

COMP90038 Algorithms and Complexity

More Divide-and-Conquer Algorithms

Douglas Pires

Lecture 12

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Divide and Conquer

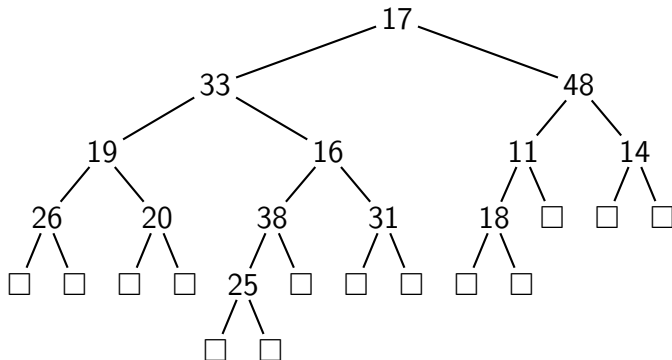
In the last lecture we studied the archetypal divide-and-conquer sorting algorithms: mergesort and quicksort.

We also introduced the powerful **master theorem**, providing solutions to a large class of recurrence relations, for free.

Now we shall look at tree traversal, and then a final example of divide-and-conquer, giving a better solution to the closest-pair problem.

Binary Trees Again

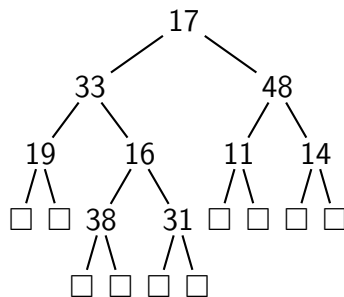
An example of a binary tree, with empty subtrees marked with \square :



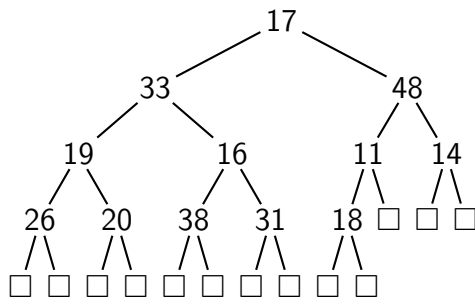
This tree has **height** 4, the empty tree having height -1.

Binary Tree Concepts

Special trees have their **external nodes** \square only at level h and $h+1$ for some h :



A **full** binary tree: Each node has 0 or 2 children.



A **complete** tree: Each level filled left to right.

Binary Tree Concepts

A non-empty tree T has a **root** T_{root} , a **left subtree** T_{left} , and a **right subtree** T_{right} .

Recursion is the natural way of calculating the **height**:

```
function HEIGHT( $T$ )  
  if  $T$  is empty then  
    return  $-1$   
  else  
    return  $\max(\text{HEIGHT}(T_{left}), \text{HEIGHT}(T_{right})) + 1$ 
```

Binary Tree Concepts

It is not hard to prove that the number x of external nodes \square is always one greater than the number n of internal nodes.

The function `HEIGHT` makes a tree comparison (empty or non-empty?) per node (internal and external), so altogether $2n + 1$ comparisons.

Binary Tree Traversal

Preorder traversal visits the root, then the left subtree, and finally the right subtree.

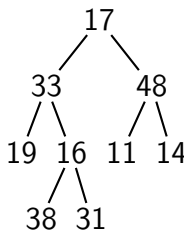
Inorder traversal visits the left subtree, then the root, and finally the right subtree.

Postorder traversal visits the left subtree, the right subtree, and finally the root.

Level-order traversal visits the nodes, level by level, starting from the root.

Binary Tree Traversal: Preorder

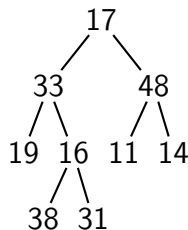
```
function PREORDERTRAVERSE( $T$ )  
  if  $T$  is non-empty then  
    visit  $T_{root}$   
    PREORDERTRAVERSE( $T_{left}$ )  
    PREORDERTRAVERSE( $T_{right}$ )
```



Visit order for the example: 17, 33, 19, 16, 38, 31, 48, 11, 14.

Binary Tree Traversal: Inorder

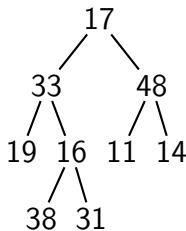
```
function INORDERTRAVERSE( $T$ )  
  if  $T$  is non-empty then  
    INORDERTRAVERSE( $T_{left}$ )  
    visit  $T_{root}$   
    INORDERTRAVERSE( $T_{right}$ )
```



Visit order for the example: 19, 33, 38, 16, 31, 17, 11, 48, 14.

Binary Tree Traversal: Postorder

```
function POSTORDERTRAVERSE( $T$ )  
  if  $T$  is non-empty then  
    POSTORDERTRAVERSE( $T_{left}$ )  
    POSTORDERTRAVERSE( $T_{right}$ )  
    visit  $T_{root}$ 
```



Visit order for the example: 19, 38, 31, 16, 33, 11, 14, 48, 17.

