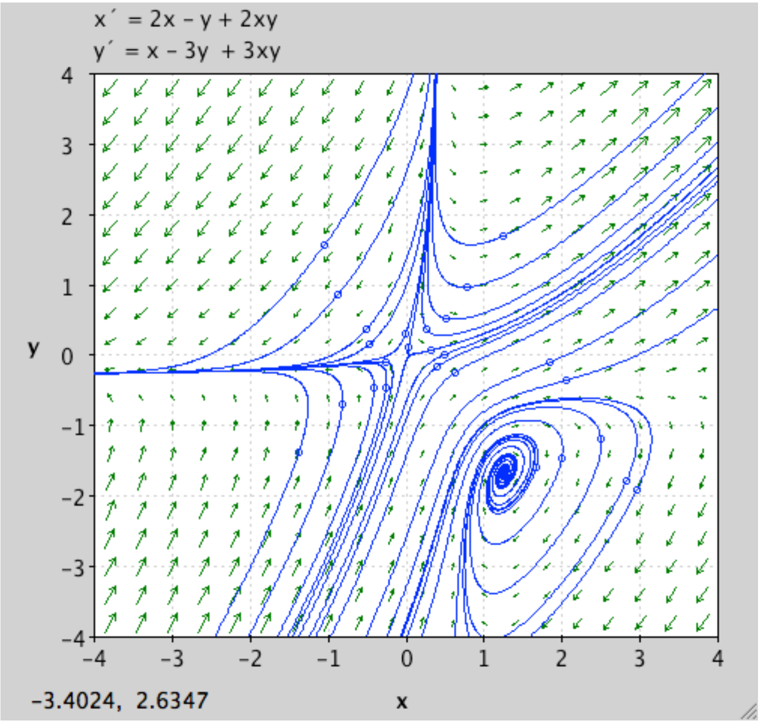
The University of Melbourne

School of Mathematics and Statistics

MAST20029
Engineering Mathematics

Semester 1, 2020



STUDENT NAME:

MAST20029 Engineering Mathematics
Semester 1 2020
Subject Organisation

MAST20029 Engineering Mathematics is a core mathematics subject that prepares students for further studies in all branches of Engineering. This subject is intended only for students pursuing an Engineering Systems major or who are enrolled in a Master of Engineering degree, who do not wish to take any further study in Mathematics and Statistics or Physics.

Students who want to supplement their Engineering Systems major with further study in Mathematics and Statistics or Physics, should seek course advice before enrolling in MAST20029. In particular, students who want to specialise in Applied Mathematics within a Mathematics and Statistics Major, should take MAST20009 Vector Calculus, MAST20026 Real Analysis and MAST20030 Differential Equations instead of MAST20029 Engineering Mathematics.

Syllabus

This subject introduces important mathematical methods required in engineering such as manipulating vector differential operators, computing multiple integrals and using integral theorems. A range of ordinary and partial differential equations are solved by a variety of methods and their solution behaviour is interpreted. The subject also introduces sequences and series including the concepts of convergence and divergence.

Topics include: Vector calculus, including Gauss' and Stokes' Theorems; sequences and series; Fourier series, Laplace transforms, including convolution; systems of homogeneous ordinary differential equations, including phase plane and linearisation for nonlinear systems; second order partial differential equations by separation of variables.

At the completion of this subject, students should be able to

- manipulate vector differential operators
- determine convergence and divergence of sequences and series
- solve ordinary differential equations using Laplace transforms
- sketch phase plane portraits for linear and nonlinear systems of ordinary differential equations
- represent suitable functions using Fourier series
- solve second order partial differential equations using separation of variables

Credit Exclusions

Students may only gain credit for one of

- MAST20029 Engineering Mathematics
- MAST20030 Differential Equations
- MAST30023 Differential Equations for Engineers (prior to 2012)
- 620-232 Mathematical Methods (prior to 2010)
- 620-234 Mathematical Methods Advanced (prior to 2009)
- 431-202 Engineering Analysis B (prior to 2009)

Note

- Students who have completed MAST20009 Vector Calculus may not enrol in MAST20029 Engineering Mathematics for credit.
- Concurrent enrolment in both MAST20029 Engineering Mathematics and MAST20009 Vector Calculus is not permitted.

Pre-requisites

One of

- MAST10006 Calculus 2
- MAST10009 Accelerated Mathematics 2
- MAST10019 Calculus Extension Studies

and one of

- MAST10007 Linear Algebra
- MAST10008 Accelerated Mathematics 1
- MAST10013 UMEP Maths for High Achieving Students
- MAST10018 Linear Algebra Extension Studies

Or

- 431-201 Engineering Analysis A (prior to 2010)

Or

- Enrolment in the Master of Engineering

Classes

The subject MAST20029 Engineering Mathematics has

- three one hour lectures per week;
- a one hour practice class per week.

Lectures and practice classes start on the first day of semester. Details of your lecture times and practice class time are given on your personal timetable in the Student Portal.

Lecturers will be available for individual consultation starting in the second week of semester. Mathematics and Statistics Learning Centre staff will be available for individual consultation starting in the third week of semester. The consultation roster will be put on the MAST20029 website.

Lecture Streams

Students are expected to attend the lectures. There are two lecture streams. Please attend the stream to which you are allocated.

- Stream 1

  Monday at 9am in Rivett Theatre, Redmond Barry Building;

  Wednesday at 9am in Rivett Theatre, Redmond Barry Building;

  Friday at 9am in Lyle Theatre, Redmond Barry Building;

  Lecturer: Associate Professor Marcus Brazil (Subject Coordinator), Room 5.11, Electrical and Electronic Engineering.

- Stream 2

  Monday at 2.15pm in Theatre A (G06), Elisabeth Murdoch Building;

  Wednesday at 2.15pm in Theatre A (G06), Elisabeth Murdoch Building;

  Friday at 2.15pm in Theatre A (G06), Elisabeth Murdoch Building.

  Lecturer: Dr Christine Mangelsdorf, Room G49, Peter Hall Building.

Textbooks and Resources

The textbook for MAST20029 is:

- E Kreyszig, *Advanced Engineering Mathematics*, 10th Edition, Wiley, 2011.

The textbook is recommended but not compulsory. It can be purchased as an e-book from Wiley Direct at `www.wileydirect.com.au`. The textbook can be borrowed from the library located in the Eastern Resource Centre. It is fine to use or refer to earlier editions of the textbook.

The Kreyszig textbook covers all topics in MAST20029 except for sequences and series.

There are many first year calculus textbooks in the ERC library that can be used as a reference for the sequence and series section of Engineering Mathematics.

Lecture Notes

All students are required to have a copy of the MAST20029 Engineering Mathematics Student Lecture Notes, which can be downloaded from the MAST20029 website.

These notes contain the theory, diagrams, and statement of the questions to be covered in lectures. Students are expected to bring these partial lecture notes to all lectures, and fill in the working of examples in the gaps provided. This can be done on a tablet or on a printed version of the notes.

Practice Class Sheets

A practice class question sheet to be worked on during the practice class will be issued at the beginning of each practice class. Students are expected to attempt the questions in groups of 2 or 3 at the whiteboards. Full solutions to the questions will be provided at the end of the practice class.

The practice class in the first week of semester covers revision material from first year mathematics subjects of most relevance to Engineering Mathematics.

You should aim to complete all the questions on the practice class sheets during semester.

Problem Sheets

All students are required to have a copy of the MAST20029 Engineering Mathematics Problem Sheet Booklet, which can be downloaded from the MAST20029 website.

This problem booklet is for you to work on at home to prepare for your practice classes, practice key concepts, and to revise for the mid-semester test and final exam. There are six problem sheets with answers in this booklet corresponding to the six major topics covered in lectures. At the end of each week you will be advised which questions should be attempted before attending your next practice class.

Whilst working through the problem sheets, make a list of any concepts or topics you are having difficulty with and ask for help during the individual consultation sessions. Tutors will not be discussing the specific questions in the problem booklet in the practice classes.

You should aim to complete all the questions in this problem booklet during semester.

Website

All material to do with the assignments, the mid-semester test, the practice classes, the consultation roster and other announcements will be available from the address:

`www.lms.unimelb.edu.au`

Expectations

In MAST20029 Engineering Mathematics you are expected to:

- Attend all lectures, and take notes during lectures.
- Attend all practice classes, participate in group work in practice classes, and complete all practice class exercises.

- Work through the problem booklet outside of class in your own time. You should try to keep up-to-date with the problem booklet questions, and aim to have attempted all questions from the problem booklet before the exam.
- Check the announcements and read the weekly modules on the LMS every week to make sure you are keeping up-to-date with the subject and are not missing any important subject information.
- Seek help when you need it during consultation sessions.

In total, you are expected to dedicate at least 140 hours to this subject, including classes, in other words, at least 10 hours per week.

MATLAB

Students are expected to use the software package MATLAB throughout the subject Engineering Mathematics to complete questions on problem sheets and assignments.

Detailed information about MATLAB is provided on pages xvi-xxiv of this document.

Assessment

The assessment is composed of three parts:

• A three-hour exam worth 70% at the end of the semester.

• Three assignments worth 5% each, due as follows:

(1) 4.00pm on Monday 30th March;

(2) 4.00pm on Monday 4th May;

(3) 4.00pm on Monday 25th May.

• A mid-semester test held during your allocated lecture time on Wednesday 22nd April, worth a total of 15%.

Hurdle requirement: *Students must pass the assessment during semester to pass the subject. That is, students must obtain a mark of at least 15% out of 30% for the combined assignment and mid-semester test mark to pass the subject.*

Assignments

• The assignments will be handed out in lectures one week before the due date. The assignments will be put on the MAST20029 website after they have been given out in lectures.

• You must attach a completed question cover sheet to the front page of every assignment.

• Assignments must be neatly handwritten in blue or black pen. Diagrams can be drawn in pencil.

• Your assignment should be placed into the appropriate assignment box, ground floor, Peter Hall building. Your tutor may be taking other subjects, so please ensure that you place the assignment in your tutor's box for MAST20029 Engineering Mathematics. A list of classes, tutors and boxes is on the noticeboard above the assignment boxes and on the MAST20029 website.

• *Extensions*

Students with medical certificates or other appropriate supporting documentation can apply to Marcus Brazil for an extension of up to 3 days after the assignment deadline by sending him a scanned copy of the documentation via email. The assignment should be submitted to the dedicated Engineering Mathematics Late assignment box.

Do not use the Student Portal to apply for special consideration for the assignments; this online application is for the final Engineering Maths exam only.

• *Late assignments*

The following applies to an assignment submitted late, where no extension has been sought and granted. Late assignments should be submitted to the dedicated Engineering Mathematics Late assignment box.

Between one day late and three days late: A mark penalty of 20% of the assignment total will be deducted from the student's result for each day the assignment is submitted late. For example, if the assignment deadline is 4.00pm on Monday, then an assignment submitted after 4.00pm on Monday and before 4.00pm on Wednesday is one day late, so 20% of the assignment total will be deducted.

More than three days late: Assignment is not accepted and a mark of zero is awarded for the assignment.

• You must complete the online plagiarism declaration on the MAST20029 website before submitting assignment 1.

• The plagiarism declaration will apply to all assignments in MAST20029 during the semester.

• Marked assignments will not be returned to students until the online plagiarism declaration is completed.

Mid-Semester Test

• *Special consideration*

Students with medical certificates or other appropriate supporting documentation can apply to Marcus Brazil for special consideration for the mid-semester test by sending him a scanned copy of the documentation via email. You must request special consideration no later than 3 days after the date of the mid-semester test. You then have a further 5 days in which to provide a medical certificate. Applications lodged after these time limits will not be accepted.

Do not use the Student Portal to apply for special consideration for the mid-semester test; this online application is for the final Engineering Maths exam only.

• The following applies to students who have applied for special consideration or who have missed the mid-semester test for any reason.

Students who sat the test but were sick during test will be allowed to sit the alternative sitting of the mid-semester test. Any student who sits the original test as well as the alternative test will be given the mark from the alternative test, even if it is lower than their mark in the original test.

Students unable to sit the test due to illness or any other acceptable reason will be allowed to sit the alternative sitting of the mid-semester test.

Students who did not sit the test and did not give any reasonable excuse will get a mark of zero for the mid-semester test.

All students wishing to take the alternative sitting of the mid-semester test must contact Marcus Brazil no later than 3 days after the date of the original test.

Special Consideration for Exam and Whole Subject

If something major goes wrong during semester or you are sick during the examination period, you should consider applying for *Special Consideration* through the Student Portal. You must submit your online special consideration application no later than 4 days after the date of the final exam in MAST20029 Engineering Mathematics. You will also need to submit the completed Health Professional Report (HPR) Form with your online application. The HPR Form can only be completed by the professional using the form provided.

For more details see the Special Consideration menu item on the website:

`http://ask.unimelb.edu.au/app/home`

Calculators, Formula Sheets and Dictionaries

Students are not permitted to use calculators, computers, dictionaries or mathomats in the mid-semester test or end of semester exam.

Students are not permitted to take formula sheets, notes or text books to the mid-semester test or end of semester exam. The formula sheet on pages xi to xv of this booklet will be provided in the mid-semester test and the end of semester exam.

Assessment in this subject concentrates on the testing of concepts and the ability to conduct procedures in simple cases. There is no formal requirement to possess a calculator for this subject. Nonetheless, there are some questions on the problem sheets for which calculator usage is appropriate. If you have a calculator or an equivalent app on your phone or laptop, then you will find it useful occasionally.

Getting Help

The first source of help is the person beside you in lectures and practice classes, who is doing the same problems as you are and having similar but perhaps not exactly the same difficulties. Remember though, that fellow students have no obligation to help you, nor you to help them. Forming a small study group of two to four people is an excellent way of sharing knowledge.

Lecturers and Mathematics and Statistics Learning Centre staff have consultation hours when they will help you on an individual basis. Attendance is on a voluntary basis. Details will be provided on the MAST20029 web site.

A Tutor on Duty service operates from 12noon to 2pm on weekdays during semester in *mathSpace*, which is located on the ground floor of the Peter Hall building. Attendance is on a voluntary basis. The tutors on duty can help you with revision of mathematical concepts from school such as

- Basic algebra and index laws
- Basic differentiation and integration
- Functions and inverses
- Trigonometric functions and their inverses
- Logarithms and exponentials
- Equations of ellipses and hyperbolae

- Graph sketching
- Elementary vector algebra
- Elementary complex arithmetic
- Elementary probability

Tutors on duty do NOT help with current assignment questions!

Lecture Outline

This is a rough guide only. The material covered in each lecture may vary a little from the following table.

Vector Calculus

1. Vector fields, div and curl operators.
2. Double integrals over general regions, change of order of integration.
3. Double integrals - change of variables, polar coordinates.
4. Triple integrals - change of variables, cylindrical coordinates.
5. Triple integrals in cylindrical coordinates and spherical coordinates.
6. Parametrisation of paths. Line integrals.
7. Work integrals. Conservative fields.
8. Integrals of scalar functions over surfaces, surface area, mass.
9. Integrals of vector functions over surfaces, flux.
10. Gauss' divergence theorem.
11. Stokes' theorem.

Systems of First Order Ordinary Differential Equations

12. Systems of linear homogenous ODEs. Solve using eigenvalues and eigenvectors.
13. 2×2 systems examples, phase space, critical points, phase portraits.
14. Phase portraits of linear systems.
15. Phase portraits of linear systems. Non-linear coupled first order ODEs.
16. Non-linear coupled first order ODEs, linearisation, phase portraits.

Laplace Transforms

17. Laplace transforms. Table of transforms.
18. Inversion of transforms using tables. Solution of ODEs (single and systems).
19. Mid-Semester test
20. S-shifting theorem. Step functions.
21. T-shifting theorem. Impulse and Dirac delta functions.
22. Convolution theorems. Solution of integral equations.

Sequences and Series

23. Revision of sequences - convergence, divergence, limits. Infinite series - partial sums.
24. Geometric series, harmonic series. Integral test. Comparison test.
25. Ratio test. Leibniz test. Power series - radius and interval of convergence.
26. Power series (continued). Taylor polynomials.
27. Taylor series, errors.

Fourier Series

28. Periodic functions, Fourier series, Euler's formulae, energy density, Parseval's identity.
29. Fourier series for odd and even functions, periodic extensions.
30. Application of Fourier series to ODEs.
31. Fourier integrals, odd/even functions, applications to ODEs.

