Chapter 9: Transfer functions

In this chapter we introduce the concept of a transfer function of a system and use it to analyse the behaviour of continuous-time and discrete-time systems.

Definition (Transfer function of a continuous-time system)

Consider the N-th order LTI continuous-time system described by the differential equation

$$\frac{d^N y(t)}{dt^N} + \sum_{i=0}^{N-1} a_i \frac{d^i y(t)}{dt^i} = \sum_{i=0}^{M} b_i \frac{d^i v(t)}{dt^i}$$

where $M < N$ and the coefficients a_i and b_i are real numbers. Assume that $v^{(i)}(0^-) = 0$ for all $i = 0,\ 1, 2, \ldots, M-1$ and $y^{(i)}(0^-) = 0$ for all $i = 0, 1, 2, \ldots, N-1$. Taking Laplace Transforms of both sides yields

$$(s^N + a_{N-1}s^{N-1} + \ldots a_1 s + a_0)Y(s) = (b_M s^M + b_{M-1}s^{M-1} + \ldots b_1 s + b_0)V(s)$$

We define the **input-output transfer function** H for the system to be the rational function

$$H(s) = \frac{Y(s)}{V(s)} = \frac{b_M s^M + b_{M-1}s^{M-1} + \ldots b_1 s + b_0}{s^N + a_{N-1}s^{N-1} + \ldots a_1 s + a_0}$$

Definition

The **numerator and denominator polynomials of** H are

$$\begin{aligned} B(s) &= b_M s^M + b_{M-1} s^{M-1} + \ldots b_1 s + b_0 \\ A(s) &= s^N + a_{N-1} s^{N-1} + \ldots a_1 s + a_0 \end{aligned}$$

Theorem (Transfer function)

Let H be the input-output transfer function of an LTI continuous-time system, let v and y be inputs and outputs to the system and let $V(s)$ and $Y(s)$ be their Laplace transforms, respectively. Then

$$Y(s) = H(s)V(s)$$

- Note that the transfer function model assumes zero initial conditions for both the input v and the output y. Thus the transfer function model will only be useful for obtaining the zero-state response of the system.

Example (Example)

An *RC* circuit has differential equation

$$\frac{dy}{dt} + \frac{1}{RC} y(t) = \frac{1}{RC} v(t)$$

where the input v is the applied voltage and the output y is the capacitor voltage. Taking Laplace transforms of both sides and using the LT pair

$$\frac{dy}{dt} \longleftrightarrow sY(s) - y(0^-)$$

we obtain

$$Y(s) = \frac{1/RC}{s + 1/RC} V(s)$$

assuming $y(0) = 0$. Hence the input-output transfer function of the system is

$$H(s) = \frac{Y(s)}{V(s)} = \frac{1/RC}{s + 1/RC}$$

Example (Example II)

If we compare the *RC* circuit differential equation

$$\frac{dy}{dt} + \frac{1}{RC} y(t) = \frac{1}{RC} v(t)$$

with the general *N*th-order differential equation

$$\frac{d^N y(t)}{dt^N} + \sum_{i=0}^{N-1} a_i \frac{d^i y(t)}{dt^i} = \sum_{i=0}^{M} b_i \frac{d^i v(t)}{dt^i}$$

we see that $N = 1$, $M = 0$, $a_0 = \frac{1}{RC}$, $a_1 = 1$ and $b_0 = \frac{1}{RC}$. Hence

$$B(s) = 1/RC, \quad A(s) = s + 1/RC$$

and so

$$H(s) = \frac{B(s)}{A(s)} = \frac{1/RC}{s + 1/RC}$$

Example (Example III)

The input-output transfer function allows us to obtain the output from any input without solving the differential equation. Suppose the input is v is u, the unit step input. Recall the LT pair

$$u(t) \longleftrightarrow \frac{1}{s}$$

The LT of y, the output from u, is

$$\begin{aligned} Y(s) &= H(s)V(s) \\ &= \left(\frac{1/RC}{s+1/RC}\right)\left(\frac{1}{s}\right) \\ &= \frac{1}{s} - \frac{1}{s+1/RC} \\ \Rightarrow \quad y(t) &= 1 - e^{-t/RC}, \quad \text{for } t \geq 0 \end{aligned}$$

Definition (Convergent, bounded and divergent signals)

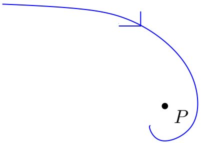
For any continuous time signal x, we say that

(a) x **converges to zero** if $|x(t)| \to 0$ as $t \to \infty$.

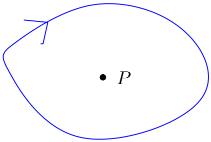
(b) x is **bounded** if there exists a constant $c \geq 0$ such that

$$|x(t)| \leq c \qquad \text{for all } t \geq 0$$

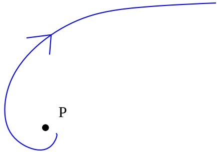
(c) x is **unbounded** or **divergent** if $|x(t)| \to \infty$ as $t \to \infty$.



(a) convergent



(b) bounded



(c) unbounded

Definition (Stability of a continuous-time system)

- An LTI system is said to be **bounded-input bounded-output stable** (BIBO stable) if bounded inputs lead to bounded outputs.
- An LTI system is BIBO stable if and only if its unit impulse response h is absolutely integrable:

$$\int_0^\infty |h(t)| \, dt < \infty$$

- An LTI system is said to be **marginally stable** if its unit impulse response h is bounded but not absolutely integrable.

