Chapter 6:
Fourier Analysis of Systems

In this chapter we develop the Fourier analysis of continuous-time and discrete-time LTI systems.

Recall from Chapter 2 that for any LTI continuous-time system, the output y from any input v can be expressed in terms of the convolution of the impulse response h as:

$$y(t) = (v \star h)(t) = \int_{-\infty}^{\infty} h(\lambda)v(t-\lambda)\,d\lambda$$

In this chapter we will assume the following is true for the impulse response of any system we consider:

Assumption

The impulse response is absolutely integrable:

$$\int_{-\infty}^{\infty} |h(t)|\,dt < \infty$$

Under this assumption the CTFT of h always exists

$$H(\omega) = \int_{-\infty}^{\infty} h(t)e^{-j\omega t}\,dt$$

Recall also that the CTFT of the convolution of two signals is the product of their CTFTs:

$$(x \star v)(t) \longleftrightarrow X(\omega)V(\omega)$$

Applying this to the impulse response $y(t) = (v \star h)(t)$ we get

$$Y(\omega) = H(\omega)V(\omega)$$

where $V(\omega)$ and $Y(\omega)$ are the CTFT of the input v and output y, respectively.

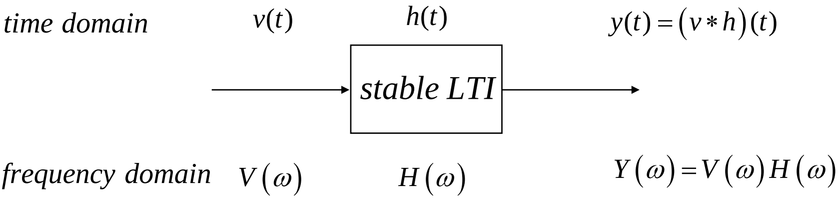
Definition

The equations

$$y(t) = (v \star h)(t) \quad \text{and} \quad Y(\omega) = H(\omega)V(\omega)$$

are called the **time-domain** and **frequency-domain** representations of the system.

We may represent relationships visually as



Theorem

For any input v, the **amplitude spectrum** *of the output y is given by*

$$|Y(\omega)| = |H(\omega)||V(\omega)|$$

and the **phase spectrum** *of the output y is given by*

$$\angle Y(\omega) = \angle H(\omega) + \angle V(\omega)$$

The CTFT $H(\omega)$ of h is called the **frequency response** of the system. It provides insight into how a system converts inputs into outputs.

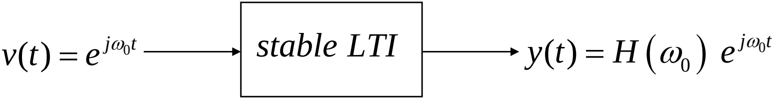
Theorem

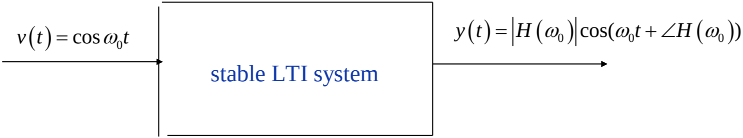
Let $H(\omega)$ be the frequency response of a stable LTI system, and let $\omega_0 \in \mathbf{R}$. Then the input $v(t) = e^{j\omega_0 t}$ has output

$$y(t) = e^{j\omega_0 t} H(\omega_0)$$

and the input $v(t) = \cos(\omega_0 t)$ has output

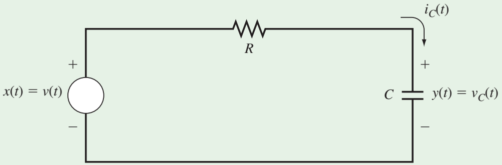
$$y(t) = |H(\omega_0)| \cos(\omega_0 t + \angle H(\omega_0))$$





Example (Frequency analysis of an RC circuit I)

In the *RC* circuit shown below, the input is the source voltage v and the output y is the capacitor voltage v_C. We assume zero initial conditions, i.e. $y(0) = 0$.