Second Order Partial Differential Equations

32. Examples and classification of second order PDEs.
33. Separation of variables for Laplace's equation.
34. Separation of variables for the wave equation.
35. Separation of variables for the diffusion equation.

MAST20029 Engineering Mathematics Formulae Sheet

1. **Change of Variable of Integration in 2D**

$$\iint_R f(x,y)\,dxdy = \iint_{R^*} f(x(u,v),y(u,v))|J(u,v)|\,dudv$$

2. **Transformation to Polar Coordinates**

$$x = r\cos\theta, \quad y = r\sin\theta, \quad J(r,\theta) = r$$

3. **Change of Variable of Integration in 3D**

$$\iiint_V f(x,y,z)\,dxdydz = \iiint_{V^*} F(u,v,w)|J(u,v,w)|\,dudvdw$$

4. **Transformation to Cylindrical Coordinates**

$$x = r\cos\theta, \quad y = r\sin\theta, \quad z = z, \quad J(r,\theta,z) = r$$

5. **Transformation to Spherical Coordinates**

$$x = r\cos\theta\sin\phi, \quad y = r\sin\theta\sin\phi, \quad z = r\cos\phi, \quad J(r,\theta,\phi) = r^2\sin\phi$$

6. **Line Integrals**

$$\int_C f(x,y,z)\,ds = \int_a^b f(x(t),y(t),z(t))\sqrt{x'(t)^2+y'(t)^2+z'(t)^2}\,dt$$

7. **Work Integrals**

$$\int_C \mathbf{F}(x,y,z)\cdot d\mathbf{r} = \int_a^b F_1\frac{dx}{dt} + F_2\frac{dy}{dt} + F_3\frac{dz}{dt}\,dt$$

8. **Surface Integrals**

$$\iint_S g(x,y,z)\,dS = \iint_R g(x,y,f(x,y))\sqrt{f_x^2+f_y^2+1}\,dxdy$$

9. **Flux Integrals** For a surface with upward unit normal,

$$\iint_S \mathbf{F}\cdot\hat{\mathbf{n}}\,dS = \iint_R -F_1 f_x - F_2 f_y + F_3\,dydx$$

10. **Gauss' (Divergence) Theorem**

$$\iiint_V \boldsymbol{\nabla} \cdot \mathbf{F}\, dV = \iint_S \mathbf{F} \cdot \hat{\mathbf{n}}\, dS$$

11. **Stokes' Theorem**

$$\iint_S (\boldsymbol{\nabla} \times \mathbf{F}) \cdot \hat{\mathbf{n}}\, dS = \int_C \mathbf{F} \cdot d\mathbf{r}$$

12. **Laplace Transforms**

1.	$f(t)$	$F(s) = \int_0^\infty f(t)e^{-st}\, dt$	(Definition of Transform)
2.	1	$\frac{1}{s}$	
3.	t^n	$\frac{n!}{s^{n+1}}$	
4.	e^{at}	$\frac{1}{s-a}$	
5.	$\sin(at)$	$\frac{a}{s^2+a^2}$	
6.	$\cos(at)$	$\frac{s}{s^2+a^2}$	
7.	$\sinh(at)$	$\frac{a}{s^2-a^2}$	
8.	$\cosh(at)$	$\frac{s}{s^2-a^2}$	
9.	$\delta(t-a)$	e^{-as}	$(a \geq 0)$
10.	$f'(t)$	$sF(s) - f(0)$	
11.	$f''(t)$	$s^2F(s) - sf(0) - f'(0)$	
12.	$f^{(n)}(t)$	$s^nF(s) - \sum_{k=0}^{n-1} s^{n-1-k} f^{(k)}(0)$	
13.	$\int_0^t f(\tau)\, d\tau$	$\frac{F(s)}{s}$	
14.	$e^{-at}f(t)$	$F(s+a)$	(s-Shifting Theorem)
15.	$f(t-a)\, u(t-a)$	$e^{-as}F(s)$	$(a > 0,$ t-Shifting Theorem)
16.	$\int_0^t f(\tau)\, g(t-\tau)\, d\tau$	$F(s)\, G(s)$	(Convolution)

13. **Standard Limits**

(i) $\displaystyle\lim_{n\to\infty}\frac{1}{n^p}=0 \qquad (p>0)$

(ii) $\displaystyle\lim_{n\to\infty} r^n=0 \qquad (|r|<1)$

(iii) $\displaystyle\lim_{n\to\infty} a^{1/n}=1 \qquad (a>0)$

(iv) $\displaystyle\lim_{n\to\infty} n^{1/n}=1$

(v) $\displaystyle\lim_{n\to\infty}\frac{a^n}{n!}=0 \qquad (\text{all } a)$

(vi) $\displaystyle\lim_{n\to\infty}\frac{\log_{\mathrm{e}} n}{n^p}=0 \qquad (p>0)$

(vii) $\displaystyle\lim_{n\to\infty}\left(1+\frac{a}{n}\right)^n=\mathrm{e}^a \qquad (\text{all } a)$

(viii) $\displaystyle\lim_{n\to\infty}\frac{n^p}{a^n}=0 \qquad (\text{all } p, a>1)$

14. **The Generalised Harmonic Series (p-Series)**

$$\sum_{n=1}^{\infty}\frac{1}{n^p} \quad \text{is} \quad \begin{cases}\text{convergent} & \text{if } p>1\\ \text{divergent} & \text{if } p\le 1\end{cases}$$

15. **Geometric Series**

$$\sum_{n=0}^{\infty} ar^n \quad \text{is} \quad \begin{cases}\text{convergent} & \text{if } |r|<1\\ \text{divergent} & \text{if } |r|\ge 1\end{cases}$$

16. **Taylor Polynomial**

$$P_n(x)=f(a)+f'(a)(x-a)+\frac{f''(a)}{2!}(x-a)^2+\cdots+\frac{f^{(n)}(a)}{n!}(x-a)^n$$

17. **The Remainder in Taylor's Theorem**

$$R_n(x)=\frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{(n+1)} \quad \text{where } c \text{ lies between } a \text{ and } x$$

18. **Fourier Series Formulae**

$$\begin{aligned} f(t) &= a_0+\sum_{n=1}^{\infty}\left[a_n\cos(n\omega t)+b_n\sin(n\omega t)\right], \qquad \omega=\frac{2\pi}{T}=\frac{\pi}{L}\\ a_0 &= \frac{1}{2L}\int_{-L}^{L} f(t)\,dt\\ a_n &= \frac{1}{L}\int_{-L}^{L} f(t)\cos(n\omega t)\,dt\\ b_n &= \frac{1}{L}\int_{-L}^{L} f(t)\sin(n\omega t)\,dt \end{aligned}$$

19. **Parseval's Identity for Energy Density**

$$\frac{1}{T}\int_{-T/2}^{T/2} f^2(t)\,dt=a_0^2+\frac{1}{2}\sum_{n=1}^{\infty}(a_n^2+b_n^2)$$

20. **Fourier Cosine Series for Even Functions**

$$\begin{aligned} f(t) &= a_0 + \sum_{n=1}^{\infty} a_n \cos(n\omega t) \\ a_0 &= \frac{1}{L}\int_0^L f(t)\,dt \\ a_n &= \frac{2}{L}\int_0^L f(t)\cos(n\omega t)\,dt \end{aligned}$$

21. **Fourier Sine Series for Odd Functions**

$$\begin{aligned} f(t) &= \sum_{n=1}^{\infty} b_n \sin(n\omega t) \\ b_n &= \frac{2}{L}\int_0^L f(t)\sin(n\omega t)\,dt \end{aligned}$$

22. **Fourier Integral Formulae**

$$\begin{aligned} f(t) &= \int_0^{\infty} A(\omega)\cos(\omega t) + B(\omega)\sin(\omega t)\,d\omega \\ A(\omega) &= \frac{1}{\pi}\int_{-\infty}^{\infty} f(t)\cos(\omega t) dt \\ B(\omega) &= \frac{1}{\pi}\int_{-\infty}^{\infty} f(t)\sin(\omega t) dt \end{aligned}$$

23. **Fourier Cosine Integrals for Even Functions**

$$\begin{aligned} f(t) &= \int_0^{\infty} A(\omega)\cos(\omega t)\,d\omega \\ A(\omega) &= \frac{2}{\pi}\int_0^{\infty} f(t)\cos(\omega t)\,dt \end{aligned}$$

24. **Fourier Sine Integrals for Odd Functions**

$$\begin{aligned} f(t) &= \int_0^{\infty} B(\omega)\sin(\omega t)\,d\omega \\ B(\omega) &= \frac{2}{\pi}\int_0^{\infty} f(t)\sin(\omega t)\,dt \end{aligned}$$

25. **Complex Exponential Formulae**

$\cosh x = \frac{1}{2}\left(e^{x} + e^{-x}\right)$ $\qquad$ $\sinh x = \frac{1}{2}\left(e^{x} - e^{-x}\right)$

$e^{ix} = \cos x + i\sin x$

$\cos z = \frac{1}{2}\left(e^{iz} + e^{-iz}\right)$ $\qquad$ $\sin z = \frac{1}{2i}\left(e^{iz} - e^{-iz}\right)$

26. **Standard Integrals**

$$\int \sin x\,dx = -\cos x + C \qquad \int \cos x\,dx = \sin x + C$$

$$\int \sec x\,dx = \log|\sec x + \tan x| + C \qquad \int \operatorname{cosec} x\,dx = \log|\operatorname{cosec} x - \cot x| + C$$

$$\int \sec^2 x\,dx = \tan x + C \qquad \int \operatorname{cosec}^2 x\,dx = -\cot x + C$$

$$\int \sinh x\,dx = \cosh x + C \qquad \int \cosh x\,dx = \sinh x + C$$

$$\int \operatorname{sech}^2 x\,dx = \tanh x + C \qquad \int \operatorname{cosech}^2 x\,dx = -\coth x + C$$

$$\int \frac{1}{\sqrt{a^2 - x^2}}\,dx = \arcsin\left(\frac{x}{a}\right) + C \qquad \int \frac{1}{\sqrt{x^2 + a^2}}\,dx = \operatorname{arcsinh}\left(\frac{x}{a}\right) + C$$

$$\int \frac{-1}{\sqrt{a^2 - x^2}}\,dx = \arccos\left(\frac{x}{a}\right) + C \qquad \int \frac{1}{\sqrt{x^2 - a^2}}\,dx = \operatorname{arccosh}\left(\frac{x}{a}\right) + C$$

$$\int \frac{1}{a^2 + x^2}\,dx = \frac{1}{a}\arctan\left(\frac{x}{a}\right) + C \qquad \int \frac{1}{a^2 - x^2}\,dx = \frac{1}{a}\operatorname{arctanh}\left(\frac{x}{a}\right) + C$$

where $a > 0$ is constant and C is an arbitrary constant of integration.

27. **Trigonometric and Hyperbolic Formulae**

$$\cos^2 x + \sin^2 x = 1 \qquad \cosh^2 x - \sinh^2 x = 1$$
$$1 + \tan^2 x = \sec^2 x \qquad 1 - \tanh^2 x = \operatorname{sech}^2 x$$
$$\cot^2 x + 1 = \operatorname{cosec}^2 x \qquad \coth^2 x - 1 = \operatorname{cosech}^2 x$$

$$\cos 2x = \cos^2 x - \sin^2 x \qquad \cosh 2x = \cosh^2 x + \sinh^2 x$$
$$\cos 2x = 2\cos^2 x - 1 \qquad \cosh 2x = 2\cosh^2 x - 1$$
$$\cos 2x = 1 - 2\sin^2 x \qquad \cosh 2x = 1 + 2\sinh^2 x$$
$$\sin 2x = 2\sin x\cos x \qquad \sinh 2x = 2\sinh x\cosh x$$

$$\sin(x+y) = \sin x\cos y + \cos x\sin y \qquad \sinh(x+y) = \sinh x\cosh y + \cosh x\sinh y$$
$$\cos(x+y) = \cos x\cos y - \sin x\sin y \qquad \cosh(x+y) = \cosh x\cosh y + \sinh x\sinh y$$

$$\sin x\sin y = \tfrac{1}{2}[\cos(x-y) - \cos(x+y)] \qquad \sinh x\sinh y = \tfrac{1}{2}[\cosh(x+y) - \cosh(x-y)]$$
$$\cos x\cos y = \tfrac{1}{2}[\cos(x-y) + \cos(x+y)] \qquad \cosh x\cosh y = \tfrac{1}{2}[\cosh(x+y) + \cosh(x-y)]$$
$$\sin x\cos y = \tfrac{1}{2}[\sin(x-y) + \sin(x+y)] \qquad \sinh x\cosh y = \tfrac{1}{2}[\sinh(x+y) + \sinh(x-y)]$$

MATLAB Assessment Requirements and Commands

1. **Using MATLAB**

   • MATLAB is installed on computers available for student use throughout the University. In particular, Engineering Mathematics students can use the Wilson Laboratory and the Nanson Laboratory in Peter Hall or any of the Engineering Faculty student computer labs in Alice Hoy, whenever the labs are vacant.

   • In Engineering Maths, we use MATLAB to perform numerical calculations and symbolic manipulations. The Symbolic Math Toolbox is used for symbolic manipulations. MATLAB (with the symbolic toolbox) is installed on all University Computers and comes with the Student Edition of MATLAB.

   • Enter commands in the MATLAB Command Window. You are in the MATLAB command window when you see the MATLAB command prompt >>.

   • To put your name and student number or comments in the command window use a percentage sign before the comment. For example: type % name, student number, comment (see sample over page).

   • To print from within the MATLAB command window, highlight what you want to print and choose Print Selection under the File menu.

   • To print a figure produced by MATLAB choose Print under the File menu on the Figure or use print screen commands.

   • Using a semi colon at the end of a MATLAB command stops the output of that command being displayed on the screen.

   • Use the up and down arrows in the command window to scroll through previously typed commands, so you can reuse them or edit them.

   • Plots and graphs appear in a separate figure window.

   • Use the help menu and search under functions: mathematics to get a list of all the mathematical function names. For example: `cosh(2*x)` is the hyperbolic cosine function $\cosh(2x)$ and `heaviside(t-pi)` is the step function $u(t-\pi)$.

2. **MATLAB Assessment Requirements**

   The requirements for MATLAB in Engineering Maths assessment are:

   • You must include your name and student number in a comment in your code. MATLAB code submitted without these details will NOT be marked.

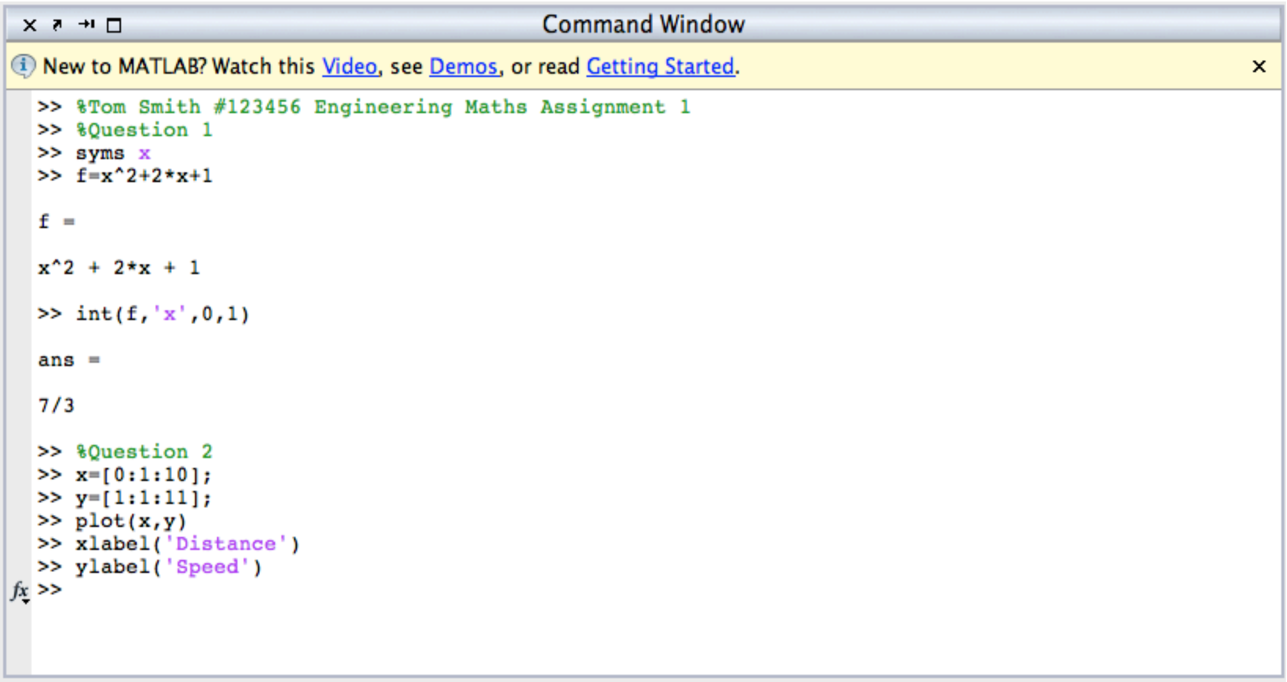
   • All MATLAB code and output must be submitted.