Remark

- *If the impulse response is unbounded, then the system is said to be* **unstable**. *This means that bounded input signals produce unbounded outputs.*
- *Marginal stability means that there exists at least one bounded input signal that yields an unbounded output.*

Theorem (Stable Transfer functions)

Let H be the transfer function of an LTI continuous-time system with numerator and denominator polynomials B and A. Assume that any common factors of B and A have been cancelled. Then

1. *The system is* **BIBO stable** *if, for all poles p of H, $Re(p) < 0$, and*
2. *The system is* **marginally stable** *if, for all poles p of H,*
   (a) *$Re(p) \leq 0$, and*
   (b) *there is at least one pole p such that $Re(p) = 0$, and p is a non-repeated pole.*

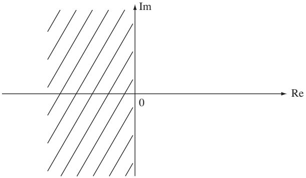


Figure: $Re(p) < 0$ is the **open left-hand complex plane (LHP)**

Example (Stable Transfer functions)

Consider the system with transfer function

$$H(s) = \frac{(s-1)(s-2)}{(s+4)^3(s-2)(s^2+9)}$$

First we cancel any common factors:

$$H(s) = \frac{s-1}{(s+4)^3(s^2+9)}$$

The poles of the denominator polynomial are

$$\{-4, -4, -4, \pm j3\}$$

The system has repeated poles in the left-hand complex plane, and a pair of non-repeated poles on the imaginary axis. So the system is **marginally stable**. *Most* bounded inputs lead to bounded outputs, but there exist *some* bounded inputs that yield unbounded outputs.

Example (Stable Transfer functions II)

To see directly why the system is marginally stable, we can use partial fractions:

$$H(s) = \frac{c_1}{s+4} + \frac{c_2}{(s+4)^2} + \frac{c_3}{(s+4)^3} + \frac{c_4}{s+j3} + \frac{\bar{c}_4}{s-j3}$$

Taking inverses we obtain the unit impulse response of the system:

$$h(t) = c_1 e^{-4t} + c_2 t e^{-4t} + \frac{1}{2} c_3 t^2 e^{-4t} + 2|c_4| \cos(3t + \angle c_4)$$

- For $t \to \infty$, the terms involving e^{-4t} will vanish, while the $\cos(3t + \angle c_4)$ term is bounded.
- Hence the impulse response is bounded, but not convergent, and the system is **marginally stable**.

Example (Stable Transfer functions III)

Lastly, let us suppose instead the transfer function is

$$H(s) = \frac{s-1}{(s+4)(s^2+9)^2}$$

In this case the poles of the denominator polynomial are $\{-4, \pm j3, \pm j3\}$. Since the system has a pair of repeated poles on the imaginary axis, it is unstable. To see why, we use partial fractions:

$$H(s) = \frac{c_1}{s+4} + \frac{c_2}{s+j3} + \frac{\bar{c}_2}{s-j3} + \frac{c_3}{(s+j3)^2} + \frac{\bar{c}_3}{(s-j3)^2}$$

Taking inverses we obtain the unit impulse response of the system:

$$h(t) = c_1 e^{-4t} + d_1 \cos(3t+\phi_1) + d_2 t \cos(3t+\phi_2)$$

For $t \to \infty$, the term $d_2 t \cos(\omega t + \phi_2)$ becomes arbitrarily large, so h is unbounded and the system is **unstable**.

Definition (Transfer function of a discrete-time system)

Consider the N-th order LTI discrete-time system described by the difference equation

$$y[n+N]+\sum_{i=0}^{N-1} a_i y[n+i]=\sum_{i=0}^{M} b_i v[n+i],$$

where $M<N$ and the coefficients a_i and b_i are real numbers. Assume that $v[i]=0$ for all $i=0,\ 1,2,\ldots,M-1$, and $y[i]=0$ for all $i=0,1,2,\ldots,N-1$. Taking z-Transforms of both sides, we obtain

$$(z^N+a_{N-1}z^{N-1}+\ldots a_1 z+a_0)Y(z)=(b_M z^M+b_{M-1}z^{M-1}+\ldots b_1 z+b_0)V(z$$

We define the **input-output transfer function** H for the system to be the rational function

$$H(z)=\frac{Y(z)}{V(z)}=\frac{b_M z^M+b_{M-1}z^{M-1}+\ldots b_1 z+b_0}{z^N+a_{N-1}z^{N-1}+\ldots a_1 z+a_0}$$

Definition

The **numerator and denominator polynomials of** H are

$$\begin{aligned} B(z) &= b_M z^M + b_{M-1} z^{M-1} + \ldots b_1 z + b_0 \\ A(z) &= z^N + a_{N-1} z^{N-1} + \ldots a_1 z + a_0 \end{aligned}$$

Remark

Let H be the input-output transfer function of an LTI discrete-time system, let v and y be inputs and outputs to the system and let V and Y be their z-Transforms, respectively. Then

$$Y(z) = H(z)V(z)$$

Note that the transfer function assumes zero initial conditions for both the input v and the output y. Thus transfer functions can only be used for obtaining the zero-state response of the system.

Definition (Stability of a discrete-time system)

- An LTI system is said to be **bounded-input bounded-output stable** (BIBO stable) if bounded inputs lead to bounded outputs.
- An LTI system is BIBO stable if and only if its unit pulse response h is absolutely summable:

$$\sum_{n=0}^{\infty} |h[n]| < \infty$$

- An LTI system is said to be **marginally stable** if its unit pulse response h is bounded but not absolutely summable.
- An LTI system is said to be **unstable** if its unit pulse response h is unbounded.