The differential equation that describes the circuit is

$$\frac{dy}{dt} + \frac{1}{RC} y(t) = \frac{1}{RC} v(t)$$

To find the frequency response $H(\omega)$, we can apply the input $v(t) = e^{j\omega t}$ with output $y(t) = H(\omega) e^{j\omega t}$.

Example (Frequency analysis of an RC circuit II)

Substituting these gives

$$\begin{aligned}
RC\frac{dy}{dt} + y(t) &= v(t) \\
\Rightarrow\ RC\frac{d}{dt}(H(\omega)e^{j\omega t}) + H(\omega)e^{j\omega t} &= e^{j\omega t} \\
\Rightarrow\ RCj\omega H(\omega)e^{j\omega t} + H(\omega)e^{j\omega t} &= e^{j\omega t} \\
\Rightarrow\ H(\omega)(RCj\omega + 1) &= 1 \\
\Rightarrow\ H(\omega) &= \frac{1}{1 + jRC\omega}
\end{aligned}$$

Hence the magnitude response and phase response of the system are

$$|H(\omega)| = \frac{1}{\sqrt{1 + (RC\omega)^2}}, \quad \angle H(\omega) = -\tan^{-1}(\omega RC)$$

Example (Frequency analysis of an RC circuit III)

Assume $RC = 1$ and the input is $v(t) = 5\cos(t)$. Then $\omega_0 = 1$, so

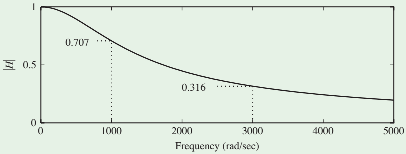
$$
\begin{aligned}
|H(1)| &= \frac{1}{\sqrt{1 + (RC\omega)^2}} \\
&= \frac{1}{\sqrt{2}} \\
\angle H(1) &= -\tan^{-1}(\omega RC) \\
&= -\tan^{-1}(1) \\
&= -\frac{\pi}{4}
\end{aligned}
$$

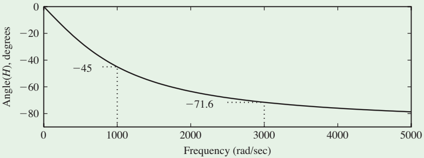
Hence

$$
\begin{aligned}
y(t) &= |H(\omega_0)| 5\cos(\omega_0 t + \angle H(\omega_0)) \\
&= \frac{5}{\sqrt{2}} \cos(t - \pi/4)
\end{aligned}
$$

Example (Frequency analysis of an RC circuit IV)

If $RC = 1/1000$, the circuit's magnitude and phase responses are





For an input $v(t) = \cos(\omega_0 t)$, the following table shows the magnitude and phase of the output y for different values of ω_0.

	$\omega_0 = 0$	$\omega_0 = 1000$	$\omega_0 = 3000$
$\lvert y \rvert$	1	0.707	0.316
$\angle y$	0	-45°	-71.6°

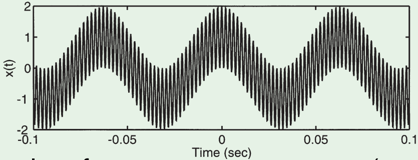
Example (Frequency analysis of an RC circuit V)

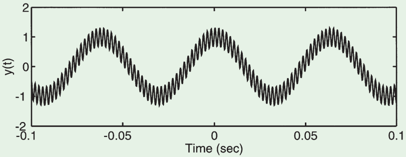
We observe that

$$|H(\omega)| \to 0 \quad \text{as} \quad \omega \to \infty$$
$$\angle H(\omega) \to -90^\circ \quad \text{as} \quad \omega \to \infty$$

The system is a **low pass filter** because high frequency signals have their amplitude attenuated by the system. We show the response to the input

$$x(t) = \cos(100t) + \cos(3000t)$$





The low frequency component ($\omega = 100$) is approximately the same, but the high frequency component ($\omega = 3000$) is strongly attenuated.

