Chapter 7:
Laplace and z-transforms

In this chapter we introduce the Laplace and z-transforms and apply them to the analysis of continuous-time and discrete-time systems.

Recall that any absolutely integrable signal $x(t)$ has CTFT

$$X(\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}\, dt$$

Similarly, the frequency response of an LTI stable system is the CTFT of its impulse response

$$H(\omega) = \int_{-\infty}^{\infty} h(t)e^{-j\omega t}\, dt$$

- We used the CTFT (and also the DTFT) to analyse the spectral content of signals, and the frequency behaviour of systems. The essential assumption was that the signals involved were absolutely integrable (or absolutely summable).
- To analyse signals that are not absolutely integrable, and unstable systems, we will need more powerful tools - the Laplace transform (in continuous-time) and the z-transform (in discrete-time).

Definition (Laplace transform)

For any continuous-time signal x we define the **Laplace transform** as

$$X(s) = \int_0^\infty x(t)e^{-st}\,dt$$

where s is a complex variable.

- The LT (Laplace transform) depends only on the value of the signal for $t \geq 0$.
- The LT is particularly useful for analysing the behaviour of systems given in terms of differential equations with specified initial conditions.
- As for the CTFT, we write

$$x(t) \longleftrightarrow X(s)$$

  to indicate a Laplace transform pair.

Definition (Region of convergence)

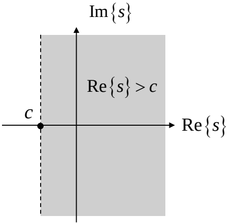
We define the **region of convergence** of the LT as the set of complex numbers s for which the transform $X(s)$ exists. This is

$$RoC(x) = \{s = \sigma + j\omega \in \mathbf{C} : x(t)e^{-\sigma t} \text{is absolutely integrable}\}$$

The RoC depends on the signal x and always has the form

$$RoC(x) = \{s \in \mathbf{C} : Re(s) > c\}$$

for some $c \in \mathbf{R}$.



Theorem

Let x be a continuous-time signal such that $x(t) = 0$ for $t < 0$. Let $X(s)$ be its Laplace transform and assume that $s = 0$ is in its region of convergence. Then the CTFT of x is given by

$$X(\omega) = X(s)|_{s=j\omega}$$

Example

Let $x(t) = e^{-bt}u(t)$ for some $b \in \mathbf{R}$. Then

$$\begin{aligned}
X(s) &= \int_0^\infty x(t)e^{-st}\,dt \\
&= \int_0^\infty e^{-bt}e^{-st}\,dt \\
&= \int_0^\infty e^{-(b+s)t}\,dt \\
&= \frac{-1}{s+b}[e^{-(b+s)t}]_{t=0}^{t=\infty}
\end{aligned}$$

Example

Writing $s = \sigma + j\omega$, we need to find

$$\lim_{t\to\infty} e^{-(b+\sigma+j\omega)t} = \lim_{t\to\infty} e^{-j\omega t} e^{-(b+\sigma)t} = 0$$

provided $b + \sigma > 0$. Then

$$X(s) = \frac{1}{s+b}$$

with $RoC(x) = \{s \in \mathbf{C} : Re(s) > -b\}$. If $b > 0$, then the point $s = 0$ is in the region of convergence, and so the CTFT of x exists and is given by

$$X(\omega) = X(s)|_{s=j\omega} = \frac{1}{j\omega + b}$$

If $b \leq 0$, then $s = 0$ is not in the $RoC(x)$ and the CTFT of x does not exist.

Similarly to the Fourier transform, the Laplace transform has many properties that allow us to compute transforms of signals without using the definition of the transform. Some of these are

Theorem (Linearity)

The Laplace Transform is linear: if $x_1(t) \longleftrightarrow X_1(s)$ and $x_2(t) \longleftrightarrow X_2(s)$, and a and b are any two scalars, then

$$ax_1(t) + bx_2(t) \longleftrightarrow aX_1(s) + bX_2(s)$$

Theorem (Time shifting)

If $x(t) \longleftrightarrow X(s)$ and $c > 0$, then

$$x(t-c)u(t-c) \longleftrightarrow X(s)e^{-cs}$$

Theorem (Time scaling)