   • The code and output must be printed from within MATLAB or take a screen shot showing your work and the MATLAB Command window heading.

   • Code typed up in MS WORD or other word processing packages will not be accepted.

   • Only submit working code. Use the Clear Command Window under the Edit menu to remove all incorrect working and copy the correct commands from the Command History Window.

   A sample of what is required for assessment is given below:


Command Window
New to MATLAB? Watch this Video, see Demos, or read Getting Started.
>> %Tom Smith #123456 Engineering Maths Assignment 1
>> %Question 1
>> syms x
>> f=x^2+2*x+1

f =

x^2 + 2*x + 1

>> int(f,'x',0,1)

ans =

7/3

>> %Question 2
>> x=[0:1:10];
>> y=[1:1:11];
>> plot(x,y)
>> xlabel('Distance')
>> ylabel('Speed')
>>

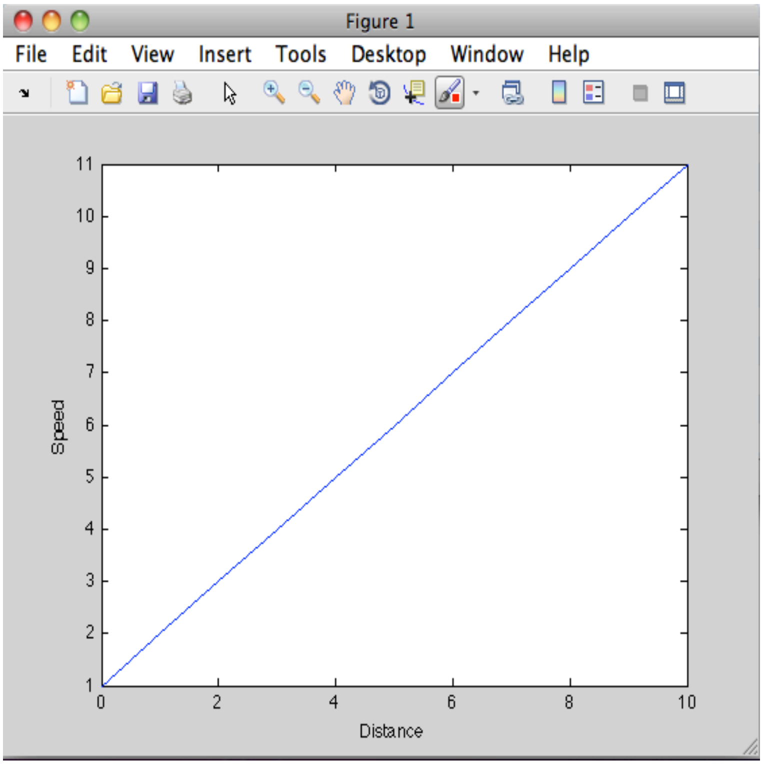


Figure 1
File Edit View Insert Tools Desktop Window Help
Speed
Distance
0 2 4 6 8 10
1 2 3 4 5 6 7 8 9 10 11

The following commands were compiled using version R2014b of MATLAB.

3. **Vectors and Matrices**

To create a row vector $\mathbf{a} = \begin{bmatrix} 1 & 5 & 2 & 3 \end{bmatrix}$ type:

```
>> a=[1 5 2 3]
```

The vector **a** will be echoed on the screen.

To create the vector **a** but not display it on the screen we use a semicolon, namely:

```
>> a=[1 5 2 3];
```

To see your vector, type

```
>> a
```

To enter a column vector, semi colons are required between the rows. To create the column vector

$$c = \begin{bmatrix} 2 \\ -5 \\ 2 \\ 6 \end{bmatrix}$$

we use:

```
>> c=[2; -5; 2; 6]
```

We can combine the row/column vector commands to define matrices. To define the matrix

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \\ 10 & 11 & 12 \end{bmatrix}$$

we type:

```
>> A=[1 2 3; 4 5 6; 7 8 9; 10 11 12]
```

To find the inverse, rank, determinant and eigenvalues of the matrix

$$M = \begin{bmatrix} 4 & 0 & 1 \\ -2 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}$$

use the following commands:

```
>> M=[4 0 1; -2 1 0; -2 0 1]
>> inv(M)
>> rank(M)
>> det(M)
>> eig(M)
```

To find the eigenvalues and eigenvectors of the matrix M, we can use the command:

```
>> [v,d]=eig(M)
```

The columns of the matrix `v` gives the eigenvectors normalized to length 1. The diagonal entries of the matrix `d` give the eigenvalues.

• It is important to be aware of the different types of matrix multiplication available.

```
>> A*A  %used to multiply scalars or multiply square matrices
>> A.*A %used to square each entry in the vector or matrix
```

For example, we can use:

```
>> M=[4 0 1; -2 1 0; -2 0 1]
>> MM=M*M
>> MMM=M.*M
```

which produces

```
>> MM =

    14     0     5
   -10     1    -2
   -10     0    -1
>> MMM =

    16     0     1
     4     1     0
     4     0     1
```

4. **Plotting Graphs**

To enter the following data

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ 0.5 & 1.5 & 2.5 & 3.5 & 4.5 & 5.5 & 4.5 & 3.5 & 2.5 & 1.5 & 0.5 \end{bmatrix}$$

into two vectors $\mathbf{x}$ and $\mathbf{y}$, we can type:

```
>> x=[0 1 2 3 4 5 6 7 8 9 10]
>> y=[0.5 1.5 2.5 3.5 4.5 5.5 4.5 3.5 2.5 1.5 0.5]
```

As x consists of the numbers 0 to 10 in steps of 1, we can also enter it as

```
>> x=[0:1:10]
```

To plot y versus x, we use the command

```
>> plot(x,y)
```

This command plots a line through the data points with x on the horizontal axis and y on the vertical axis. The plot appears in a separate figure window.

To plot the data with red stars for each point (no line), use

```
>> plot(x,y,'r*')
```

For a list of plotting colours and line types, type help plot. I To plot more than one line on the same graph, use the hold (on/off) function as follows.

```
>> x=[0:1:10];
>> y=[0.5 1.5 2.5 3.5 4.5 5.5 4.5 3.5 2.5 1.5 0.5];
>> z=[3 2 7 3.5 1 5.2 6 8 .9 .3 5];
>> p1=plot(x,y,'r-')
>> hold on
>> p2=plot(x,z,'k-.')
>> hold off
```

We can also add a title and axis labels to the graph using the following commands:

```
>> title('Plot of Speed vs. Distance')
>> xlabel('Distance')
>> ylabel('Speed')
```

To plot curves $y = f(x)$, we can parametrise the curve using the variable t. For example, to plot $y = x^2$ from $(-1, 1)$ to $(2, 4)$, we can use

```
>>t=[-1:.01:2];
>>plot(t,t.^2)
```

An easy way to sketch a parametrised curve in 2D is to use the `ezplot` command. To plot $y = x^2$ from $(-1, 1)$ to $(2, 4)$, we can use

```
>>ezplot('t','t^2',[-1,2])
```

This command automatically labels the x and y axes and puts a heading on the graph of $x = t$ and $y = t^2$.

To plot a curve in 3D we can use the `plot3` command. Plot the helix with parametrisation

$$\mathbf{r}(t) = \cos t\mathbf{i} + \sin t\mathbf{j} + 2t\mathbf{k}$$

from $(1, 0, 0)$ to $(1, 0, 4\pi)$.

```
>>t= 0:pi/50:2*pi;
>>plot3(cos(t),sin(t),2*t)
```

5. **Differential Equations**

We can use the `dsolve` command to solve single differential equations and systems of differential equations. We can find the general solution or the solution subject to initial conditions.

To find the general solution of the second order differential equation

$$y'' + y' = 2x$$

we use:

```
>>y=dsolve('D2y+Dy=2*x','x')
```

To find the solution of the differential equation subject to the initial conditions $y(0) = 1,\ y'(0) = 0$, we use:

```
>>y=dsolve('D2y+Dy=2*x','y(0)=1','Dy(0)=0','x')
```

To find the general solution of the system of differential equations

$$\begin{aligned} \frac{dx}{dt} &= x + 3y \\ \frac{dy}{dt} &= 2x + 2y \end{aligned}$$

we use:

```
>> [x,y]=dsolve('Dx=x+3*y','Dy=2*x+2*y')
```

To find the solution of the system subject to the initial conditions $x(0) = 0,\ y(0) = 1$, we use:

```
>> [x,y]=dsolve('Dx=x+3*y','Dy=2*x+2*y','x(0)=0','y(0)=1')
```

6. **Symbolic Toolbox Calculations**

To do calculations on symbolic expressions, we use the MATLAB symbolic toolbox. To see a list of the commands available type help symbolic. To use commands from the toolbox you need to first declare the names of your variables, for example to declare the variable x we type:

```
>> syms x
```

To differentiate and integrate the function $f(x) = \dfrac{x}{1+x^2}$ we use the `diff` and `int` commands.

```
>> syms x
>> f=x/(1+x^2)
>> diff(f)
>> int(f)
```

We can evaluate definite integrals by adding terminals to the `int` command:

```
>> syms x
>> f=x/(1+x^2)
>> int(f,'x',0,1)
```

We can also evaluate double and triple integrals using the `int` command multiple times. To evaluate

$$\int_0^1 \int_0^{x^2} xy^2 \; dy \; dx$$

we use

```
>> syms x y
>> int(int(x*y^2,'y',0,x^2),'x',0,1)
```

To evaluate

$$\int_0^1 \int_0^{z^2} \int_0^{y^2} xyz\; dx\; dy\; dz$$

we use

```
>> syms x y z
>> int(int(int(x*y*z,'x',0,y^2),'y',0,z^2),'z',0,1)
```

We can also factorise polynomials over the real numbers:

```
>> syms x
>> f=x^2+2*x+1
>> factor(f)
```

or expand them:

```
>> syms x
>> g=(x+1)^2
>> expand(g)
```

To simplify an expression we use the simplify command:

```
>> syms x
>> simplify(sin(x)^2 + cos(x)^2)
```

which gives 1.

To compute the Laplace transform of $f(t) = \sin(3t)$ we use

```
>> syms t
>> laplace(sin(3*t))
```

To compute the inverse Laplace transform of $F(s) = \dfrac{1}{(s-2)(s-1)}$ we use

```
>> syms s
>> F=1/(s-2)/(s-1)
>> ilaplace(F)
```

To compute the limit $\lim_{n\to\infty} \dfrac{\log_e(n^2)}{n}$ we use the `limit` command:

```
>> syms n
>> a=log(n^2)/n
>> limit(a,n,inf)
```

To compute the Taylor polynomial of a function we use the `taylor` command. The word 'ExpansionPoint' can be omitted for a function of one variable. To compute the 4^{th} degree polynomial (5 terms in the polynomial including the constant term) about $a = -1$ for $f(x) = xe^x$ we can use either of the commands below.

```
>> syms x f
>> f=x*exp(x)
>> taylor(f,x,'ExpansionPoint',-1,'Order',5)
>> taylor(f,x,-1,'Order',5)
```

The `diff` command gives $f^{(5)}(x)$, which is needed for the remainder function

$$R_4(x) = \frac{f^{(5)}(c)(x - x_0)}{5!}.$$

```
>> syms x f
>> f=x*exp(x)
>> diff(f,5)
```

Taylor polynomials can be sketched using `taylortool`. To start the interactive package type:

```
>> taylortool
```

You will need to enter the function f, number of terms N, expansion point a and the x-interval required in the display window.

7. **Phase Portraits**

Phase portraits can be plotted by running a MATLAB m-file. The latest m-file for MATLAB 2014b is available on the MAST20029 website. For earlier MATLAB versions, download the MATLAB m-file from the website

`http://math.rice.edu/%7Edfield/index.html`

8. **Fourier Series**

Consider the saw tooth wave defined by

$$f(t) = \begin{cases} 0, & -\pi < t < 0 \\ t, & 0 < t < \pi \end{cases}$$

with $f(t) = f(t + 2\pi)$.

We can use MATLAB to see how the Fourier series

$$f(t) = a_0 + \sum_{n=1}^{N} (a_n \cos(n\omega t) + b_n \sin(n\omega t))$$

approximates the function.

Below is code to plot the original function as well as the Fourier series approximation for the case $N = 5$ in the same picture. For a better approximation, replace 5 in the "for" loop by a larger integer. Comments have been added to some lines to explain the code.

```
>> t=-pi:0.01:0;
>> x=zeros(size(t));  %define zero vector with same size as t
>> plot(t,x)
>> axis([-4 4 -1 4])
>> hold on
>> t=0:0.01:pi;
>> y=t;
>> plot(t,y)
>> t=-pi:0.01:pi;
>> f=pi/4;           %a_0 term
>> for n=1:5       %compute sin and cos terms
sinterm = sin(n*t)*(-1)^(n+1)/n;          %nth sin term
if n/2 == round(n/2)      %check if n is even
costerm = 0;       %nth cos term (n even)
else       %case where n is odd
costerm = -(2/pi)*cos(n*t)/n^2;
end
f = f + sinterm + costerm;      %add terms to f
end
>> plot(t,f)
```

Alternatively, you can sum series $\sum_{n=a}^{b} f$ using the symbolic toolbox commands

```
>> syms n
>> symsum(f,n,a,b)
```

where f defines the terms in the series with respect to the symbolic variable n.

MAST20029 Engineering Mathematics
Sheet 1: Vector Calculus

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1. Consider the velocity field:

$$\mathbf{v}(x, y) = -2y\mathbf{i} + 2x\mathbf{j}$$

   (a) Sketch the velocity field at the points $(1, 0)$, $(2, 0)$, $(0, 1)$, $(0, 2)$, $(-1, 0)$, $(-2, 0)$, $(0, -1)$ and $(0, -2)$.

   (b) Find $\boldsymbol{\nabla} \cdot \mathbf{v}$ and $\boldsymbol{\nabla} \times \mathbf{v}$.

2. Consider the velocity field:

$$\mathbf{v}(x, y) = x^2 y\mathbf{i} - 2y^2\mathbf{j}$$

   (a) Sketch the velocity field at the points $(0, -1)$, $(-1, 1)$, $(-2, -1)$, $(1, 2)$, $(0, 1)$, $(2, 1)$, and $(1, 1)$.

   (b) Find $\boldsymbol{\nabla} \cdot \mathbf{v}$ and $\boldsymbol{\nabla} \times \mathbf{v}$.

3. Find the divergence of the following vector fields:

   (a) $\mathbf{V}(x, y, z) = yz\mathbf{i} + xz\mathbf{j} + xy\mathbf{k}$

   (b) $\mathbf{V}(x, y, z) = x\mathbf{i} + (y + \cos x)\mathbf{j} + (z + e^{xy})\mathbf{k}$

4. Find the curl of the following vector fields:

   (a) $\mathbf{V}(x, y, z) = yz\mathbf{i} + xz\mathbf{j} + xy\mathbf{k}$

   (b) $\mathbf{V}(x, y, z) = (x^2 + y^2 + z^2)(3\mathbf{i} + 4\mathbf{j} + 5\mathbf{k})$

5. Evaluate the following double integrals:

   (a) $\displaystyle\int_0^3 \int_0^x 4 - y^2 \, dy dx$ (b) $\displaystyle\int_1^{10} \int_0^{1/y} y e^{xy} \, dx dy$

6. Using double integrals, find the area of the

   (a) region enclosed by $x = 1 - y^2$ and $y = -x - 1$

   (b) ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ if $a > 0$, $b > 0$

7. For each of the following double integrals:

(a) $\displaystyle\int_0^1 \int_y^{\sqrt{y}} dxdy$

(b) $\displaystyle\int_0^2 \int_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}} y\, dxdy$

(c) $\displaystyle\int_0^2 \int_{y^2}^4 ye^{x^2}\, dxdy$

(d) $\displaystyle\int_0^1 \int_1^{e^x} dydx$

(i) Sketch the region of integration in the xy-plane.

