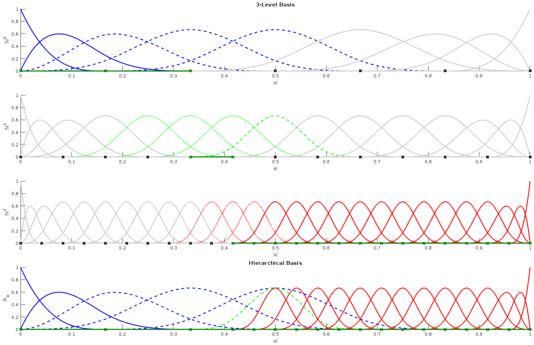
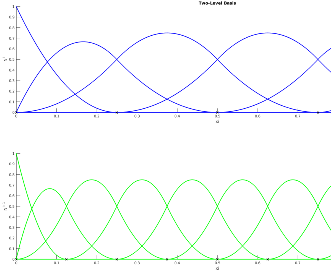
Multi-level by knot insertion

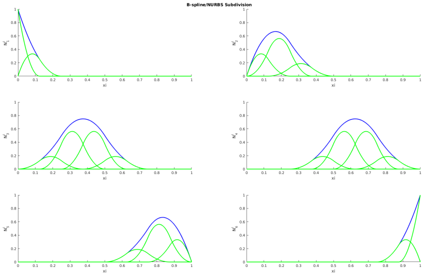
- Consider the multi-level for IGA:
  - Start from a base knot-vector.
  - Super-imposition of knot-vectors obtained by knot insertion.
- Knot insertion defines a space containing the space defined by the previous knot-vector.
  - "Nested spaces"
- Recursively: it contains the space defined by **all** previous knot vectors.



Multi-level by knot insertion

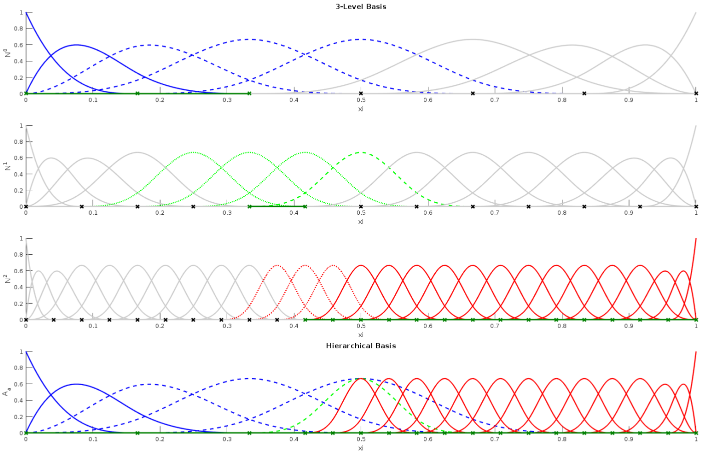
- $\implies$ Linear combination of basis function of one level can represent all the basis functions of all the level below it.
- $N^1 = M^{12} N^2$.
- M^{12} is a standard knot insertion matrix available in the literature.





Multi-level by knot insertion

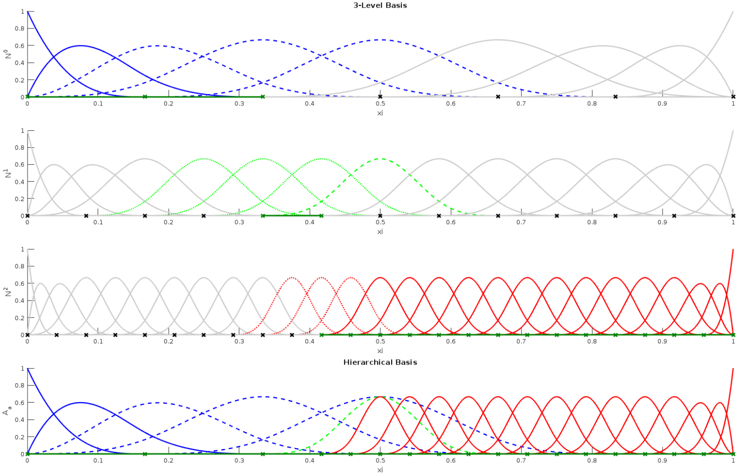
- The M^{mn} operator can be localized to each knot-span
  - "Element point of view"
- $N^1_e = M^{12}_e N^2_e$.