If $x(t) \longleftrightarrow X(s)$ and $a > 0$, then

$$x(at) \longleftrightarrow \frac{1}{a} X\left(\frac{s}{a}\right)$$

Theorem (Modulation)

Let $x(t) \longleftrightarrow X(s)$, let $a \in \mathbf{C}$, let $\omega \in \mathbf{R}$ and let N be a positive integer. Then

$$\begin{aligned}
x(t)e^{at} &\longleftrightarrow X(s-a) \\
x(t)\cos(\omega t) &\longleftrightarrow \frac{1}{2}[X(s+j\omega)+X(s-j\omega)] \\
x(t)\sin(\omega t) &\longleftrightarrow \frac{j}{2}[X(s+j\omega)-X(s-j\omega)] \\
t^N x(t) &\longleftrightarrow (-1)^N \frac{d^N}{ds^N} X(s)
\end{aligned}$$

Theorem (Convolution)

If $x(t) \longleftrightarrow X(s)$ *and* $v(t) \longleftrightarrow V(s)$, *then*

$$(x \star v)(t) \longleftrightarrow X(s)V(s)$$

Theorem (Derivative)

If $x(t) \longleftrightarrow X(s)$, *then*

$$\begin{aligned} \dot{x}(t) &\longleftrightarrow sX(s) - x(0^-) \\ \ddot{x}(t) &\longleftrightarrow s^2X(s) - sx(0^-) - \dot{x}(0^-) \end{aligned}$$

where $x(0^-) = \lim_{t \to 0, t<0} \; x(t)$ *and* $\dot{x}(0^-) = \lim_{t \to 0, t<0} \; \dot{x}(t)$

Theorem (Integral)

If $x(t) \longleftrightarrow X(s)$, *then*

$$\int_0^t x(\lambda)\, d\lambda \longleftrightarrow \frac{X(s)}{s}$$

Example

Let $x(t) = \delta(t)$. Then its LT is

$$\begin{aligned} X(s) &= \int_0^\infty x(t)e^{-st}\,dt \\ &= \int_0^\infty \delta(t)e^{-st}\,dt \\ &= e^0 \\ &= 1 \end{aligned}$$

by the Sifting theorem. Since this holds for any complex number s, we see that the RoC(x) is **C**.

Example

Let $x(t) = tu(t)$. Then its LT is

$$X(s) = \int_0^\infty te^{-st}\,dt = -\left[\frac{st+1}{s^2 e^{st}}\right]_0^\infty$$

using integration by parts. Writing $s = \sigma + j\omega$, we need to find

$$\lim_{t\to\infty} \frac{st+1}{s^2 e^{st}} = \lim_{t\to\infty} \frac{(\sigma + j\omega)t + 1}{(\sigma + j\omega)^2 e^{(\sigma + j\omega)t}} = 0$$

provided $\sigma > 0$. (Here we have used $\lim_{t\to\infty} \frac{\alpha t}{e^{\alpha t}} = 0$ for $\alpha > 0$). Hence

$$X(s) = \frac{1}{s^2}$$

with $RoC(x) = \{s \in \mathbf{C} : Re(s) > 0\}$. We note that $s = 0$ is not in the $RoC(x)$ and hence x does not have a CTFT.

Example

Show that for any integer $N \geq 1$ and any $b \in \mathbf{R}$,

$$t^N e^{-bt} u(t) \longleftrightarrow \frac{N!}{(s+b)^{N+1}}$$

Introduce

$$y(t) = e^{-bt} u(t), \quad Y(s) = \frac{1}{s+b}, \quad \text{and } x(t) = t^N y(t)$$

Then by modulation

$$\begin{aligned} X(s) &= (-1)^N \frac{d^N}{ds^N} Y(s) \\ &= (-1)^N \frac{d^N}{ds^N} \left(\frac{1}{s+b} \right) \\ &= \frac{N!}{(s+b)^{N+1}} \end{aligned}$$

Definition (Inverse Laplace transform)

For any continuous-time signal x with LT $X(s)$, we define the **inverse Laplace transform** as

$$x(t) = \frac{1}{j2\pi} \int_{c-j\infty}^{c+j\infty} X(s)e^{st}\, ds$$

- The integral is evaluated on the contour (path) $s = c + j\omega$ in the complex plane, between $s = c - j\infty$ and $s = c + j\infty$. The contour must lie within the region of convergence of the LT of x.
- Evaluation of this integral involves techniques of complex integration, which are outside the scope of ELEN 30012.
- Hence we will use alternative methods to obtain inverse Laplace transforms.

