7.1 (a) Let $v(t) = e^{j\omega_0 t}$ be the input to an LTI system with frequency response $H(\omega)$. Show that the output is

$$y(t) = e^{j\omega_0 t} H(\omega_0).$$

Proof: Since the system is LTI,

$$y(t) = (v * h)(t)$$

$$= \int_{-\infty}^{\infty} v(t-\lambda)\, h(\lambda)\, d\lambda$$

$$= \int_{-\infty}^{\infty} e^{j\omega_0 (t-\lambda)}\, h(\lambda)\, d\lambda$$

$$= \int_{-\infty}^{\infty} e^{j\omega_0 t}\, e^{-j\omega_0 \lambda}\, h(\lambda)\, d\lambda$$

$$= e^{j\omega_0 t} \int_{-\infty}^{\infty} h(\lambda) e^{-j\omega_0 \lambda}\, d\lambda$$

$$= e^{j\omega_0 t} H(\omega_0).$$

as required.

72 We have $H(\omega) = \dfrac{1}{j\omega + 1}$

(a) For the input $v_1(t) = \cos(t)$, $w_0 = 1$

we have $A_k^v = \begin{cases} 1, & k=1 \\ 0, & \text{otherwise} \end{cases}$

and $\theta_k^v = 0$ for all k.

Let y_1 be the output of the system. Then

$$\boxed{A_k^y = A_k^v \, | H(k w_0) |}$$

$$\begin{aligned} \text{So } A_1^y &= |H(w_0)| \\ &= |H(1)| \\ &= \left| \frac{1}{j+1} \right| \\ &= \frac{1}{\sqrt{2}} \end{aligned}$$

$$\boxed{\theta_k^y = \theta_k^v + \angle H(k w_0)}$$

$$\begin{aligned} \text{So } \theta_1^y &= 0 + \angle H(1) \\ &= 0 + \angle \left(\frac{1}{j+1} \right) \\ &= -\angle (j+1) \\ &= -\pi/4 \end{aligned}$$

$$\begin{aligned} \text{So } y_1(t) &= A_1^y \cos(t + \theta_1^y) \\ &= \frac{1}{\sqrt{2}} \cos\left(t - \frac{\pi}{4}\right). \end{aligned}$$

(b) For $v_2(t) = \cos\left(t + \frac{\pi}{4}\right)$
$= v_1(t + \pi/4)$

by time-invariance

$$\begin{aligned} y_2(t) &= \text{output from } v_2(t) \\ &= y_1(t + \pi/4) \\ &= \frac{1}{\sqrt{2}} \cos(t). \end{aligned}$$

(73) An LTI continuous time system has frequency response

$$H(\omega) = \frac{j\omega}{j\omega + 2}.$$

(a) Then $|H(\omega)| = \dfrac{|j\omega|}{|2+j\omega|}$

$$= \frac{|\omega|}{\sqrt{4+\omega^2}}$$

and

$$\angle H(\omega) = \angle(j\omega) - \angle(j\omega+2)$$

Case (1): Assume $\omega > 0$. Then

$$\angle j\omega = \frac{\pi}{2}$$

$$\angle(2+j\omega) = \tan^{-1}\left(\frac{\omega}{2}\right)$$

So $\angle H(\omega) = \dfrac{\pi}{2} - \tan^{-1}\left(\dfrac{\omega}{2}\right).$

Case (2): Assume $\omega < 0$: Then $\angle(j\omega) = -\dfrac{\pi}{2}$

$$\angle(2+j\omega) = \tan^{-1}\left(\frac{\omega}{2}\right)$$

So $\angle H(\omega) = -\dfrac{\pi}{2} - \tan^{-1}\left(\dfrac{\omega}{2}\right).$