Multi-level Operator

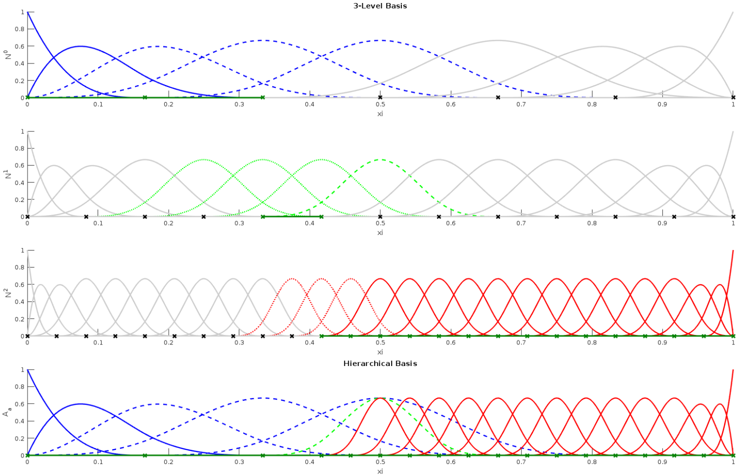
These properties are exploited to ease the implementation of multi-level refinements

- First: define active leaf-elements on each levels



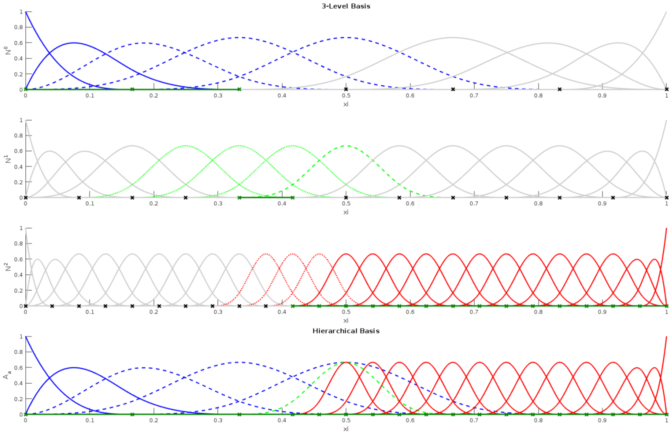
Multi-level Operator

- Second: identify active (dashed and solid) and non-active (dotted and gray) basis functions



Multi-level Operator

- For each (leaf-)element, compute the linear operator that relates the non-zero functions on the element (active and non-active) to the non-zero active functions of **all** previous levels

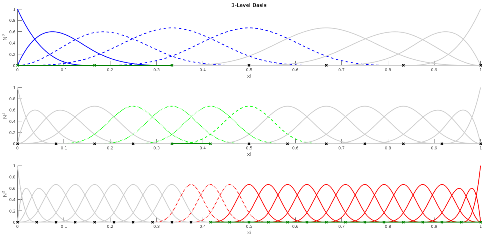


Multi-level Operator

E.g., for the 5th element from left

$$\underset{6x1}{N_e} = \begin{bmatrix} \underset{3x1}{N_e^1} \\ \underset{1x1}{N_e^2} \\ \underset{2x1}{N_e^3} \end{bmatrix} = \begin{bmatrix} \underset{3x4}{M_e^{13}} \underset{4x1}{\tilde{N}_e^3} \\ \underset{1x4}{M_e^{12}} \underset{4x1}{\tilde{N}_e^3} \\ \underset{2x4}{I_{active}^3} \underset{4x1}{\tilde{N}_e^3} \end{bmatrix} = \begin{bmatrix} \underset{3x4}{M_e^{13}} \\ \underset{1x4}{M_e^{12}} \\ \underset{2x4}{I_{active}^3} \end{bmatrix} \underset{4x1}{\tilde{N}_e^3} = \underset{6x4}{M_e} \underset{4x1}{\tilde{N}_e^3}$$