Definition (Rational functions)

Let X be a complex function in the form

$$X(s) = \frac{B(s)}{A(s)}$$

where B and A are real polynomials in the complex variable s, given by

$$\begin{aligned} B(s) &= b_M s^M + b_{M-1} s^{M-1} + \ldots b_1 s + b_0 \\ A(s) &= a_N s^N + a_{N-1} s^{N-1} + \ldots a_1 s + a_0 \end{aligned}$$

where M and N are positive integers and the coefficients of B and A are all real numbers. We say that $X(s)$ is a **rational function of** s. Assuming that $b_M \neq 0$ and $a_N \neq 0$, the **degree** of the polynomials B and A are M and N, respectively. We refer to B and A as the **numerator polynomial** and **denominator polynomial** of X, respectively. Finally we say that X is **proper** if $M < N$.

Definition

Let $A(s)$ be a real polynomial in s and let $p \in \mathbf{C}$ be such that

$$A(p) = 0$$

We say that p **is root** or **zero** of A. We define the **multiplicity** of p to be the largest integer m such that

$$A(s) = (s-p)^m H(s)$$

where H is a polynomial in s and $H(p) \neq 0$.

Example

Let

$$A(s) = (s-3)(s+j5)^3(s-j5)^3$$

Then $s_1 = 3$, $s_2 = -j5$ and $s_3 = j5$ are roots of A with multiplicities $m_1 = 1$ and $m_2 = m_3 = 3$.

Theorem (Fundamental Theorem of Algebra I)

Every real polynomial of degree $N \geq 1$ has a factorisation involving only linear factors in $\mathbf{C}$.

This is equivalent to:

Theorem (Fundamental Theorem of Algebra II)

Let A be a real polynomial of degree $N \geq 1$. Then A has factorisation

$$A(s) = a_N(s - p_1)(s - p_2)\dots(s - p_N)$$

for some $a_N \in \mathbf{R}$ and some (not necessarily distinct) $p_1, p_2, \dots, p_N \in \mathbf{C}$.

Theorem (Complex roots)

Complex roots of a real polynomial occur in complex-conjugate pairs.

This equivalent to the following: If A is a real polynomial and $A(p) = 0$, then $A(\bar{p}) = 0$ also.

Definition (Poles of a rational function)

Let X be a rational function of s and let its denominator polynomial A have factorisation

$$A(s) = a_N(s-p_1)(s-p_2)\dots(s-p_N)$$

Then the roots $p_1, p_2, \dots, p_N$ of A are said to be the **poles of X.**

Theorem (Partial Fractions with distinct poles)

Let X be a proper rational function of s and assume its denominator polynomial has N distinct real or complex poles $p_1, p_2, \dots, p_N$. Then X has partial fraction expansion

$$X(s) = \frac{c_1}{s-p_1} + \frac{c_2}{s-p_2} + \dots + \frac{c_N}{s-p_N}$$

where the constants c_i are given by

$$c_i = [(s-p_i)X(s)]_{s=p_i}, \quad i = 1, 2, \dots, N$$

Remark (Partial Fractions with distinct poles)

- *'X has distinct poles' means that the poles all have multiplicty of 1, which implies that $p_i \neq p_j$ whenever $i \neq j$.*
- *The contants c_i are called the* **residues** *of X. They are real or complex according to whether the corresponding pole p_i is real or complex, and $c_i = \bar{c}_j$ if $p_i = \bar{p}_j$.*
- *Taking the inverse LT we obtain the time-domain signal*

$$x(t) = c_1 e^{p_1 t} + c_2 e^{p_2 t} + \cdots + c_N e^{p_N t}, \quad t \geq 0$$

Note: the solution for x depends only upon the poles of X.

