved	small
apple	1	0	1	1	0	?
banana	0	1	0	1	1	?
cherry	1	0	1	1	0	1

The number of differing elements in two 'strings' of equal length.

## Comparing Nominal Features: Hamming Distance

instance	features					
	red	yellow	round	sweet	curved	small
apple	1	0	1	1	0	?
banana	0	1	0	1	1	?
cherry	1	0	1	1	0	1

The number of differing elements in two 'strings' of equal length.

$$d(\text{apple}, \text{banana}) = 4$$

## Comparing Nominal Features: Simple Matching Distance

instance	features					
	red	yellow	round	sweet	curved	small
apple	1	0	1	1	0	?
banana	0	1	0	1	1	?
cherry	1	0	1	1	0	1

The number of matching features divided by the number of all features in the sample

$$d = 1 - \frac{k}{m}$$

- $d$ : distance
- $k$ : number of matching features
- $m$ : total number of features



## Comparing Nominal Features: Simple Matching Distance

instance	features					
	red	yellow	round	sweet	curved	small
apple	1	0	1	1	0	?
banana	0	1	0	1	1	?
cherry	1	0	1	1	0	1

The number of matching features divided by the number of all features in the sample

$$d = 1 - \frac{k}{m}$$

- $d$ : distance
- $k$ : number of matching features
- $m$ : total number of features

$$d(\text{apple}, \text{banana}) = 1 - \frac{2}{6} = \frac{4}{6}$$



## Comparing Nominal Feature Vectors: Jaccard Distance

instance	features					
	red	yellow	round	sweet	curved	small
apple	1	0	1	1	0	?
banana	0	1	0	1	1	?
cherry	1	0	1	1	0	1

Jaccard *similarity*  $J$ : intersection of two **sets** divided by their union.

$$d = 1 - J$$

$$= 1 - \frac{|A \cap B|}{|A \cup B|}$$

$$= 1 - \frac{|A \cap B|}{|A| + |B| - |A \cap B|}$$



## Comparing Nominal Feature Vectors: Jaccard Distance

instance	features					
	red	yellow	round	sweet	curved	small
apple	1	0	1	1	0	?
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$$d = 1 - J$$

$$= 1 - \frac{|A \cap B|}{|A \cup B|}$$

$$= 1 - \frac{|A \cap B|}{|A| + |B| - |A \cap B|}$$

$$d(\text{apple}, \text{banana}) = 1 - \frac{1}{5} = \frac{4}{5}$$



## Manhattan Distance (or: L1 distance)

- Given two instances  $a$  and  $b$ , each with a set of numerical features, e.g.,  
 $a = [2.0, 1.4, 4.6, 5.5]$   
 $b = [1.0, 2.4, 6.6, 2.5]$
- Their distance  $d$  is the sum of absolute differences of each feature

$$d(a, b) = \sum_{i=1}^m |a_i - b_i| \quad (1)$$

## Example

$$\begin{aligned} d(a, b) &= |2.0 - 1.0| + |1.4 - 2.4| + |4.6 - 6.6| + |5.5 - 2.5| \\ &= 1 + 1 + 2 + 3 \\ &= 7 \end{aligned}$$



# Comparing Numerical Feature Vectors: Euclidean Distance

## Euclidean Distance (or: L2 distance)

- Given two instances  $a$  and  $b$ , each with a set of numerical features, e.g.,  
 $a = [2.0, 1.4, 4.6, 5.5]$   
 $b = [1.0, 2.4, 6.6, 2.5]$
- Their distance  $d$  is the distance in Euclidean space (2-dimensional space). Defined as the squared root of the sum of squared differences of each feature

$$d(a, b) = \sqrt{\sum_{i=1}^m (a_i - b_i)^2} \quad (2)$$

## Example

$$\begin{aligned} d(a, b) &= \sqrt{(2.0 - 1.0)^2 + (1.4 - 2.4)^2 + (4.6 - 6.6)^2 + (5.5 - 2.5)^2} \\ &= \sqrt{1 + 1 + 4 + 9} = \sqrt{15} \\ &= 3.87 \end{aligned}$$





## Cosine Distance

- Cosine similarity = cosine of angle between two vectors (= inner product of the *normalized* vectors)
- Cosine distance  $d$ : one minus cosine similarity

$$\cos(a, b) = \frac{a \cdot b}{|a||b|} = \frac{\sum_i a_i b_i}{\sqrt{\sum_i a_i^2} \sqrt{\sum_i b_i^2}}$$

$$d(a, b) = 1 - \cos(a, b)$$

- Cosine distance is **normalized by the magnitude** of both feature vectors, i.e., we can compare instances of different magnitude
  - word counts: compare long vs short documents
  - pixels: compare high vs low resolution images



