informs
https://pubsonline.informs.org/journal/mnsc

MANAGEMENT SCIENCE
Articles in Advance, pp. 1–24
ISSN 0025-1909 (print), ISSN 1526-5501 (online)

Best of Both Worlds Ad Contracts: Guaranteed Allocation and Price with Programmatic Efficiency

Maxime C. Cohen,[a,*] Antoine Désir,[b] Nitish Korula,[c] Balasubramanian Sivan[d]

[a] Desautels Faculty of Management, McGill University, Montreal, Quebec H3A 1G5, Canada; [b] Technology and Operations Management, INSEAD, 77305, Fontainebleau, France; [c] Google, New York 10011; [d] Google Research, New York 10011
*Corresponding author

Contact: maxccohen@gmail.com, https://orcid.org/0000-0002-2474-3875 (MCC); antoine.desir@insead.edu, https://orcid.org/0000-0001-5654-3301 (AD); nitish@google.com (NK); balusivan@google.com (BS)

Received: October 27, 2020
Revised: March 2, 2022; July 15, 2022
Accepted: August 12, 2022
Published Online in Articles in Advance: September 19, 2022

https://doi.org/10.1287/mnsc.2022.4542

Abstract. Buying display ad impressions via real-time auctions comes with significant allocation and price uncertainties. We design and analyze a contract that mitigates this uncertainty risk by providing guaranteed allocation and prices while maintaining the efficiency of buying in an auction. We study how risk aversion affects the desire for guarantees and how to price a guaranteed allocation. We propose to augment the traditional auction with a programmatic purchase option (which we call a *Market-Maker contract*) that removes allocation and price uncertainties. Instead of participating in the auction, advertisers can secure impressions in advance at a fixed premium price offered by the Market-Maker. It is then the responsibility of the Market-Maker to procure these impressions by bidding in the auction. We model buyers as risk-averse agents and analyze the equilibrium outcome when buyers face two purchase options (auction and Market-Maker contract). We derive analytical expressions for the Market-Maker price that reveal insightful relationships with uncertainties in the auction price and buyers' risk levels. We also show the existence of a Market-Maker price that simultaneously improves the seller's revenue and the sum of buyers' utilities. As a building block to our analysis, we establish the truthfulness of the multiunit auction when buyers have nonquasilinear utilities because of risk aversion. Recently, the Google's Display & Video 360 platform started offering a product akin to Market-Maker called "Guaranteed Packages," which was inspired by this paper.

History: Accepted by Gabriel Weintraub, revenue management and market analytics.
Funding: The authors thank Google Research for its generous support.

Keywords: online ads • guaranteed contracts • risk aversion • auctions • revenue management

1. Introduction

We propose and study an advance selling contract for internet advertising. When sellers (publishers of internet content who sell advertising space) offer this contract, we show that it leads to a win-win outcome so that both sellers and buyers (advertisers who purchase the right to display their ad) are better off. Traditionally, display advertising space has been sold in two main ways. First, one way is through *reservation contracts* sold in advance, where an advertiser enters into an agreement with a publisher by paying a fixed price for its ads to be shown to a specified volume of users who are visiting the publisher's website, often satisfying certain additional criteria. For example, Nike may pay $50,000 to have its ads shown to 5 million espn.com users who are based in the United States and frequently visit the basketball section of the website. Reservation contracts have guaranteed spend from the advertiser and a guaranteed number of impressions (an impression occurs when an ad is shown to a user). Second, another way is through *real-time bidding* in programmatic advertising, in which advertisers and publishers meet through an *ad exchange* platform (e.g., Google's DoubleClick Ad Exchange). When a user visits the publisher's website, the exchange requests real-time bids from advertisers and runs an auction, awarding the ad slot to the highest bidder. This is usually a first-price or second-price auction for each impression, with no guarantee to either the publisher or the advertisers.

Although reservation contracts solve the problem of allocation and price uncertainties for advertisers who are willing to pay a premium, they introduce a problem of their own: substantial overhead costs and inefficiencies associated with striking and managing these contracts both for publishers and for advertisers (see, for example, Medium 2018). These inefficiencies include lengthy human negotiations between publishers and advertisers on price and volume discounts. In addition, the advertiser needs to manage such processes with *all* the publishers and to follow-up with individual publishers when contracts underdeliver. Similarly, publishers need to manage this process with all the advertisers.

Thus, scaling this process to a large number of buyers and sellers is impractical. This motivates the central question studied in this paper: design and formally analyze an ad-buying mechanism that provides the efficiencies of programmatic advertising (real-time bidding in auctions) as well as the allocation and price guarantees of reservation contracts—allowing us to ultimately achieve the best of both worlds.

1.1. Risk Aversion

In this paper, we propose to analyze the benefits of providing allocation and price guarantees under the assumption that buyers are risk averse. Indeed, risk aversion is one of the potential explanations for the fact that buyers pay significantly higher prices for the same ad impression purchased via reservation contracts. More specifically, the difference in prices between auctions and reservation contracts cannot be explained by higher-"quality" ads being delivered via reservation contracts (e.g., better ad placements or targeted to users who are more likely to click on the ad). It turns out that even after controlling for content-level and ad slot-level features, reservation contract prices are significantly higher. Besides, via features such as Header Bidding, almost all ad slots are available in both reservations and auctions. If anything, auction buyers have more ability to cherry-pick valuable users relative to reservation buyers who typically purchase large volumes of bundled impressions (and hence, have a limited ability to target specific audiences). Nevertheless, transaction volumes via these high-priced reservation contracts form a large fraction of the overall display ad transactions. Moreover, apart from internet advertising, risk aversion has been used to explain the high premiums that exist in the cloud-computing marketplace, which also supports two purchase mechanisms (guaranteed versus spot market). As Hoy et al. (2016, p. 74) puts it, "It is easy to imagine that a company would have a soft budget set aside for computational costs. They would then spend freely within the confines of this budget and extend the budget cautiously when necessary to meet their computing needs. This type of behavior suggests a tendency towards risk aversion on the part of the clients. As the budget is freely available, clients might prefer to 'overspend' to guarantee the required resources at the on-demand price." The same situation prevails in the display ads market, where several large advertisers operate under a fixed budget for their marketing campaigns and are willing to pay a premium to avoid uncertainties in the volume of impressions. Although it is quite possible that there may be alternative possible explanations beyond risk aversion for these behaviors, in this paper, we assume that the buyers are risk averse, and we study the benefits of price guarantees under risk aversion. Our results suggest that it would be interesting to empirically validate the risk aversion assumption and also, determine the exact degree of risk aversion from data.

1.2. Market-Maker: High-Level Description

In this paper, we introduce and study the provision of a *Market-Maker contract* as an alternative purchase option for buyers, beyond the traditional option of real-time bidding in first-price or second-price auctions. The Market-Maker contract can be seen as either completely replacing reservation contracts or complementing reservation contracts for advertisers who want to opt out of the high reservation costs. A Market-Maker contract quotes a price higher than the expected auction price (the expectation is over all the possible sources of uncertainty in the auction). Buyers can then choose either to pay this higher price or to take their chance in the auction. Risk-averse buyers concerned with possibly not receiving impressions at all or paying an unpredictable high price may thus prefer to pay a premium over the expected auction price in exchange of certainty in allocation and price. A Market-Maker contract guarantees (like reservation contracts) that buyers will receive the impressions they paid for.

1.2.1. Novelty.

The idea of offering a guaranteed contract in uncertain markets is not novel (see, e.g., Ghosh et al. 2009, Yang et al. 2010, Sayedi 2018). The Market-Maker contract studied in this paper differs from a standard guaranteed reservation contract in two important ways. The main novelty in a Market-Maker contract is that the ad exchange becomes the single point of sales for impressions. This is ensured by requiring that the entity offering the Market-Maker contract *purchases impressions in the ad exchange by bidding in the auction on behalf of its buyers*, even if it has to pay a higher price than the amount charged to the buyers. This implies that the overhead of dealing with numerous publishers and advertisers is eliminated. Furthermore, because all the impressions are sold via the exchange, publishers need not to worry about which impressions to sell in the exchange versus allocate to reservation contracts. Because the capability to carve out impressions, which a publisher alone can do, is not necessary anymore when all impressions are sent to the exchange, any third party can offer a Market-Maker contract, thereby *democratizing the concept of reservation contracts*. In fact, Google's Display & Video 360 platform started offering a Market-Maker style product, called "Guaranteed Packages," inspired by the ideas underlying this research. The second difference between a Market-Maker contract and reservations is that we require that no buyer who can afford the Market-Maker (and would have preferred to purchase a Market-Maker contract over participating in the auction) should go empty handed because impressions were sold out. In today's market, it is possible for a publisher to underprice its reservation contracts and hence, leave several buyers empty handed. We highlight that this problem cannot be solved by simply raising the Market-Maker or reservation-contract price until supply meets demand. Indeed, when buyers face two purchase options, increasing the price of one option could result in nontrivial changes in the equilibrium outcome. Finally, a great advantage of Market-Maker contracts over reservation contracts is that they can be offered by a third party, such as a demand-side platform, whereas reservation contracts can only be offered by publishers. Thus, one way to interpret a Market-Maker contract is as an insurance product, wherein the entity offering the Market-Maker contract charges a premium to bear the risk.

1.3. Contributions

We next summarize our contributions.