(ii) Obtain an equivalent double integral with the order of integration reversed.

(iii) Evaluate the reversed integral.

(iv) Show how you would use MATLAB to evaluate the integral.

8. Let R be the region in the xy-plane bounded by the circle $x^2 + y^2 = 4$.

(a) Sketch the region R and describe it in terms of polar coordinates.

(b) By changing the variables to polar coordinates, evaluate

$$\iint_R 10 - 2x^2 - 2y^2\, dydx$$

9. Let R be the region bounded by the circles $x^2 + y^2 = 1$ and $x^2 + y^2 = 4$.

(a) Sketch the region R and describe it in terms of polar coordinates.

(b) By changing the variables to polar coordinates, evaluate

$$\iint_R 3x + 8y^2\, dydx$$

10. Evaluate the following double integrals by changing the variables to polar coordinates.

(a) $\displaystyle\int_{-2}^2 \int_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}} \sin(x^2 + y^2)\, dxdy$

(b) $\displaystyle\int_0^2 \int_0^{\sqrt{4-x^2}} e^{x^2+y^2}\, dydx$

11. Let the solid region V be the tetrahedron bounded by the planes

$$x = 0, \quad y = 0, \quad z = 0 \quad \text{and } 3 = 3x + 3y + z$$

(a) Sketch V and describe it in cartesian coordinates.

(b) Find the volume of V.

12. Let the solid region V be the tetrahedron bounded by the planes

$$x = 0, \quad y = 0, \quad z = 0 \quad \text{and } 6 = 6x + 3y + z$$

(a) Sketch V and describe it in cartesian coordinates.

(b) Find the volume of V.

13. Let V be the solid region bounded above by the paraboloid $z = 4 - x^2 - y^2$ and below by the plane $z = 0$.

(a) Sketch the region V and describe it in cylindrical coordinates.

(b) Use cylindrical coordinates to find

$$\iiint_V y \, dzdydx$$

(c) Use MATLAB to evaluate the triple integral in part (b).

14. Let V be the solid region such that $x^2 + y^2 \leq 4$, $x \geq 0$, and bounded by the planes $z = 0$ and $z = 4$.

(a) Sketch the region V and describe it in cylindrical coordinates.

(b) If the density (mass per unit volume) of V is given by

$$\rho(x, y, z) = xz$$

use cylindrical coordinates to find the mass of V.

15. Let B be the region within the cylinder $x^2 + y^2 = 1$ that is above the xy plane and below the cone $z = \sqrt{x^2 + y^2}$.

(a) Sketch the region B.

(b) Using cylindrical coordinates, evaluate

$$\iiint_B z \, dxdydz$$

16. Let V be the solid region inside the sphere $x^2+y^2+z^2 = a^2$ that lies above the xy-plane.

(a) Sketch the region V and describe it in spherical coordinates.

(b) Find the volume of V.

17. Let V be the solid region such that $x^2 + y^2 + z^2 \leq 4$ and $y \geq 0$.

   (a) Sketch the region V and describe it in spherical coordinates.

   (b) If the density (mass per unit volume) of V is given by

   $$\rho(x, y, z) = x^2 + y^2$$

   use spherical coordinates to find the mass of V.

18. Let V be the solid region bounded above by the sphere $r = 1$ and below by the cone $\phi = \pi/3$.

   (a) Sketch the region V and describe it in spherical coordinates.

   (b) Find the volume of V.

   (c) Check your answer to part (b) using MATLAB.

19. Give parametrisations for the following curves in terms of a parameter t, with t increasing. Sketch the curve, using arrows to show the direction for increasing t.

   (a) The straight line from $(-1,\ 2)$ to $(3,\ 3)$.

   (b) The portion of the circle $x^2 + y^2 = 4$ traversed anticlockwise from $(2, 0)$ to $(0, 2)$.

   (c) The portion of the circle $x^2 + y^2 = 4$ traversed clockwise from $(-2, 0)$ to $(0, 2)$.

   (d) The part of the ellipse $\dfrac{x^2}{4} + \dfrac{y^2}{9} = 1$ that lies above the line $y = 0$, traversed clockwise.

   (e) The part of the parabola $y = 2x - x^2$ from $(0, 0)$ to $(2, 0)$.

   (f) The part of the parabola $y = 2x - x^2$ from $(2, 0)$ to $(0, 0)$.

20. Let C be the part of the circle $x^2 + y^2 = 9$ that lies between the points $(0, 3)$ and $(3, 0)$, oriented clockwise.

   (a) Evaluate

   $$\int_C x^2 y \, ds$$

   (b) Show how you would use MATLAB to sketch this curve.

21. Let C be the curve $y = x^{3/2}$ from the point $(1, 1)$ to $(4, 8)$.

   (a) Find a parametrisation for C.

   (b) Find the length of C.

   (c) Show how you would use MATLAB to sketch this curve.

22. Consider the helix with parametrisation

$$\mathbf{r}(t) = 2\cos t\mathbf{i} + 2\sin t\mathbf{j} + t\mathbf{k}$$

Let C be the coil from $(-2, 0, \pi)$ to $(2, 0, 2\pi)$.

(a) Sketch this part of the helix, using arrows to show the direction for increasing t.

(b) Suppose the density (mass per unit length) of the coil is

$$\rho(x, y, z) = y^2 + 2z$$

Find the mass of the coil.

23. Let C be the curve with parametrisation

$$x(t) = t^3, \quad y(t) = -t, \quad z(t) = t^2, \qquad 1 \leq t \leq 2$$

and let the force field be

$$\mathbf{F}(x, y, z) = x\mathbf{i} - yz\mathbf{j} + z^2\mathbf{k}$$

Find the work done by $\mathbf{F}$ in moving a particle along C.

24. Consider the force field

$$\mathbf{F}(x, y) = (xy + 2y^2)\mathbf{i} + (3x^2 + y)\mathbf{j}$$

Find the work done by $\mathbf{F}$ to move a particle from (0,0) to (1,1) along

(a) $y = x^2$

(b) the y-axis to $(0, 1)$ and then along the line $y = 1$

(c) path (b) and then returns to the origin along path (a).

25. Consider the following vector fields $\mathbf{F}$. Which vector fields are conservative? For each conservative vector field, find a scalar function ϕ such that $\mathbf{F} = \nabla\phi$.

(a) $\mathbf{F}(x, y, z) = 2x\mathbf{i} + 3y\mathbf{j} + 4z\mathbf{k}$

(b) $\mathbf{F}(x, y) = y\cos x\mathbf{i} + x\sin y\mathbf{j}$

(c) $\mathbf{F}(x, y, z) = 2xye^z\mathbf{i} + x^2e^z\mathbf{j} + (x^2ye^z + z^2)\mathbf{k}$

26. In each case, show that $\mathbf{F}$ is a conservative vector field and find a scalar function ϕ such that $\mathbf{F} = \nabla\phi$. Evaluate $\displaystyle\int_C \mathbf{F} \cdot d\mathbf{r}$ along paths joining $(1, -2, 1)$ to $(3, 1, 4)$.

(a) $\mathbf{F}(x, y, z) = (2xyz + \sin x)\mathbf{i} + x^2z\mathbf{j} + x^2y\mathbf{k}$

(b) $\mathbf{F}(x, y, z) = (2xy + z^3)\mathbf{i} + x^2\mathbf{j} + 3xz^2\mathbf{k}$

27. Evaluate $\int_{\mathbf{c}} \mathbf{F} \cdot d\mathbf{r}$ along the given path.

(a) $\mathbf{F}(x, y, z) = e^x \sin y\mathbf{i} + e^x \cos y\mathbf{j} + z^2\mathbf{k}$
$\mathbf{c} = (\sqrt{t}, t^3, e^{\sqrt{t}}), \qquad 0 \leq t \leq 1.$

(b) $\mathbf{F}(x, y, z) = (xy^2 + 3x^2y)\mathbf{i} + (x^3 + x^2y)\mathbf{j}$
$\mathbf{c}$ is the curve consisting of line segments from (1,1) to (0,2) to (3,0).

28. Let the surface S be the part of the paraboloid

$$z = 16 - x^2 - y^2$$

that lies above the xy-plane. Sketch S and find its surface area.

29. Let the surface S be the part of the cone

$$z = \sqrt{x^2 + y^2}$$

that lies between the xy-plane and the plane $z = 4$. Sketch S and find its surface area.

30. Let the surface S be the part of the plane $2x - y + z = 3$ that lies above the triangle in the xy-plane that is bounded by the lines $y = 0$, $x = 1$ and $y = x$. Find the total mass of S if its density (mass per unit area) is given by

$$\rho(x, y, z) = xy + z$$

31. Let the surface S be the part of the paraboloid $z = x^2 + y^2$ that lies between the planes $z = 4$ and $z = 9$. Sketch S and then evaluate

$$\iint_S \frac{xy}{z}\, dS$$

32. Let S be the part of the cone $z^2 = x^2 + y^2$ that lies between the planes $z = 1$ and $z = 2$. Let S be oriented with outward unit normal. Find the flux of the vector field

$$\mathbf{F}(x, y, z) = -z^2\mathbf{k}$$

across S.

33. Let S be the surface defined by the unit sphere $x^2 + y^2 + z^2 = 1$, and let S be oriented with outward unit normal. Find the flux of the vector field

$$\mathbf{F}(x, y, z) = z\mathbf{k}$$

across S.

34. Let S be the part of the paraboloid $z = 1 - x^2 - y^2$ that lies above the xy-plane, and let S be oriented with outward unit normal. Find the flux of the vector field

$$\mathbf{F}(x, y, z) = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$$

across S.

35. Use Gauss' theorem to find the flux of the vector field

$$\mathbf{F}(x, y, z) = x^2y\mathbf{i} + 2xz\mathbf{j} + yz^3\mathbf{k}$$

across the surface S of the cuboid V defined by $0 \le x \le 1$, $0 \le y \le 2$, and $0 \le z \le 3$. Let S be oriented with outward unit normal.

36. Use Gauss' theorem to find the flux of the vector field

$$\mathbf{F}(x, y, z) = y\mathbf{i} + x\mathbf{j} + z\mathbf{k}$$

across the surface S of the region V formed by the portion of the cylinder $x^2 + y^2 \le 4$ that lies between the plane $z = 0$ and the paraboloid $z = x^2 + y^2$. Let S be oriented with outward unit normal.

37. Use Gauss' theorem to find the flux of the vector field

$$\mathbf{F}(x, y, z) = x\mathbf{i} + y\mathbf{j} + z^2\mathbf{k}$$

across the surface S of the sphere with radius a, centred at the origin, and oriented with outward unit normal.

38. Use Gauss' theorem to find the flux of the field

$$\mathbf{F}(x, y, z) = x^3z\mathbf{i} + y^3z\mathbf{j} + xy\mathbf{k}$$

across the surface S of the sphere with equation $x^2+y^2+z^2 = 9$, oriented with outward unit normal.

39. Use Gauss' theorem to find the flux of the vector field

$$\mathbf{F}(x, y, z) = y\mathbf{i} + z\mathbf{j} + z\mathbf{k}$$

across the surface S of the tetrahedron V with sides formed by the planes $x = 0$, $y = 0$, $z = 0$ and $x + 2y + 3z = 6$. Let S be oriented with outward unit normal.

40. Use Stokes' theorem to evaluate

$$\int_C \mathbf{F} \cdot d\mathbf{r}$$

where the vector field is

$$\mathbf{F}(x, y, z) = y\mathbf{i} - x\mathbf{j} + yx\mathbf{k}$$

and C is the boundary of the disk S defined by the equations $x^2 + y^2 = 1$ and $z = 1$. Assume S is oriented with upward unit normal.

41. Verify Stokes' theorem when the vector field is

$$\mathbf{F}(x, y, z) = (x - y)^2\mathbf{i} + 2z\mathbf{j} + x^2\mathbf{k}$$

and S is the cone $z = \sqrt{x^2 + y^2}$ with the circle $x^2 + y^2 = 4$, $z = 2$ as its boundary. Let S be oriented with outward unit normal.

42. Verify Stokes' theorem when the vector field is

$$\mathbf{F}(x, y, z) = y\mathbf{i} - x\mathbf{j} + yx\mathbf{k}$$

and S is the paraboloid $z = x^2 + y^2$ with the circle $x^2 + y^2 = 1$, $z = 1$ as its boundary. Let S be oriented with outward unit normal.

43. Let S be the surface given by

$$z = 3 - \sqrt{x^2 + y^2},\ 1 \le z \le 3.$$

Assume S is oriented using the outward unit normal.

(a) Sketch the surface S.

(b) Let $\mathbf{F}(x, y, z) = -yz^3\mathbf{i} + 3x\mathbf{j} + x^5\mathbf{k}$. Evaluate the surface integral

$$\iint_S (\nabla \times \mathbf{F}) \cdot \hat{\mathbf{n}}\, dS$$

using Stokes' theorem and

(i) an appropriate line integral;

(ii) the simplest surface for S.

ADDITIONAL EXERCISES
Kreyszig 10th Edition

Problem Set 9.4 (page 380): Q. 15 - 20

Problem Set 9.8 (page 405): Q. 1 - 6, 8

Problem Set 9.9 (page 408): Q. 2, 4 - 8

Problem Set 10.1 (page 418): Q. 2 - 11

Problem Set 10.3 (page 432): Q. 2 - 8, 9 - 11

Problem Set 10.6 (page 450): Q. 1 - 10, 12 - 16

Problem Set 10.7 (page 457): Q. 1 - 8, 9 - 18

Problem Set 10.9 (page 468): Q. 1 - 10, 13 - 20

MAST20029 Engineering Mathematics
Sheet 2: Systems of First Order ODE's and Phase Plane

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1. (a) Find the eigenvalues and eigenvectors of the matrix

$$\begin{bmatrix} 4 & -3 \\ 5 & -4 \end{bmatrix}$$

   Use MATLAB to verify your answers.

   (b) Using part (a), find the general solution of the system

$$\frac{dx}{dt} = 4x - 3y, \qquad \frac{dy}{dt} = 5x - 4y$$

   (c) Verify your answer to part (b) by showing that it satisfies the original system of differential equations.

2. (a) Using eigenvalues and eigenvectors, find the general solution to

$$\frac{dx}{dt} = x - y, \qquad \frac{dy}{dt} = x + y$$

   (b) Verify your answer to part (a) by showing that it satisfies the original system of differential equations.

3. Using eigenvalues and eigenvectors, find the general solution of the following coupled differential equations.

   (a) $\begin{aligned} \dot{x} &= x + 9y \\ \dot{y} &= -x - 5y \end{aligned}$ (b) $\begin{aligned} \dot{x} &= x + y \\ \dot{y} &= -x + 3y \end{aligned}$

4. Using eigenvalues and eigenvectors, solve the following systems of first order differential equations subject to the given initial conditions.

   (a) $\begin{aligned} \dot{x} &= x + y \\ \dot{y} &= 4x - 2y \\ x(0) &= 1 \\ y(0) &= 2 \end{aligned}$ (b) $\begin{aligned} \dot{x} &= x + 2y \\ \dot{y} &= -\tfrac{1}{2}x + y \\ x(0) &= 2 \\ y(0) &= 3 \end{aligned}$

   In each case, use MATLAB to check your solution.

5. Consider the following linear systems and their general solutions:

(a) $\dfrac{dx}{dt} = 6x - 2y, \quad \dfrac{dy}{dt} = 4x + 2y$

with general solution

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \alpha_1 \begin{pmatrix} \cos(2t) \\ \cos(2t) + \sin(2t) \end{pmatrix} e^{4t} + \alpha_2 \begin{pmatrix} \sin(2t) \\ \sin(2t) - \cos(2t) \end{pmatrix} e^{4t}$$

(b) $\dfrac{dx}{dt} = 5x + 2y, \quad \dfrac{dy}{dt} = 4x + 3y$

with general solution

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \alpha_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{7t} + \alpha_2 \begin{pmatrix} 1 \\ -2 \end{pmatrix} e^{t}$$

(c) $\dfrac{dx}{dt} = 3x - y, \quad \dfrac{dy}{dt} = 6x - 4y$

with general solution

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \alpha_1 \begin{pmatrix} 1 \\ 6 \end{pmatrix} e^{-3t} + \alpha_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{2t}$$

(d) $\dfrac{dx}{dt} = -2x - y, \quad \dfrac{dy}{dt} = x$

with general solution

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \alpha_1 \begin{pmatrix} -1 \\ 1 \end{pmatrix} e^{-t} + \alpha_2 \begin{pmatrix} -t + 1 \\ t \end{pmatrix} e^{-t}$$

For each of the linear systems

(i) Sketch the phase portrait near the critical point at the origin.