# Comparing Numerical Feature Vectors: Cosine distance

## Example

$$\cos(a, b) = \frac{a \cdot b}{|a||b|} = \frac{\sum_i a_i b_i}{\sqrt{\sum_i a_i^2} \sqrt{\sum_i b_i^2}}$$

$$d(a, b) = 1 - \cos(a, b)$$

feature	doc1	doc2	doc3
word1	200	300	50
word2	300	200	40
word3	200	100	25

$$\cos(\text{doc1}, \text{doc2}) = \frac{200 \times 300 + 300 \times 200 + 200 \times 100}{\sqrt{200^2 + 300^2 + 200^2} \sqrt{300^2 + 200^2 + 100^2}} = 0.93$$

$$d(\text{doc1}, \text{doc2}) = 0.07$$

$$\cos(\text{doc2}, \text{doc3}) = \frac{300 \times 50 + 200 \times 40 + 100 \times 25}{\sqrt{300^2 + 200^2 + 100^2} \sqrt{50^2 + 40^2 + 25^2}} = 0.99$$

$$d(\text{doc2}, \text{doc3}) = 0.01$$



# Comparing Ordinal Feature Vectors

## Normalized Ranks

- sort values, and return a rank  $r \in \{0 \dots m\}$
- map ranks to evenly spaced values between 0 and 1

$$z = \frac{r}{m}$$

- compute a distance function for numeric features (e.g., Euclidean distance)

## Example: Customer ratings

1. Sorted ratings: { -2: 😞, -1: 😞, 0: 😐, 1: 😊, 2: 😄 }  
2. Ranks: { 0, 1, 2, 3, 4 }

feature	A	B
safety	0	2
comfortable	-2	1
convenient	-1	2



# Comparing Ordinal Feature Vectors

## Normalized Ranks

- sort values, and return a rank  $r \in \{0 \dots m\}$
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## Example: Customer ratings

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2. Ranks: { 0, 1, 2, 3, 4 }

feature	A	B
safety	0	2
comfortable	-2	1
convenient	-1	2



feature	A	B
safety	2/4	4/4
comfortable	0	3/4
convenient	1/4	4/4



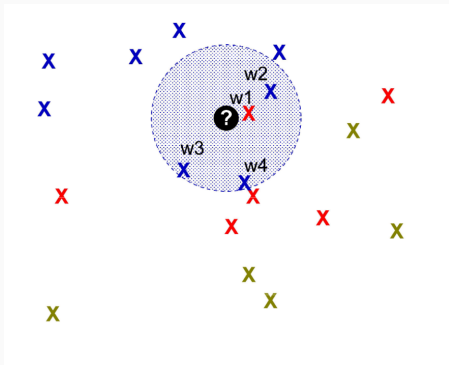
# Four problems

1. How to represent each data point?
2. How to measure the distance between data points?
3. **What if the neighbors disagree?**
4. How to select  $K$ ?



# Majority Voting

Equal weights (=majority vote)

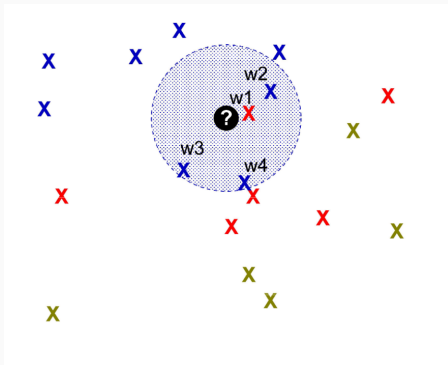


**Voting Example** ( $k=4$ )

- $w_1 = w_2 = w_3 = w_4 = 1$

# Majority Voting

Equal weights (=majority vote)



## Voting Example ( $k=4$ )

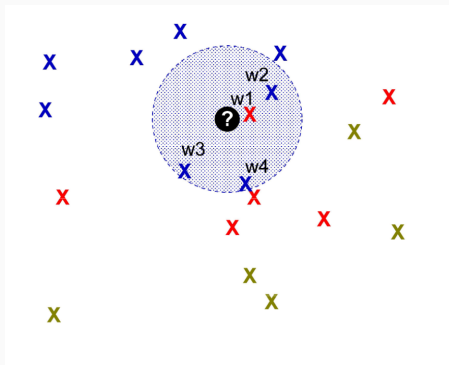
- $w_1 = w_2 = w_3 = w_4 = 1$
- red: 1  
  **blue:**  $1+1+1=3$