- Equilibrium outcome analysis. We consider a stylized setting where buyers participate in a single-shot multiunit auction. In other words, we abstract the auction process (repeated first- or second-price auctions with budget constraints) into a single-shot multiunit auction. This simplification is made for tractability purposes, and we discuss some assumptions under which it is reasonable and also, examine how the results degrade as we relax the assumption in a natural way. Moreover, the multiunit auction arises in other settings, such as spectrum auctions (Milgrom 2004). The introduction of a new alternative for buyers, namely the Market-Maker contract, alters the market outcome. What is the equilibrium under these two buying alternatives? We study this question for two widely used risk aversion utility models: the constant absolute risk aversion (CARA) model and the mean-variance utility model (precise definitions can be found in Section 3). Under some mild assumptions, we show that the auction process admits a similar equilibrium distribution of clearing prices before introducing the Market-Maker option. Interestingly, this requires proving that the multiunit auction is truthful in the presence of risk aversion, which is an interesting result in itself (see Theorem 4). Armed with this equivalence, we analyze the joint equilibrium in the presence of the Market-Maker contract. In particular, we characterize the range of feasible Market-Maker prices that enable the seller to guarantee allocation to all buyers who choose the Market-Maker contract in equilibrium.[1]
- Pareto improvement in seller's revenue and sum of buyers' utilities. We analyze the equilibrium outcome and prove the existence of a Market-Maker price that *simultaneously* improves the seller's revenue and the sum of buyers' utilities. This result holds irrespective of whether the population is homogeneous or heterogeneous in the risk aversion level. When the population is heterogeneous, there exists a range of optimal prices. Under a homogeneous population, however, we show that there is a unique optimal Market-Maker price and derive its analytical expression for both utility models (CARA and mean-variance). Importantly, the premium charged by the Market-Maker over the expected auction clearing price admits a crisp characterization. Specifically, the premium has a simple, insightful relationship with the inherent auction uncertainty, hence offering valuable pricing guidance to the Market-Maker provider.
- Practical impact. As discussed, Google's Display & Video 360 platform started offering a product akin to Market-Maker called "Guaranteed Packages," which was inspired by the research presented in this paper.

1.3.1. Informing the Pricing of Reservation Contracts.

Although our results are for a Market-Maker contract, they also provide pricing guidance for a publisher who offers reservation contracts. Specifically, our results on pricing a Market-Maker contract can be interpreted as answering the question of how one should price a reservation contract when it coexists with the auction. For example, the Market-Maker price we derive can act as a guidance, or as a starting point, in the price negotiation with advertisers. A Market-Maker contract can inform the pricing of reservation contracts because for pricing purposes, it does not matter whether the contract provider bids in the auction to procure impressions (like in the Market-Maker) or uses a different mechanism to carve out impressions for buyers (like in reservation contracts). The only relevant feature is the fact that a contract (either Market-Maker or reservation) coexists with the auction. In fact, the main differences between the Market-Maker contract and a reservation contract are operational and in terms of execution. From the perspective of pricing analysis, the same analysis holds for both contracts.

1.3.2. Structure of the Paper.

We review the related literature in Section 2. We discuss in detail our model and assumptions in Section 3. Section 4 presents a key step of our analysis: deriving the equilibrium distribution over clearing prices when there is only a single purchase option. Our main results on the impact of offering a Market-Maker contract are reported in Sections 5 and 6 (for homogeneous and heterogeneous settings, respectively). In Section 7, we present computational experiments on commonly used distributions to illustrate the lifts in the seller's revenue and in buyers' utilities generated by adding a Market-Maker contract. Finally, our conclusions are reported in Section 8. Most proofs of the technical results are relegated to the appendix.

2. Related Literature

This paper is related to the topic of advance selling from marketing and operations management, where the key message is that advance selling helps the seller increase its profit by offering a *discounted* price to buyers who commit to purchase in advance. In contrast, we show that when buyers are risk averse, advance selling with a *marked-up* price allows for the increase of both the seller's revenue and the buyers' utilities. This paper is also related to various other research streams as discussed next.

2.1. Marketing

In the marketing community, the topic of advance selling has received great attention in the last two decades (see, e.g., Shugan and Xie 2000, 2005). Shugan and Xie (2000) show that advance selling allows sellers to increase their profits. More precisely, it is shown that even when buyers are homogeneous in the advance-period purchase, advance selling can achieve first-degree price discrimination. Subsequently, Shugan and Xie (2005) extend the treatment to competitive environments and show that the relative profit advantage from advance selling in a competitive market can be higher as in a monopoly.

2.2. Operations Management

In the operations management community, advance selling was also extensively studied. Prasad et al. (2011) consider advance selling in a newsvendor setting and examine the price and inventory decisions to conclude that advance selling is not always optimal. Cho and Tang (2013) consider a supply chain setting with a manufacturer that produces and sells a seasonal product to a retailer under uncertain supply and demand. Cachon (2004) studies how inventory risk allocation impacts supply chain efficiency under advance purchase discount contracts. Boyacı and Özer (2010) consider a model that uses advance sales information to make capacity decisions. The authors derive a threshold policy to determine when the firm will stop acquiring advance sales information and show that advance selling can significantly increase profits. Caldentey and Vulcano (2007) study a similar problem in the context of an online multiunit auction. In their model, the seller faces a Poisson arrival stream of consumers who can get the product from the auction or from a list price channel. Our work differs by explicitly modeling the risk aversion of the buyers in the utility function. In addition, we focus on studying how to design and set the price of the advance selling option. Araman and Popescu (2010) study the ad allocation problem in the context of TV broadcasting. The authors consider the problem of how to allocate limited advertising space between up-front contracts (i.e., sales at the start of the season) and the scatter market (i.e., a spot market that sells ads for the next few days) under audience or supply uncertainty. Lastly, Gao et al. (2019) study the competition between two service providers that differ in their pricing menu. The first firm offers the same service to all customers at a fixed price. If customers choose the second firm, their service is tied to a bid they submit: the higher the bid, the lower the waiting time. This paper differs from ours in two fundamental ways. First, the authors use a queuing approach where the price impacts the service through customer waiting times. Second, the motivations are quite different as the equilibrium in Gao et al. (2019) follows from customer heterogeneity, whereas our results are driven by buyers' risk aversion.

2.3. Dual-Selling Channels

Iyer et al. (2012) study the coexistence of two buying options, but the equilibrium in their paper is driven by information asymmetry as opposed to risk. Several other papers also study the problem of having dual-selling channels between reservation contracts and real-time bidding (i.e., auctions). Athey et al. (2013) and Sayedi (2018) conclude that both channels will likely continue to coexist, as advertisers and publishers have incentives to leverage the specific features of each channel (e.g., fine-grained targeting, ad customization, and unbundling the impressions). Another closely related stream of work is auctions with buy prices (see, e.g., Budish and Takeyama 2001, Hidvegi et al. 2006, Kirkegaard and Overgaard 2008, Reynolds and Wooders 2009, Shunda 2009), which consider adding a "buy-it-now" option to the English auction mechanism and study the revenue impact in the presence of risk-averse buyers. Although the idea of offering two purchase options is common, our work differs from this stream in three important ways. First, we design the optimal Market-Maker contract as the solution to a constrained optimization problem, the constraint being on the Market-Maker price to be such that any buyer who prefers buying a Market-Maker contract is not left empty handed because of insufficient supply. This constraint significantly restricts the range of prices one can offer (whereas the aforementioned stream of work does not impose such a constraint). Second, we consider two risk aversion models: CARA and mean-variance utility. To our knowledge, the latter model, despite being popular in many applications, has rarely been studied in the context of auctions. Third, our analysis is not focused on the seller's revenue but also, considers the impact on buyers' utilities. Specifically, we show that adding a Market-Maker contract yields a Pareto improvement for both the seller and the buyers.

2.4. Computer Science and Economics

Yang et al. (2010) study the problem of allocating inventory using a multiobjective optimization formulation when display ads can be sold both via guaranteed contracts and real-time auctions. Turner (2012) solves the ad network's single-period planning problem that allocates guaranteed impressions as a transportation problem with a quadratic objective. These two papers do not investigate the pricing of the contracts but rather, focus on the complementary inventory allocation problem. Several papers study display ad allocation strategies under the presence of both contract-based advertisers and spot market advertisers. Given that advertisers can cherry-pick impressions in real-time bidding, contract-buying advertisers are often left with lower-value impressions. This adverse selection issue can yield reduced revenue to contract-buying advertisers. Ghosh et al. (2009) address this issue by proposing bidding strategies where the publisher directly bids in the auction on behalf of contract-based advertisers while randomly varying the bid to get a representative allocation in reservation contracts. This method allows them to maintain the quality of the impressions assigned to reservation contracts.

As mentioned, Hoy et al. (2016) independently considered the problem of guaranteed and spot markets coexisting in the context of cloud computing. Their model assumes that the seller is offering both options and explicitly sets aside inventory for the guaranteed purchasers. In contrast, our Market-Maker contract cannot set aside inventory, as it is not necessarily operated by the seller. Instead, the Market-Maker places its bids in the auction to purchase inventory for its buyers and hence, incurs a loss with some probability (although it earns a positive profit in expectation). Consequently, for markets running first-price or second-price auctions, this service can be offered by any third party willing to accept the arbitrage risk, although it is likely to be the exchange, passing on a large portion of the profit to publishers. Wang and Chen (2012), Chen et al. (2014), and Chen (2016) study the pricing aspect of a programmatic version of reservation contracts. More precisely, Chen et al. (2014) and Chen (2016) both discuss dynamic models for selling guaranteed impressions. In their model, advertisers arrive sequentially, and the publisher decides which ones to accept based on a dynamic control policy. Similar to our paper, advertisers are assumed to be risk averse, and their utility is modeled using the mean-variance model. The authors consider the case where all the unallocated advertisers participate in a single-shot auction, whereas we provide a formal reduction to that setting. Moreover, they do not formally characterize the equilibrium outcome. Finally, their proposed control policy is dynamic in nature, and the advertisers are allocated on a first come, first served basis. This is different to the spirit of our proposed solution, where any buyer who prefers buying a Market-Maker contract is not left empty handed. Mirrokni and Nazerzadeh (2017) consider preferred deals and show how to design them to approximately optimize revenue. A preferred deal allows a buyer to bid on an inventory before it is exposed to the general set of buyers in the open auction. A buyer is required to purchase at least a specific fraction of the inventory at a prenegotiated price. In the multibuyer case, a preferred deal contract can also specify priorities for different buyers and expose inventory in the order of priority. Their work compares the revenue obtained by a preferred deal contract relative to extracting the entire surplus. They show that preferred deals can extract at least one-third of the surplus and empirically establish that this fraction can be much higher. The main differences from our work are that (a) buyers are risk neutral and that (b) there is no comparison between both purchase options (guaranteed contracts and auction). Balseiro et al. (2011) study the joint optimization problem of earning short-term revenue from the ad exchange and delivering high-quality impressions to reservations. The main differences with our work are that (a) buyers are risk neutral and that (b) their work does not address the pricing of reservation contracts. Instead, they study how to reserve impressions that are sold in the ad exchange so as to not underdeliver reservation contracts while also maximizing the ad exchange revenue. Fu et al. (2013) study prior independent revenue maximization in the presence of risk-averse buyers and show that a first-price auction yields a constant factor approximation to revenue when agent valuations are independent and identically distributed. Revenue maximization with risk-averse buyers is an important problem even without having two purchasing options (see, e.g., Maskin and Riley 1984). Wang and Chen (2012) take the perspective of a publisher and study numerically the pricing of reservation contracts when the auction price is modeled as a Brownian motion. Finally, the recent work in Chawla et al. (2018) characterizes revenue-optimal mechanisms in the presence of risk-averse buyers using a model based on prospect theory to capture risk aversion.