Theorem (Response to Periodic Inputs)

Let $H(\omega)$ be the frequency response of a stable LTI system, and let v be a periodic signal with Fourier series

$$v(t) = a_0 + \sum_{k=1}^{\infty} A_k \cos(k\omega_0 t + \theta_k)$$

Then the output y from the input v is

$$y(t) = a_0 H(0) + \sum_{k=1}^{\infty} A_k |H(k\omega_0)| \cos(k\omega_0 t + \theta_k + \underline{/H(k\omega_0)})$$

If we use A_k^y and θ_k^y to denote the Fourier coefficients of y, and likewise let A_k^v and θ_k^v be the coefficients for v, then

$$\begin{aligned} A_k^y &= A_k^v |H(k\omega_0)| \\ \theta_k^y &= \theta_k^v + \underline{/H(k\omega_0)} \end{aligned}$$

Corollary (Response to Periodic Inputs)

Let $H(\omega)$ be the frequency response of a stable LTI system, and let v be a periodic signal with complex Fourier series

$$v(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t}$$

Then the output y from the input v is

$$y(t) = \sum_{k=-\infty}^{\infty} c_k H(k\omega_0) e^{jk\omega_0 t}$$

If we use c_k^y and c_k^v to denote the Fourier coefficients of y and v, then

$$c_k^y = c_k^v H(k\omega_0)$$

Theorem (Response to non-periodic Inputs)

Let $H(\omega)$ be the frequency response of a stable LTI system, and let v be a non-periodic signal with CT Fourier transform $V(\omega)$. Then the output y from the input v has CTFT

$$Y(\omega) = H(\omega)V(\omega)$$

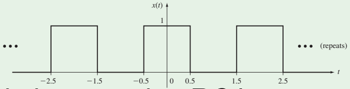
and can be obtained by taking the inverse Fourier transform

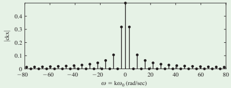
$$y(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} H(\omega)V(\omega)e^{j\omega t}\, d\omega$$

In some cases the integral may be difficult to compute, so we often work with tables of Fourier transform pairs.

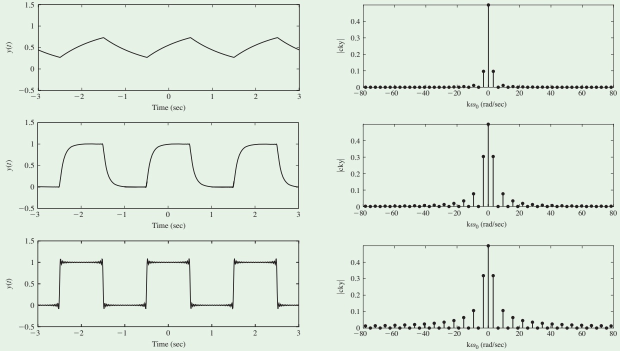
Example (Lowpass filter response)

Consider the rectangular pulse train with amplitude spectrum below.





We apply this input to the *RC* lowpass filter circuit with parameters (i) $RC = 1$, (ii) $RC = 1/10$, and (iii) $RC = 1/100$:

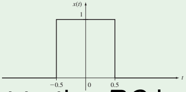


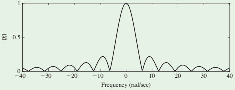
Remark (Lowpass filter response)

- *The pulse train input signal has discontinuities and hence its spectrum contains many high-frequency components.*
- *A lowpass RC circuit filters (attenuates) higher frequencies, but the circuit with* $RC = 1$ *does so much more sharply than the circuit with* $RC = 1/100$*.*
- *The circuit with* $RC = 1/100$ *passes the rectangular pulse with little distortion. Some fuzziness is apparent near the points of discontinuity due to the loss of high frequency spectrum.*
- *The circuit with* $RC = 1$ *distorts the signal substantially.*
- *We say that the circuit with* $RC = 1/100$ *has greater* **bandwidth** *because it passes much more of the frequency content of the input signal to the output signal.*