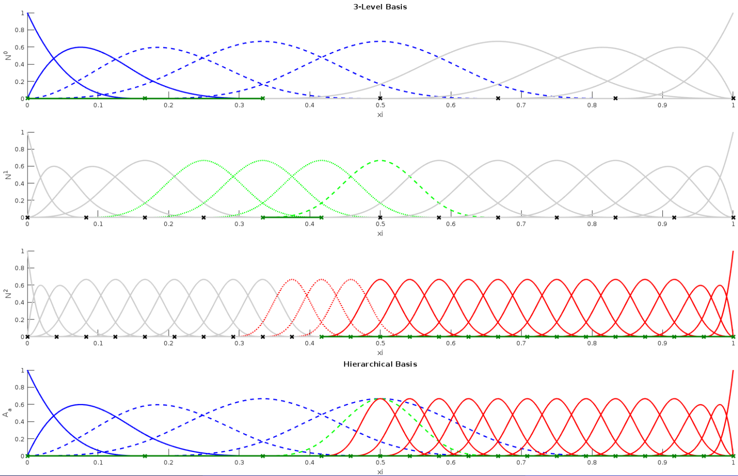
- N_e^3: *active* functions (solid and dashed) on element
- $\tilde{N}_e^3$: *active and non-active* functions (solid, dashed, dotted) on element
- I_{active}^3: selects the active basis functions $N_e^3 = I_{active}^3 \tilde{N}_e^3$



$$M_e = \begin{bmatrix} 0.0156 & 0 & 0 & 0 \\ 0.4844 & 0.3125 & 0.1562 & 0.0625 \\ 0.4844 & 0.6250 & 0.6875 & 0.6250 \\ 0.1250 & 0.5000 & 0.7500 & 0.5000 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

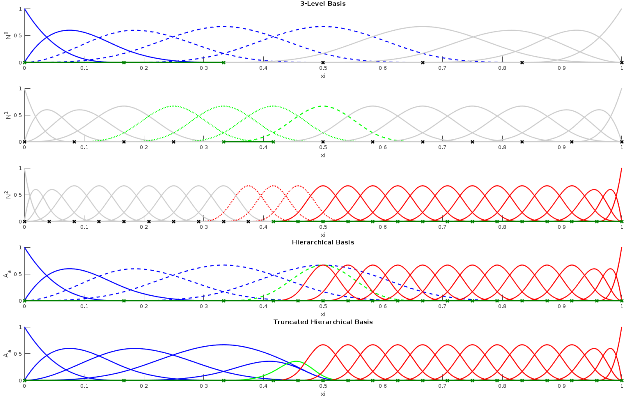
Multi-level Bèzier Extraction Operator

- To ease implementation on FEM codes, combine with Bèzier Extraction Operator C
  - $N_e^{all} = M_e N_e^3 = M_e C_e^3 B_e = K_e B_e$.



Multi-level Bèzier Extraction Operator

- Truncation can be included in the operator M_e by considering in each level just the linear combination of the dotted functions.
- I.e., the rows and columns of active functions are set to zero in each level.



$$M_e = \begin{bmatrix} 0.0156 & 0 & 0 & 0 \\ 0.4844 & 0.3125 & 0.1562 & 0.0625 \\ 0.4844 & 0.6250 & 0.6875 & 0.6250 \\ 0.1250 & 0.5000 & 0.7500 & 0.5000 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

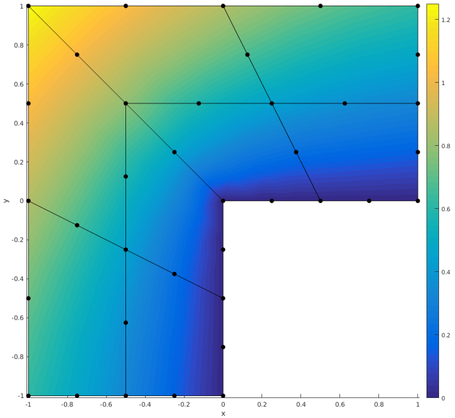
$$M_e^{trunc} = \begin{bmatrix} 0.0156 & 0 & 0 & 0 \\ 0.4687 & 0.25 & 0 & 0 \\ 0.3906 & 0.25 & 0 & 0 \\ 0.1250 & 0.50 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