(ii) Discuss the type and stability of the critical point.

6. Consider the following linear systems:

(a) $\dfrac{dx}{dt} = -y, \quad \dfrac{dy}{dt} = x$

(b) $\dfrac{dx}{dt} = -10x - y, \quad \dfrac{dy}{dt} = 15x - 2y$

For each of the linear systems

(i) Obtain the general solution.

(ii) Sketch the phase portrait near the critical point at the origin.

(iii) Discuss the type and stability of the critical point at the origin.

(iv) Find the solution satisfying $x(0) = 2$ and $y(0) = 7$. Use MATLAB to sketch the corresponding orbit.

7. Consider the following linear systems:

(a) $\dfrac{dx}{dt} = -6x - 2y, \quad \dfrac{dy}{dt} = 4x - 2y$

(b) $\dfrac{dx}{dt} = -2x, \quad \dfrac{dy}{dt} = -2y$

For each of the linear systems

(i) Obtain the general solution.

(ii) Sketch the phase portrait near the critical point at the origin.

(iii) Discuss the type and stability of the critical point at the origin.

(iv) Find the solution satisfying $x(0) = 0$ and $y(0) = 2$.

8. Consider the nonlinear system

$$\frac{dx}{dt} = y, \quad \frac{dy}{dt} = x^2 - 4x + y$$

(a) Find the critical points.

(b) Determine whether or not the linear system can be used to approximate the non-linear system near each critical point.

(c) If the linear system can be used to approximate the non-linear system, determine the type and stability of the critical point.

(d) Use MATLAB to sketch the global phase portrait for the nonlinear system in the region $-4 \leq x \leq 8$ and $-4 \leq y \leq 4$.

9. Consider the nonlinear system

$$\frac{dx}{dt} = y, \quad \frac{dy}{dt} = -\sin x - y$$

(a) Find the critical points.

(b) Determine whether or not the linear system can be used to approximate the non-linear system near each critical point.

(c) If the linear system can be used to approximate the non-linear system, determine the type and stability of the critical point.

(d) Use MATLAB to sketch the global phase portrait for the nonlinear system in the region $-10 \leq x \leq 10$ and $-4 \leq y \leq 4$.

10. Consider the nonlinear system

$$\frac{dx}{dt} = y, \qquad \frac{dy}{dt} = -x + (1 - x^2)y$$

(a) Find the critical points.

(b) Determine whether or not the linear system can be used to approximate the non-linear system near each critical point.

(c) If the linear system can be used to approximate the non-linear system, determine the type and stability of the critical point.

(d) Use MATLAB to sketch the global phase portrait for the nonlinear system in the region $-4 \le x \le 4$ and $-4 \le y \le 4$.

11. Consider the nonlinear system

$$\frac{dx}{dt} = y, \qquad \frac{dy}{dt} = -x^3 + 2x$$

(a) Find the critical points.

(b) Determine whether or not the linear system can be used to approximate the non-linear system near each critical point.

(c) If the linear system can be used to approximate the non-linear system, determine the type and stability of the critical point.

(d) Use MATLAB to sketch the global phase portrait for the nonlinear system in the region $-4 \le x \le 4$ and $-4 \le y \le 4$.

ADDITIONAL EXERCISES
Kreyszig 10th Edition

Problem Set 4.3 (page 147): Q. 1 - 6, 8, 10 - 15

Problem Set 4.4 (page 151): Q. 1 - 10

Problem Set 4.5 (page 159): Q. 4 - 8

MAST20029 Engineering Mathematics
Sheet 3: Laplace Transforms

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1. Using the integral definition, find the Laplace transforms of

   (a) $f(t) = \begin{cases} kt/c, & 0 \leq t \leq c \\ 0, & t > c \end{cases}$ (b) $f(t) = \begin{cases} k, & 0 < a \leq t \leq b \\ 0, & \text{elsewhere} \end{cases}$

   (c) $f(t) = \sinh(at)$ (d) $f(t) = \cos(at)$

2. Use the tables to find the Laplace transforms of

   (a) $f(t) = 2t^2 + 6 - 3e^{4t}$ (b) $f(t) = A\sin(\omega t) + B\cosh(kt)$

   (c) $f(t) = t^4 + t^5$ (d) $f(t) = \cos^2 t$

   In each case use MATLAB to verify your answers.

3. Use the tables to find the inverse Laplace transforms of

   (a) $F(s) = \dfrac{3}{s^2}$ (b) $F(s) = \dfrac{4}{s^2 + 16}$

   (c) $F(s) = \dfrac{s}{s^2 + 9}$ (d) $F(s) = \dfrac{5s}{s^2 - 25}$

   In each case use MATLAB to verify your answers.

4. Use partial fractions to find the inverse Laplace transforms of

   (a) $F(s) = \dfrac{s}{s^2 + 3s - 4}$ (b) $F(s) = \dfrac{1}{(s-2)(s+1)}$

5. Using the s-shifting theorem, find the Laplace transforms of

   (a) $f(t) = 3te^{2t}$ (b) $f(t) = 5e^{-6t}\sin(4t)$

   (c) $f(t) = 2e^{-t}\sinh(3t)$ (d) $f(t) = 2\sinh t \cos t$

6. Use the s-shifting theorem to find the inverse Laplace transforms of

   (a) $F(s) = \dfrac{3}{(s+2)^2}$ (b) $F(s) = \dfrac{1}{(s-2)^3}$

   (c) $F(s) = \dfrac{1}{(s-4)^2 + 1}$ (d) $F(s) = \dfrac{s+2}{(s+2)^2 + 4}$

7. Using the t-shifting theorem, find the Laplace transforms of

   (a) $f(t) = \cosh(t-1)u(t-1)$ (b) $f(t) = tu(t-\pi)$

   Use MATLAB to verify your answer to part (b).

8. Consider the following functions:

(a) $f(t) = \begin{cases} e^t & \text{if} \quad 0 \le t < 2 \\ 0 & \text{elsewhere} \end{cases}$

(b) $f(t) = \begin{cases} \cos(\pi t) & \text{if} \quad 1 \le t < 4 \\ 0 & \text{elsewhere} \end{cases}$

(c) $f(t) = \begin{cases} 1 & \text{if} \quad 0 \le t < 5 \\ t & \text{if} \quad 5 \le t < 10 \\ 0 & \text{if} \quad t \ge 10 \end{cases}$

(d) $f(t) = \begin{cases} 0 & \text{if} \quad 0 \le t < 1 \\ \sin(\pi t) & \text{if} \quad 1 \le t < 2 \\ 0 & \text{if} \quad 2 \le t < 3 \\ e^{-2t} & \text{if} \quad t \ge 3 \end{cases}$

Represent f using step functions. Hence find the Laplace transform of f.

9. Use the t-shifting theorem to find the inverse Laplace transforms of

(a) $F(s) = \dfrac{e^{-3s}}{s}$ (b) $F(s) = \left(\dfrac{5s+4}{s^2}\right) e^{-2s}$

10. Find the inverse Laplace transforms of

(a) $F(s) = \dfrac{e^{-3s}}{s+8}$ (b) $F(s) = \dfrac{e^{-2s}}{s^2 - 3s - 4}$

(c) $F(s) = \dfrac{s+6}{(s+2)^2+4}$ (d) $F(s) = \dfrac{2}{s(s^2+4)}$

(e) $F(s) = 2 + \dfrac{e^{-s}}{s^2 - 16}$ (f) $F(s) = \dfrac{4}{s^2 + 6s + 13}$

11. Given $y(0) = 4$, and $\dot{y}(0) = 3$, use Laplace transforms to solve

$$\ddot{y} - 3\dot{y} + 2y = 4t$$

12. Consider the dynamical system described by

$$\ddot{y} + 4\dot{y} + 4y = 4$$

with $y(0) = \dot{y}(0) = 0$. Find $y(t)$ using Laplace transforms.
Check your solution to the differential equation using MATLAB.

13. Use Laplace transforms to solve the system

$$\frac{dx}{dt} = -y, \qquad \frac{dy}{dt} = x$$

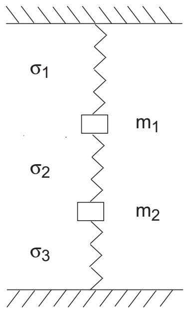
for $t \ge 0$, with $x(0) = 1$, $y(0) = 0$.

14. Use Laplace transforms to solve the system

$$\frac{dx}{dt} = x + y, \qquad \frac{dy}{dt} = -x - y$$

for $t \geq 0$, with $x(0) = 1$, $y(0) = 0$.

15. A mechanical system consists of three springs, σ_1, σ_2 and σ_3 of negligible mass, equal natural length and spring constant 1N/m, and two objects, m_1 and m_2 of mass 1 kg each.



Assuming that the damping is negligible, and that the departure of m_1 from its equilibrium position is $y_1(t)$, and that of m_2 is $y_2(t)$ at time t. The equations of motion are given by the following system of second order ordinary differential equations:

$$\begin{aligned} y_1'' &= y_2 - 2y_1 \\ y_2'' &= y_1 - 2y_2 \end{aligned}$$

Assume the initial conditions are

$$\begin{aligned} y_1(0) &= 1, \qquad y_1'(0) = \sqrt{3} \\ y_2(0) &= 1, \qquad y_2'(0) = -\sqrt{3} \end{aligned}$$

Use Laplace transforms to solve the system of differential equations subject to the given initial conditions.

16. An object with mass m receives 11 impulses of strength p at 1 second intervals at $t = 0$, $t = 1$, $t = 2$, ..., $t = 10$. The differential equation describing the motion of this object is

$$m\frac{dv}{dt} = p\sum_{k=0}^{10} \delta(t-k)$$

If the object is initially at rest, find its velocity at time $t \geq 0$.

17. Use the convolution theorem to find the Laplace transforms of

(a) $f(t) = \int_0^t e^{\tau}\,(t-\tau)\,d\tau$

(b) $f(t) = \int_0^t \sin(t-\tau)e^{3\tau}\,d\tau$

(c) $f(t) = \int_0^t \delta(t-\tau)\sinh(2\tau)\,d\tau$

(d) $f(t) = \int_0^t (1+\tau)(t-\tau)^2\,d\tau$

18. Solve the integral-differential equation

$$\int_0^t y(\tau)\,d\tau - y'(t) = t$$

for $t \geq 0$, with $y(0) = 4$.

19. In a series RLC-circuit with resistance $R = 2\Omega$, inductance $L = 1H$, capacitance $C = 0.25F$ and voltage $v = 50V$, the current i at time t is determined by the integral-differential equation

$$2i + \frac{di}{dt} + 4\int_0^t i(\tau)\,d\tau = 50$$

Solve the integral-differential equation to find the current, if initially there is no current in the circuit.

20. Consider an RC circuit, with resistance R and capacitance C. The voltage v at time t is always zero, except from $t = a$ to $t = b$ when $v = v_0$ is a constant. The current i at any time is given by the differential equation

$$Ri + \frac{q}{C} = v$$

where q is the charge on the capacitor at any time and

$$v(t) = v_0[u(t-a) - u(t-b)]$$

Since $i = \dfrac{dq}{dt}$, the differential equation becomes

$$Ri + \frac{1}{C}\int_0^t i(\tau)\,d\tau = v_0[u(t-a) - u(t-b)]$$

Solve the integral equation for the current.

ADDITIONAL EXERCISES
Kreyszig 10th Edition

Problem Set 6.1 (page 210): Q. 1 - 8, 25 - 45

Problem Set 6.2 (page 216): Q. 1 - 21, 23 - 29

Problem Set 6.3 (page 223): Q. 2 - 27

Problem Set 6.4 (page 230): Q. 3 - 12

Problem Set 6.5 (page 237): Q. 1 - 14, 17 - 25

MAST20029 Engineering Mathematics
Sheet 4: Sequences and Series

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1. Use standard limits to determine whether the following sequences are convergent or divergent, and find the limit if it exists.

   (a) $a_n = \dfrac{6}{4^n}$

   (b) $a_n = 5$

   (c) $a_n = \left(\dfrac{n+2}{n}\right)^n$

   (d) $a_n = \dfrac{\log_e(n^2)}{n}$

   (e) $a_n = \dfrac{5^n}{n!}$

   (f) $a_n = \dfrac{n^2}{2^n}$

2. Which of the following sequences converge? Find their limits if they converge. For part (a) and (b), verify your answer using MATLAB.

   (a) $a_n = \dfrac{n^2}{3n^2+4n+1}$

   (b) $a_n = \dfrac{2n^2+6n+2}{3n^3-n^2-n}$

   (c) $a_n = n^{(-1)^n}$

   (d) $a_n = \cos(n\pi)$

3. Decide which of the following sequences converge. Find their limits if they do.

   (a) $a_n = \dfrac{n^n}{(n+3)^{n+1}}$

   (b) $a_n = \left(1 - \dfrac{1}{n^2}\right)^n$

   (c) $a_n = \dfrac{\log_e(n+1)}{n}$

   (d) $a_n = \dfrac{n(n+1)^{n+1}}{(n+2)^{n+2}}$

4. Find the limit (if it exists) of each of the sequences whose nth term is given below.

   (a) $\dfrac{5^n - 2^n - 9}{n^3 + 5^n + n}$

   (b) $\exp\left(\dfrac{n^2-2}{2n^2+1}\right)$

   (c) $\dfrac{e^n}{3n+4}$

   (d) $n\sin\left(\dfrac{\pi}{n}\right)$

   (e) $\log_e(2n) - \log_e(3n+2)$

   (f) $n\left[1 - \cos\left(\dfrac{2}{n}\right)\right]$

5. Let $a_n = \dfrac{3}{4^n}$, $n \geq 1$.

   (a) Is the sequence $\{a_n\}$ convergent?

   (b) Calculate the first four partial sums of the sequence $\{a_n\}$

   (c) Show that the series $\displaystyle\sum_{n=1}^{\infty} a_n$ converges and find the sum.

6. Let $a_n = \dfrac{n-1}{n}$, $n \geq 1$.

   (a) Is the sequence $\{a_n\}$ convergent?

   (b) Calculate the first four partial sums of the sequence $\{a_n\}$

   (c) Show that the series $\displaystyle\sum_{n=1}^{\infty} a_n$ diverges.

7. In this question, we determine the sum of a telescoping series.
   Let $a_n = \dfrac{1}{n(n+1)}$, $n \geq 1$.
   (a) Write a_n in terms of partial fractions.
   (b) Using part (a), calculate the nth partial sum s_n of the sequence $\{a_n\}$.
   (c) Show that the series $\displaystyle\sum_{n=1}^{\infty} a_n$ converges and find the sum.