- *For complex c_i, we obtain a real expression for $x(t)$ using*

Theorem

Let $p = \sigma + j\omega \in \mathbf{C}$ for some $\sigma \in \mathbf{R}$ and $\omega \in \mathbf{R}$. Let $c \in \mathbf{C}$ and $t \in \mathbf{R}$. Then

$$ce^{pt} + \bar{c}e^{\bar{p}t} = 2|c|e^{\sigma t}\cos(\omega t + \angle c)$$

Example (Partial Fractions with distinct poles I)

Find x if its LT is the rational function

$$X(s) = \frac{s^2 - 2s + 1}{s^3 + 3s^2 + 4s + 2}$$

Use the fact that

$$s^3 + 3s^2 + 4s + 2 = (s^2 + 2s + 2)(s+1) = (s+1-j)(s+1+j)(s+1)$$

The roots (poles) of X are $p_1 = -1 + j,\ p_2 = -1 - j,\ p_3 = -1$. We compute the residues of X:

$$\begin{aligned} c_1 &= [(s - p_1)X(s)]_{s=p_1} \\ &= \left.\frac{s^2 - 2s + 1}{(s+1+j)(s+1)}\right|_{s=-1+j} \\ &= \frac{-3}{2} + j2 \end{aligned}$$

Example (Partial Fractions with distinct poles II)

$$
\begin{aligned}
c_2 &= \bar{c}_1 \\
c_3 &= [(s-p_3)X(s)]_{s=p_3}, \\
&= \left.\frac{s^2-2s+1}{s^2+2s+2}\right|_{s=-1} \\
&= 4 \\
\text{Hence } X(s) &= \frac{c_1}{s-p_1} + \frac{\bar{c}_1}{s-\bar{p}_1} + \frac{c_3}{s-p_3}
\end{aligned}
$$

Taking the inverse LT gives

$$
\begin{aligned}
x(t) &= [c_1 e^{p_1 t} + \bar{c}_1 e^{\bar{p}_1 t} + c_3 e^{p_3 t}]u(t) \\
&= [2|c_1|e^{-t}\cos(t+\angle c_1) + c_3 e^{-t}]u(t) \\
&= [5e^{-t}\cos(t+2.214) + 4e^{-t}]u(t)
\end{aligned}
$$

Theorem (Partial Fractions with repeated real poles)

Let X be a proper rational function and assume its denominator polynomial has a real pole p_1 with multiplicity r, and the remaining $N-r$ poles (denoted by $p_{r+1}, p_{r+2}, \ldots, p_N$) of X are distinct. Then $X(s)$ has partial fraction expansion

$$X(s) = \frac{c_1}{s-p_1} + \frac{c_2}{(s-p_1)^2} + \cdots + \frac{c_r}{(s-p_1)^r} + \frac{c_{r+1}}{s-p_{r+1}} + \cdots + \frac{c_N}{s-p_N}$$

where the residues c_i are given by

$$\begin{aligned} c_{r-i} &= \frac{1}{i!}\left[\frac{d^i}{ds^i}[(s-p_1)^r X(s)]\right]_{s=p_1}, \quad i = 0,1,2,\ldots,r-1 \\ c_i &= [(s-p_i)X(s)]_{s=p_i}, \quad i = r+1, r+2, \ldots, N \end{aligned}$$

Note that $0! = 1$ by definition.

Example (Partial Fractions with repeated real poles)

Find x if its LT is the rational function

$$X(s) = \frac{5s-1}{(s-2)(s+1)^2}$$

The poles of X are $p_1 = -1$ with multiplicity $r = 2$, and $p_3 = 2$.