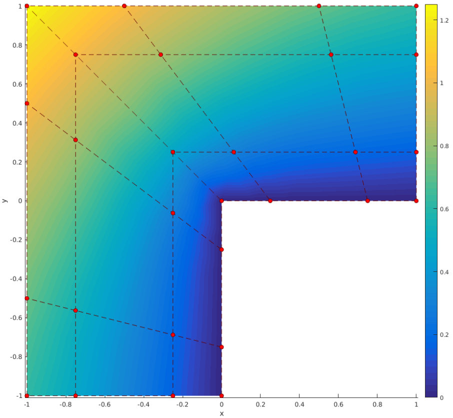
Advantages

- Shape evaluation just in the leaf-span level.
  - No evaluation is need in any other level.
- Removes concatenation of mappings of mappings of mappings
  - Just direct mapping from the leaf-span to the physical space.
  - Important for high-order PDEs
- Total element-point-of-view also for the multi-level. This eases the implementation in existing FE codes.
- Approach valid for every overlay of nested spaces.
  - E.g. for p-FEM.
- Eases solution evaluation, e.g. for material non-linearities, deformation gradient
- I think it easily allows for anisotropic refinements
- Domain distribution in parallel codes
- Same properties and function of Bèzier extraction, but generalized to multi-level

Example: (Bèzier) Control mesh of L-shape domain

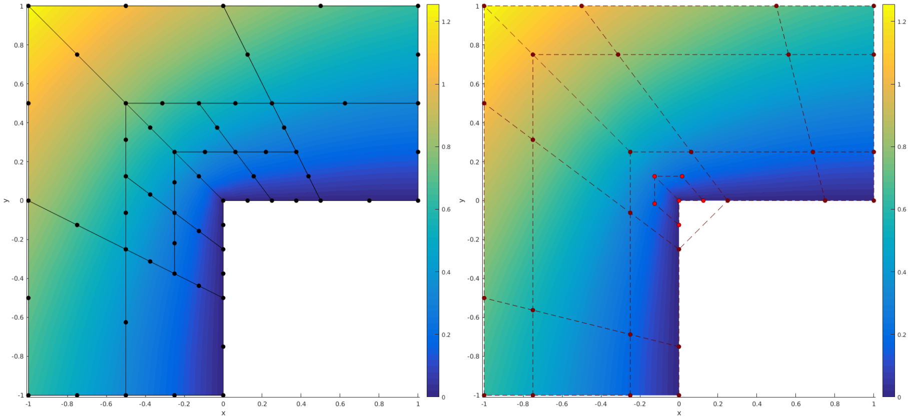
Temperature, control mesh and Bèzier control mesh obtained by the multi-level Bèzier extraction operator ($p = 2$)



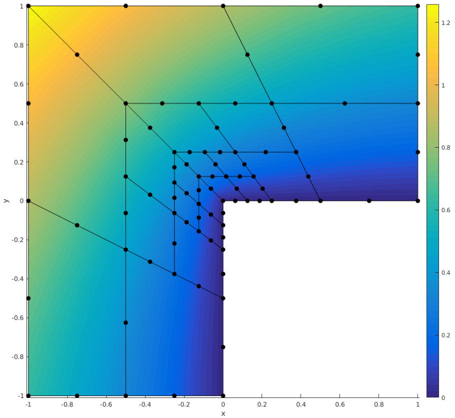


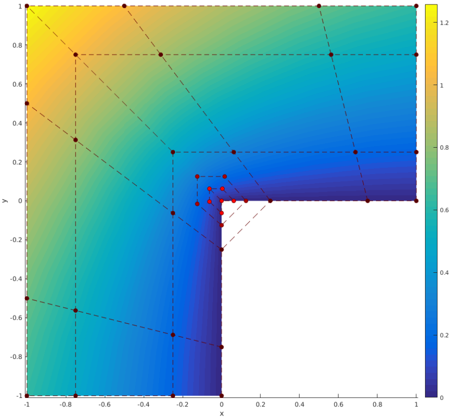
Example: Heat conduction on L-shape domain

Temperature, control mesh and Bèzier control mesh obtained by the multi-level Bèzier extraction operator ($p = 2$)