8. Use the integral test to determine whether the following series converge or diverge.

   (a) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2+4}$ (b) $\displaystyle\sum_{n=2}^{\infty} \frac{1}{\sqrt{n-1}}$

   (c) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{(n+1)^2}$ (d) $\displaystyle\sum_{n=1}^{\infty} \frac{2}{2n-1}$

9. Use the comparison test to determine whether the following series converge or diverge.

   (a) $\displaystyle\sum_{n=1}^{\infty} \frac{2n}{n^2+1}$ (b) $\displaystyle\sum_{n=1}^{\infty} \frac{n}{(n+1)(n^2+3)}$

   (c) $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n^3-1}$ (d) $\displaystyle\sum_{n=1}^{\infty} \frac{10^n}{n+4^{2n}}$

10. Use the ratio test to determine whether the following series converge or diverge.

   (a) $\displaystyle\sum_{n=1}^{\infty} \frac{n!}{n^n}$ (b) $\displaystyle\sum_{n=1}^{\infty} \frac{(3n)!}{(n!)^3}$

   (c) $\displaystyle\sum_{n=1}^{\infty} \frac{2^n}{n+1}$ (d) $\displaystyle\sum_{n=1}^{\infty} \frac{10^n}{n+4^{2n}}$

11. Use the alternating series test to determine whether the following series converge or diverge.

(a) $\displaystyle\sum_{n=1}^{\infty}(-1)^n 2^n$

(b) $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^n}{n^{1/3}}$

(c) $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^n n}{n^2+1}$

(d) $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^{n+1}3^n}{4n^2}$

12. Determine whether the following series converge or diverge.

(a) $\displaystyle\sum_{n=0}^{\infty}\frac{1}{4^n}$

(b) $\displaystyle\sum_{n=1}^{\infty}2^{-n}3^{n-1}$

(c) $\displaystyle\sum_{n=1}^{\infty}\frac{4n}{7n+1}$

(d) $\displaystyle\sum_{n=3}^{\infty}\frac{(-1)^{n+1}\log_e n}{n}$

(e) $\displaystyle\sum_{n=1}^{\infty}\frac{\sqrt{n}}{n^2+2}$

(f) $\displaystyle\sum_{n=1}^{\infty}\frac{\sin^2 n}{n^2}$

(g) $\displaystyle\sum_{n=1}^{\infty}\frac{2^n}{n^3}$

(h) $\displaystyle\sum_{n=1}^{\infty}\frac{e^n}{\sqrt{n!}}$

13. Find (i) the radius of convergence and (ii) the interval of convergence for each of the following power series.

(a) $\displaystyle\sum_{n=0}^{\infty}\frac{2^n x^n}{n+1}$

(b) $\displaystyle\sum_{n=1}^{\infty}\frac{x^n}{\log_e(n+1)}$

(c) $\displaystyle\sum_{n=1}^{\infty}\frac{(3x+6)^n}{n!}$

(d) $\displaystyle\sum_{n=0}^{\infty}\frac{(-1)^n(x+1)^n}{(n+1)^2}$

14. For each of the following functions, write down the Taylor polynomial for the given values of a and n and give an expression for the remainder $R_n(x)$.

Also show how you would use the MATLAB command **taylor** to verify your answer.

(a) $f(x)=\sin x \qquad a=\pi/2 \qquad n=3$

(b) $f(x)=\sqrt{x} \qquad a=4 \qquad n=3$

(c) $f(x)=xe^x \qquad a=-1 \qquad n=4$

15. Consider the function

$$f(x) = x \sin x, \;\; x \in \mathbb{R}$$

(a) Find the quadratic Maclaurin polynomial for f.

(b) Write down an expression for the error in terms of x. In what interval does the unknown constant c lie?

(c) Check your answer for the Maclaurin polynomial and derivative function needed to calculate the error, using MATLAB.

(d) Use the MATLAB command **taylortool** to compare f with the Maclaurin polynomial for f of varying degrees.

16. Consider the function

$$f(x) = \sinh x, \quad x \in \mathbb{R}$$

(a) Determine the cubic Maclaurin polynomial for f.

(b) Determine an upper bound for the error on the interval $|x| \leq 1$.

17. For what values of x does the cubic Maclaurin polynomial for $\cos x$ have an error of

(a) less than .01

(b) less than .01 of $|x|$?

18. Consider the function

$$f(x) = \sqrt{1+x}, \quad x \geq -1$$

(a) Determine the linear Maclaurin polynomial for f.

(b) Give an expression for the error in terms of x when $|x| < 0.19$.

(c) Estimate $\sqrt{1.1}$ and determine the size of the error.

19. Consider the function

$$f(x) = \frac{1}{1-x}, \quad x \neq 1$$

(a) By finding the Maclaurin series for f, show that

$$f(x) = \sum_{n=0}^{\infty} x^n$$

(b) Find the radius of convergence and interval of convergence for the series in part (a).

20. Consider the function

$$f(x) = \log_e(1 + x), \quad x > -1$$

(a) Find the Maclaurin series for f.

(b) Find the radius of convergence and interval of convergence for the series in part (a).

(c) Use your Maclaurin series to find value of the alternating series

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$$

ADDITIONAL EXERCISES

For additional questions on sequences and series look in any first year Calculus textbook available in the ERC library.

MAST20029 Engineering Mathematics
Sheet 5: Fourier Series and Fourier Integrals

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1. Consider the waveform

$$f(t) = e^t, \qquad -\pi < t < \pi$$

where $f(t) = f(t + 2\pi)$.

   (a) Sketch the graph of $f(t)$ for $-3\pi \le t \le 3\pi$.

   (b) Determine the $n = 0, 1, 2, 3$ terms for the Fourier series of f.

   (c) Write down a general expression for the Fourier series of f.

2. A sinusoidal voltage $\sin t$ is passed through a half-wave rectifier to produce a periodic output voltage with period 2π such that

$$E(t) = \begin{cases} 0, & -\pi < t < 0 \\ \sin t, & 0 < t < \pi \end{cases}$$

   (a) Sketch the graph of $E(t)$ for $-3\pi \le t \le 3\pi$.

   (b) Determine the $n = 0, 1, 2, 3, 4$ terms for the Fourier series of E.

   (c) Use Parseval's Identity to find the first five non zero terms of the energy density of E.

   (d) Calculate the energy density of E.

3. Consider the function

$$f(t) = \begin{cases} 1, & 0 < t < 1 \\ 2, & 1 < t < 2 \end{cases}$$

   (a) Sketch $f_e(t)$, the even periodic extension of f in the range $-4 \le t \le 4$.

   (b) Determine the first four non zero terms of the Fourier cosine series for f_e.

   (c) What values does the Fourier series for f_e converge to if $t = 0$ and $t = 2$?

   (d) Sketch $f_o(t)$, the odd periodic extension of f in the range $-4 \le t \le 4$.

   (e) Determine the first four non zero terms of the Fourier sine series for f_o.

   (f) What values does the Fourier series for f_o converge to if $t = 0$ and $t = 2$?

4. Consider the function

$$f(t) = t^2 \qquad \text{for } 0 < t < \pi$$

   (a) Sketch $f_e(t)$, the even periodic extension of f in the range $-3\pi \le t \le 3\pi$.

   (b) Determine the first four non zero terms of the Fourier cosine series for f_e.

   (c) Sketch $f_o(t)$, the odd periodic extension of f in the range $-3\pi \le t \le 3\pi$.

   (d) Determine the first four non zero terms of the Fourier sine series for f_o.

5. Consider the function

$$f(t) = \begin{cases} -(\pi + t), & -\pi < t < 0 \\ \pi - t, & 0 < t < \pi \end{cases}$$

(a) Obtain a general Fourier series representation for f, if f is periodic with period 2π.

(b) Using part (a), find a particular solution for the non-homogenous ordinary differential equation

$$-y'' + y = f$$

(c) Find the general solution for the differential equation in part (b).

6. Consider the function

$$f(t) = \frac{t^2}{4} \quad \text{for } -\pi < t < \pi.$$

(a) Obtain a general Fourier series representation for f, if f is periodic with period 2π.

(b) Using part (a), find a particular solution to the differential equation

$$y'' + y' + y = f$$

that describes the oscillations of a damped mechanical system subject to the periodic driving force f.

(c) Using part (a), evaluate the sum

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$$

7. Consider the function

$$f(t) = \begin{cases} 0, & t < 0 \\ \pi/2, & t = 0 \\ \pi e^{-t}, & t > 0 \end{cases}$$

(a) Sketch $f(t)$ for all real values of t.

(b) Obtain the Fourier integral representation for f.

8. Consider the function

$$f(t) = \begin{cases} 1, & 0 \le t \le 1 \\ 2, & 1 < t \le 4 \\ 0, & t > 4 \end{cases}$$

(a) Sketch $f(t)$.

(b) Obtain the Fourier sine integral representation for f.

(c) Obtain the Fourier cosine integral representation for f.

(d) What values do the Fourier integrals in (b) and (c) converge to if $t = 1$?

(e) What values do the Fourier integrals in (b) and (c) converge to if $t = 4$?

(f) What values do the Fourier integrals in (b) and (c) converge to if $t = 0$?

9. Consider the function

$$f(t) = \begin{cases} \frac{\pi}{2}\cos t, & 0 \le t < \pi/2 \\ \\ 0, & t \ge \pi/2 \end{cases}$$

(a) Sketch $f_e(t)$, the even extension of f, for all real values of t.

(b) Obtain the Fourier cosine integral representation for f_e.

10. Consider the function

$$f(t) = -e^{-2t} \qquad \text{for } t > 0$$

(a) Sketch $f_o(t)$, the odd extension of f, for all real values of t.

(b) Obtain the Fourier sine integral representation for f_o.

11. Consider the inhomogeneous differential equation

$$y'' + y = f$$

where

$$f(t) = \begin{cases} \sin t, & |t| \le \pi \\ 0, & |t| > \pi \end{cases}$$

(a) Find the Fourier integral representation for f.

(b) Using part (a), find a particular solution to the differential equation.

12. Consider an infinitely long beam on an elastic foundation with deflection u satisfying the nonhomogeneous differential equation

$$EI\frac{d^4u}{dx^4} + ku = y$$

where the constant k is the "foundation modulus" (that is, the spring stiffness per unit x length) and EI the "flexural rigidity" of the beam and y is a prescribed loading (force per unit x length) given by

$$y(x) = \begin{cases} \omega_0, & |x| < 1 \\ \omega_0/2, & |x| = 1 \\ 0, & |x| > 1 \end{cases}$$

(a) Find the Fourier integral representation for y.

(b) Using part (a), find a particular solution to the differential equation.

ADDITIONAL EXERCISES
Kreyszig 10th Edition

Problem Set 11.1 (page 482): Q. 6 - 10, 12 - 21

Problem Set 11.2 (page 490): Q. 8 - 17

Problem Set 11.7 (page 517): Q. 7, 8, 16,17

MAST20029 Engineering Mathematics
Sheet 6: Second Order Partial Differential Equations

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1. State the order and the number of independent variables for the following PDE's. Decide whether the equation is linear or nonlinear, and homogeneous or inhomogeneous.

   (a) $\phi\dfrac{\partial^2\phi}{\partial x^2}+\dfrac{\partial\phi}{\partial y}=1$

   (b) $\phi^2\dfrac{\partial^2\phi}{\partial x^2}+\dfrac{\partial^2\phi}{\partial y\partial z}=1$

   (c) $\dfrac{\partial^4\phi}{\partial x^4}+\dfrac{\partial^2\phi}{\partial y^2}=0$

   (d) $\dfrac{\partial^2\phi}{\partial x^2}+\dfrac{\partial^3\phi}{\partial y^2\partial z}=\phi^2$

2. Consider the second order wave equation

$$\frac{\partial^2\phi}{\partial t^2}=36\frac{\partial^2\phi}{\partial x^2}$$

   for $0<x<\pi$ and $t>0$, subject to the boundary and initial conditions

$$\begin{aligned}\phi(0,t)&=0\\ \phi(\pi,t)&=0\\ \phi(x,0)&=\sin(2x)\\ \frac{\partial\phi}{\partial t}(x,0)&=\sin x\end{aligned}$$

   (a) Using the method of separation of variables, show that the wave equation reduces to two ordinary differential equations (ODE's) of the form:

$$\begin{aligned}X''(x)-\lambda X(x)&=0\\ T''(t)-36\lambda T(t)&=0\end{aligned}$$

   where λ is the separation constant.

   (b) By solving the ODE's in part (a) for the case $\lambda<0$, determine the solution of the wave equation subject to the given initial and boundary conditions.

   You may assume that solving the ODE's in part (a) for the cases $\lambda>0$ and $\lambda=0$ leads to trivial solutions. *You do not need to work through these two cases.*

3. Using separation of variables, solve the wave equation

$$\frac{\partial^2 \phi}{\partial x^2} - \frac{\partial^2 \phi}{\partial t^2} = 0$$

for $\{(x,t) : 0 < x < 1, t > 0\}$, subject to the boundary conditions

$$\begin{aligned} \phi(0,t) &= 0 \\ \phi(1,t) &= 0 \\ \frac{\partial \phi}{\partial t}(x,0) &= 0 \\ \phi(x,0) &= \sin(2\pi x) \end{aligned}$$

4. Solve Laplace's equation

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0$$

for $\{(x,y) : 0 < x < L, 0 < y < L\}$, subject to the boundary conditions

$$\begin{aligned} \phi(x,0) &= 0 \\ \frac{\partial \phi}{\partial x}(0,y) &= 0 \\ \frac{\partial \phi}{\partial x}(L,y) &= 0 \\ \phi(x,L) &= EL + \cos\left(\frac{\pi x}{L}\right) \end{aligned}$$

where E is a constant.

5. Solve Laplace's equation

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0$$

in the region

$$\{(x,y) : 0 < x < \pi,\ 0 < y < 1\}$$

subject to the boundary conditions

$$\phi(x,y) = \begin{cases} 0 & \text{at } x = 0 \\ 0 & \text{at } x = \pi \\ \sin x + \sin(2x) & \text{at } y = 0 \\ 2\sin x & \text{at } y = 1 \end{cases}$$

6. Solve Laplace's equation

$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = 0$$

in the region

$$\{(x,y) : 0 < x < 1,\ 0 < y < 1\}$$

subject to the boundary conditions

$$\phi(x,y) = \begin{cases} 0 & \text{at } x = 0 \\ 0 & \text{at } x = 1 \\ \sin(\pi x) & \text{at } y = 0 \\ \sin(2\pi x) & \text{at } y = 1 \end{cases}$$

7. Consider the heat equation

$$\frac{\partial\phi}{\partial t} = 4\frac{\partial^2\phi}{\partial x^2}$$

for $\{(x,t) : 0 < x < 1, t > 0\}$, subject to the boundary and initial conditions

$$\begin{aligned} \phi(0,t) &= 0 \\ \phi(1,t) &= 0 \\ \phi(x,0) &= \sin(3\pi x) \end{aligned}$$

(a) Using the method of separation of variables, show that the heat equation reduces to two ordinary differential equations (ODE's) of the form:

$$\begin{aligned} X''(x) - \lambda X(x) &= 0 \\ T'(t) - 4\lambda T(t) &= 0 \end{aligned}$$

where λ is the separation constant.

(b) By solving the ODE's in part (a) for the case $\lambda < 0$, determine the solution of the heat equation subject to the given initial and boundary conditions.

You may assume that solving the ODE's in part (a) for the cases $\lambda > 0$ and $\lambda = 0$ leads to trivial solutions. *You do not need to work through these two cases.*

8. (a) Determine the Fourier sine series for the function

$$f(x) = \begin{cases} 1 & 0 < x \le \frac{L}{2} \\ 0 & \frac{L}{2} < x \le L \end{cases}$$

(b) Using your answer to part (a), solve the diffusion equation

$$\frac{\partial \phi}{\partial t} = \frac{\partial^2 \phi}{\partial x^2}$$

for $\{(x,t) : 0 < x < L, \quad t > 0\}$ subject to the boundary conditions

$$\begin{aligned} \phi(0,t) &= 0 \\ \phi(L,t) &= 0 \\ \phi(x,0) &= f(x) \end{aligned}$$

9. (a) Determine the Fourier cosine series for the function

$$g(x) = x(L-x), \qquad 0 < x < L$$

(b) Using your answer to part (a), solve the diffusion equation

$$\frac{\partial \phi}{\partial t} = \frac{\partial^2 \phi}{\partial x^2}$$

for $\{(x,t) : 0 < x < L, \quad t > 0\}$ subject to the boundary conditions

$$\begin{aligned} \frac{\partial \phi}{\partial x}(0,t) &= 0 \\ \frac{\partial \phi}{\partial x}(L,t) &= 0 \\ \phi(x,0) &= g(x) \end{aligned}$$

ADDITIONAL EXERCISES
Kreyszig 10th Edition

Problem Set 12.1 (page 542): Q. 2 - 13

Problem Set 12.3 (page 551): Q. 5 - 14

MAST20029 Engineering Mathematics
Sheet 1 Numerical Answers

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1. (b) $\boldsymbol{\nabla} \cdot \mathbf{v} = 0$, $\boldsymbol{\nabla} \times \mathbf{v} = 4\mathbf{k}$

2. (b) $\boldsymbol{\nabla} \cdot \mathbf{v} = 2xy - 4y$, $\boldsymbol{\nabla} \times \mathbf{v} = -x^2\mathbf{k}$

3. (a) 0
   (b) 3

4. (a) $\mathbf{0}$
   (b) $(10y - 8z)\mathbf{i} - (10x - 6z)\mathbf{j} + (8x - 6y)\mathbf{k}$