# Weighted KNN: Inverse Distance

Inverse Distance

$$w_j = \frac{1}{d_j + \epsilon}$$

with  $\epsilon \approx 0$ , e.g.,  $1e-10$



**Voting Example** ( $k=4$ )

- $d_1=0$ ;  $d_2=1$ ;  $d_3=d_4=1.5$
- $\epsilon = 1e-5$

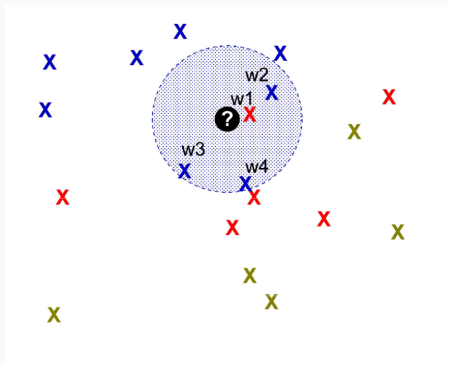


# Weighted KNN: Inverse Distance

Inverse Distance

$$w_j = \frac{1}{d_j + \epsilon}$$

with  $\epsilon \approx 0$ , e.g.,  $1e-10$



**Voting Example** ( $k=4$ )

- $d_1=0$ ;  $d_2=1$ ;  $d_3=d_4=1.5$
- $\epsilon = 1e-5$

red:  $\frac{1}{0+\epsilon} = 100000$

blue:  $\frac{1}{1+\epsilon} + \frac{1}{1.5+\epsilon} + \frac{1}{1.5+\epsilon} = 1.0 + 0.67 + 0.67 = 2.34$

# Weighted K-NN: Inverse Linear Distance

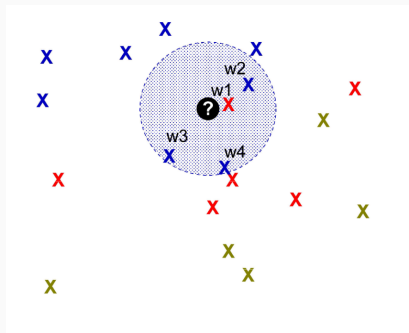
Inverse Linear distance

$$w_j = \frac{d_k - d_j}{d_k - d_1}$$

$d_1 = \min d$  among neighbors

$d_k = \max d$  among neighbors

$d_j = \text{distance of } j\text{th neighbor}$



**Voting Example** ( $k=4$ )

- $d_1=0$ ;  $d_2=1$ ;  $d_3=d_4=1.5$

# Weighted K-NN: Inverse Linear Distance

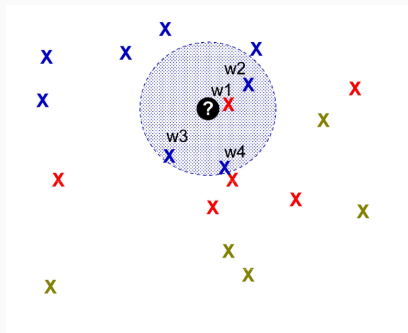
Inverse Linear distance

$$w_j = \frac{d_k - d_j}{d_k - d_1}$$

$d_1 = \min d$  among neighbors

$d_k = \max d$  among neighbors

$d_j = \text{distance of } j\text{th neighbor}$



**Voting Example (k=4)**

- $d_1=0$ ;  $d_2=1$ ;  $d_3=d_4=1.5$

red:  $\frac{1.5-0}{1.5-0} = 1$

blue:  $\frac{1.5-1}{1.5-0} + \frac{1.5-1.5}{1.5-0} + \frac{1.5-1.5}{1.5-0} = 0.3 + 0 + 0 = 0.3$

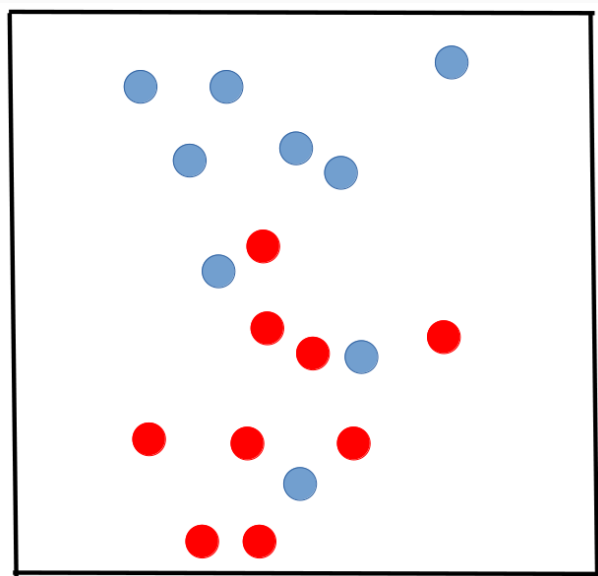


## Four problems

1. How to represent each data point?
2. How to measure the distance between data points?
3. What if the neighbors disagree?
4. **How to select  $K$ ?**