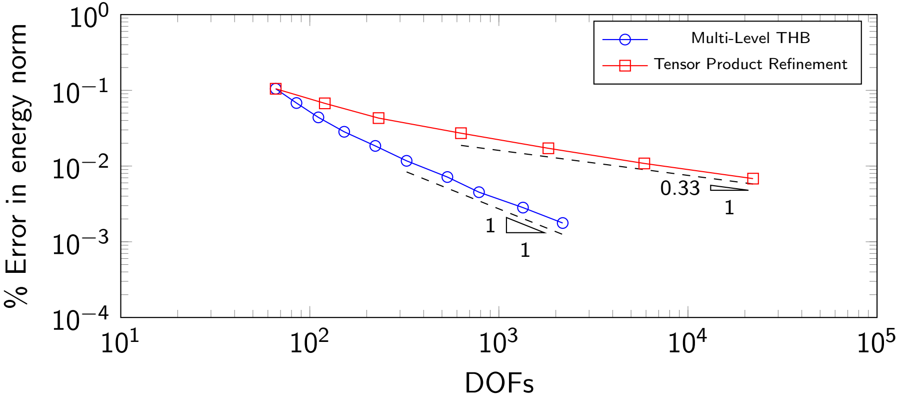
Temperature, control mesh and Bèzier control mesh obtained by the multi-level Bèzier extraction operator ($p = 2$)





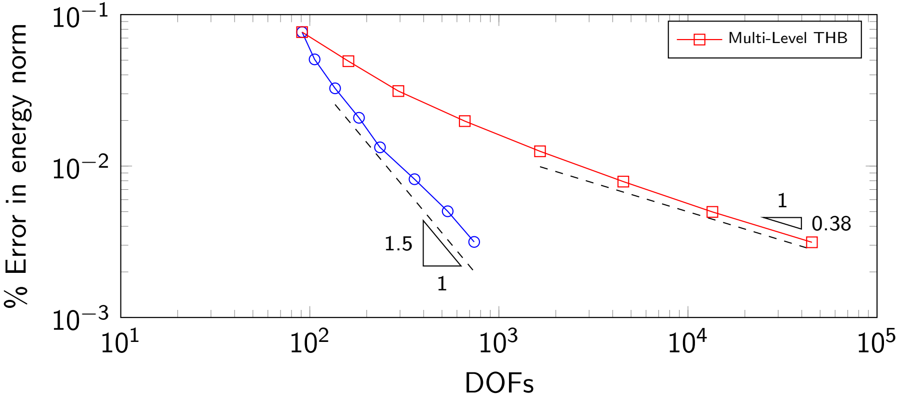
Example: Heat conduction on L-shape domain

- $p = 2$



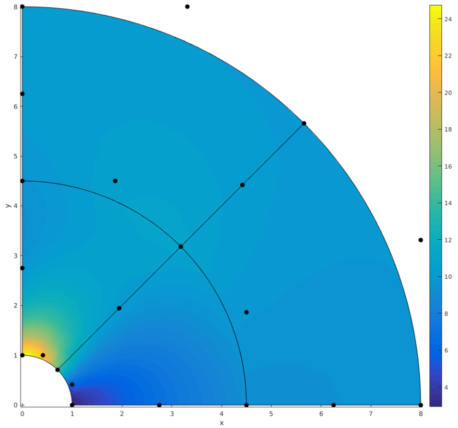
Example: Heat conduction on L-shape domain