5. (a) $\dfrac{45}{4}$
   (b) $9(e - 1)$

6. (a) $\dfrac{9}{2}$
   (b) πab

7. (a)(ii) $\displaystyle\int_0^1 \int_{x^2}^{x} dydx$

   (iii) $\dfrac{1}{6}$

   (iv)

```
>> syms x y
>> int(int(1,'y',x^2,x),'x',0,1)
```

   (b)(ii) $\displaystyle\int_{-2}^{2} \int_0^{\sqrt{4-x^2}} y\,dydx$

   (iii) $\dfrac{16}{3}$

   (iv)

```
>> syms x y
>> int(int(y,'y',0,sqrt(4-x^2)),'x',-2,2)
```

   (c)(ii) $\displaystyle\int_0^4 \int_0^{\sqrt{x}} ye^{x^2}\,dydx$

   (iii) $\dfrac{1}{4}\left(e^{16} - 1\right)$

   (iv)

```
>> syms x y
>> int(int(y*exp(x^2),'y',0,sqrt(x)),'x',0,4)
```

(d)(ii) $\displaystyle\int_1^e \int_{\log_e y}^1 dx dy$

(iii) $e-2$

(iv)

```
>> syms x y
>> int(int(1,'x',log(y),1),'y',1,exp(1))
```

8. (a) $\begin{array}{rcl} 0 \leq & r & \leq 2 \\ 0 \leq & \theta & \leq 2\pi \end{array}$

   (b) 24π

9. (a) $\begin{array}{rcl} 1 \leq & r & \leq 2 \\ 0 \leq & \theta & \leq 2\pi \end{array}$

   (b) 30π

10. (a) $\pi[1-\cos(4)]$

    (b) $\dfrac{\pi}{4}(e^4-1)$

11. (a) $\begin{array}{ccccc} 0 & \leq & z & \leq & 3-3x-3y \\ 0 & \leq & y & \leq & 1-x \\ 0 & \leq & x & \leq & 1 \end{array}$

    (b) $\dfrac{1}{2}$

12. (a) $\begin{array}{ccccc} 0 & \leq & z & \leq & 6-6x-3y \\ 0 & \leq & y & \leq & 2-2x \\ 0 & \leq & x & \leq & 1 \end{array}$

    (b) 2

13. (a) $\begin{array}{ccccc} 0 & \leq & z & \leq & 4-r^2 \\ 0 & \leq & r & \leq & 2 \\ 0 & \leq & \theta & \leq & 2\pi \end{array}$

    (b) 0

    (c)
```
>> syms r t z
>> int(int(int(r^2*sin(t),'z', 0, 4-r^2),'r',0,2), 't', 0, 2*pi)
```

14. (a) $\begin{array}{rcl} 0 \leq & z & \leq 4 \\ 0 \leq & r & \leq 2 \\ -\pi/2 \leq & \theta & \leq \pi/2 \ \text{ OR } \ 0 \leq \theta \leq \pi/2, \ 3\pi/2 \leq \theta \leq 2\pi \end{array}$

    (b) $\dfrac{128}{3}$

15. (b) $\dfrac{\pi}{4}$

16. (a) $$\begin{array}{rcl} 0 \leq & r & \leq a \\ 0 \leq & \phi & \leq \pi/2 \\ 0 \leq & \theta & \leq 2\pi \end{array}$$

(b) $\dfrac{2\pi a^3}{3}$

17. (a) $$\begin{array}{rcl} 0 \leq & r & \leq 2 \\ 0 \leq & \theta & \leq \pi \\ 0 \leq & \phi & \leq \pi \end{array}$$

(b) $\dfrac{128\pi}{15}$

18. (a) $$\begin{array}{rcl} 0 \leq & r & \leq 1 \\ 0 \leq & \theta & \leq 2\pi \\ 0 \leq & \phi & \leq \frac{\pi}{3} \end{array}$$

(b) $\dfrac{\pi}{3}$

(c)
```
>> syms r t p
>> int(int(int(r^2*sin(p),'r',0,1),'t',0,2*pi),'p',0,pi/3)
```

19. Many answers are possible for each parametrisation.

(a) $x(t) = -1 + 4t,\ y(t) = 2 + t$ for $0 \leq t \leq 1$

(b) $x(t) = 2\cos t,\ y(t) = 2\sin t$ for $0 \leq t \leq \frac{\pi}{2}$

(c) $x(t) = 2\cos t,\ y(t) = -2\sin t$ for $\pi \leq t \leq \frac{3\pi}{2}$

(d) $x(t) = 2\cos t,\ y(t) = -3\sin t$ for $\pi \leq t \leq 2\pi$

(e) $x(t) = t,\ y(t) = 2t - t^2$ for $0 \leq t \leq 2$

(f) $x(t) = 2 - t,\ y(t) = 2(2 - t) - (2 - t)^2 = 2t - t^2$ for $0 \leq t \leq 2$

20. (a) 27

(b)
```
>> t=[0:.01:pi/2];
>> x = 3*sin(t);
>> y = 3*cos(t);
>> plot(x,y)
```

21. (a) $x(t) = t,\ y(t) = t^{3/2}$ for $1 \le t \le 4$

    (b) $\frac{1}{27}(40^{3/2} - 13^{3/2})$

    (c)
```
>> t=[1:.01:4];
>> x = t;
>> y = t.^(3/2);
>> plot(x,y)
```

22. (b) $\sqrt{5}\left(3\pi^2 + 2\pi\right)$

23. $\dfrac{195}{4}$

24. (a) $\dfrac{53}{20}$

    (b) 3

    (c) $\dfrac{7}{20}$

25. (a) yes, $\phi = x^2 + \frac{3}{2}y^2 + 2z^2 + c$

    (b) no

    (c) yes, $\phi = x^2ye^z + \frac{1}{3}z^3 + c$

26. (a) $\phi = x^2yz - \cos x + c,\ \ 38 - \cos(3) + \cos(1)$

    (b) $\phi = xz^3 + x^2y + c,\ \ 202$

27. (a) $e\sin(1) + \dfrac{1}{3}e^3 - \dfrac{1}{3}$

    (b) $-\dfrac{3}{2}$

28. $\dfrac{(65^{3/2} - 1)\pi}{6}$

29. $16\sqrt{2}\pi$

30. $\dfrac{9\sqrt{6}}{8}$

31. 0

32. $\dfrac{15\pi}{2}$

33. $\dfrac{4\pi}{3}$

34. $\dfrac{3\pi}{2}$

35. 60

36. 8π

37. $\dfrac{8\pi a^3}{3}$

38. 0

39. 6

40. -2π

41. 0

42. 2π

43. (b) 16π

MAST20029 Engineering Mathematics
Sheet 2 Numerical Answers

---

1. (a) eigenvalues $1, -1$ and eigenvectors $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$, $\begin{pmatrix} 3 \\ 5 \end{pmatrix}$

```
>>m=[4 -3; 5 -4];
>> [t,e]=eig(m)
t =
[ 0.7071, 0.5145]
[ 0.7071, 0.8575]
e =
[ 1,  0]
[ 0, -1]
```

(b) $\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \alpha_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^t + \alpha_2 \begin{pmatrix} 3 \\ 5 \end{pmatrix} e^{-t}$

2. (a) $\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \alpha_1 \begin{pmatrix} \cos t \\ \sin t \end{pmatrix} e^t + \alpha_2 \begin{pmatrix} \sin t \\ -\cos t \end{pmatrix} e^t$

3. (a) $\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \alpha_1 \begin{pmatrix} 3 \\ -1 \end{pmatrix} e^{-2t} + \alpha_2 \begin{pmatrix} 3t+1 \\ -t \end{pmatrix} e^{-2t}$

(b) $\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \alpha_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{2t} + \alpha_2 \begin{pmatrix} t-1 \\ t \end{pmatrix} e^{2t}$

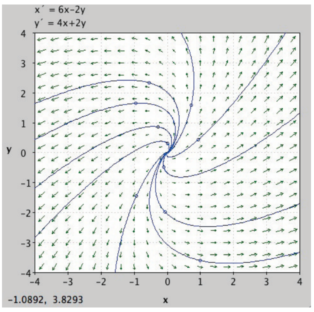
4. (a) $x(t) = -\frac{1}{5}e^{-3t} + \frac{6}{5}e^{2t}, \qquad y(t) = \frac{4}{5}e^{-3t} + \frac{6}{5}e^{2t}$

```
>> [x,y]=dsolve('Dx=x+y','Dy=4*x-2*y','x(0)=1','y(0)=2')
```

(b) $x(t) = 2e^t \cos t + 6e^t \sin t, \qquad y(t) = -e^t \sin t + 3e^t \cos t$

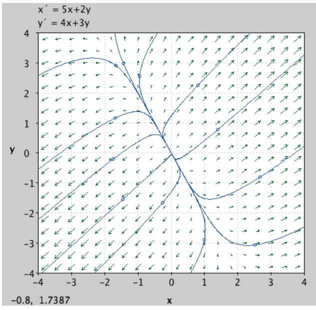
```
>> [x,y]=dsolve('Dx=x+2*y','Dy=-0.5*x+y','x(0)=2','y(0)=3')
```

5. (a) (i)



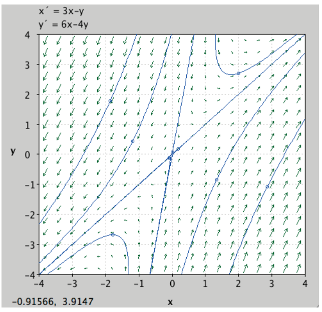
(ii) The origin is an unstable spiral, oriented anticlockwise.

(b) (i)



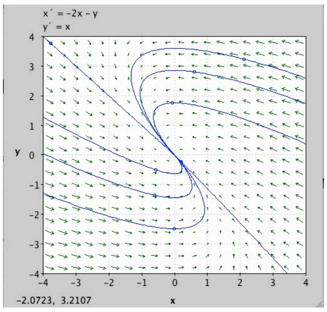
(ii) The origin is an unstable node.

(c) (i)



(ii) The origin is an unstable saddle.

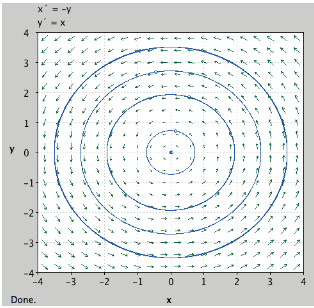
(d) (i)



(ii) The origin is an asymptotically stable node.

6. (a) (i) $\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \alpha_1 \begin{pmatrix} \cos t \\ \sin t \end{pmatrix} + \alpha_2 \begin{pmatrix} \sin t \\ -\cos t \end{pmatrix}$

(ii) The orbits are circles, oriented anticlockwise.



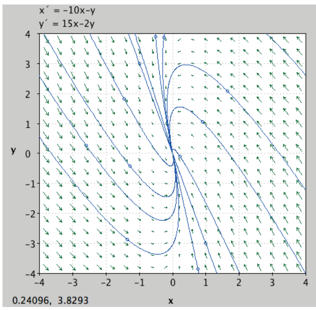
(iii) The origin is a stable centre.

(iv) $\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = 2 \begin{pmatrix} \cos t \\ \sin t \end{pmatrix} - 7 \begin{pmatrix} \sin t \\ -\cos t \end{pmatrix}$

```
>> ezplot('2*cos(t)-7*sin(t)','2*sin(t)+7*cos(t)',([-10,10]))
>> axis([-10 10 -10 10])
```

(b) (i) $\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \alpha_1 \begin{pmatrix} 1 \\ -3 \end{pmatrix} e^{-7t} + \alpha_2 \begin{pmatrix} 1 \\ -5 \end{pmatrix} e^{-5t}$

(ii)



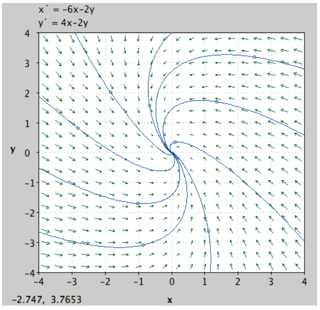
(iii) The origin is an asymptotically stable node.

(iv) $\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \frac{17}{2} \begin{pmatrix} 1 \\ -3 \end{pmatrix} e^{-7t} - \frac{13}{2} \begin{pmatrix} 1 \\ -5 \end{pmatrix} e^{-5t}$

```
>> ezplot('17*exp(-7*t)/2-13*exp(-5*t)/2','-51*exp(-7*t)/2
+65*exp(-5*t)/2',([-10 10]))
>> axis([-3 3 -3 3])
```

7. (a) (i) $\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \alpha_1 \begin{pmatrix} \cos(2t) \\ \sin(2t) - \cos(2t) \end{pmatrix} e^{-4t} + \alpha_2 \begin{pmatrix} \sin(2t) \\ -(\sin(2t) + \cos(2t)) \end{pmatrix} e^{-4t}$

(ii) The orbits are anticlockwise spirals.

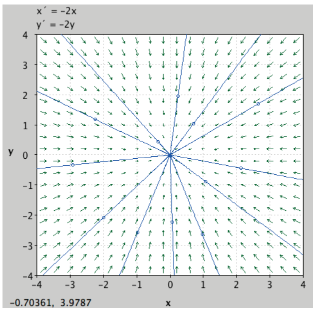


(iii) The origin is asymptotically stable.

(iv) $\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} -2\sin(2t) \\ 2\sin(2t) + 2\cos(2t) \end{pmatrix} e^{-4t}$

(b) (i) $\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \alpha_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{-2t} + \alpha_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{-2t}$

(ii)



(iii) The origin is an asymptotically stable star node.

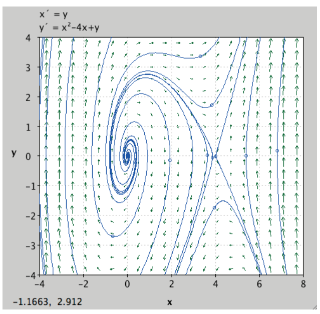
(iv) $\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = 2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{-2t}$

8. (a) $(0, 0)$ and $(4, 0)$.

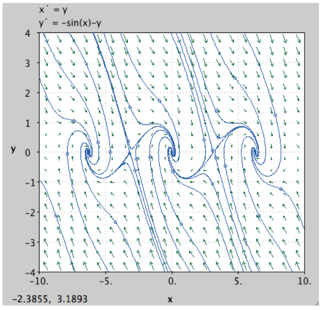
(b) can use linear system to approximate non-linear system near $(0, 0)$ and $(4, 0)$.

(c) (0,0) is an unstable spiral. (4,0) is an unstable saddle.

