THE UNIVERSITY OF MELBOURNE

Semester 2 Assessment
MID-SEMESTER TEST 2017

Department of Electrical and Electronic Engineering
ELEN 30012 SIGNALS AND SYSTEMS

Time allowed: 50 minutes
This paper has 4 pages including the 1-page Formulae Sheet

Authorised materials:

Only the following calculators may be used:

- Casio FX82 (any suffix)
- Casio FX100 (any suffix)

Instructions to invigilators:
Script books are to be collected at the end of the Test.

Instruction to students:

Attempt **ALL** questions.

The questions carry weight in proportion to the marks in brackets after the question numbers. These marks total 50 marks. You must show your work in order to receive credit!

Answer all questions and show all working in the script book provided.

Question 1 (15 marks)

Consider the continuous-time system with input-output trajectories (v, y) defined by

$$y(t) = \int_0^t \lambda v^2(\lambda)\ d\lambda$$

(a) Is the system linear? Justify your answer with a proof or counterexample.

(b) Is the system time-invariant? Justify your answer with a proof or counterexample.

(c) Is the system causal? Give a reason for your answer.

(d) Does the system have memory? Give a reason for your answer.

Question 2 (15 marks)

A bank loan account with monthly repayments can be described as a discrete-time system with difference equation

$$y[n+1] = (1+r)y[n] + v[n+1], \qquad\qquad y[-1] = \frac{y_0}{1+r}.$$

Here input value $v[n] > 0$ is the repayment at the beginning of month n, and the output of the system at time index n is the account balance $y[n]$ at the beginning of month n. The monthly interest rate is $r > 0$, and y_0 represents the balance of the loan account just prior to the repayment at month $n = 0$.

(a) Assume the loan account receives repayments of v, defined by

$$v[n] = \begin{cases} 0 & n < 0, \\ 1 & 0 \leqslant n < N_0, \\ 0 & N_0 \leqslant n. \end{cases}$$

Solve the difference equation to find the monthly account balance for $0 \leq n < N_0$. Simplify your answer by using the sum of a geometric series:

$$\sum_{n=0}^{N-1} \alpha^n = \frac{1-\alpha^N}{1-\alpha}$$

(b) Hence find the account balance for $n \geq N_0$.

Question 3 (8 marks)

Let $N \geq 1$ be any integer. Use Mathematical Induction to prove that for any $\alpha \in \mathbf{R}$,

$$\sum_{n=0}^{N-1} \alpha^n = \frac{1-\alpha^N}{1-\alpha}$$

Question 4 (12 marks)

(a) Find the inverse Fourier transform of

$$Y(\omega) = \cos(4\omega)$$

(b) Hence find the inverse Fourier transform of

$$G(\omega) = \cos(2\omega)\cos(4\omega)$$

You may use the following Fourier transform pairs

$$\begin{aligned} 1 &\longleftrightarrow 2\pi\delta(\omega) \\ X(t) &\longleftrightarrow 2\pi x(-\omega) \\ x(t)\cos(\omega_0 t) &\longleftrightarrow \frac{1}{2}[X(\omega+\omega_0) + X(\omega-\omega_0)] \\ x \star y &\longleftrightarrow X(\omega)Y(\omega) \end{aligned}$$

ELEN 30012 Signals and Systems
Formulae for the Mid-Test

1. **Trigonometric formulae**

$$
\begin{aligned}
\sin(\theta \pm \alpha) &= \sin(\theta)\cos(\alpha) \pm \cos(\theta)\sin(\alpha) \\
\cos(\theta \pm \alpha) &= \cos(\theta)\cos(\alpha) \mp \sin(\theta)\sin(\alpha) \\
e^{j\theta} &= \cos(\theta) + j\sin(\theta) \\
\cos(\theta) &= \frac{1}{2}\left(e^{j\theta} + e^{-j\theta}\right) \\
\sin(\theta) &= \frac{1}{2j}\left(e^{j\theta} - e^{-j\theta}\right)
\end{aligned}
$$

2. **Convolution**

$$(x \star v)[n] = \sum_{i=-\infty}^{\infty} x[i]v[n-i], \qquad (x \star v)(t) = \int_{-\infty}^{\infty} x(\lambda)v(t-\lambda)\, d\lambda$$

3. **Integration Methods**

$$\int_a^b v(x)u'(x)\,dx = [u(x)v(x)]_a^b - \int_a^b v'(x)u(x)\,dx, \qquad \int_a^b f(u(x))u'(x)\,dx = \int_{u(a)}^{u(b)} f(u)\,du$$

4. **Fourier Series Formulae**

$$
\begin{aligned}
x(t) &= a_0 + \sum_{k=1}^{\infty}(a_k\cos(k\omega_0 t) + b_k\sin(k\omega_0 t)), \qquad \omega_0 = \frac{2\pi}{T} \\
& a_0 = \frac{1}{T}\int_0^T x(t)\,dt,\ a_k = \frac{2}{T}\int_0^T x(t)\cos(k\omega_0 t)\,dt,\ b_k = \frac{2}{T}\int_0^T x(t)\sin(k\omega_0 t)\,dt \\
x(t) &= a_0 + \sum_{k=1}^{\infty} A_k\cos(k\omega_0 t + \theta_k), \qquad A_k = \sqrt{a_k^2 + b_k^2}, \qquad \theta_k = \begin{cases} \tan^{-1}\left(\frac{-b_k}{a_k}\right), & a_k \geq 0 \\ \pi + \tan^{-1}\left(\frac{-b_k}{a_k}\right), & a_k < 0 \end{cases} \\
x(t) &= \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t}, \qquad c_k = \frac{1}{T}\int_0^T x(t)e^{-jk\omega_0 t}\,dt, \qquad k = 0, \pm1, \pm2,\ \ldots
\end{aligned}
$$

$$
\begin{aligned}
c_0 &= a_0, \qquad c_k = \frac{1}{2}(a_k - jb_k), \qquad c_{-k} = \frac{1}{2}(a_k + jb_k),\ k = 1, 2,\ \ldots \\
a_0 &= c_0, \qquad a_k = c_k + c_{-k}, \qquad b_k = j(c_k - c_{-k}),\ k = 1, 2,\ \ldots
\end{aligned}
$$

$$|c_k| = \frac{1}{2}A_k, \qquad \angle c_k = \theta_k,\ k = 1, 2,\ \ldots$$

5. **Transforms**

$$
\begin{aligned}
X(\omega) &= \int_{-\infty}^{\infty} x(t)e^{-j\omega t}\,dt, \\
x(t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)e^{j\omega t}\,d\omega
\end{aligned}
$$