So $|H(\omega)| = \begin{cases} \dfrac{\omega}{\sqrt{4+\omega^2}}, & \omega > 0 \\ \dfrac{-\omega}{\sqrt{4+\omega^2}}, & \omega < 0 \end{cases}$

$$\angle H(\omega) = \begin{cases} \dfrac{\pi}{2} - \tan^{-1}\left(\dfrac{\omega}{2}\right), & \omega > 0 \\ -\dfrac{\pi}{2} - \tan^{-1}\left(\dfrac{\omega}{2}\right), & \omega < 0. \end{cases}$$

(73) Plot graphs (without MATLAB)

For $|H(\omega)|$ we have

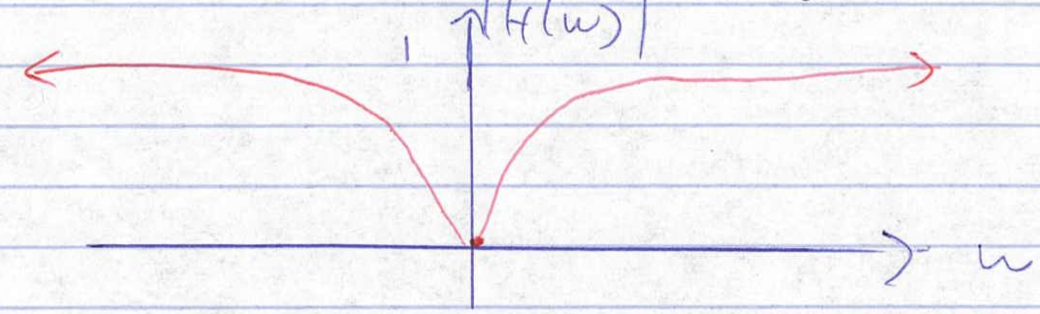
(i) $|H(0)| = 0$

(ii) even function as $|\omega|$ is even,

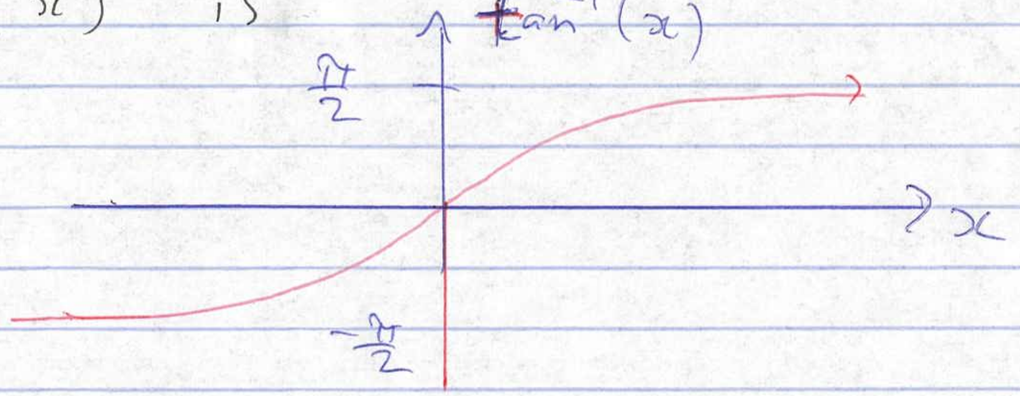
(iii) As $\omega \to \infty$, $|H(\omega)| \to \frac{\omega}{\sqrt{\omega^2}} = 1$