Theorem (Stable Transfer functions)

Let H be the transfer function of an LTI discrete-time system with numerator and denominator polynomials B and A. Assume that any common factors of B and A have been cancelled. Then

1. *The system is* **BIBO stable** *if, for every pole p of H, $|p| < 1$, and*
2. *The system is* **marginally stable** *if, for all poles p of H,*
   (a) $|p| \leq 1$, *and*
   (b) *there is at least one pole p such that $|p| = 1$, and p is a non-repeated pole.*

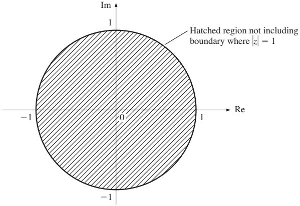


Figure: $|z| < 1$ is the **open unit disk** in the complex plane

Example

Consider the difference equation

$$y[n+4]+\frac{11}{2}y[n+3]+\frac{47}{18}y[n+2]+\frac{11}{18}y[n+1]+\frac{5}{18}y[n]=v[n+2]+v[n+1]$$

Assuming zero initial conditions, the left shift theorem gives the zT pair

$$x[n+q]\longleftrightarrow z^q X(z)$$

Applying this to the difference equation gives

$$Y(z)\left[z^4+\frac{11}{2}z^3+\frac{47}{18}z^2+\frac{11}{18}z+\frac{5}{18}\right]=V(z)[z^2+z]$$

and so we obtain input-output transfer function

$$H(z)=\frac{Y(z)}{V(z)}=\frac{z^2+z}{z^4+\frac{11}{2}z^3+\frac{47}{18}z^2+\frac{11}{18}z+\frac{5}{18}}$$

Example

Factorising the denominator polynomial gives

$$\frac{H(z)}{z} = \frac{z+1}{(z+.5)(z+5)(z^2+1/9)}$$

The poles of the denominator polynomial are $\{-0.5, -5, \pm j/3\}$. The pole at $z=-5$ is outside the unit disk, and hence is the system is **unstable**. To see why, we use partial fractions:

$$\begin{aligned}
\frac{H(z)}{z} &= \frac{c_0}{z} + \frac{c_1}{z+5} + \frac{c_2}{z+.5} + \frac{c_3}{z+j/3} + \frac{\bar{c}_3}{z-j/3} \\
H(z) &= c_0 + \frac{c_1 z}{z+5} + \frac{c_2 z}{z+.5} + \frac{c_3 z}{z+j/3} + \frac{\bar{c}_3 z}{z-j/3} \\
\Rightarrow h[n] &= c_0\delta[n] + c_1(-5)^n + c_2(-0.5)^n + 2|c_3|\left(\frac{1}{3}\right)^n \cos(\frac{\pi n}{2} + \angle c_3)
\end{aligned}$$

As $n \to \infty$, $|c_1(-5)^n| \to \infty$, so h is unbounded and the system is **unstable**.

Theorem (Step response of a continuous-time system)

Let H be the transfer function of an LTI continuous-time system with numerator and denominator polynomials B and A, and assume that any common factors of B and A have been canceled. Let v be the unit-step function, so that $V(s) = 1/s$. Let y be the output from the input v so that

$$Y(s) = \frac{B(s)V(s)}{A(s)} = \frac{B(s)}{sA(s)}$$

Let E be a polynomial function in s such that

$$Y(s) = \frac{E(s)}{A(s)} + \frac{H(0)}{s}$$

Then the system **step response** *is given by*

$$y(t) = y_1(t) + H(0), \quad t \geq 0.$$

where y_1 is the inverse LT of $E(s)/A(s)$.

Remark

- *For stable systems, the term $y_1(t)$ converges to zero. We say that y_1 is the* **transient response** *of the system. The constant $H(0)$ is the* **steady-state** *value of the step response.*
- *Recall the RC circuit example had transfer function*

$$H(s) = \frac{1/RC}{s + 1/RC} = \frac{Y(s)}{U(s)}$$

so $H(0) = 1$. The step input u with $U(s) = 1/s$ yields output

$$\begin{aligned} Y(s) &= \frac{1}{s} - \frac{1}{s + 1/RC} \\ \Rightarrow y(t) &= 1 - e^{-t/RC} \\ &= H(0) + y_1(t) \end{aligned}$$

Hence $H(0) = 1$ is the steady-state response, and $y_1(t) = -e^{-t/RC}$ is the transient response.

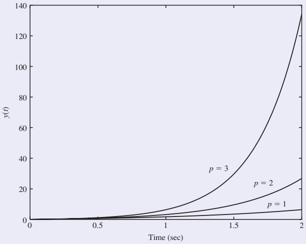
Case (First-order step response)

For a general first-order transfer function $H(s) = \frac{k}{s-p}$, the step response is $y(t) = -\frac{k}{p}(1 - e^{pt})$. The response is shown for the cases

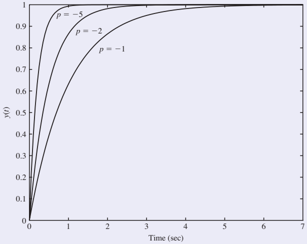
(a) *the poles are unstable, e.g. $p = 1$, $p = 2$ and $p = 3$.*

(b) *the poles are stable, e.g. $p = -1$, $p = -2$ and $p = -5$.*

For the unstable poles, the output becomes arbitrarily large (unbounded). For the stable poles, the output converges to $H(0) = -\frac{k}{p}$.