3. Model

In this section, we present our model and assumptions. We begin here with an informal high-level description to give the gist of our model. We consider a two-period model with periods P1 and P2. In P1, there is a deterministic buying process, and in P2, there is a randomized buying process (representing auctions). In P1, buyers make the decision of whether to buy in P1 with certainty or take a chance in P2. The auction process in P2 is a single-shot multiunit auction. This simplified model can be thought as a reduced form of the actual market, which is a sequence of repeated first- or second-price auctions. In Section 4, we discuss our choice and provide some assumptions under which this is a reasonable approximation. Furthermore, to gain analytical tractability, we impose assumptions on the model in Section 3.2.

3.1. Notation and Time Line

A seller intends to sell I identical units of a good available for delivery at a future date, which we refer to as period P2. There are N buyers (where N can potentially be a random variable) whose values for consuming the good are drawn from a joint distribution $F(\cdot)$, which is not necessarily independent across bidders. Buyer i has a demand of k_i units of the inventory. In the current period P1, buyers' values are drawn from the joint distribution $F(\cdot)$, and then, the seller offers two purchase options to every buyer: (a) purchase each unit at a fixed price p_M (referred to as the Market-Maker contract price) in the current period P1 or (b) take a chance and participate in an auction process in the later period P2, where the buyer may or may not get allocated depending on the overall demand (in this case, the buyer will pay the realization of the auction clearing price). In period P1, each buyer has to choose whether to opt for the Market-Maker contract available in P1 or wait for the auction in P2. While making this decision in P1, each buyer knows (i) their own value per unit of good and (ii) the distribution of v_{I+1}, which is the $(I+1)$ th highest value (including the nonunit demand multiplicities) with probability distribution function (pdf) $g(\cdot)$ (we assume that this is common knowledge). The distribution of v_{I+1} captures the randomness in buyers' values as well as the potential uncertainty in the number of buyers.

3.1.1. Buyer Decision Process. Every buyer decides between the two purchase options by comparing his or her utility from each option in equilibrium (the utility functions are defined formally in Section 3.3). The equilibrium here refers to the set of buyers who choose each purchase option and how they bid in the auction *such that all buyers are mutually best responding*: that is, no buyer can benefit from a unilateral deviation. Characterizing the equilibrium and using this characterization to set the Market-Maker price in order to optimize revenue and analyze its impact on related objectives, such as welfare and efficiency, are the focus of this paper.

3.1.2. Model Is Oblivious to the Auction in P2. We remark that the two-period model presented is oblivious to the exact auction process and dynamics in P2. Although the exact auction in P2 certainly influences how buyers respond in P1 (and in P2), the description of the two-period model itself does not depend on the precise auction process used in P2. Importantly, the information that buyers know when making decisions in P1 (namely deciding whether to opt for the Market-Maker or wait for the auction in P2) is oblivious to the type of auction run in P2.

3.1.3. Auction Process in P2. The auction process used in period P2 depends on the application. In internet ads, the auction process consists of a sequence of second-price auctions or a sequence of first-price auctions, as is common in most ad exchanges. In other one-off sales settings, such as cloud compute spot markets, spectrum, or timber auctions, the auction process in P2 is a single-shot multiunit auction. In this paper, we assume that the auction process in P2 is a single-shot multiunit auction, namely an auction selling I identical units of a good in a single-shot fashion using the Vickrey–Clarke–Groves (VCG) auction. As discussed, this is a simplification of the actual ads market, which is made for tractability purposes. Indeed, this provides a simpler language and framework and allows us to draw further insights. We discuss this topic in more details in Section 4.

3.2. Assumptions and Analysis Road Map

We begin by discussing our assumptions.

Assumption 1 (Identical Units). *All the units being sold are identical*.

When focusing on a certain segment of inventory (e.g., relevant demographics of the target audience and user interests), we assume that all the units within that segment are equally valued by a buyer. Indeed, reservation contracts in practice usually specify a fixed price for a large volume of impressions for a specified targeting criteria. Of course, different buyers can have different values, possibly correlated. In the context of display advertising, although it is true that advertisers often create numerous contracts (each focusing on a different segment and a different piece of inventory), different contract categories are typically allocated different budgets. Because different campaigns are usually run by different teams with their own budgets, they typically each solve their own subproblems, and thus, there is often no joint risk behavior across all segments but only segment-specific ones. This motivates us to focus our analysis on a per-segment basis.

Assumption 2 (Large Market). *The market is large enough so that no single buyer's demand is large enough relative to the total inventory value* (*i.e.*, $k_j \ll I$ *for all* j). *In particular, therefore*, $v_{I+1} + \cdots + v_{I+k_j} \approxeq k_j \cdot v_{I+1}$.

When all the demands are single unit (i.e., $k_j = 1$ for all j), the multiunit auction reduces to the I-highest bidders each being allocated a good and paying v_{I+1}. When the k_j's could be larger than one, the multiunit auction corresponds to the VCG auction where the allocated buyer-item pairs are still the ones with values $v_1, \ldots, v_I$, and each buyer j pays his or her externality, namely $v_{I+1} + \cdots + v_{I+k_j}$. Assumption 2 simplifies the VCG auction's payment to be approximately $k_j \cdot v_{I+1}$. Consequently, whether the k_j's are all equal to one (i.e., unit demand), the multiunit auction's payment is assumed to be v_{I+1} per unit. Importantly, this allows us to have all further discussions on a per-unit basis, even if k_j's are larger than one. Moreover, this implies that the VCG payment will be identical to the $(I+1)$ th price auction's payment, namely each buyer i will pay for every allocated unit the value of the highest-valued buyer j whose demand is not fully allocated. For this reason, we will often refer to the single-shot multiunit auction as the $(I+1)$ th price auction.

3.2.1. Analysis Road Map. For a buyer who signs a Market-Maker contract at price p_M, we denote his or her utility per unit of good by $U_M(v, p_M)$. Note that there is no randomness in this option for the buyer, and hence, there is no expectation over his or her utility. For a buyer who participates in the auction, we denote the resulting *expected* utility per unit of good by $U_A(v)$, where the expectation is over the distribution of auction clearing prices in equilibrium. The question is what exactly the distribution is over clearing prices in equilibrium. We answer this question in two steps.

1. We derive the equilibrium distribution over clearing prices when there is only one purchase option, namely the auction process in P2 (i.e., there is no Market-Maker purchase option). We refer to this as the *single-period equilibrium or P2 equilibrium*. In particular, we show that the equilibrium distribution over clearing prices is exactly the distribution of v_{I+1}, namely $g(\cdot)$.

2. We use the derivation from step 1 to derive the *joint equilibrium* or the *two-period equilibrium* when the two purchase options are offered across two periods (auction in P2 and Market-Maker in P1). Recall that the decision of which option to choose is made in P1.

3.3. Risk-Averse Utility Models

To proceed further, we need to specify the expression for utilities in the presence of risk aversion. After introducing our risk aversion models and defining utilities, we return in Section 4 to deriving the clearing price distribution mentioned in step 1 in Section 3. We finally complete step 2 in Sections 5 and 6.

We consider two popular utility models to capture buyers' risk aversion. Specifically, the utility of a buyer with value v, per unit of good received, takes one of the following forms:

$$U_A(v) = \begin{cases} \mathbf{E}_p\left[(v-p)^+\right] - \beta \cdot \mathsf{Var}_p\left[(v-p)^+\right] & \text{(Mean-variance model)}, \\ \mathbf{E}_p\left[1 - e^{-\alpha \cdot (v-p)^+}\right] & \text{(CARA model)}, \end{cases} \tag{1}$$

where $\alpha \geq 0$ and $\beta \geq 0$ are the parameters of each model that capture the buyer's risk aversion and p is the auction clearing price. The operators $\mathbf{E}[\cdot]$ and $\mathsf{Var}[\cdot]$ refer to the expectation and variance over the distribution of clearing prices. As mentioned earlier, we will later derive explicitly this distribution over clearing prices. The CARA and mean-variance classes of utility models are commonly used in the literature and aim to capture buyers' risk aversion. The mean-variance utility model is widely used in finance (e.g., portfolio optimization) and in marketing applications. See Levy and Markowitz (1979) and Markowitz (2014) for more details on the mean-variance model.[2] The CARA model is a commonly used risk aversion model in a large number of applications (see, e.g., Arrow 1971, Pratt 1992, Rabin 2000).

Because there is no uncertainty in allocation or in price when a buyer opts for the Market-Maker contract, the utility of a buyer with value v facing a Market-Maker contract with price p_M is

$$U_M(v, p_M) = \begin{cases} (v - p_M)^+ & \text{(Mean-variance model)}, \\ 1 - e^{-\alpha \cdot (v - p_M)^+} & \text{(CARA model)}. \end{cases} \tag{2}$$

Note that the expression in Equation (2) coincides with Equation (1) when there is no price uncertainty (i.e., when the distribution over p is a point mass at $p = p_M$ (in the mean-variance model, the variance term disappears when there is no uncertainty over p)). For conciseness, we will drop p_M from the argument of U_M when it is clear from the context.