As $\omega \to -\infty$, $|H(\omega)| \to \frac{-\omega}{\sqrt{\omega^2}} = 1$

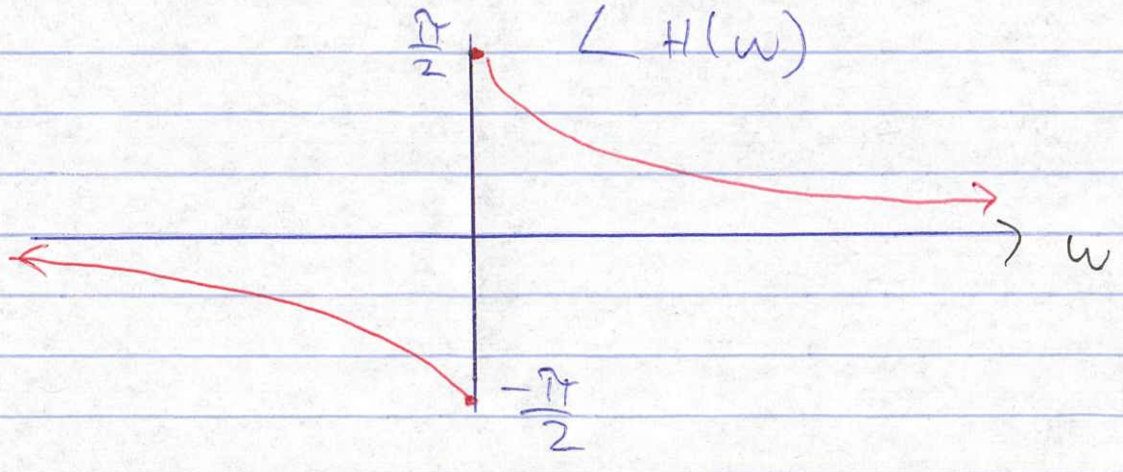
Plot:



For $\angle H(\omega)$, we know that graph of $\tan^{-1}(x)$ is



So plot of $\angle H(\omega)$ is



Consider the RL circuit

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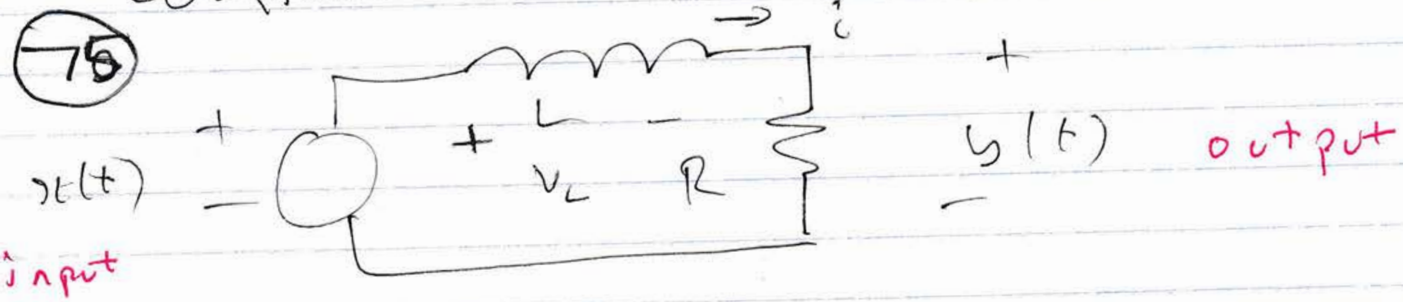

with $R = 5\,\Omega$.

(a) Find the frequency response of the circuit.

From KVL: $x(t) = L\frac{di}{dt} + Ri(t)$

OHMS LAW: $y(t) = R\,i$

$\Rightarrow \frac{dy}{dt} = R\frac{di}{dt}$.

So $\boxed{x(t) = \frac{L}{R}\frac{dy}{dt} + y(t)}$.

Assume $x(t) = e^{j\omega t}$

then $y(t) = H(\omega)e^{j\omega t}$

$\frac{dy}{dt} = j\omega H(\omega)e^{j\omega t}$

Substitute:

$$e^{j\omega t} = \frac{L}{R}j\omega H(\omega)e^{j\omega t} + H(\omega)e^{j\omega t}$$

rearranging gives

$$e^{j\omega t} = e^{j\omega t}\left(\frac{L}{R}j\omega + 1\right)H(\omega)$$

$$\Rightarrow H(\omega) = \frac{1}{1 + \frac{jL\omega}{R}} = \frac{5}{5 + jL\omega}$$

using $R = 5\,\Omega$.