Compute residues:
$$\begin{aligned}
c_1 &= \frac{1}{1!}\left[\frac{d}{ds}[(s-p_1)^r X(s)]\right]_{s=p_1} \\
&= \left[\frac{d}{ds}\left[(s+1)^2 X(s)\right]\right]_{s=-1} \\
&= \left[\frac{d}{ds}\left[\frac{5s-1}{s-2}\right]\right]_{s=-1} \\
&= \left.\frac{-9}{(s-2)^2}\right|_{s=-1} \\
&= -1
\end{aligned}$$

Example (Partial Fractions with repeated real poles II)

$$
\begin{aligned}
c_2 &= \frac{1}{0!}\left[(s-p_1)^r X(s)\right]_{s=p_1} \\
&= \Big[(s+1)^2 X(s)\Big]_{s=-1} \\
&= \left.\frac{5s-1}{(s-2)}\right|_{s=-1} \\
&= 2 \\
c_3 &= \left[(s-p_3)X(s)\right]_{s=p_3} \\
&= \left[(s-2)X(s)\right]_{s=2} \\
&= \left.\frac{5s-1}{(s+1)^2}\right|_{s=2} \\
&= 1
\end{aligned}
$$

Example (Partial Fractions with repeated real poles III)

Hence

$$\begin{aligned} X(s) &= \frac{c_1}{s+1} + \frac{c_2}{(s+1)^2} + \frac{c_3}{(s-2)} \\ &= \frac{-1}{s+1} + \frac{2}{(s+1)^2} + \frac{1}{(s-2)} \end{aligned}$$

Taking the inverse LT we obtain

$$x(t) = (-e^{-t} + 2te^{-t} + e^{2t})u(t)$$

We used the LT pairs

$$e^{-bt} \longleftrightarrow \frac{1}{s+b} \quad \text{and} \quad t^N e^{-bt} \longleftrightarrow \frac{N!}{(s+b)^{N+1}}$$

Recall that any absolutely summable discrete-time signal $x[n]$ has DTFT

$$X(\omega) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\Omega n}$$

Similarly, the frequency response of an LTI stable discrete-time system is the DTFT of its unit pulse response

$$H(\Omega) = \sum_{n=-\infty}^{\infty} h[n]e^{-j\Omega n}$$

To analyse signals that are not absolutely summable, and unstable systems, we introduce the z-transform:

Definition (z-Transform)

For any discrete-time signal x we define the **z-transform** (zT) as

$$X(z) = \sum_{n=0}^{\infty} x[n]z^{-n}$$

where z is a complex variable.

Definition (Region of convergence)

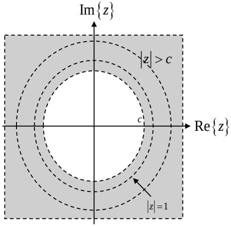
We define the **region of convergence** of the zT as the set of complex numbers z for which the transform $X(z)$ exists. This is

$$RoC(x) = \{z \in \mathbf{C} \setminus \{0\} : x[n]z^{-n} \text{ is absolutely summable}\}$$

The RoC depends on the signal x and always has the form

$$RoC(x) = \{z \in \mathbf{C} : |z| > c\}$$

for some $c \geq 0$.



Theorem

Let x be a discrete-time signal such that $x[n] = 0$ for $n < 0$. Let $X(z)$ be its z-transform and assume that $z = 1$ is in its region of convergence. Then the DTFT of x is given by

$$X(\Omega) = X(z)|_{z=e^{j\Omega}}$$

Example

Let $x[n] = a^n u[n]$ for some $a \in \mathbf{R}$. Then

$$\begin{aligned} X(z) &= \sum_{n=0}^{\infty} x[n]z^{-n} \\ &= \sum_{n=0}^{\infty} a^n z^{-n} \\ &= \sum_{n=0}^{\infty} (az^{-1})^n \end{aligned}$$

Example (continued)

We know that for a geometric series with common ratio r

$$\sum_{n=0}^{N} r^n = \frac{1 - r^{N+1}}{1 - r}, \quad \text{and} \quad \sum_{n=0}^{\infty} r^n = \frac{1}{1 - r}$$

provided $0 < |r| < 1$. Hence

$$X(z) = \sum_{n=0}^{\infty} (az^{-1})^n = \frac{1}{1 - az^{-1}}$$

provided $|az^{-1}| < 1$. This is equivalent to $|z| > |a|$, and hence with $RoC(x) = \{z \in \mathbf{C} : |z| > |a|\}$. If $|a| < 1$, then the point $z = 1$ is in the region of convergence, and so the DTFT of x exists and is given by

$$X(\Omega) = X(z)|_{z=e^{j\Omega}} = \frac{1}{1 - ae^{-j\Omega}}$$

Similarly to the Laplace transform, the z-transform has many properties that allow us to compute transforms of signals without using the definition of the transform. Some of these are