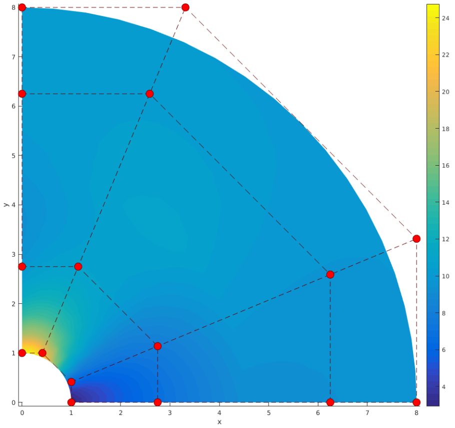
- $p = 3$



Example: (Linear) elastic plate with circular hole

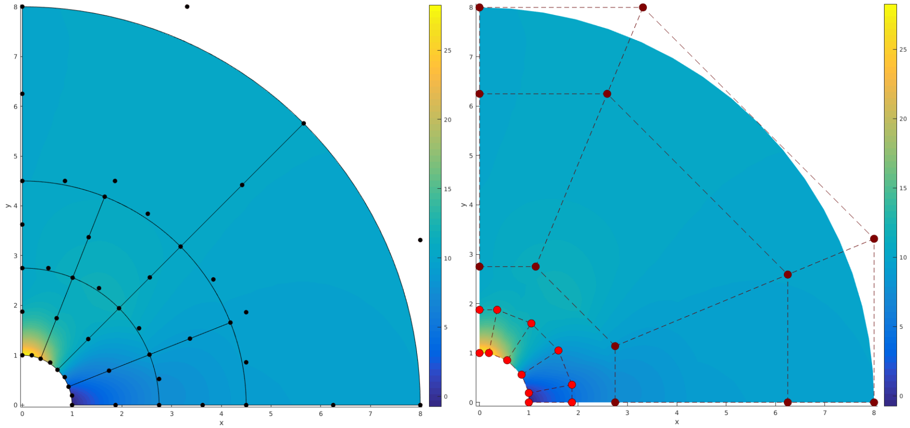
σ_{XX}, control mesh and Bèzier control mesh obtained by the multi-level Bèzier extraction operator ($p = 2$)





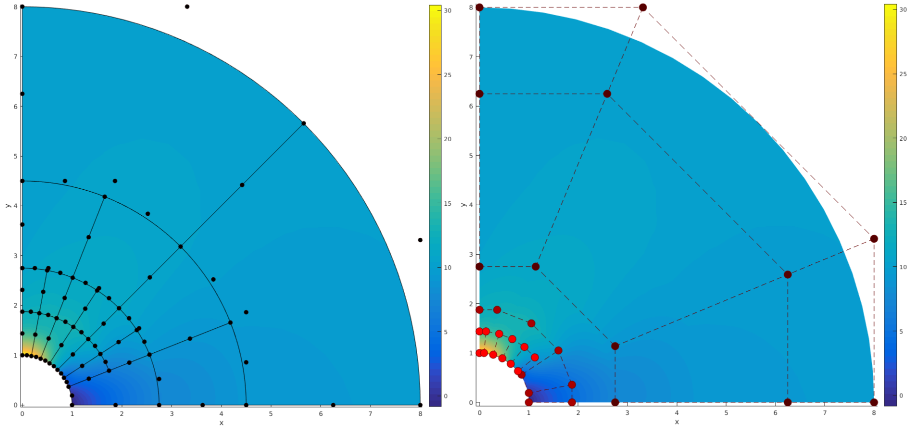
Example: (Linear) elastic plate with circular hole

Control mesh and Bèzier control mesh obtained by the multi-level Bèzier extraction operator ($p = 2$)



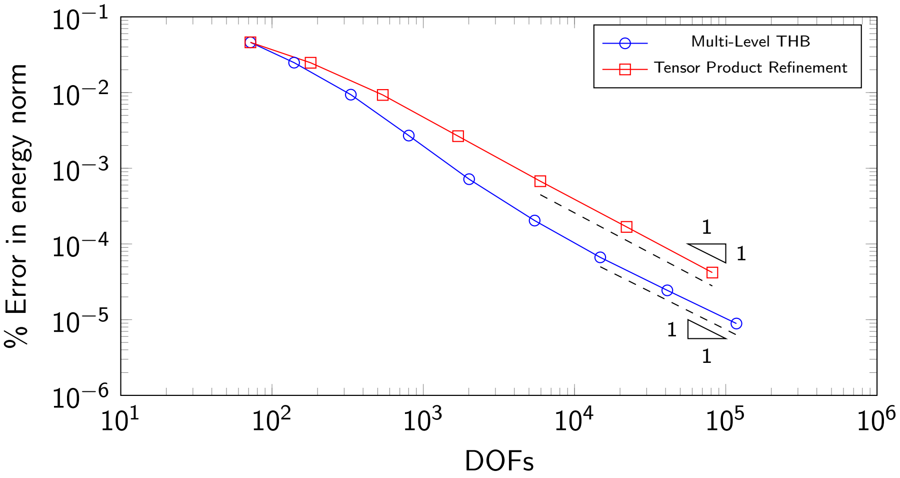
Example: (Linear) elastic plate with circular hole

Control mesh and Bèzier control mesh obtained by the multi-level Bèzier extraction operator ($p = 2$)



Example: (Linear) elastic plate with circular hole

- $p = 2$



Example: (Linear) elastic plate with circular hole

- $p = 3$