(a) unstable poles



(b) stable poles

Case (Second-order step response)

Consider the second-order transfer function

$$H(s) = \frac{k}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

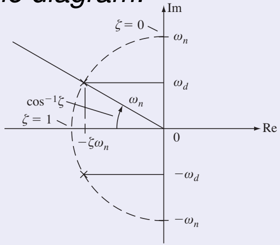
The real parameters ζ and ω_n are respectively called the **damping ratio** *and* **natural frequency**. *Assuming $\zeta > 0$ and $\omega_n > 0$ implies the system is stable. The poles are*

$$p_1, p_2 = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1}$$

- *When $\zeta > 1$, the poles are real and distinct (Overdamped).*
- *When $\zeta = 1$, the poles are real and repeated (Critically damped).*
- *When $\zeta < 1$, the poles are complex-conjugate pairs (Underdamped).*

Case (ζ, ω_n and the pole locations II)

For $0 \le \zeta < 1$, the poles are complex-conjugate pairs and their locations are shown in the diagram:



- *The distance of the poles from the origin is ω_n.*
- *The phase of the poles is $\pi \pm \cos^{-1}(\zeta)$.*
- *Smaller values of ζ yield poles are closer to the imaginary axis. This corresponds to a slower transient response, and explains why ζ is called the* **damping ratio**.

Case (Overdamped response: $\zeta > 1$)

With $\zeta > 1$ we have two distinct real poles p_1, p_2 and the transform of the step response is

$$\begin{aligned} Y(s) &= \frac{k}{s(s-p_1)(s-p_2)} \\ \Rightarrow y(t) &= \frac{k}{p_1 p_2}(k_1 e^{p_1 t} + k_2 e^{p_2 t} + 1), \quad t \geq 0 \end{aligned}$$

Here

$$\begin{aligned} y_{tr}(t) &= \frac{k}{p_1 p_2}(k_1 e^{p_1 t} + k_2 e^{p_2 t}) \quad \textit{is the transient response} \\ y_{ss} &= \frac{k}{p_1 p_2} \quad \textit{is the steady-state value} \end{aligned}$$

Example ($\zeta > 1$)

Suppose the LT of the step response is

$$Y(s) = \frac{2}{s(s+1)(s+2)} = \frac{-2}{s+1} + \frac{1}{s+2} + \frac{1}{s}$$

with system poles $p = -1$ and $p = -2$. Then the step response is

$$y(t) = -2e^{-t} + e^{-2t} + 1, \quad t \geq 0$$

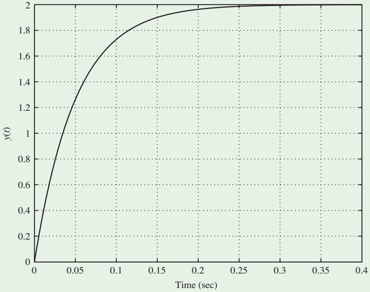
where

$$\begin{aligned} y_{tr}(t) &= -2e^{-t} + e^{-2t} \quad \text{is the transient response} \\ y_{ss} &= 1 \quad \text{is the steady-state value} \end{aligned}$$

Note that $e^{-2t} << e^{-t}$ so that the contribution to the output from the pole $p_2 = -2$ is much smaller than from pole $p_1 = -1$.

Example ($\zeta > 1$)

As a result the output looks similar to the graph of a step response from a first order system



We say that the pole at $p_2 = -1$ is the **dominant pole.** The dominant pole is the one that is closer to the imaginary axis, as it has the larger time constant.

Case (Critically damped response: $\zeta = 1$)

With $\zeta = 1$ *we have repeated real poles* $p_1 = p_2$ *and the transform of the step response is*

$$\begin{aligned} Y(s) &= \frac{k}{s(s+\omega_n)^2} \\ \Rightarrow y(t) &= \frac{k}{\omega_n^2}(1-(1+\omega_n t)e^{-\omega_n t}), \quad t \geq 0 \end{aligned}$$

Here

$$\begin{aligned} y_{tr}(t) &= -\frac{k}{\omega_n^2}(1+\omega_n t)e^{-\omega_n t} \quad \textit{is the transient response} \\ y_{ss} &= \frac{k}{\omega_n^2} \quad \textit{is the steady-state value} \end{aligned}$$

Example ($\zeta = 1$)

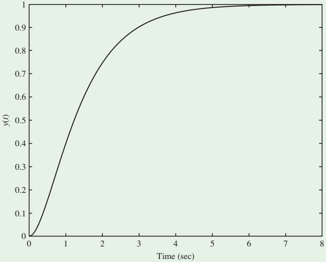
Suppose the LT of the step response is

$$Y(s) = \frac{4}{s(s+2)^2} = \frac{-1}{s+2} + \frac{-2}{(s+2)^2} + \frac{1}{s}$$

with system repeated poles at $p = -2$. Then the step response is

$$y(t) = 1 - (1+2t)e^{-2t}, \quad t \geq 0$$

The response is similar to a first-order system with pole at $p = -2$.



Case (Underdamped response: $0 < \zeta < 1$)

With $0 < \zeta < 1$ we have complex-conjugate poles $p_1, p_2 = -\zeta\omega_n \pm j\omega_d$, where $\omega_d = \omega_n\sqrt{1-\zeta^2}$. The transform of the step response is

$$\begin{aligned} Y(s) &= \frac{k}{s((s+\zeta\omega_n)^2+\omega_d^2)} \\ \Rightarrow y(t) &= \frac{k}{\omega_n^2}\left[1 - \frac{\omega_n}{\omega_d}e^{-\zeta\omega_n t}\sin(\omega_d t+\phi)\right], \quad t \geq 0 \end{aligned}$$

where $\phi = \tan^{-1}(\omega_d/\zeta\omega_n)$.