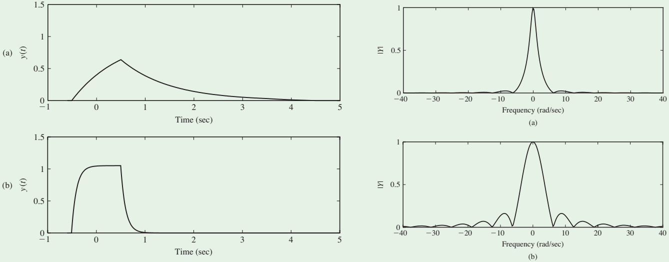
Example (Lowpass filter response II)

Consider the single pulse $x(t) = p_1(t)$ with amplitude spectrum below.





We apply this input to the RC lowpass filter circuit with parameters (i) $RC = 1$, and (ii) $RC = 1/10$:



As expected the greater bandwidth of the circuit with $RC = 1/10$ passes more of the frequency spectrum and has less signal distortion.

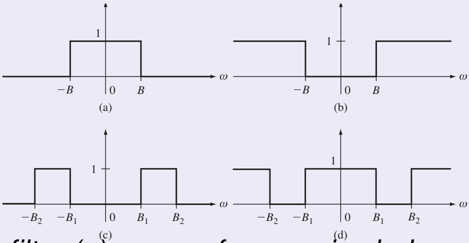
Definition (Ideal filters)

An **ideal filter** completely attenuates signals of the form $v(t) = A\cos(\omega t)$ for ω in a certain range of frequency values, while passing without attenuation signals outside of the specified range. There are four main types:

- **Ideal lowpass**: $|H(\omega)| = \begin{cases} 1, & -B \le \omega \le B \\ 0, & |\omega| > B \end{cases}$
- **Ideal highpass**: $|H(\omega)| = \begin{cases} 0, & -B \le \omega \le B \\ 1, & |\omega| > B \end{cases}$
- **Ideal bandpass**: $|H(\omega)| = \begin{cases} 1, & B_1 \le |\omega| \le B_2 \\ 0, & otherwise \end{cases}$
- **Ideal bandstop**: $|H(\omega)| = \begin{cases} 0, & B_1 \le |\omega| \le B_2 \\ 1, & otherwise \end{cases}$

where B, B_1 and B_2 are positive real numbers.

Remark (Ideal filters)



- *The lowpass filter (a) passes frequencies below a threshold and blocks those above the threshold value.*
- *The highpass filter (b) passes those above the threshold and blocks those below.*
- *The bandpass filter (c) passes all the frequencies within a certain band and blocks those outside.*
- *The bandstop filter (d) (also called a* **notch** *filter) blocks those frequencies inside a certain range and passes all frequencies outside the range.*

Remark (Ideal filters)

- *The* **passband** *of filter is the set of all input frequencies passed by the filter without attenuation, and the* **stopband** *is the range of input frequencies that are completely attenuated.*
- *The width of the passband is the* **bandwidth** *of the filter.*
- *The lowpass filter is equal to the rectangular pulse function*

$$H(\omega) = p_{2B}(\omega)e^{-j\omega c}$$

  for some $c \in \mathbf{R}$. We can obtain the impulse response of the filter as

$$h(t) = \frac{B}{\pi}\,\mathrm{sinc}\left(\frac{B(t-c)}{\pi}\right)$$

  Since $h(t) \neq 0$ for $t < 0$, the filter is not causal, and hence cannot be physically realized, hence the name 'ideal filter'.
- *Building physically realizable filters that approximate ideal filters is an important problem area.*

Definition (Linear phase)

An LTI system is called a **linear phase system** if its frequency response H can be written as

$$H(\omega) = K(\omega)e^{-j\omega t_d},$$

where K is a real-valued function and $t_d \in \mathbf{R}$ is a positive constant.