3.4. Efficiency, Welfare, and Revenue

To measure the impact of adding a Market-Maker contract, we consider three metrics: efficiency, welfare, and revenue (apart from the sum of buyers' utilities). The *efficiency* of an outcome corresponds to the sum of valuations of the allocated buyers. The *welfare* is the sum of utilities of all buyers and the seller's revenue (note that under nonquasilinear utility models, *efficiency* and *welfare* are different). Finally, the *revenue* is defined as the sum of prices paid to the seller.

4. Equilibrium Clearing Prices

In this section, we return to step 1 mentioned in Section 3. More specifically, we analyze the equilibrium outcome in the single-shot multiunit auction (i.e., the $(I+1)$ th unit auction) when there is uncertainty in v_{I+1} under risk aversion. When buyers are risk neutral or alternatively, when there is no uncertainty in clearing prices, it is well known that the $(I+1)$ th price auction is truthful, regardless of what amount of uncertainty there is about v_{I+1}. What happens in the presence of risk? Does an equilibrium exist, and if it does, is truthful reporting an equilibrium? Here, a distinction has to be made between universal truthfulness (UT) and truthfulness in expectation (TIE), which are two well-studied notions from the literature (Nisan et al. 2007). An auction is UT if reporting the true values is a dominant strategy for the agents under each possible outcome (in this case, it corresponds to each possible value of v_{I+1}). An auction is TIE if reporting the true values is a dominant strategy in expectation over the random outcomes of the mechanism. Usually, UT implies TIE. Indeed, if the utility is maximized under each outcome, it is also maximized in expectation over outcomes, and this is the case for the CARA model. Under the mean-variance model, however, the utility for a distribution over outcomes is not equal to the expected utility over individual outcomes (one can see this from the expression of the mean-variance utility model in Equation (1)). In other words, the preferences over random outcomes under the mean-variance model do not all satisfy the von Neumann and Morgenstern (1953) (VNM) utility axioms. Nevertheless, as already discussed, the mean-variance model is an important model that is commonly used in the literature. Several theories exist (e.g., prospect theory) to describe how VNM axioms are often violated by real-life decision makers. As a consequence of calculus of expectations not holding, TIE under the mean-variance model does not follow from UT. In fact, UT under the mean-variance model is not meaningful (it is straightforwardly true because for each individual outcome (i.e., for each possible value of v_{I+1}), the $(I+1)$ th-unit auction is truthful—because the mean-variance utility reduces to the standard quasilinear utility given that the variance of a single outcome is zero). On the other hand, proving TIE under the mean-variance model happens to be quite intricate. Unlike standard arguments, we do not show that the utility decreases by misreporting. Instead, we show that when the utility under true reporting is positive, then it decreases by misreporting. Additionally, when the utility under true reporting is nonpositive, then it remains nonpositive by misreporting.

Theorem 1 (Truthfulness Under Risk Aversion). *Under the CARA and mean-variance utility models, the $(I+1)$th price auction is both universally truthful and truthful in expectation. Let U_{truth} (U_{lie}) be the utility in the auction under truthful reporting (misreporting).*

1. *If $U_{\text{truth}} > 0$, then $U_{\text{truth}} \geq U_{\text{lie}}$.*
2. *If $U_{\text{truth}} \leq 0$, then $U_{\text{lie}} \leq 0$.*

The proof of Theorem 1 can be found in Appendix B. We highlight the following subtlety. We do not claim that the utility never increases by misreporting. Instead, we show that if the true utility is positive, misreporting never yields a utility higher than the true value. If the true utility is negative, misreporting could yield a utility higher than the true value, but it remains negative. An agent with negative utility simply does not participate in the auction and earns zero utility instead. A negative utility means that any amount of uncertainty in the outcome is enough to cause a net disutility, so that such agents simply do not participate in the auction. Importantly, the Market-Maker contract helps such buyers by offering them a risk-free option, which is worth considering and hence, helps increase the net buyer participation. We note that the proof for the CARA model is straightforward, whereas the proof for the mean-variance model is more intricate. Under the latter utility model, we can write the utility earned by a buyer when reporting her or his true valuation as $U_{\text{truth}} = \mu_{\text{truth}} - \beta \cdot \text{Var}_{\text{truth}}$ and the utility when misreporting as $U_{\text{lie}} = \mu_{\text{lie}} - \beta \cdot \text{Var}_{\text{lie}}$. Given the structure of the utility, a buyer could misreport her or his valuation so that $\mu_{\text{truth}} \geq \mu_{\text{lie}}$ but $\text{Var}_{\text{truth}} \geq \text{Var}_{\text{lie}}$. In other words, a buyer could prefer a situation with a lower expected surplus but with less variability. We formally show that this situation never happens. In particular, a key step of the proof is to show that the relative change in the variance part of the utility is always upper bounded by the relative change in the mean part of the utility. More precisely, we show that when $\text{Var}_{\text{truth}} \geq \text{Var}_{\text{lie}}$, we must have

$$\frac{\text{Var}_{\text{truth}} - \text{Var}_{\text{lie}}}{\text{Var}_{\text{truth}}} \leq \frac{\mu_{\text{truth}} - \mu_{\text{lie}}}{\mu_{\text{truth}}}.$$

The proof leverages this inequality combined with carefully manipulating the expressions of the utility under both truthful reporting and misreporting.

Theorem 1 demonstrates the robustness of the multiunit auction in the presence of risk aversion under the utility models we consider. Specifically, we show that it is never beneficial for risk-averse buyers to report a value that is different from their true valuation.

4.1. Repeated Auction Process

For internet ads, one would ideally model the auction process in P2 by using a sequence of first- or second-price auctions. However, analyzing the equilibrium in such repeated settings is quite challenging, as discussed in Section 1.3. This is why we consider a reduced form of a single-shot multiunit auction in this paper. This provides a simpler language along with analytical tractability while still allowing us to draw interesting insights. Additionally, we show in Appendix A that if P2 consists of a sequence of first- or second-price auctions under the assumption that buyers can quickly learn the precise value of v_{I+1} when they enter the auction process in P2, then there exists an equilibrium that is the same as when the auction process is a single-shot multiunit auction. This assumption on buyer knowledge refinement over time stems from empirical evidence in major ad exchanges, hence supporting the fact that our reduced form in P2 provides a reasonable simplification of the actual ad market. We further discuss in Appendix F how our results degrade as we relax the assumption in a natural manner.

In the next two sections, we study the joint equilibrium when all the buyers have the same degree of risk aversion (Section 5) and then, in the presence of heterogeneous risk aversion levels (Section 6).

5. Homogeneous Risk Aversion

In this section, we consider the setting where all the buyers exhibit the same degree of risk aversion (i.e., the same value of α or β in the CARA and mean-variance models). Although this may not always be the case, studying this setting already provides useful insights, and many of these carry over for heterogeneous degrees of risk aversion, which we study in Section 6.

Our first result in Theorem 2 establishes an important property in any equilibrium induced by the coexistence of the Market-Maker contract and the auction mechanism. Specifically, we show that if a buyer with valuation v opts for the Market-Maker contract in P1, then all the buyers with valuation above v will also choose the Market-Maker contract in P1. We then show in Corollary 1 that for any price p_M quoted by the Market-Maker, there exists a threshold value $\overline{v}(p_M) > p_M$ such that buyers will select the Market-Maker contract if and only if their valuation exceeds $\overline{v}(p_M)$.

This result may seem counterintuitive at first. Why would only high-value agents—who do not seem to have any allocation uncertainty—select the Market-Maker contract? For example, buyers with valuation between p_M and $\overline{v}(p_M)$ do not opt for the Market-Maker contract, despite having a higher allocation uncertainty than buyers with value above $\overline{v}(p_M)$. To develop an intuition on this finding, we note that low-value agents can better tolerate price uncertainty because when the price is too high, they simply do not get allocated. Consider, for instance, an auction that clears at a price of \$2 with probability 0.5 and at a price of \$0 with probability 0.5. In this case, a (risk-neutral) buyer with a valuation of \$1 prefers participating in the auction because it gives an expected utility of \$0.5 rather than buying a Market-Maker contract in P1 for \$1, which gives an expected utility of \$0. On the other hand, a (risk-neutral) buyer with a valuation of \$3 is indifferent between the Market-Maker contract and the auction. Our results formalize the intuition that buyers with lower valuations are less likely to pay a premium for a Market-Maker contract in the presence of risk aversion. Importantly, this result holds for both the CARA and the mean-variance models, highlighting its robustness.

Theorem 2 (Market-Maker Equilibrium Monotonicity). *Under both the CARA and mean-variance models in any two-period equilibrium induced by a Market-Maker price p_M, if there exists a value $\tilde{v}$ such that buyers with valuation $\tilde{v}$ purchase the Market-Maker contract in P1 (rather than waiting to participate in the auction in P2), all buyers with valuations $v \geq \tilde{v}$ will also purchase the Market-Maker contract in P1.*

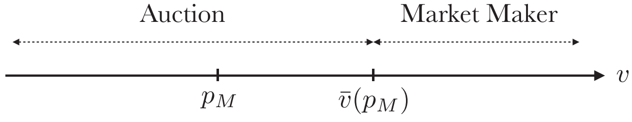
The equilibrium notion in Theorem 2 is the traditional Nash equilibrium. The proof of Theorem 2 can be found in Appendix C. The next corollary immediately follows.

Corollary 1. *Under both the CARA and mean-variance models in any two-period equilibrium induced by a Market-Maker price p_M, there exists a threshold $\overline{v}(p_M) > p_M$ such that all buyers with valuations $v \geq \overline{v}(p_M)$ prefer to select the Market-Maker contract in P1, whereas all other buyers strictly prefer participating in the auction in P2.*

Note that $\overline{v}(p_M) > p_M$ is implied by the fact that a buyer with valuation $v = p_M$ earns zero utility from choosing the Market Maker contract in P1 and would thus prefer to take a chance in the auction in P2. Figure 1 illustrates this equilibrium behavior.