- *The steady-state value is $y_{ss} = \frac{k}{\omega_n^2}$.*
- *The transient response is $y_{tr}(t) = -\frac{k}{\omega_n^2}\frac{\omega_n}{\omega_d}e^{-\zeta\omega_n t}\sin(\omega_d t+\phi)$. This is an exponentially decaying sinusoid.*

Example ($0 < \zeta < 1$)

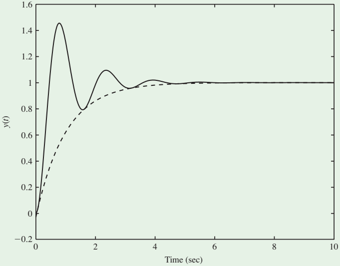
Suppose the LT of the step response is

$$Y(s) = \frac{17}{s(s^2 + 2s + 17)}$$

with system poles $p_1, p_2 = -1 \pm j4$. The step response is

$$y(t) = 1 - \frac{\sqrt{17}}{4} e^{-t} \sin(4t + 1.326), \quad t \geq 0$$

The response is an oscillating and decaying sinusoid. The dashed line shows the exponential part of the transient.

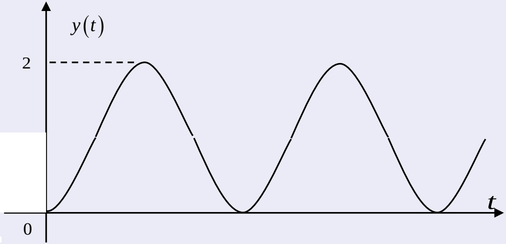


Case (Undamped response: $\zeta = 0$)

With $\zeta = 0$ we have purely imaginary poles $p_1, p_2 = \pm j\omega_n$. The transform of the step response is

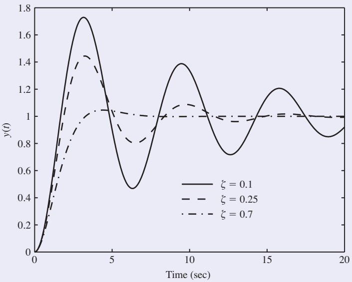
$$\begin{aligned} Y(s) &= \frac{k}{s(s^2+\omega_n^2)} \\ \Rightarrow y(t) &= \frac{k}{\omega_n^2}\left(1-\cos(\omega_n t)\right), \quad t \geq 0 \end{aligned}$$

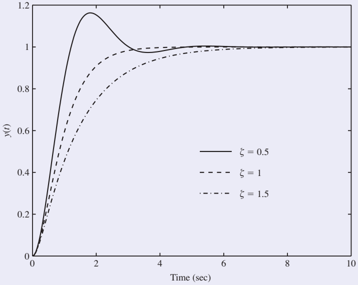
Since the system poles lie on the imaginary axis, the system is **marginally stable**. *The step response is bounded but not convergent.*



Case (ζ and the oscillations in the transient response)

These graphs show how the oscillations in the transient vary with ζ:





- *For $0 \leq \zeta < 1$, there are oscillations in the transient, with smaller ζ giving rise to larger oscillations.*
- *For $\zeta \geq 1$, the poles are real and there is no overshoot of the steady-state value.*

Remark (Limitations of the transient analysis)

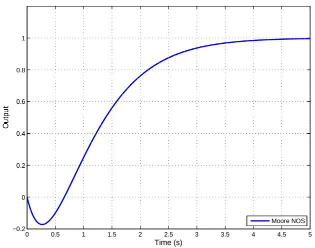
The above discussion assumed the transfer function was of the form

$$H(s) = \frac{k}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

The analysis is not applicable if the numerator is not constant.
Consider

$$H(s) = \frac{s-2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

The numerator polynomial $B(s) = s - 2$ has a zero at $s = 2$. Such systems always exhibit **undershoot** *in their transient response.*



Theorem (Sinusoidal response of a continuous time system)

Let H be the transfer function of an LTI continuous-time system with numerator and denominator polynomials B and A, and assume that B and A have no common factors. Let $v(t) = \cos(\omega_0 t)u(t)$ be a sinusoidal input of arbitrary frequency ω_0. Let y be the output from input v. Then

$$Y(s) = \frac{B(s)V(s)}{A(s)} = \frac{sB(s)}{A(s)(s^2+\omega_0^2)}$$

Let γ be a polynomial in s such that

$$Y(s) = \frac{\gamma(s)}{A(s)} + \frac{c}{s-j\omega_0} + \frac{\bar{c}}{s+j\omega_0}$$

Then the system **sinusoidal response** *is given by*

$$y(t) = y_1(t) + |H(j\omega_0)|\cos(\omega_0 t + \underline{/H(j\omega_0)}), \quad t \geq 0$$

where y_1 is the inverse LT of $\gamma(s)/A(s)$.