Note that then its phase response is a linear function of frequency, i.e.

$$\angle H(\omega) = -\omega t_d \text{ modulo } \pi.$$

Suppose that $x(t) = A_0 \cos(\omega_0 t + \Theta_0) + A_1 \cos(\omega_1 t + \Theta_1)$ is input to a linear phase system. Then the response is (verify this for yourself!)

$$y(t) = A_0 K(\omega_0) \cos(\omega_0(t - t_d) + \Theta_0) + A_1 K(\omega_1) \cos(\omega_1(t - t_d) + \Theta_1).$$

Thus linear phase systems are shifting input sinusoids of different frequencies by the *same* time constant t_d.

The Fourier analysis of discrete-time systems is quite similar to that of continuous-time systems:
Recall that the output y from any input v can be expressed in terms of the convolution of the unit pulse response h as:

$$y[n] = (v \star h)[n] = \sum_{i=-\infty}^{\infty} h[i]v[n-i]$$

We need to assume for any LTI discrete-time system

Assumption

The unit pulse response is absolutely summable:

$$\sum_{n=-\infty}^{\infty} |h[n]| < \infty$$

Under this assumption the DTFT of h always exists

$$H(\Omega) = \sum_{n=-\infty}^{\infty} h[n]e^{-j\Omega n}$$

Recall also that the DTFT of the convolution of two signals is the product of their DTFTs:

$$(x \star v)[n] \longleftrightarrow X(\Omega)V(\Omega)$$

Applying this to the unit pulse response $y[n] = (v \star h)[n]$ we get

$$Y(\Omega) = H(\Omega)V(\Omega)$$

where $V(\Omega)$ and $Y(\Omega)$ are the DTFT of the input v and output y, respectively.

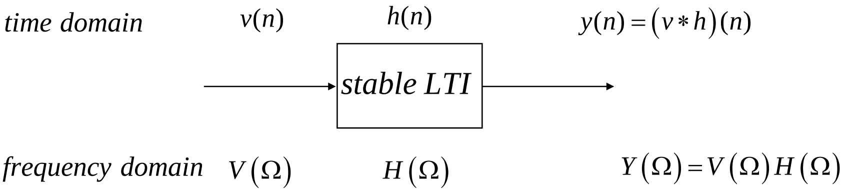
Definition

The equations

$$y[n] = (v \star h)[n] \quad \text{and} \quad Y(\Omega) = H(\Omega)V(\Omega)$$

are called the **time-domain** and **frequency-domain** representations of the system.

We may represent relationships visually as



Theorem

For any input v, the **amplitude spectrum** *of the output y is given by*

$$|Y(\Omega)| = |H(\Omega)||V(\Omega)|$$

and the **phase spectrum** *of the output y is given by*

$$\angle Y(\Omega) = \angle H(\Omega) + \angle V(\Omega)$$

The DTFT $H(\Omega)$ of h is called the **frequency response** of the system. It is periodic with period 2π.

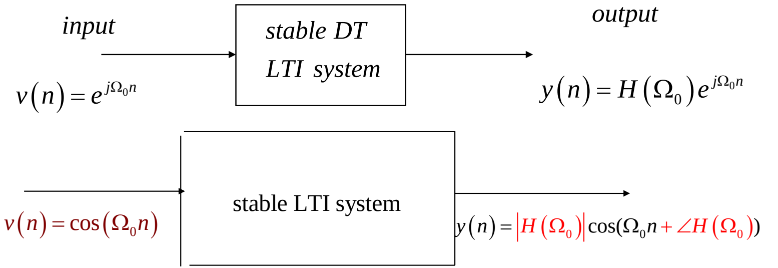
Theorem

Let $H(\Omega)$ be the frequency response of a stable LTI discrete-time system, and let $\Omega_0 \in \mathbf{R}$. Then the input $v[n] = e^{j\Omega_0 n}$ has output

$$y[n] = e^{j\Omega_0 n} H(\Omega_0)$$

and the input $v[n] = \cos(\Omega_0 n)$ has output

$$y[n] = |H(\Omega_0)| \cos(\Omega_0 n + \angle H(\Omega_0))$$



Example (Moving Average filter I)