Remark 1. The properties derived in Theorem 2 and Corollary 1 hold only when a two-period equilibrium exists for the Market-Maker price p_M under consideration. Whether an equilibrium exists or not is not addressed here and will be investigated in Proposition 1. In particular, it is possible that for some Market-Maker prices, more than I buyers would select the Market-Maker contract. In this case, the Market-Maker would default, so that such Market-Maker prices will not induce a two-period equilibrium. In Theorem 2 and Corollary 1, our focus was on identifying structural properties of an equilibrium when it exists for a given Market-Maker price p_M. This is why for any Market-Maker price p_M, the utility of selecting the Market-Maker option was assumed to be $(v - p_M)^+$ (given by Equation (2)) without worrying about the feasibility of the Market-Maker truly delivering the goods (i.e., we did not worry about the fact that the Market-Maker would default if more than I buyers choose it; regardless of the number of buyers that opt for the Market-Maker, we assume that it can deliver a utility of $(v - p_M)^+$ in Theorem 2 and Corollary 1).

Figure 1. Equilibrium Behavior When Both the Market-Maker Contract and the Auction Are Offered



Following Remark 1, the natural question is whether a two-period equilibrium exists. We next show that among all possible Market-Maker prices, the ones above a certain minimum price p_M^* (defined formally here) indeed result in equilibria; in particular, they satisfy the constraint that no more than I buyers select the Maker-Maker.

Proposition 1 (Market-Maker Price Characterization). *Let $p_{\max}$ be the highest auction clearing price (in the support of the distribution of clearing prices) in a single-period ($P2$) equilibrium where there is no Market-Maker contract. Let p_M^* be the Market-Maker price such that $\overline{v}(p_M^*) = p_{max}$ (see Corollary 1). Then, we have the following.*

1. *A Market-Maker price will result in an equilibrium where at most, I buyers opt for the Market-Maker contract only if it is at least p_M^*.*
2. *A Market-Maker price will result in at least one buyer opting for the Market-Maker contract only if it is at most p_M^*.*

Moreover, a closed-form expression for p_M^* is given by

$$p_M^* = \begin{cases} \mu_A + \beta \cdot \sigma_A^2 & (\textit{Mean-variance model}), \\ \dfrac{1}{\alpha} \cdot \log\left(\mathbf{E}_p[e^{\alpha \cdot p}]\right) & (\textit{CARA model}). \end{cases} \tag{3}$$

Here, μ_A and σ_A are the mean and standard deviation of the auction clearing price, respectively, in the single-period ($P2$) equilibrium, where the Market-Maker contract is not offered. Likewise, the expectation $\mathbf{E}_p$ in the CARA model is over the distribution of auction clearing prices in the single-period equilibrium, where the Market-Maker contract is not offered.

The proof of Proposition 1 can be found in Appendix D. The first part of the proposition filters out, among the possible Market-Maker prices, the ones that violate the constraint ensuring that no more than I buyers opt for the Market-Maker. This is done by imposing a lower bound p_M^* on the Market-Maker price. The second part of the proposition shows that, surprisingly, any Market-Market price strictly higher than p_M^* will result in an equilibrium where no buyer selects the Market-Maker contract. This, in turn, implies that there is a unique equilibrium where at least one buyer selects for the Market-Maker contract.

Furthermore, we derive a closed-form expression for the unique Market-Maker price. This expression sheds light on the markup over the expected auction clearing price μ_A that buyers need to pay to eliminate their allocation and price uncertainties. In the mean-variance model, this markup has a crisp relationship with the inherent auction uncertainty by being proportional to the variance of the auction clearing price (it is exactly β times the variance σ_A^2). In the CARA model, although the relationship to μ_A is not easy to carve out, one can see that the price markup is nonnegative (i.e., $p_M^* \geq \mu_A$, where $\mu_A = \mathbf{E}_p[p]$) because of the concavity of the $\log(\cdot)$ function. In addition, for both models, the value of p_M^* is naturally capped at p_{max} as $\overline{v}(p_M^*) = p_{\max}$. A direct consequence of the proof of Proposition 1 is the following characterization of buyers' equilibrium behavior.

Corollary 2. *When offering a Market-Maker contract at price p_M^*, buyers with valuations $v < p_{\max}$ strictly prefer to participate in the auction in $P2$, whereas buyers with valuations $v \geq p_{\max}$ weakly prefer to choose the Market-Maker contract in $P1$.*

It follows from the proof of Proposition 1 that all the buyers with valuations $v \geq p_{max}$ are indifferent between the Market-Maker contract and the auction. There are two options here. One is that we assume that such indifferent buyers opt for the Market-Maker contract, given its advantages of having a fixed price as opposed to a variable auction clearing price that could be as high as $p_{\max}$. Two is that we assume that buyers are breaking ties randomly. Our insights still continue to hold. Specifically, one can show that the allocation remains the same and that the revenue strictly increases irrespective of the tiebreaking rule. The only difference would be a reduction in the revenue increase for the seller in Theorem 3. We are now ready to state the main result of this section, which shows that adding the Market-Maker contract is beneficial.

Theorem 3 (Market-Maker Pareto Improvement). *When offering the Market-Maker contract at price p_M^*, both the seller's revenue and the welfare strictly increase. Moreover, the efficiency remains optimal, and each buyer's utility remains the same.*

We note that under homogeneous risk aversion, the Market-Maker contract does not change the allocation but just eliminates price uncertainty. One may wonder if eliminating price uncertainty alone is a good reason for buyers to pay a premium for Market-Maker. This is resolved by realizing that most buyers operate under a fixed budget for their advertising campaign. Under a fixed budget, any price uncertainty translates to allocation uncertainty. Indeed, market fluctuations could raise the price for the desired volume of impressions beyond the budget, thereby leading to an allocation shortfall. Such buyers are thus often willing to afford the Market-Maker premium.

Proof of Theorem 3. We first present the proof for the revenue. We will assume in the proof that indifferent buyers opt for the Market-Maker contract, but as discussed earlier, even with random tiebreaking the insights continue to hold. Because only the buyers with valuations $v \geq p_{\max}$ opt for the Market-Maker contract in P1, the auction clearing price (in P2) remains the same as in the situation where no Market-Maker contract was offered (i.e., this is because the highest-valued $\ell \leq I$ buyers select the Market-Maker contract in P1 and $I - \ell$ buyers participate in the auction in P2; the auction clearing price is still the $(I+1)$ th highest value for all $\ell \leq I$), and thus, the set of allocated buyers in the presence of both purchase options remains exactly the same as when just the auction purchase option was presented. As a result, the revenue generated by the buyers with valuations $v < p_{max}$ (who buy via auction) also remains the same. However, the expected revenue generated by the buyers with valuations higher than $p_{\max}$ increases from μ_A to p^*_M. Because for both utility models, $p^*_M > \mu_A$, the seller's expected revenue strictly increases.

We now show the claim for the welfare, namely the sum of buyers' and seller's utilities. Because the auction clearing price is unaffected, the utility of the auction buyers remains the same. In addition, the buyers of the Market-Maker contract are also unaffected as they are indifferent between the Market-Maker and auction purchase options. Because the revenue (which is seller's utility) strictly increases, so does the welfare.

Finally, the outcome is efficient: that is, the highest-valued I bidders are allocated. Because the items are still allocated to the same set of buyers after introducing the Market-Maker contract and the auction was efficient to begin with, the optimal efficiency is maintained. □

We end this section by showing that the optimal Market-Maker price p^*_M increases with the risk aversion parameter (α or β) for both utility models and with the variance of the auction clearing price σ^2_A for the mean-variance model.

Proposition 2 (Market-Maker Price as a Function of Risk). *The revenue-optimal Market-Maker price p^*_M is an increasing function of the risk aversion parameter (α in the CARA model and β in the mean-variance model). Further, in the mean-variance model, p^*_M is an increasing function of the variance of the auction clearing price (σ^2_A).*

The proof of Proposition 2 can be found in Appendix E. This result shows that when buyers are more risk averse or when the variability in the auction clearing price is higher, a Market-Maker contract can take advantage of the situation by increasing its premium.

In conclusion, we have shown that adding a Market-Maker contract eliminates price uncertainty for high-valuation buyers ($v \geq p_{max}$) while simultaneously increasing revenue and welfare and maintaining efficiency. In Section 7, we show computationally that for several popular distributions and reasonable risk parameters, this revenue lift is often significant.

6. Heterogeneous Risk Aversion

So far, we assumed that all buyers have the same risk aversion parameter (α or β). In practice, different buyers may behave differently with regard to the way they perceive risk. In this section, we study the setting where buyers have a different risk aversion level and examine the impact of adding a Market-Maker contract. We assume that there are k populations of buyers with $\alpha_1 \leq \alpha_2 \leq \ldots \leq \alpha_k$ ($\beta_1 \leq \beta_2 \leq \ldots \leq \beta_k$) for CARA (mean-variance), where each population is present with proportion $0 \leq \rho_i \leq 1, i = 1, \ldots, k$ ($\sum_{i=1}^k \rho_i = 1$) among the N buyers. Note that for the mean-variance model, a higher value of β implies a higher risk aversion, and likewise, for the CARA model, a higher value of α implies a higher risk aversion.

Theorem 4 (Heterogeneous Risk Aversion). *Under heterogeneous risk aversion, the equilibrium Market-Maker price is not necessarily unique. Instead, there exists a range of equilibrium prices that depends on the range of risk parameters. There always exists at least one price, $p^*_{M,k}$, that strictly increases* (i) *the seller's revenue and* (ii) *the welfare. In addition, each buyer's utility stays the same, and the efficiency remains optimal.*

Remark 2. Recall that under risk neutrality, efficiency and welfare coincide. In the presence of risk aversion, efficiency is not a measure of interest for the seller/buyer/central planner. Instead, the revenue, buyers' utilities, and welfare become the most relevant metrics. Although Theorem A.1 proves that each buyer's utility stays the same under the price $p^*_{M,k}$, an illustrative example (see Section 7) shows that one can slightly decrease the price and strictly improve the sum of buyers' utilities.