Remark

- *For stable systems, the* **transient response** *term $y_1(t)$ converges to zero. The* **steady-state response** *is*

$$y_{ss}(t) = |H(j\omega_0)| \cos(\omega_0 t + \underline{/H(j\omega_0)}), \quad t \geq 0$$

- *The steady-state response is also a sinusoidal signal, of the same frequency as the input. It is scaled in magnitude by the amount $|H(j\omega_0)|$ and phase-shifted by $\angle H(j\omega_0)$.*
- *Recall that if $H(\omega)$ is the frequency response of a stable LTI system, then the input $v(t) = \cos(\omega_0 t)$ has output*

$$y(t) = |H(\omega_0)| \cos(\omega_0 t + \underline{/H(\omega_0)})$$

These results are in agreement because $H(\omega) = H(s)|_{s=j\omega} = H(j\omega)$ when H is a stable LTI system There is no transient term because $\cos(\omega_0 t)$ is applied from $t = -\infty$.

Example (Resonance)

Consider the system with transfer function

$$H(s) = \frac{64}{s^2+64}$$

The system is marginally stable with purely imaginary poles $p = \pm j8$. Applying a sinusoidal input $v(t) = \cos(8t)u(t)$, the output has LT

$$\begin{aligned}
Y(s) &= \frac{64}{(s^2+64)}\frac{s}{(s^2+64)} \\
&= \frac{-j8}{4}\left(\frac{1}{(s-j8)^2} - \frac{1}{(s+j8)^2}\right) \\
\Rightarrow y(t) &= -j2\left(te^{j8t} - te^{-j8t}\right) = 4t\sin(8t), \quad t \geq 0
\end{aligned}$$

Clearly $y(t) \to \infty$ as $t \to \infty$. This is called **resonance**, where a bounded input produces an unbounded output. This can only occur because the system is marginally stable.

Theorem (Response of a discrete-time system)

Let H be the transfer function of a stable LTI discrete-time system.

1. *The system* **step response** *to $v[n] = u[n]$ is*

$$y[n] = y_{tr}[n] + H(1), \quad n \geq 0.$$

where y_{tr} is the **transient response**, *and $H(1)$ is the* **steady-state value**.

2. *The system* **sinusoidal response** *to $v[n] = \cos(\Omega_0 n)u[n]$ is*

$$y[n] = y_{tr}[n] + y_{ss}[n], \quad n \geq 0.$$

where y_{tr} is the **transient response**, *and the* **steady-state response** *is*

$$y_{ss}[n] = |H(e^{j\Omega_0})| \cos(\Omega_0 n + \angle H(e^{j\Omega_0})), \quad n \geq 0$$

Example (Example: Step response)

Consider the discrete-time system with transfer function

$$H(z) = \frac{1}{z-a}, \quad \text{where } |a| < 1$$

Suppose the input is the unit step input. Recalling the zT pair $b^n u[n] \longleftrightarrow \frac{z}{z-b}$ and choosing $b = 1$ we see that $u[n] \longleftrightarrow \frac{z}{z-1}$. Hence the zT of the output is

$$\begin{aligned} Y(z) &= H(z)U(z) \\ &= \left(\frac{1}{z-a}\right)\left(\frac{z}{z-1}\right) \\ &= \frac{1}{1-a}\left(\frac{-z}{z-a} + \frac{z}{z-1}\right) \\ \Rightarrow \quad y[n] &= \frac{1}{1-a} + \frac{a^n}{1-a}, \quad \text{for } n \geq 0 \\ &= y_{ss} + y_{tr}[n] \end{aligned}$$

Consider the N-th order LTI continuous-time system described by the differential equation

$$\frac{d^N y}{dt^N} + \sum_{i=0}^{N-1} a_i \frac{d^i y}{dt^i} = \sum_{i=0}^{N-1} c_i \frac{d^i v}{dt^i}$$

where the coefficients a_i and c_i are real numbers. Recall that a **state vector** for a system is a vector of the form

$$x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_N(t) \end{bmatrix}$$

- Together with a known input $v(t)$, the state vector contains sufficient information at any time t to enable the future outputs $y(t)$ to be uniquely determined.
- The variables $x_1, \ldots, x_N$ are called the **state variables** of the system, and N is the **dimension** of the state.

Definition (State representation for an N-th order system I)

Let H be the input-output transfer function of a proper LTI continuous-time system

$$H(s) = \frac{Y(s)}{V(s)}$$

Let A and C be the real polynomials

$$\begin{aligned} C(s) &= c_{N-1}s^{N-1} + \cdots + c_1 s + c_0 \\ A(s) &= s^N + a_{N-1}s^{N-1} + \cdots + a_1 s + a_0 \end{aligned}$$

Then the transfer function H can be expressed as the rational function

$$H(s) = \frac{C(s)}{A(s)}$$

Introduce the signal w such that $C(s)W(s) = Y(s)$:

$$c_0 w(t) + c_1 \frac{dw}{dt} + \cdots + c_{N-1} \frac{d^{N-1}w}{dt^{N-1}} = y(t)$$

Definition (State representation for a N-th order system II)

and also define state variables

$$
\begin{aligned}
x_i(t) &= \frac{dw^{i-1}}{dt^{i-1}}, \quad i = 1, 2, \ldots, N \\
\text{then } x_{i+1}(t) &= \frac{dx_i}{dt}, \quad i = 1, 2, \ldots, N-1 \\
\text{Hence } y(t) &= c_0 x_1(t) + c_1 x_2(t) + \cdots + c_{N-1} x_N(t) \\
\text{Also } V(s) &= \frac{V(s)Y(s)}{Y(s)} \\
&= \frac{A(s)Y(s)}{C(s)} \\
&= A(s)W(s) \\
\text{Hence } v(t) &= a_0 w(t) + a_1 \frac{dw}{dt} + \cdots + a_{N-1} \frac{d^{N-1}w}{dt^{N-1}} + \frac{d^N w}{dt^N} \\
&= a_0 x_1(t) + a_1 x_2(t) + \cdots + a_{N-1} x_N(t) + \dot{x}_N(t)
\end{aligned}
$$