Theorem (Linearity)

The z-transform is linear: if $x_1[n] \longleftrightarrow X_1(z)$ *and* $x_2[n] \longleftrightarrow X_2(z)$*, and a and b are any two scalars, then*

$$ax_1[n] + bx_2[n] \longleftrightarrow aX_1(z) + bX_2(z)$$

Theorem (Right Time shift)

If $x[n] \longleftrightarrow X(z)$ *and* $q > 0$*, then*

$$x[n-q]u[n-q] \longleftrightarrow X(z)z^{-q}$$

Theorem (Modulation)

Let $x[n] \longleftrightarrow X(z)$, *let* $a \in \mathbf{C}$, *and let* $\Omega \in \mathbf{R}$. *Then*

$$
\begin{aligned}
x[n]a^n &\longleftrightarrow X\left(\frac{z}{a}\right) \\
x[n]\cos(\Omega n) &\longleftrightarrow \frac{1}{2}[X(ze^{j\Omega}) + X(ze^{-j\Omega})] \\
x[n]\sin(\Omega n) &\longleftrightarrow \frac{j}{2}[X(ze^{j\Omega}) - X(ze^{-j\Omega})] \\
nx[n] &\longleftrightarrow -z\frac{d}{dz}X(z)
\end{aligned}
$$

Theorem (Convolution)

If $x[n] \longleftrightarrow X(z)$ *and* $v[n] \longleftrightarrow V(z)$, *then*

$$(x \star v)[n] \longleftrightarrow X(z)V(z)$$

Theorem (Left Time Shift)

If $x[n] \longleftrightarrow X(z)$*, and* $q > 0$*, then*

$$x[n+q] \longleftrightarrow z^q X(z) - x[0]z^q - x[1]z^{q-1} - \cdots - x[q-1]z$$

Example

Let $x[n] = \delta[n]$. Then its zT is

$$\begin{aligned} X(z) &= \sum_{n=0}^{\infty} x[n]z^{-n} \\ &= \sum_{n=0}^{\infty} \delta[n]z^{-n} \\ &= z^0 \\ &= 1 \end{aligned}$$

by the Sifting theorem. Since this holds for any complex number z, we see that the RoC(x) is $\mathbf{C} \setminus \{0\}$.

Example

Let $x[n] = a^n p_3[n]$ for some $a \in \mathbf{R}$. To find its zT, we first recall that

$$p_3[n] = \begin{cases} 1, & -1 \leq n \leq 1 \\ 0, & \text{otherwise} \end{cases} = u[n+1] - u[n-2]$$

We saw earlier that zT of $u[n]$ is $U(z) = \frac{1}{1-z^{-1}}$. Hence by left and right time-shifting

$$\begin{aligned} P_3(z) &= zU(z) - u[0]z - U(z)z^{-2} \\ &= \frac{z}{1-z^{-1}} - z - \frac{z^{-2}}{1-z^{-1}} \\ &= 1 + z^{-1} \end{aligned}$$

Since $x[n] = a^n p_3[n]$, by modulation we have

$$X(z) = P_3\left(\frac{z}{a}\right) = 1 + \frac{a}{z}$$

Example

Show that any $p \in \mathbf{R}$,

$$np^n u[n] \longleftrightarrow \frac{pz}{(z-p)^2}$$

Introduce

$$y[n] = p^n u[n] \quad \text{and } x[n] = ny[n]$$

Recall that $u[n] \longleftrightarrow \frac{1}{1-z^{-1}} = U(z)$. Then modulating u with p^n gives

$$\begin{aligned} Y(z) &= U\left(\frac{z}{p}\right) = \frac{z}{z-p} \\ \Rightarrow X(z) &= -z\frac{d}{dz}\left(\frac{z}{z-p}\right) \quad \text{by modulating } y \text{ with } n \\ &= -z\left[\frac{z-p-z}{(z-p)^2}\right] \\ &= \frac{pz}{(z-p)^2} \end{aligned}$$

Definition (Inverse z-transform)

For any discrete-time signal x with z-transform $X(z)$, we define the **inverse z-transform** as

$$x[n] = \frac{1}{j2\pi} \int_C X(z) z^{n-1} \, dz$$

- The integral is evaluated on any circular contour C within the region of convergence of $X(z)$.
- As with the inverse LT, evaluation of this complex integral is outside the scope of ELEN 30012.
- Hence we will use partial fractions to obtain inverse z-transforms.