The proof of Theorem 4 can be found in Appendix G. Theorem 4 shows the existence of at least one Market-Maker price, denoted by $p^*_{M,k}$, that yields a Pareto improvement. This price corresponds to the unique Market-Maker price that we would get from Proposition 1 if we assume that all buyers originate from population k (i.e., with the highest risk aversion parameter) and can be calculated using Equation (3) with $\alpha = \alpha_k$ (or $\beta = \beta_k$). The range of possible prices is of the form $[q, p^*_{M,k}]$, where q corresponds to the smallest price point where the Market-Maker contract does not default. We next discuss an interesting effect of introducing heterogeneity in risk aversion.

6.1. Homogeneous Vs. Heterogeneous: Change in Allocation

A notable aspect of the setting under heterogeneous risk aversion is that there exist Market-Maker prices for which the allocation may not be efficient. More precisely, a buyer with a lower valuation from a higher risk-averse population may be allocated, whereas a buyer with a higher valuation from a lower risk-averse population is not. Namely, the I items are not necessarily allocated to the I buyers with the highest valuations. To see this, consider the realization that the auction clearing price is $p_{\max}$ and the Market-Maker price $p^*_{M,k}$: that is, the optimal price using Equation (3) assuming all buyers are from population k.

Recall that there are exactly I buyers with valuation above $p_{\max}$ when $p_{\max}$ is the clearing price. However, only $I \cdot \rho_k$ of them are from population k and will opt for the Market-Maker contract (using Corollary 1), whereas the remaining $I \cdot (1 - \rho_k)$ buyers who are from population $\ell < k$ will participate and win in the auction. Indeed, because buyers from population k are indifferent between the auction and the Market-Maker with price $p^*_{M,k}$, by using the result of Proposition 2, buyers from population $\ell < k$ (who are less risk averse) will only accept a smaller price and hence, strictly prefer the auction. Consequently, if we decrease the Market-Maker price by a small amount, to $p^*_{M,k} - \epsilon$, those buyers from population $\ell < k$ will still strictly prefer the auction, whereas the ones from population k with valuation just below p_{max}—who were not allocated earlier when $p_{\max}$ was the clearing price—will now switch to the Market-Maker contract and get allocated. For each buyer from population k with valuation below p_{max} who switches to the Market-Maker contract, some buyer from population ℓ (with $\ell < k$) with valuation above p_{max} will now be unallocated. This follows from the fact that there are only I units available.

This discussion conveys that Market-Maker prices that fall strictly below $p^*_{M,k}$ will result in a loss of efficiency (under heterogeneous risk aversion). The next question relates to the impact of the Market-Maker contract on the sum of buyers' utilities. We present an answer to this question under some additional assumptions. We will further investigate this question computationally in Section 7.

Theorem 5 (Impact on Sum of Buyers' Utilities). *Consider a heterogeneous population of buyers. If the valuation function has an increasing failure rate (i.e., $f(x)/[1 - F(x)]$ is increasing)*[3] *and the support of the clearing price distribution consists of two points, then under the mean-variance model, the sum of buyers' utilities increases as the Market-Maker price decreases (as long as the Market-Maker does not default).*

The proof of Theorem 5 can be found in Appendix H. Note that we formally prove the result of Theorem 5 only for the mean-variance model. Indeed, under the CARA model, comparing utilities of buyers with different risk aversion parameters does not provide any insight because the different utilities are not comparable quantities. More precisely, the mean-variance model preserves the same unit as expected utility, and hence, we can compare buyers' utilities. On the other hand, the CARA model transforms all utilities into the $[0, 1]$ space, making comparisons meaningless.

7. Computational Experiments

In this section, we consider the setting with a standard multiunit auction mechanism. Our goal is to illustrate and quantify the results presented in Sections 5 and 6.

7.1. Homogeneous Risk Aversion

In Section 5, we have shown that there exists a unique Market-Maker price p^*_M. In addition, we characterized this optimal price in closed form for the CARA and mean-variance utility models. We also demonstrated that adding a Market-Maker contract increases the seller's revenue without affecting buyers' utilities (as it does not modify the allocation). More precisely, the expected revenue increase amounts to $\overline{F}(p_{max}) \cdot (p_M - \mu_A)$. Our next goal is to show that the revenue improvement generated by adding a Market-Maker contract can often be significant.

We use several distributions for the buyers' valuations and the number of buyers N. Specifically, we consider that F is uniform in $[0, 1]$, exponential with mean 0.5, and follows a two-point distribution (equal to 0.2 with probability 5/8 and to 1 with probability 3/8). For N, we consider a discrete two-point distribution (with equal probabilities) and a normal distribution.

Several previous studies have considered the task of estimating risk aversion. Ang (2014) claims that most individuals have risk aversions between 1 and 10. Typically, risk aversion estimates are obtained from experimental and survey evidence. By considering real financial choices, Paravisini et al. (2016) examined actual financial decisions made by investors in an online peer-to-peer lending platform. The authors estimated that investors have risk aversions around three. Consequently, we consider an instances of the CARA model with $\alpha = 1$ and $\alpha = 3$ (using $I = 1{,}000$). Note that in the homogeneous setting, there exists a one-to-one correspondence between the mean-variance model and the CARA model. For a given α, one can find a corresponding value of β that yields the same optimal Market-Maker price p^*_M and hence, the same market equilibrium outcome. Consequently, the results in Table 1 can correspond to either utility model.

In Table 1, we generate 500 independent instances and compute the expected seller's revenue improvement obtained by adding a Market-Maker contract $\left(\text{i.e., } \frac{R(p^*_P, p^*_M)}{R(p^*_P, \infty)}\right)$ for settings with an average number of buyers equal to $1.2I$. Note that in all the cases we considered, the Market-Maker contract allows us to

Table 1. Expected Revenue Improvement for the Setting with Homogeneous Risk Aversion Under Different Distributions (Parameters: $I = 1{,}000$, $\mathbb{E}[N] = 1.2I$, $\mathbb{E}[v] = 0.5$, and $\alpha = 1$ or $\alpha = 3$)

Distributional assumptions		Revenue improvement	
F	N	$\alpha = 1$	$\alpha = 3$
Uniform	Discrete two point in $\{I, 1.4I\}$	1.068	1.201
Exponential	Discrete two point in $\{I, 1.4I\}$	1.160	1.390
Two point	Discrete two point in $\{I, 1.4I\}$	1.133	1.340
Exponential	Normal with $(1.2I, 0.1I)$	1.044	1.116

generate a significant revenue increase. More precisely, when $\alpha = 1$, the revenue improvement ranges between 4.4% and 16%, and when $\alpha = 3$, the revenue improvement is between 11.6% and 39%. In Figure A.2 (see Appendix I), we vary the risk aversion parameter α for the CARA model and compute the revenue improvement generated by adding a Market-Maker contract. For the specific instance considered in Figure A.2, when α varies between one and five, the revenue improvement ranges between 8% and 33% (2% and 7.5%) when N follows a two-point (normal) distribution. As discussed before, for the mean-variance model, the revenue improvement is linear with respect to the risk aversion parameter β.

7.2. Heterogeneous Risk Aversion

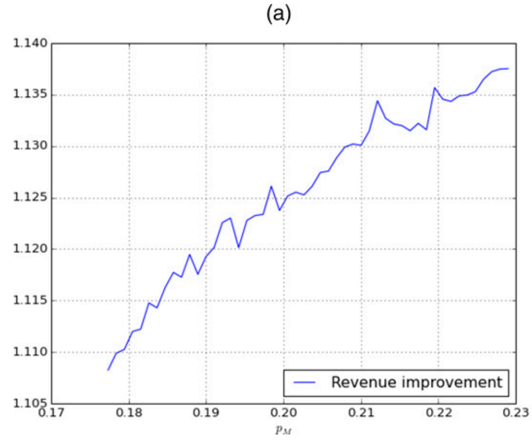
We next quantify the impact of adding a Market-Maker contract under the mean-variance utility model when buyers have heterogeneous risk aversion. We note that for the CARA model, comparing utilities of buyers with different risk aversion parameters does not provide any insight, as the different utilities are not comparable quantities. More precisely, the CARA model transforms all utilities into the $[0, 1]$ space, making comparisons meaningless. On the other hand, the mean-variance model preserves the same unit as expected utility, so that one can compare buyers' utilities. Consequently, this section focuses on the mean-variance model.

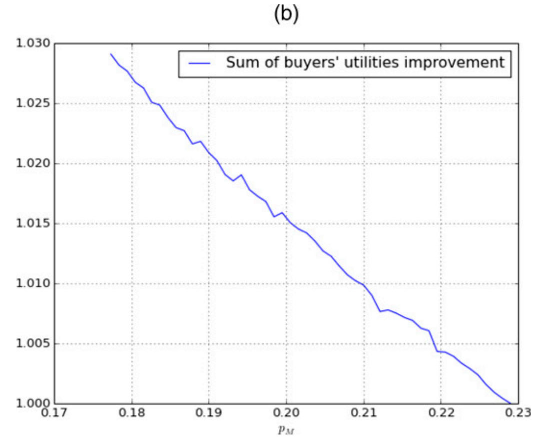
We consider a setting with two populations of buyers $\beta_1 = 0.5$ and $\beta_2 = 4$: that is, type 2 buyers are more risk averse. We assume that the proportions are $\rho_1 = \rho_2 = 0.5$ and consider the distribution of valuations F to be uniform $[0, 1]$. Additionally, we assume that N can take the two values $\{I, 1.4I\}$ with equal probabilities (we observed similar qualitative results when N follows a normal distribution).