Definition (State representation for an N-th order system III)

If we write these in matrix form we obtain the **state space representation in controller canonical form**:

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \vdots \\ \dot{x}_N(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ -a_0 & -a_1 & -a_2 & \dots & -a_{N-1} \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_N(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} v(t)$$

$$y(t) = \begin{bmatrix} c_0 & c_1 & \dots & c_{N-1} \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_N(t) \end{bmatrix}$$

If we introduce state matrices

$$
\begin{aligned}
A &= \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ -a_0 & -a_1 & -a_2 & \dots & -a_{N-1} \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} \\
C &= \begin{bmatrix} c_0 & c_1 & \dots & c_{N-1} \end{bmatrix}
\end{aligned}
$$

then we can write the system in compact matrix form as

$$
\begin{aligned}
\dot{x}(t) &= Ax(t) + Bv(t) \\
y(t) &= Cx(t)
\end{aligned}
$$

For brevity we sometimes use (A, B, C) to denote the state matrices in the system representation.

Example

Consider the system continuous-time system described by

$$\frac{d^2y}{dt^2} + 4\frac{dy}{dt} - 8y(t) = -\frac{dv}{dt} + v(t)$$

Assuming zero initial conditions we have the Laplace transform pairs

$$\frac{dx}{dt} \longleftrightarrow sX(s), \quad \frac{d^2x}{dt^2} \longleftrightarrow s^2X(s)$$

which gives

$$Y(s)[s^2 + 4s - 8] = V(s)[-s + 1]$$

so the input-output transfer function for the system is

$$H(s) = \frac{Y(s)}{V(s)} = \frac{-s+1}{s^2+4s-8}$$

Example

To obtain a state space representation in controller canonical form, we introduce polynomials

$$\begin{aligned} C(s) &= c_0 + c_1 s = 1 - s \\ A(s) &= a_0 + a_1 s + s^2 = -8 + 4s + s^2 \end{aligned}$$

and let the implicit variable $w(t)$ be such that $C(s)W(s) = Y(s)$, which is equivalent to

$$w(t) - \frac{dw}{dt} = y(t)$$

Next we define state variables

$$x_1(t) = w(t),\ x_2(t) = \dot{x}_1(t),$$

Hence

$$y(t) = x_1(t) - x_2(t)$$

Example

Also $V(s) = A(s)W(s)$, so

$$\begin{aligned} v(t) &= a_0 w(t) + a_1 \dot{w}(t) + \ddot{w}(t) \\ &= a_0 x_1(t) + a_1 x_2(t) + \dot{x}_2(t) \end{aligned}$$

which yields

$$\dot{x}_2(t) = v(t) + 8x_1(t) - 4x_2(t)$$

Hence the state equations are

$$\begin{aligned} \begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} &= \begin{bmatrix} 0 & 1 \\ 8 & -4 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} v(t) \\ y(t) &= \begin{bmatrix} 1 & -1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} \end{aligned}$$

Definition (State representation for a proper N-th order discrete-time system I)

Consider the N-th order LTI discrete-time system described by the difference equation

$$y[n+N]+\sum_{i=0}^{N-1} a_i y[n+i]=\sum_{i=0}^{N-1} c_i v[n+i], \quad \text{for } n \geq -N$$

where the coefficients a_i and c_i are real numbers. Let H be the transfer function from the input y to the output v, so that

$$H(z)=\frac{Y(z)}{V(z)}$$

Let A and C be the real polynomials

$$\begin{aligned} C(z) &= c_{N-1}z^{N-1}+\ldots c_1 z+c_0 \\ A(z) &= z^N+a_{N-1}z^{N-1}+\ldots a_1 z+a_0 \end{aligned}$$

Definition (State representation for an N-th order system II)

The transfer function H can be expressed as the rational function

$$H(z) = \frac{C(z)}{A(z)}$$

Introduce the variable w such that $C(z)W(z) = Y(z)$:

$$c_{N-1}w[n+N-1] + \ldots c_1 w[n+1] + c_0 w[n] = y[n]$$

and also define state variables

$$\begin{aligned} x_i[n] &= w[n+i-1], \quad i = 1,2,\ldots,N \\ \text{then } x_i[n+1] &= x_{i+1}[n], \quad i = 1,2,\ldots,N-1 \end{aligned}$$

It can be shown that $V(z) = A(z)W(z)$, and hence

$$v[n] = a_0 x_1[n] + a_1 x_2[n] + \cdots + a_{N-1}x_N[n] + x_N[n+1]$$

Definition (State representation for an N-th order discrete-time system III)

Writing these in matrix form we obtain the **state representation in controller canonical form**:

$$
\begin{aligned}
x[n+1] &= \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ -a_0 & -a_1 & -a_2 & \dots & -a_{N-1} \end{bmatrix} \begin{bmatrix} x_1[n] \\ x_2[n] \\ \vdots \\ x_N[n] \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} v[n] \\
y[n] &= \begin{bmatrix} c_0 & c_1 & \dots & c_{N-1} \end{bmatrix} \begin{bmatrix} x_1[n] \\ x_2[n] \\ \vdots \\ x_N[n] \end{bmatrix}
\end{aligned}
$$

Example (State representation for a discrete-time system I)