Theorem (Partial Fractions with distinct nonzero poles)

Let X be a rational function of z and assume its denominator polynomial has N distinct nonzero poles $p_1, p_2, \ldots, p_N$. Then X has partial fraction expansion

$$\frac{X(z)}{z} = \frac{c_0}{z} + \frac{c_1}{z-p_1} + \frac{c_2}{z-p_2} + \cdots + \frac{c_N}{z-p_N}$$

where the constants c_i are given by

$$c_0 = X(0), \quad c_i = \left[(z-p_i)\frac{X(z)}{z}\right]_{z=p_i}, \quad i = 1, 2, \ldots, N$$

Remark

- *Unlike the LT, in this theorem we have not required that X be a proper rational function. We accommodate the case where the numerator and denominator polynomials of X have the same degree by working with $X(z)/z$ instead of $X(z)$.*

Remark

- *Multiplying through by z gives*

$$X(z) = c_0 + \frac{c_1 z}{z - p_1} + \frac{c_2 z}{z - p_2} + \cdots + \frac{c_N z}{z - p_N}$$

- *Taking the inverse zT we obtain the time-domain signal x with*

$$x[n] = c_0 \delta[n] + c_1 p_1^n + c_2 p_2^n + \cdots + c_N p_N^n, \quad n \geq 0$$

- *For complex c_i, we obtain a real expression for $x[n]$ using*

Theorem

Let $p = \sigma e^{j\Omega} \in \mathbf{C}$ for some real numbers σ and Ω, and let $c \in \mathbf{C}$. Then

$$cp^n + \bar{c}\bar{p}^n = 2|c|\sigma^n \cos(\Omega n + \underline{/c})$$

Example (Partial Fractions with distinct poles I)

Find x if its zT is the rational function

$$X(z) = \frac{z^3+1}{(z+1/2+j\sqrt{3}/2)(z+1/2-j\sqrt{3}/2)(z-2)}$$

The roots (poles) of X are
$p_1 = -1/2 - j\sqrt{3}/2,\ p_2 = -1/2 + j\sqrt{3}/2,\ p_3 = 2$. We compute the residues of $X(z)/z$:

$$\begin{aligned}
c_0 &= X(0) \\
&= -1/2 \\
c_1 &= [(z-p_1)X(z)/z]_{z=p_1} \\
&= 0.429 + j0.0825 \\
c_2 &= \bar{c}_1 \\
c_3 &= [(z-p_3)X(z)/z]_{z=p_3} \\
&= 0.643
\end{aligned}$$

Example (Partial Fractions with distinct poles II)

$$\begin{aligned}
\text{Hence } \frac{X(z)}{z} &= \frac{c_0}{z} + \frac{c_1}{z-p_1} + \frac{\bar{c}_1}{z-\bar{p}_1} + \frac{c_3}{z-p_3} \\
\Rightarrow X(z) &= c_0 + \frac{c_1 z}{z-p_1} + \frac{\bar{c}_1 z}{z-\bar{p}_1} + \frac{c_3 z}{z-p_3} \\
\text{Also } |p_1| &= 1 = \sigma, \quad \angle p_1 = \pi + \tan^{-1}\frac{0.866}{0.5} = \frac{4\pi}{3}\ rad = \Omega \\
|c_1| &= 0.437, \quad \angle c_1 = \tan^{-1}\left(\frac{0.0825}{0.429}\right) = 0.19\ rad
\end{aligned}$$