In Figure 2, we generate 1,000 independent samples and plot the expected revenue improvement obtained by adding a Market-Maker contract $\left(\text{i.e., } \frac{R(p_P^*, p_M^*)}{R(p_P^*, \infty)}\right)$ as well as the expected improvement in the sum of buyers' utilities. We observe that decreasing the Market-Maker price below $p_{M,2}^*$ increases the sum of buyers' utilities. This implies that there exists a range of prices for which we obtain a strict Pareto improvement for both the seller and the buyers. As discussed before, the maximal meaningful value of p_M is such that type 2 buyers (i.e., the more risk-averse buyers) are indifferent between the Market-Maker contract and the auction. This maximum price is denoted by $p_{M,2}^*$ and corresponds to the rightmost point on the x axis in Figure 2. We next decrease the Market-Maker price p_M below $p_{M,2}^*$. Consequently, some type 2 buyers will now strictly prefer the Market-Maker contract, whereas some type 1 buyers will lose their allocation when $N = 1.4I$ realizes (because the auction clearing price increases). In other words, risk-averse buyers secure allocations via the Market-Maker contract at the expense of some type 1 buyers who now lose the auction when $N = 1.4I$ realizes. Therefore, as p_M decreases, type 2 buyers are better off, whereas type 1 buyers are worse off. Nevertheless, the overall impact of adding the Market-Maker contract is positive: that is, the sum of buyers' utilities (across both populations) increases. Note that in Figure 2, we decrease p_M until the point at which the Market-Maker defaults.

One can see that we obtain a Pareto improvement in the seller's revenue and in the sum of buyers' utilities. In the instances we considered, the relative revenue improvement ranges between 10.5% and 14.0%, and the relative sum of buyers' utilities improvement reaches 3%.

Figure 2. (Color online) Improvements Obtained by Adding the Market-Maker Contract for the Setting with Heterogeneous Risk Aversion Under the Mean-Variance Model (Parameters: F Is Uniform $[0, 1]$, N Is Discrete in $\{I, 1.4I\}$, $\rho_1 = \rho_2 = 0.5$, $\beta_1 = 0.5$, and $\beta_2 = 4$)





Notes. (a) Seller's revenue. (b) Sum of buyers' utilities.

In Figure A.3 (see Appendix I), we plot the buyers' utilities for each population separately as a function of p_M. As explained before, when p_M decreases, the sum of type 2 buyers' utilities (more risk averse) increases, whereas the sum of type 1 buyers' utilities decreases. When $p_M = p^*_{M,2}$, because the allocation does not change, adding the Market-Maker contract does not affect the sum of buyers' utilities. One can see that, as expected, the gain in utility earned by type 2 buyers dominates the loss suffered by type 1 buyers. Thus, adding a Market-Maker contract has an overall positive effect on the population of buyers.

8. Conclusion

In this paper, we introduce and study a framework to mathematically analyze the benefits of allocation and price guarantees for risk-averse internet advertisers. In particular, we consider introducing a programmatic purchase option referred to as a *Market-Maker contract* to the real-time auction ad-selling mechanism. A Market-Maker contract guarantees allocation and fixed price to its buyers. We show that adding a Market-Maker contract to a single-shot multiunit auction, which can be seen as a reduced-form market, can benefit both the seller and the buyers. It eliminates allocation and price uncertainties for risk-averse buyers who are willing to pay a premium over the expected auction clearing price to hedge against uncertainty. On the methodological side, our equilibrium analysis characterizes buyers' equilibrium behavior when facing two alternatives and uncovers the truthfulness of the multiunit auction under the mean-variance utility model.

We assumed a fixed inventory (number of impressions) of I units. Although this assumption often holds because of reasonably sharp concentration bounds, one can wonder what happens when there is inventory uncertainty. In this case, one can implement the Market-Maker contract using the same principle of setting its price according to the worst-case scenario. When I is deterministic, our results suggest to price the Market-Maker contract based on the highest realized number of buyers (and we still prove its profitability). We can now modify this price to handle the combination of the highest number of buyers with the smallest number of impressions. Alternatively, we can price according to a value I^*, which is slightly above the smallest possible value, so that the probability that the realized inventory exceeds I^* is high enough. To accommodate the rare events in which the Market-Maker defaults, the contract can be amended with an underdelivery penalty.

Acknowledgments

The authors thank the department editor (Gabriel Weintraub), the associate editor, and the three anonymous referees for their insightful comments, which have helped to improve this paper. Part of this work was completed while the first two authors were hosted by Google Research in New York City.

Appendix A. Repeated Auction Process

In the paper, we assume that the auction process in P2 is a single-shot multiunit auction. In the online ads markets, the auction often consists of a sequence of first- or second-price auctions. In this section, we show that this reduced form is a reasonable simplification in certain cases. Indeed, we show that when buyers can quickly discover the clearing price, then the equilibrium outcome is the same as when P2 consists of a single-shot multiunit auction.

Assumption A.1 (Accurate Predictability for Near Future). *The auction process is a repeated first-price or repeated second-price auction. Additionally, although the buyers only know the distribution of v_{I+1} when they make the decision in period P1, they know the precise value of v_{I+1} when they enter the auction process in period P2.*

This assumption on buyer knowledge refinement over time (as time passes from P1 to P2) stems from empirical evidence in major ad exchanges. More specifically, the prediction accuracy when making a decision in P1 is significantly lower relative to the accuracy of predicting for the next day once the auction process starts in P2. In fact, predictions for several relevant metrics for a period far in the future are usually limited to only knowing the distribution. However, predictions for a period close in time become quite accurate. Although Assumption A.1 states that soon after P2 begins, all the information that can be learned is instantly learned, we further discuss how our results degrade as we relax this assumption in a natural manner (see Remark A.1).

We now show that in the single-period P2 equilibrium, the distribution of auction clearing prices is exactly the distribution of v_{I+1}. In particular, we identify a bidding equilibrium whose outcome is such that the I-highest valued bidders (including nonunit demand multiplicities) receive the item and each buyer pays v_{I+1} per unit. To prove this result, we rely on Assumption A.1 stating that the buyers know the precise value of v_{I+1} when they enter the auction process in period P2. Note also that in our model in both the repeated auction setting and the single-shot auction setting, the buyers know their values in advance (i.e., in P1), and further, in the repeated auction setting, the value remains constant over time. We acknowledge that this is an assumption we are making for analytical tractability and is a simplification of real RTB (Real Time Bidding) markets.

Theorem A.1. *In a repeated first-price auction, every bidding equilibrium will result in the I-highest valued buyers (including multiplicities in demand) being allocated and paying v_{I+1} per unit of good (plus a vanishingly small $\epsilon > 0$). In a repeated second-price auction, every bidding equilibrium where no bidder overbids will result in the I-highest valued buyers (including multiplicities in demand) being allocated; further, there exists an equilibrium where the allocated buyers pay v_{I+1} per unit of good.*

Proof of Theorem A.1. Recall that by Assumption A.1, buyers know the precise value of v_{I+1} in period P2 before entering the auction process.

First, we show the existence of an equilibrium for both the repeated first-price and the repeated second-price auctions.

Consider the following bidding strategy for buyers. Each buyer j with a value strictly higher than v_{I+1} bids $v_{I+1}+\epsilon$ (for a small $\epsilon>0$) in every repeated auction until they get allocated k_j goods in total, and bid zero afterward. Buyers with value at most v_{I+1} bid their true value. In this bidding strategy profile, no buyer can benefit from a unilateral deviation. Indeed, buyers with value at most v_{I+1} cannot afford to pay more than v_{I+1} per item and hence, cannot earn a higher utility by bidding more than v_{I+1}. They do not earn any higher utility by bidding lower than v_{I+1} either, as they already do not get allocated when bidding v_{I+1}. The remaining buyers, who bid $v_{I+1}+\epsilon$, get allocated and pay at most $v_{I+1}+\epsilon$ in either a first- or second-price auction. These buyers do not benefit from raising their bids. They cannot reduce their bids below v_{I+1} either and still get allocated. This immediately follows for the case of $k_j=1$. For larger k_j, one might wonder whether the buyer could reduce their bid and get *reduced number of units* (i.e., lesser than k_j). However, this is not possible because of Assumption 2; the market is large, and hence, no single agent's demand can affect the price in a meaningful way (i.e., the $I+1$th unit's price is quite close to $I+k_j$ th unit's price). So, reducing their bid will make them not get any allocation.

Second, we show that in both repeated first-price and repeated second-price auctions, any equilibrium allocation outcome will always have the I-highest valued bidders being allocated. Suppose on the contrary that there is an equilibrium where a bidder who is not one of the I-highest valued buyers gets allocated. Given that such a bidder's value is at most v_{I+1}, she or he could not have paid more than v_{I+1} in equilibrium. This means that one of the I-highest valued bidders with value $v>v_{I+1}$ who did not get allocated (and hence, earns zero utility in the contrary equilibrium being considered) could have raised his or her bid to $v_{I+1}+\epsilon$ (where $\epsilon>0$ is small so that $v_{I+1}+\epsilon<v$). It would thus allow this bidder to be allocated and to earn a positive utility. This contradicts the fact that every bidder was best responding in equilibrium and hence, concludes the argument.

Third, we show that in any equilibrium of the repeated first-price auction, all winners pay exactly v_{I+1} (plus a small ϵ). To see this, note that no winning bidder will pay more than the $(I+1)$ th highest bid (plus a small ϵ). Indeed, they could have lowered their bid to the $(I+1)$ th highest bid (plus a small ϵ) and still be allocated while earning a higher utility. On the other hand, the bidder with value v_{I+1} will bid exactly v_{I+1} in equilibrium. Indeed, she or he will not bid a value higher than v_{I+1} because it leads to negative utility. Furthermore, if she or he bids any lower than v_{I+1} in equilibrium, say $v_{I+1}-c$ for $c>0$, then by the previous discussion, the I-highest valued bidders will bid $v_{I+1}-c+\epsilon$, which would not be an equilibrium given that the $(I+1)$ th highest-valued bidder could raise his or her bid to $v_{I+1}-c+2\epsilon$, get allocated, and earn positive utility (as $c-2\epsilon>0$). This proves the uniqueness of the payment scheme.