(5)

(b) Suppose the input is

$$x(t) = 10 \, |\sin(377t)| \quad \text{for } 0 \le t < \frac{\pi}{377}$$

and $x(t) = x(t + \pi/377)$.

Find the dc term in the complex Fourier series for $y(t)$.

Solution : Here $T = \pi/377$

and $w_0 = \frac{2\pi}{T} = 754$ rad/s

Also

$$c_0^x = \frac{1}{T}\int_0^T |\sin(377t)| \, dt$$

$$= \frac{20}{\pi}.$$

So the dc (constant) term in the output $y(t)$ is

$$c_0^y = H(0) \, c_0^x$$

$$= \left(\frac{5}{5+0}\right)\left(\frac{20}{\pi}\right)$$

$$= \frac{20}{\pi}.$$

Note this does not depend on L.

(77) Consider the 2-point MAV filter

$$y[n] = \frac{1}{2}\left[v[n] + v[n-1]\right]$$

Obtain the frequency response and show that it has linear phase for $0 < \Omega < \pi$.

Solution: We saw in Lectures that

(a)
$$H(\Omega) = \frac{1}{2}\left[\frac{1-e^{-j2\Omega}}{1-e^{-j\Omega}}\right]$$

$$= \left(\frac{1}{2}\right)\left[\frac{(e^{-j\Omega})\left(\frac{1}{2j}\right)\left[e^{j\Omega}-e^{-j\Omega}\right]}{(e^{-j\Omega/2})\left(\frac{1}{2j}\right)\left[e^{j\Omega/2}-e^{-j\Omega/2}\right]}\right]$$

$$= \left(\frac{1}{2}\right)\frac{\sin(\Omega)}{\sin(\Omega/2)}\, e^{-j\Omega/2}$$

(b)

Assume $0 \le \Omega \le \pi$:

Then
$$\angle H(\Omega) = \angle \frac{\sin(\Omega)}{\sin(\Omega/2)} + \angle e^{-j\Omega/2}$$

$$= \angle (a) + \angle e^{-j\Omega/2} \quad \text{for some } a > 0$$

$$= -\frac{\Omega}{2}.$$

Assume: $\pi \le \Omega < 2\pi$.

Then
$$\angle H(\Omega) = \angle \frac{\sin(\Omega)}{\sin(\Omega/2)} + \angle e^{-j\Omega/2}$$

$$= \angle (-a) + \angle e^{-j\Omega/2} \quad \text{for some } a > 0$$

affine, not linear in Ω $\longrightarrow$
$$= \pi - \frac{\Omega}{2}.$$

Thus for $0 < \Omega < \pi$

$$\angle H(\Omega) = -\Omega t_d, \quad \text{where } t_d = \frac{1}{2}$$

So it has linear phase for these Ω.

(80) We know that

$$H(\Omega) = \frac{3\sin(1.5\Omega)}{(3-e^{-j\Omega})\sin(\Omega/2)}$$

$$= H_1(\Omega)\, H_2(\Omega)$$

where $H_1(\Omega) = FT(h_1[n])$

$$= FT\left(\left(\frac{1}{3}\right)^n u[n]\right)$$

$$= \frac{1}{1-\left(\frac{1}{3}\right)e^{-j\Omega}}$$

$$= \frac{3}{3-e^{-j\Omega}}$$

Hence $H_2(\Omega) = \dfrac{\sin(1.5\Omega)}{\sin(\Omega/2)}$.

Recall that

$$p_L[n] \longleftrightarrow \frac{\sin\left[\left(q+\frac{1}{2}\right)\Omega\right]}{\sin(\Omega/2)}$$

We see that here $q = 1$

so $L = 2q+1$

$= 3$

Hence $h_2[n] = p_3[n]$.