Preorder Traversal Using a Stack

We could also implement preorder traversal of T by maintaining a **stack** explicitly.

```
push( $T$ )  
while the stack is non-empty do  
     $T \leftarrow pop$   
    visit  $T_{root}$   
    if  $T_{right}$  is non-empty then  
        push( $T_{right}$ )  
    if  $T_{left}$  is non-empty then  
        push( $T_{left}$ )
```

In an implementation, the elements placed on the stack would not be whole trees, but pointers to the corresponding internal nodes.

Tree Traversal Using a Queue: Level-Order

Level-order traversal results if we replace the stack with a **queue**.

inject(T)

while the queue is non-empty **do**

$T \leftarrow \text{eject}$

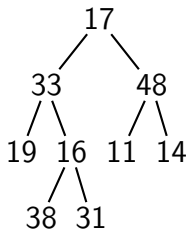
visit T_{root}

if T_{left} is non-empty **then**

inject(T_{left})

if T_{right} is non-empty **then**

inject(T_{right})



Visit order for the example: 17, 33, 48, 19, 16, 11, 14, 38, 31.

The Closest Pair Problem Revisited

In Lecture 5 we gave a brute-force algorithm for the closest pair problem: Given n points in the Cartesian plane, find a pair with minimal distance.

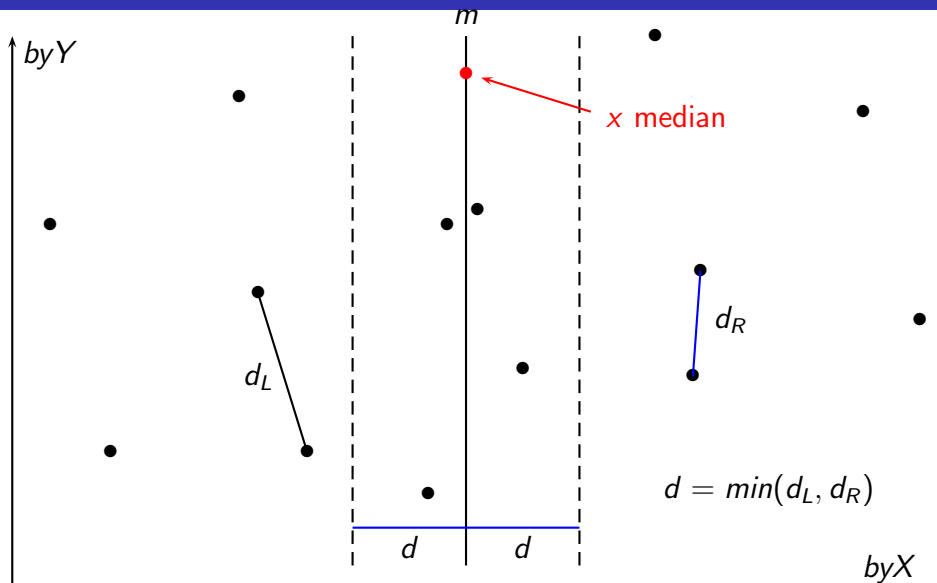
The brute-force method had complexity $\Theta(n^2)$. We can use divide-and-conquer to do better, namely $\Theta(n \log n)$.

First, sort the points by x value and store the result in array *byX*.

Also sort the points by y value and store the result in array *byY*.

Now we can identify the x median, and recursively process the set P_L of points with lower x values, as well as the set P_R with higher x values.

The Closest Pair Problem Revisited



The Closest Pair Problem Revisited

The recursive calls will identify d_L , the shortest distance for pairs in P_L , and d_R , the shortest distance for pairs in P_R .

Let m be the x median and let $d = \min(d_L, d_R)$. This d is a candidate for the smallest distance.

But d may not be the global minimum—there could be some close pair whose points are on opposite sides of the median line $x = m$.

For candidates that may improve on d we only need to look at those in the band $m - d \leq x \leq m + d$.

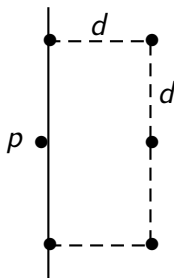
So pick out, from array byY , each point p with x -coordinate between $m - d$ and $m + d$, and keep these in array S .

For each point in S , consider just its “close” neighbours.

The Closest Pair Problem Revisited

The following calculates the smallest distance and leaves the (square of the) result in *minsq*.

It can be shown that the while loop can execute **at most 5 times for each *i* value**—see diagram.



$minsq \leftarrow d^2$

copy all points of *byY* with $|x - m| < d$ to array *S*

$k \leftarrow |S|$

for $i \leftarrow 0$ to $k - 2$ **do**

$j \leftarrow i + 1$

while $j \leq k - 1$ and $(S[j].y - S[i].y)^2 < minsq$ **do**

$minsq \leftarrow \min(minsq, (S[j].x - S[i].x)^2 + (S[j].y - S[i].y)^2)$

$j \leftarrow j + 1$

Coming Up Next

More sorting ...