Consider the N-point moving-average filter

$$y[n] = \frac{1}{N}[v[n] + v[n-1] + v[n-2] + \cdots + v[n-N+1]]$$

The unit impulse response is

$$h[n] = \frac{1}{N}[\delta[n] + \delta[n-1] + \delta[n-2] + \cdots + \delta[n-N+1]]$$

Taking the DTFT gives

$$\begin{aligned} H(\Omega) &= \frac{1}{N}[1 + e^{-j\Omega} + \cdots + e^{-j(N-1)\Omega}] \\ &= \frac{1}{N}\left[\frac{1 - e^{-jN\Omega}}{1 - e^{-j\Omega}}\right] \end{aligned}$$

Example (Moving Average filter II)

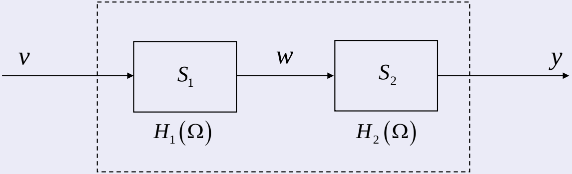
Alternatively, we can apply the input $v[n] = e^{j\Omega n}$ and obtain the output response $y[n] = H(\Omega)e^{j\Omega n}$. Substituting these gives

$$\begin{aligned}
H(\Omega)e^{j\Omega n} &= \frac{1}{N}[e^{j\Omega n} + e^{j\Omega(n-1)} + e^{j\Omega(n-2)} + \cdots + e^{j\Omega(n-N+1)}] \\
\Rightarrow H(\Omega) &= \frac{1}{N}[e^{j\Omega n} + e^{j\Omega(n-1)} + e^{j\Omega(n-2)} + \cdots + e^{j\Omega(n-N+1)}]e^{-j\Omega n} \\
&= \frac{1}{N}[1 + e^{-j\Omega} + \cdots + e^{-j(N-1)\Omega}] \\
&= \frac{1}{N}\left[\frac{1 - e^{-jN\Omega}}{1 - e^{-j\Omega}}\right]
\end{aligned}$$

It can be shown that MVA filter has linear phase of over a range of frequencies - see the Problem Booklet.

Definition (Cascaded Systems)

Let S_1 be a system that maps input v to outputs w, while S_2 maps inputs w to y:



We say that the overall system S that maps input v to y is the **cascaded connection** of the systems S_1 and S_2.

Theorem (Cascaded Systems)

If systems S_1 and S_2 have frequency responses $H_1(\Omega)$ and $H_2(\Omega)$, then the cascaded system S has frequency response

$$H(\Omega) = H_1(\Omega)H_2(\Omega)$$

Definition (System Inverse)

For cascaded systems S_1 and S_2, we say that system S_2 is the **inverse system** of S_1 if the cascaded system S is the **identity system**, i.e.

$$Sv = v$$

Theorem

Let systems S_1 and S_2 have frequency responses $H_1(\Omega)$ and $H_2(\Omega)$, and unit pulse responses h_1 and h_2, respectively. Then S_2 is the inverse of S_1 if

$$H_1(\Omega)H_2(\Omega) = 1 \tag{1}$$

Equivalently, S_2 is the inverse of S_1 if

$$(h_1 \star h_2)[n] = \delta[n] \tag{2}$$

To show that two systems cancel each other out (i.e. their cascade is the identity system), we can use either (1) in the frequency domain, or else (2) in the time domain.