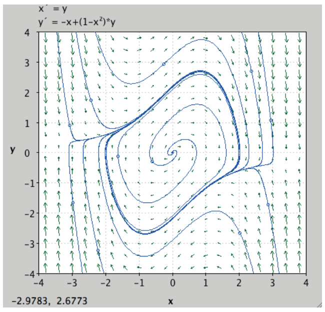
(d)



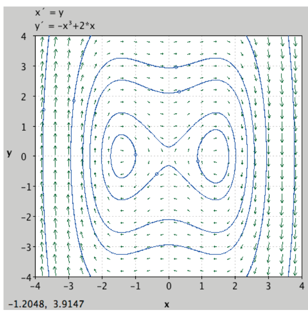
9. (a) $(n\pi, 0)$ where $n = 0, \pm 1, \pm 2 \ldots$
   (b) can use linear system to approximate non-linear system near $(n\pi, 0)$ where $n = 0, \pm 1, \pm 2 \ldots$
   (c) Critical points at $(0, 0)$ and $(n\pi, 0)$ for n even, are asymptotically stable spirals. Critical points at $(n\pi, 0)$ for n odd, are unstable saddles.
   (d)



10. (a) $(0, 0)$
    (b) can use linear system to approximate non-linear system near $(0, 0)$
    (c) The origin is an unstable spiral.
    (d)



11. (a) $(0, 0), (\sqrt{2}, 0), (-\sqrt{2}, 0)$
    (b) can use linear system to approximate non-linear system near (0,0) only.
    (c) 0,0) is an unstable saddle.
    (d)



MAST20029 Engineering Mathematics
Sheet 3 Numerical Answers

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1. (a) $\dfrac{k}{s^2c}(1 - e^{-sc} - cse^{-sc})$

   (b) $\dfrac{k}{s}(e^{-sa} - e^{-sb})$

   (c) $\dfrac{a}{s^2 - a^2}$

   (d) $\dfrac{s}{s^2 + a^2}$

2. (a) $\dfrac{4}{s^3} + \dfrac{6}{s} - \dfrac{3}{s-4}$

   (b) $\dfrac{A\omega}{s^2 + \omega^2} + \dfrac{Bs}{s^2 - k^2}$

   (c) $\dfrac{24}{s^5} + \dfrac{120}{s^6}$

   (d) $\dfrac{s^2 + 2}{s(s^2 + 4)}$

```
>> syms t w A B k
>> laplace(2*t^2+6-3*exp(4*t))
>> laplace(A*sin(w*t)+B*cosh(k*t))
>> laplace(t^4+t^5)
>> laplace((cos(t))^2)
```

3. (a) $3t$

   (b) $\sin(4t)$

   (c) $\cos(3t)$

   (d) $5\cosh(5t)$

```
>> syms s
>> ilaplace(3/s^2)
>> ilaplace(4/(s^2+16))
>> ilaplace(s/(s^2+9))
>> ilaplace(5*s/(s^2-25))
```

4. (a) $\frac{1}{5}\left(4e^{-4t} + e^{t}\right)$

   (b) $\frac{1}{3}\left(e^{2t} - e^{-t}\right)$

5. (a) $\dfrac{3}{(s-2)^2}$

   (b) $\dfrac{20}{(s+6)^2+16}$

   (c) $\dfrac{6}{(s+1)^2-9}$

   (d) $\dfrac{s-1}{(s-1)^2+1} - \dfrac{s+1}{(s+1)^2+1}$

6. (a) $3te^{-2t}$

   (b) $\dfrac{t^2e^{2t}}{2}$

   (c) $e^{4t}\sin t$

   (d) $e^{-2t}\cos(2t)$

7. (a) $\dfrac{se^{-s}}{s^2-1}$

   (b) $\dfrac{e^{-\pi s}}{s^2} + \dfrac{\pi e^{-\pi s}}{s}$

```
>> syms t
>> laplace(t*heaviside(t-pi))
```

8. (a) $\dfrac{1-e^{2-2s}}{s-1}$

   (b) $\dfrac{s}{s^2+\pi^2}\left(-e^{-s}-e^{-4s}\right)$

   (c) $\dfrac{1}{s} + \left(\dfrac{1}{s^2}+\dfrac{4}{s}\right)e^{-5s} - \left(\dfrac{1}{s^2}+\dfrac{10}{s}\right)e^{-10s}$

   (d) $\dfrac{-\pi}{s^2+\pi^2}\left(e^{-s}+e^{-2s}\right) + \dfrac{e^{-6-3s}}{s+2}$

9. (a) $u(t-3)$

   (b) $5u(t-2)+4(t-2)u(t-2)$

10. (a) $u(t-3)e^{-8(t-3)}$

    (b) $\dfrac{1}{5}u(t-2)(e^{4(t-2)}-e^{-(t-2)})$

    (c) $e^{-2t}[\cos(2t)+2\sin(2t)]$

    (d) $\dfrac{1}{2}[1-\cos(2t)]$

    (e) $2\delta(t)+\dfrac{1}{4}\sinh[4(t-1)]u(t-1)$

    (f) $2\sin(2t)e^{-3t}$

11. $y(t) = 3+2t+e^t$

12. $y(t) = 1 - e^{-2t} - 2te^{-2t}$

```
>> y=dsolve('D2y+4*Dy+4*y=4','y(0)=0','Dy(0)=0')
```

13. $x(t) = \cos t, \quad y(t) = \sin t$

14. $x(t) = 1 + t, \quad y(t) = -t$

15. $y_1(t) = \cos t + \sin(\sqrt{3}t), \quad y_2(t) = \cos t - \sin(\sqrt{3}t)$

16. $v(t) = \dfrac{p}{m} \displaystyle\sum_{k=0}^{10} u(t-k)$

17. (a) $\dfrac{1}{s^2(s-1)}$

    (b) $\dfrac{1}{(s-3)(s^2+1)}$

    (c) $\dfrac{2}{s^2-4}$

    (d) $\dfrac{2}{s^4} + \dfrac{2}{s^5}$

18. $y(t) = 1 + \dfrac{3}{2}(e^t + e^{-t})$

19. $i(t) = \dfrac{50}{\sqrt{3}} e^{-t} \sin(\sqrt{3}t)$

20. $i(t) = \dfrac{v_0}{R}\left[e^{-(t-a)/(RC)}u(t-a) - e^{-(t-b)/(RC)}u(t-b)\right]$

MAST20029 Engineering Mathematics
Sheet 4 Numerical Answers

1. (a) 0
   (b) 5
   (c) e^2
   (d) 0
   (e) 0
   (f) 0

2. (a) $\dfrac{1}{3}$

```
>> syms n
>> an = n^2/(3*n^2+4*n+1)
>> limit(an,n,inf)
```

   (b) 0

```
>> syms n
>> an = (2*n^2+6*n+2)/(3*n^3-n^2-n)
>> limit(an,n,inf)
```

   (c) divergent
   (d) divergent

3. (a) 0
   (b) 1
   (c) 0
   (d) $\dfrac{1}{e}$

4. (a) 1
   (b) $\sqrt{e}$
   (c) divergent
   (d) π
   (e) $\log_e\left(\dfrac{2}{3}\right)$
   (f) 0

5. (a) converges to 0
   (b) $\dfrac{3}{4}, \dfrac{15}{16}, \dfrac{63}{64}, \dfrac{255}{256}$
   (c) geometric series $|r| < 1$. Sum is 1.

6. (a) converges to 1
   (b) $0, \frac{1}{2}, \frac{7}{6}, \frac{23}{12}$
   (c) Divergent as terms a_n do not approach 0.

7. (a) $a_n = \frac{1}{n} - \frac{1}{n+1}$
   (b) $s_n = 1 - \frac{1}{n+1}$
   (c) $s = 1$

8. (a) convergent
   (b) divergent
   (c) convergent
   (d) divergent

9. (a) divergent
   (b) convergent
   (c) convergent
   (d) convergent

10. (a) convergent
    (b) divergent
    (c) divergent
    (d) convergent

11. (a) divergent
    (b) convergent
    (c) convergent
    (d) divergent

12. (a) convergent
    (b) divergent
    (c) divergent
    (d) convergent
    (e) convergent
    (f) convergent
    (g) divergent
    (h) convergent

13. (a) (i)$R = \frac{1}{2}$ (ii) $-\frac{1}{2} \leq x < \frac{1}{2}$

(b) (i) $R = 1$ (ii) $-1 \leq x < 1$

(c) (i) $R = \infty$ (ii) $x \in \mathbf{R}$

(d) (i) $R = 1$ (ii) $-2 \leq x \leq 0$

14. (a) $P_3(x) = 1 - \frac{1}{2}(x - \pi/2)^2$

$R_3(x) = \frac{\sin(c)(x - \pi/2)^4}{24}$ where c lies between $\pi/2$ and x

```
>> syms x
>> f= sin(x)
>> taylor(f,x,pi/2,'Order',4)
```

(b) $P_3(x) = 2 + \frac{1}{4}(x - 4) - \frac{1}{64}(x - 4)^2 + \frac{1}{512}(x - 4)^3$

$R_3(x) = \frac{-5(x - 4)^4}{128c^{\frac{7}{2}}}$ where c lies between 4 and x

```
>> syms x
>> f= sqrt(x)
>> taylor(f,x,4,'Order',4)
```

(c) $P_4(x) = \frac{-1}{e} + \frac{(x + 1)^2}{2e} + \frac{(x + 1)^3}{3e} + \frac{(x + 1)^4}{8e}$

$R_4(x) = \frac{(5 + c)e^c}{120}(x + 1)^5$ where c lies between -1 and x

```
>> syms x
>> f= x*exp(x)
>> taylor(f,x,-1,'Order',5)
```

15. (a) $P_2(x) = x^2$

(b) $|R_2(x)| = \frac{1}{6}\,|3\sin(c) + c\cos(c)|\, x^3$ where c lies between 0 and x

(c)
```
>> syms x
>> f=x*sin(x)
>> taylor(f,x,0,'Order',3)
>> diff(f,3)
>> taylortool('x*sin(x)')
```

16. (a) $P_3(x) = x + \frac{x^3}{6}$

(b) $|R_3| < \frac{\sinh(1)}{24} < \frac{1}{16}$

17. (a) $|x| < \sqrt[4]{\frac{6}{25}}$

(b) $|x| < \sqrt[3]{\frac{6}{25}}$

18. (a) $1 + \dfrac{x}{2}$

    (b) $|R_1(x)| < \dfrac{x^2}{8(0.81)^{1.5}} < \dfrac{x^2}{5}$

    (c) $\sqrt{1.1} \approx 1.05, \quad |R_1| = \dfrac{1}{800(1+c)^{1.5}} < \dfrac{1}{800}$

19. (b) $R = 1, \quad -1 < x < 1$

20. (a) $f(x) = \displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}\ x^n}{n}$

    (b) $R = 1, \quad -1 < x \leq 1$

    (c) $\log_e(2)$

MAST20029 Engineering Mathematics
Sheet 5 Numerical Answers

1. (b) $f(t) = \frac{1}{\pi}(e^{\pi} - e^{-\pi})(\frac{1}{2} - \frac{1}{2}\cos t + \frac{1}{5}\cos(2t) - \frac{1}{10}\cos(3t) + \ldots$
$+\frac{1}{2}\sin t - \frac{2}{5}\sin(2t) + \frac{3}{10}\sin(3t) + \ldots)$

   (c) $f(t) = \frac{1}{\pi}(e^{\pi} - e^{-\pi})\left\{\frac{1}{2} + \sum_{n=1}^{\infty}\frac{(-1)^n}{1+n^2}[\cos(nt) - n\sin(nt)]\right\}$

2. (b) $E(t) = \frac{1}{\pi} + \frac{1}{2}\sin t - \left(\frac{2}{3\pi}\right)\cos(2t) - \left(\frac{2}{15\pi}\right)\cos(4t) - \ldots$

   (c) $\frac{1}{\pi^2} + \frac{1}{2}\left[\left(\frac{1}{2}\right)^2 + \left(\frac{2}{3\pi}\right)^2 + \left(\frac{2}{15\pi}\right)^2 + \left(\frac{2}{35\pi}\right)^2 + \ldots\right]$

   (d) $\frac{1}{4}$

3. (b) $f_e(t) = \frac{3}{2} - \frac{2}{\pi}\cos\left(\frac{\pi t}{2}\right) + \frac{2}{3\pi}\cos\left(\frac{3\pi t}{2}\right) - \frac{2}{5\pi}\cos\left(\frac{5\pi t}{2}\right) + \ldots$

   (c) $1, 2$ respectively

   (e) $f_o(t) = \frac{6}{\pi}\sin\left(\frac{\pi t}{2}\right) - \frac{2}{\pi}\sin(\pi t) + \frac{6}{3\pi}\sin\left(\frac{3\pi t}{2}\right) + \frac{6}{5\pi}\sin\left(\frac{5\pi t}{2}\right) + \ldots$

   (f) $0, 0$ respectively

4. (b) $f_e(t) = \frac{\pi^2}{3} - 4\cos t + \cos(2t) - \frac{4}{9}\cos(3t) + \ldots$

   (d) $f_o(t) = \left(\frac{2\pi^2 - 8}{\pi}\right)\sin t - \pi\sin(2t) + \left(\frac{18\pi^2 - 8}{27\pi}\right)\sin(3t) - \frac{\pi}{2}\sin(4t) + \ldots$

5. (a) $f(t) = 2\sum_{n=1}^{\infty}\frac{\sin(nt)}{n}$

   (b) $y_p(t) = 2\sum_{n=1}^{\infty}\frac{\sin(nt)}{n(n^2+1)}$

   (c) $y(t) = Ae^t + Be^{-t} + 2\sum_{n=1}^{\infty}\frac{\sin(nt)}{n(n^2+1)}$

6. (a) $f(t) = \frac{\pi^2}{12} + \sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos(nt)$

   (b) $y_p(t) = \frac{\pi^2}{12} - \frac{3}{52}\cos(2t) + \cdots - \sin(t) + \frac{1}{26}\sin(2t) - \ldots$

   (c) $-\frac{\pi^2}{12}$

7. (b) $f(t) = \displaystyle\int_0^\infty \frac{\cos(\omega t) + \omega \sin(\omega t)}{1+\omega^2}\, d\omega$

8. (b) $f_o(t) = \displaystyle\int_0^\infty \frac{2}{\omega\pi}\left[1 + \cos\omega - 2\cos(4\omega)\right]\sin(\omega t)\, d\omega$

   (c) $f_e(t) = \displaystyle\int_0^\infty \frac{2}{\omega\pi}\left[2\sin(4\omega) - \sin\omega\right]\cos(\omega t)\, d\omega$

   (d) $\frac{3}{2}$

   (e) 1

   (f) $0, 1$ respectively

9. (b) $f_e(t) = \displaystyle\int_0^\infty \frac{\cos(\pi\omega/2)\cos(\omega t)}{1-\omega^2}\, d\omega$

10. (b) $f_o(t) = \dfrac{-2}{\pi}\displaystyle\int_0^\infty \frac{\omega \sin(\omega t)}{\omega^2+4}\, d\omega$

11. (a) $f(t) = \dfrac{2}{\pi}\displaystyle\int_0^\infty \frac{\sin(\omega\pi)}{1-\omega^2}\sin(\omega t)\, d\omega$

    (b) $y_p(t) = \dfrac{1}{\pi}\displaystyle\int_0^\infty \frac{2\sin(\omega\pi)\sin(\omega t)}{(1-\omega^2)^2}\, d\omega$

12. (a) $y(x) = \dfrac{2\omega_0}{\pi}\displaystyle\int_0^\infty \frac{\sin\omega}{\omega}\cos(\omega x)\, d\omega$

    (b) $y_p(x) = \displaystyle\int_0^\infty \frac{2\omega_0}{\pi\omega}\frac{\sin\omega}{(EI\omega^4 + k)}\cos(\omega x)\, d\omega$

MAST20029 Engineering Mathematics
Sheet 6 Numerical Answers

1. (a) Second order, 2 variables, nonlinear, inhomogeneous.
   (b) Second order, 3 variables, nonlinear, inhomogeneous.
   (c) Fourth order, 2 variables, linear, homogenous.
   (d) Third order, 3 variables, nonlinear, homogenous.

2. (b) $\phi(x,t) = \sin(2x)\cos(12t) + \dfrac{1}{6}\sin x \sin(6t)$

3. $\phi(x,t) = \sin(2\pi x)\cos(2\pi t)$

4. $\phi(x,y) = Ey + \dfrac{1}{\sinh(\pi)}\cos\left(\dfrac{\pi x}{L}\right)\sinh\left(\dfrac{\pi y}{L}\right)$

5. $\phi(x,y) = \sin x[\alpha_1 \sinh y + \cosh y] + \sin(2x)[\alpha_2 \sinh(2y) + \cosh(2y)]$
   where $\alpha_1 = \dfrac{2-\cosh(1)}{\sinh(1)}$, $\alpha_2 = \dfrac{-\cosh(2)}{\sinh(2)}$

6. $\phi(x,y) = \sin(\pi x)[\alpha_1 \sinh(\pi y) + \cosh(\pi y)] + \alpha_2 \sin(2\pi x)\sinh(2\pi y)$
   where $\alpha_1 = \dfrac{-\cosh(\pi)}{\sinh(\pi)}$ and $\alpha_2 = \dfrac{1}{\sinh(2\pi)}$

7. (b) $\phi(x,t) = \sin(3\pi x)\, e^{-36\pi^2 t}$

8. (a) $f(x) = \displaystyle\sum_{n=1}^{\infty} \frac{2}{n\pi}[1-\cos(n\pi/2)]\sin\left(\frac{n\pi x}{L}\right)$
   (b) $\phi(x,t) = \displaystyle\sum_{n=1}^{\infty} \frac{2}{n\pi}[1-\cos(n\pi/2)]\sin\left(\frac{n\pi x}{L}\right)\exp\left[-\left(\frac{n\pi}{L}\right)^2 t\right]$

9. (a) $g(x) = \dfrac{L^2}{6} - \displaystyle\sum_{n=1}^{\infty} \frac{L^2}{(n\pi)^2}\cos\left(\frac{2n\pi x}{L}\right)$
   (b) $\phi(x,t) = \dfrac{L^2}{6} - \displaystyle\sum_{n=1}^{\infty} \frac{L^2}{(n\pi)^2}\cos\left(\frac{2n\pi x}{L}\right)\exp\left[-\left(\frac{2n\pi}{L}\right)^2 t\right]$