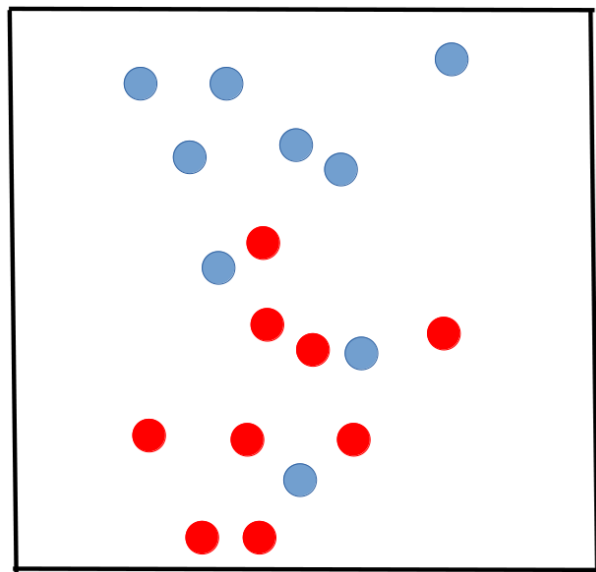
## Selecting the value of $K$

$K=1$



## Selecting the value of $K$

$K=3$



# Selecting the value of $K$

## Small $K$

- jagged decision boundary
- we capture noise
- lower classifier performance

Draw validation error:

## Large $K$

- smooth decision boundary
- danger of grouping together unrelated classes
- also: lower classifier performance!
- **what if  $K == N$ ?** ( $N$ =number of training instances)



**What if more than  $K$  neighbors have the same (smallest) distance?**

- select at random
- change the distance metric

**What if two classes are equally likely given the current neighborhood?**

- avoid an even  $K$
- random tie breaking
- include  $K + 1$ th neighbor
- use class with highest prior probability





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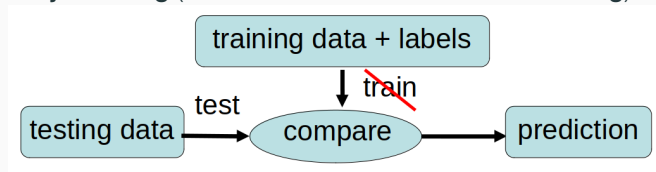
## Pros

- Intuitive and simple
- No assumptions
- Supports classification and regression
- **No training**: new data  $\rightarrow$  evolve and adapt immediately

## Cons

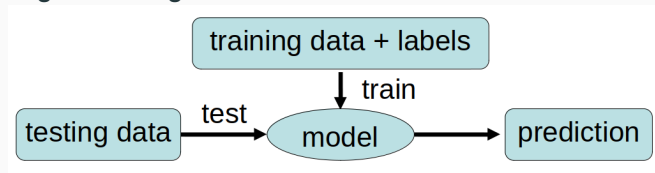
- How to decide on best distance functions?
- How to combine multiple neighbors?
- How to select  $K$ ?
- Expensive with large (or growing) data sets

## Lazy Learning (also known as **Instance-based Learning**)



- **store** the training data
- **fixed** distance function
- **fixed** prediction rule (majority, weighting, ...)
- **compare** test instances with stored instances
- **no learning**

## Eager Learning



- **train** a **model** using labelled training instances
- the model will **generalize** from seen data to unseen data
- use the model to **predict** labels for test instances
- we will look at a variety of **eager** models and their learning algorithms over the next couple of weeks

## Today... Our first machine learning algorithm

- K-nearest neighbors
- Application to classification
- Application to regression

Also: the topic of your **first assignment!**

**Next: Probabilities (recap) and probabilistic modeling**

- *Data Mining: Concepts and Techniques*, 2nd ed., Jiawei Han and Micheline Kamber, Morgan Kaufmann, 2006. Chapter 2, Chapter 9.5.
- *The elements of statistical learning*, 2nd ed., Trevor Hastie, Jerome Friedman and Robert Tibshirani. New York: Springer series in statistics, 2001. Chapter 2.3.2