Obtain a state representation for LTI discrete-time system described by the difference equation

$$y[n+3] = 8v[n+2] - 4v[n+1], \quad \text{for } n \geq -3$$

Let A and C be the real polynomials

$$\begin{aligned} C(z) &= 8z^2 - 4z \\ A(z) &= z^3 \end{aligned}$$

Then the input-output transfer function for the system is

$$H(z) = \frac{Y(z)}{V(z)} = \frac{8z^2 - 4z}{z^3}$$

Introduce the variable w such that $C(z)W(z) = Y(z)$:

$$8w[n+2] - 4w[n+1] = y[n]$$

Example (State representation for a discrete-time system II)

Also introduce the state variables

$$
\begin{aligned}
x_1[n] &= w[n], \quad x_2[n] = w[n+1], \quad x_3[n] = w[n+2] \\
\text{then} \quad x_1[n+1] &= x_2[n] \quad \text{and } x_2[n+1] = x_3[n] \\
y[n] &= -4x_2[n] + 8x_3[n] \\
V(z) &= A(z)W(z) \\
\Rightarrow v[n] &= w[n+3] = x_3[n+1]
\end{aligned}
$$

Hence we obtain the state representation

$$
\begin{aligned}
\begin{bmatrix} x_1[n+1] \\ x_2[n+1] \\ x_3[n+1] \end{bmatrix} &= \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1[n] \\ x_2[n] \\ x_3[n] \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} v[n] \\
y[n] &= \begin{bmatrix} 0 & -4 & 8 \end{bmatrix} \begin{bmatrix} x_1[n] \\ x_2[n] \\ x_3[n] \end{bmatrix}
\end{aligned}
$$

Next we consider how to obtain the transfer function from any given state representation. Suppose the system is

$$\begin{aligned}\dot{x}(t) &= Ax(t)+Bv(t)\\ y(t) &= Cx(t)\end{aligned}$$

Taking Laplace transforms in the first state equation, and assuming zero initial conditions, gives

$$\begin{aligned}sX(s) &= AX(s)+BV(s)\\ \Rightarrow (sI-A)X(s) &= BV(s)\\ \Rightarrow X(s) &= (sI-A)^{-1}BV(s)\end{aligned}$$

where I denotes the $N\times N$ identity matrix. Taking Laplace transforms in the second state equation gives

$$\begin{aligned}Y(s) &= CX(s)\\ &= C(sI-A)^{-1}BV(s)\\ &= [C(sI-A)^{-1}B]V(s)\end{aligned}$$

The above analysis leads to:

Theorem

Consider an LTI system with either continuous-time or discrete-time state representation given by

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bv(t) \\ y(t) &= Cx(t) \end{aligned} \quad \textit{or} \quad \begin{aligned} x[n+1] &= Ax[n] + Bv[n] \\ y[n] &= Cx[n] \end{aligned}$$

Then the corresponding input-output transfer function for the system is

$$H(s) = C(sI - A)^{-1}B \quad \textit{or} \quad H(z) = C(zI - A)^{-1}B$$

Note that while an LTI system has many different state representations, its transfer function is unique. Since the poles of the system are the roots of the denominator polynomial of the transfer function, we obtain:

Theorem

The poles of a LTI continuous-time or discrete-time system are those $s \in \mathbf{C}$ (or $z \in \mathbf{C}$) such that

$$\det(sI - A) = 0 \quad \textit{or} \quad \det(zI - A) = 0$$

Recall the definition of an eigenvalue of a square matrix:

Definition

Let A be a square matrix. Then $\lambda \in \mathbf{C}$ is an eigenvalue of A if and only if

$$\det(\lambda I - A) = 0$$

Hence we see that

Theorem

For any LTI continuous-time or discrete-time system with state representation (A, B, C), the poles of the system are equal to the eigenvalues of A.

Example (Transfer function from the State representation I)

To obtain the transfer function of the system

$$\begin{aligned}
\dot{x}(t) &= \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} x(t) + \begin{bmatrix} 1 \\ 2 \end{bmatrix} v(t) \\
y(t) &= \begin{bmatrix} 3 & 4 \end{bmatrix} x(t) \\
\text{we compute} \quad H(s) &= C(sI - A)^{-1}B \\
&= \begin{bmatrix} 3 & 4 \end{bmatrix} \begin{bmatrix} s & -1 \\ 2 & s+3 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 2 \end{bmatrix} \\
&= \begin{bmatrix} 3 & 4 \end{bmatrix} \frac{1}{\det(sI - A)} \begin{bmatrix} s+3 & 1 \\ -2 & s \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} \\
&= \begin{bmatrix} 3 & 4 \end{bmatrix} \frac{1}{\det(sI - A)} \begin{bmatrix} s+5 \\ -2+2s \end{bmatrix} \\
&= \frac{11s+7}{(s+1)(s+2)}
\end{aligned}$$

Example (Transfer function from the State representation II)

We see that the poles of the system are $p_1 = -1$ and $p_2 = -2$. To compare with the eigenvalues of A, we compute

$$\det(\lambda I - A) = \det \begin{bmatrix} \lambda & -1 \\ 2 & \lambda + 3 \end{bmatrix} = (\lambda + 1)(\lambda + 2)$$

As expected, the eigenvalues are -1 and -2.

Theorem

- *An LTI continuous-time system with state representation* (A, B, C) *is* **stable** *if and only if the eigenvalues of A are in the left-hand complex plane.*
- *An LTI discrete-time system with state representation* (A, B, C) *is* **stable** *if and only if the eigenvalues of A are inside the unit disk.*