Hence taking the inverse zT gives

$$\begin{aligned}
x[n] &= [c_0\delta[n] + c_1 p_1^n + \bar{c}_1 \bar{p}_1^n + c_3 p_3^n]u[n] \\
&= [c_0\delta[n] + 2|c_1|\sigma^n \cos(\Omega n + \angle c_1) + c_3 p_3^n]u[n] \\
&= [-0.5\delta[n] + 0.874\cos(4\pi n/3 + 0.19) + 0.643(2)^n]u[n]
\end{aligned}$$

Theorem (Partial Fractions with repeated real poles)

Let X be a rational function of z and assume its denominator polynomial has a real nonzero pole p_1 with multiplicity r, and the remaining $N-r$ poles (denoted by $p_{r+1}, p_{r+2}, \ldots, p_N$) of X are distinct and nonzero. Then $X(z)/z$ has partial fraction expansion

$$\frac{X(z)}{z} = \frac{c_0}{z} + \frac{c_1}{z-p_1} + \frac{c_2}{(z-p_1)^2} + \cdots + \frac{c_r}{(z-p_1)^r} + \frac{c_{r+1}}{z-p_{r+1}} + \cdots + \frac{c_N}{z-p_N}$$

where the residues c_i are given by

$$\begin{aligned} c_0 &= X(0) \\ c_{r-i} &= \frac{1}{i!}\left[\frac{d^i}{dz^i}\left[(z-p_1)^r \frac{X(z)}{z}\right]\right]_{z=p_1}, \quad i = 0, 1, 2, \ldots, r-1 \\ c_i &= \left[(z-p_i)\frac{X(z)}{z}\right]_{z=p_i}, \quad i = r+1, r+2, \ldots, N \end{aligned}$$

Example (Partial Fractions with repeated real poles)

Find x given that its zT is such that

$$\frac{X(z)}{z} = \frac{6z^2+2z-1}{(z+1)(z-1)^2}$$

The roots (poles) of X are $p_1 = 1$ with multiplicity $r = 2$, and $p_3 = -1$.

$$\begin{aligned}
\text{Compute residues:} c_0 &= X(0) = 0 \\
c_1 &= \frac{1}{1!}\left[\frac{d}{dz}[(z-p_1)^r X(z)/z]\right]_{z=p_1} \\
&= \left[\frac{d}{dz}\left[(z-1)^2 X(z)/z\right]\right]_{z=1} \\
&= \left[\frac{d}{dz}\left[\frac{6z^2+2z-1}{(z+1)}\right]\right]_{z=1} \\
&= \left.\frac{(z+1)(12z+2)-(6z^2+2z-1)(1)}{(z+1)^2}\right|_{z=1}
\end{aligned}$$

Example (Partial Fractions with repeated real poles II)

$$
\begin{aligned}
c_1 &= 5.25 \\
c_2 &= \frac{1}{0!}\left[(z-p_1)^r X(z)/z\right]_{z=p_1} \\
&= \Big[(z-1)^2 X(z)/z\Big]_{z=1} \\
&= \left.\frac{6z^2+2z-1}{z+1}\right|_{z=1} \\
&= 3.5 \\
c_3 &= \left[(z-p_3)X(z)/z\right]_{z=p_3} \\
&= \left[(z+1)X(z)/z\right]_{z=-1} \\
&= \left.\frac{6z^2+2z-1}{(z-1)^2}\right|_{z=-1} \\
&= 0.75
\end{aligned}
$$

Example (Partial Fractions with distinct poles II)

$$\text{Hence } \frac{X(z)}{z} = \frac{c_0}{z} + \frac{c_1}{z-p_1} + \frac{c_2}{(z-p_1)^2} + \frac{c_3}{z-p_3}$$

$$\Rightarrow X(z) = \frac{5.25z}{z-1} + \frac{3.5z}{(z-1)^2} + \frac{0.75z}{z+1}$$

Hence taking the inverse zT gives

$$\begin{aligned} x[n] &= [5.25(1)^n + 3.5n(1)^n + 0.75(-1)^n]u[n] \\ &= [5.25 + 3.5n + 0.75(-1)^n]u[n] \end{aligned}$$

We used the zT pairs

$$p^n u[n] \longleftrightarrow \frac{z}{z-p} \quad \text{and} \quad np^n u[n] \longleftrightarrow \frac{pz}{(z-p)^2}$$