Note that the proof is agnostic to the risk-averse utility model. This is because in period P2, the buyer knows the precise value of v_{I+1} and hence, does not face any uncertainty because of Assumption A.1, making the exact risk-averse utility model irrelevant. All we need is that the expression for risk-averse utility under no uncertainty is an increasing function of the value and a decreasing function of the price. Any reasonable utility model satisfies this property. □

The absence of equilibrium uniqueness for repeated second-price auctions is not surprising. Even in a (complete information) single-shot second-price auction, the equilibrium is not unique. For example, there exist some implausible equilibria where the highest-valued bidder bids their true value and every other bidder bids zero. Nevertheless, essentially all the literature on second-price auctions works with the canonical equilibrium where bidders bid truthfully, and the highest bidder pays the second-highest value. Similarly in our setting, under the canonical equilibrium for the repeated second-price auction, all the I highest bidders pay v_{I+1} per unit of good.

Remark A.1. Even when buyers do not have sufficient capability (or data) to accurately predict v_{I+1} (i.e., cases when Assumption A.1 does not hold), we show that our results degrade as a function of the amount of uncertainty. In particular, in Appendix F, we relax Assumption A.1 and assume that buyers only know v_{I+1} approximately. We then formally show that the structure of the results presented in Theorem A.1 and Corollary 2 still holds but degrades with the level of uncertainty in knowledge of v_{I+1}.

Appendix B. Truthfulness Under Risk Aversion: Proof of Theorem 1

Theorem 1 (Truthfulness Under Risk Aversion). *Under the CARA and mean-variance utility models, the $(I+1)$th price auction is both universally truthful and truthful in expectation. Let U_{truth} (U_{lie}) be the utility in the auction under truthful reporting (misreporting).*

1. *If $U_{\text{truth}}>0$, then $U_{\text{truth}}\geq U_{\text{lie}}$.*
2. *If $U_{\text{truth}}\leq 0$, then $U_{\text{lie}}\leq 0$.*

We next prove the result for each utility model.

B.1. Proof for the CARA Model

We consider a particular buyer with valuation v. To prove the result, we show that such a buyer will never benefit from misreporting her or his true valuation. For each single outcome given by some clearing price p, the utility of a buyer with valuation v is $1-e^{-\alpha\cdot(v-p)^+}$. This expression is an increasing function of the traditional quasilinear utility, and thus, bidding truthfully is a dominant strategy. This shows that under the CARA model, the $(I+1)$ th price auction is universally truthful. Because the utility for a distribution over outcomes is equal to the expected utility over individual outcomes (for the CARA model), this also implies truthfulness in expectation (see also Nautz and Wolfstetter 1997).

B.2. Proof for the Mean-Variance Model

As mentioned, universal truthfulness is not meaningful for the mean-variance model; it is straightforward because the utility for an individual outcome is the standard quasilinear utility (the variance term vanishes to zero), and the truthfulness of the $(I+1)$ th price auction under standard quasilinear utility is immediate. For truthfulness in expectation, we first write the utility earned by a buyer with valuation v when reporting her or his true valuation as $U_{\text{truth}}=\mu_{\text{truth}}-\beta\cdot\text{Var}_{\text{truth}}$, where $\mu_{\text{truth}}=\mathbf{E}_p[(v-p)^+]$ and $\text{Var}_{\text{truth}}=\text{Var}_p\ [(v-p)^+]$ are given by

$$\mu_{\text{truth}} = \int_0^v (v-p)\cdot h(p)\cdot dp,$$
$$\mathsf{Var}_{\text{truth}} = \int_0^\infty [(v-p)^+ - \mu_{\text{truth}}]^2 \cdot h(p)\cdot dp.$$

Here, we note that we do not assume that the bidders are in an equilibrium nor that the distribution over clearing prices is $g(\cdot)$. In particular, we do not impose any assumption on the way that the other buyers bid; regardless of their bidding behavior, we let $h(\cdot)$ be the distribution over clearing prices induced by this bidding behavior. In more detail, the bidding strategy profile of the remaining bidders, the randomness in their valuations, and the randomness in the number of buyers together determine the distribution of the auction clearing price $h(\cdot)$ as seen by the focal bidder. We highlight that both μ_{truth} and $\mathsf{Var}_{\text{truth}}$ are computed over all cases, even when the buyer is not allocated: that is, even when $v < p$ (here, p denotes the clearing price). This is why we have $(v-p)^+ = \max\{v-p, 0\}$ inside the integral defining $\mathsf{Var}_{\text{truth}}$. We next consider two cases depending on whether the buyer underbids or overbids.

B.2.1. Buyer Underbids. Suppose that the buyer decides to report some value $v' < v$. We prove the result in three steps. First, we derive closed-form expressions for the difference in the mean and variance incurred by misreporting. Second, we show an inequality relating these differences. Third, we use this inequality to complete the proof.

Step 1 (Rewriting the utilities when misreporting). The buyer's utility when reporting v' can be written as $U_{\text{lie}} = \mu_{\text{lie}} - \beta\cdot\mathsf{Var}_{\text{lie}}$, where μ_{lie} and $\mathsf{Var}_{\text{lie}}$ are given by

$$\mu_{\text{lie}} = \mu_{\text{truth}} - \int_{v'}^v (v-p)\cdot h(p)\cdot dp,$$
$$\begin{aligned}\mathsf{Var}_{\text{lie}} &= \int_0^{v'} [(v-p)-\mu_{\text{lie}}]^2\cdot h(p)\cdot dp + \int_{v'}^\infty (0-\mu_{\text{lie}})^2\cdot h(p)\cdot dp\\ &= \int_0^\infty [(v-p)^+ - \mu_{\text{lie}}]^2 \cdot h(p)\cdot dp\\ &\quad + \int_{v'}^v [(0-\mu_{\text{lie}})^2 - ((v-p)-\mu_{\text{lie}})^2]\cdot h(p)\cdot dp. \end{aligned} \tag{B.1}$$

Let $\delta = \int_{v'}^v (v-p)\cdot h(p)\cdot dp$ (i.e., $\mu_{\text{lie}} = \mu_{\text{truth}} - \delta$). We next rewrite the expression of $\mathsf{Var}_{\text{lie}}$. We have

$$\begin{aligned}\int_0^\infty [(v-p)^+ - \mu_{\text{lie}}]^2\cdot h(p)\cdot dp &= \int_0^\infty [(v-p)^+ - \mu_{\text{truth}} + \delta]^2\cdot h(p)\cdot dp\\ &= \int_0^\infty \{[(v-p)^+ - \mu_{\text{truth}}]^2\\ &\quad + 2\cdot\delta\cdot[(v-p)^+ - \mu_{\text{truth}}] + \delta^2\}\cdot h(p)\cdot dp\\ &= \mathsf{Var}_{\text{truth}} + 2\cdot\delta\cdot\left(\int_0^\infty (v-p)^+\cdot h(p)\cdot dp\right)\\ &\quad - 2\cdot\delta\cdot\mu_{\text{truth}} + \delta^2\\ &= \mathsf{Var}_{\text{truth}} + 2\cdot\delta\cdot\mu_{\text{truth}} - 2\cdot\delta\cdot\mu_{\text{truth}} + \delta^2\\ &= \mathsf{Var}_{\text{truth}} + \delta^2, \end{aligned} \tag{B.2}$$

where the third equality follows from the fact that $\int_0^\infty h(p)\cdot dp = 1$. Thus, by substituting (B.2) into (B.1) and using $\mu_{\text{lie}} = \mu_{\text{truth}} - \delta$, we obtain

$$\begin{aligned}\mathsf{Var}_{\text{lie}} &= \mathsf{Var}_{\text{truth}} + \delta^2 + \int_{v'}^v \{(\mu_{\text{truth}}-\delta)^2 - [(v-p)-(\mu_{\text{truth}}-\delta)]^2\}\\ &\quad \cdot h(p)\cdot dp\\ &= \mathsf{Var}_{\text{truth}} + \delta^2 - \int_{v'}^v (v-p)\cdot[v-p-2\cdot(\mu_{\text{truth}}-\delta)]\cdot h(p)\cdot dp\\ &= \mathsf{Var}_{\text{truth}} - \left(\int_{v'}^v h(p)\cdot(v-p)^2\cdot h(p)\cdot dp\right) + 2\cdot\delta\cdot\mu_{\text{truth}} - \delta^2\\ &\qquad \left(\text{because } \delta = \int_{v'}^v (v-p)\cdot h(p)\cdot \mathrm{dp}.\right)\end{aligned}$$

So far, we manipulated the expressions of μ_{lie} and $\mathsf{Var}_{\text{lie}}$ to relate them to μ_{truth} and $\mathsf{Var}_{\text{truth}}$. In particular, we have shown that

$$\mu_{\text{lie}} = \mu_{\text{truth}} - \delta \text{ and } \mathsf{Var}_{\text{lie}} = \mathsf{Var}_{\text{truth}} - \Delta,$$

where

$$\delta = \int_{v'}^v (v-p)\cdot h(p)\cdot dp \text{ and}$$
$$\Delta = \int_{v'}^v (v-p)^2\cdot h(p)\cdot dp - 2\cdot\delta\cdot\mu_{\text{truth}} + \delta^2.$$

Note that $\delta \geq 0$, but Δ can be of any sign.

Step 2 (Showing a useful inequality). We next show the following inequality that relates δ and Δ:

$$\frac{\Delta}{\mathsf{Var}_{\text{truth}}} \leq \frac{\delta}{\mu_{\text{truth}}}. \tag{B.3}$$

To do so, we compute the following:

$$\begin{aligned}\Delta\cdot\frac{\mu_{\text{truth}}}{\delta} &= \frac{\mu_{\text{truth}}}{\delta}\cdot\left[\int_{v'}^v (v-p)^2\cdot h(p)\cdot dp - 2\cdot\delta\cdot\mu_{\text{truth}} + \delta^2\right]\\ &= \underbrace{\frac{\mu_{\text{truth}}}{\delta}\cdot\left(\int_{v'}^v (v-p)^2\cdot h(p)\cdot dp\right)}_{=\Gamma} - 2\cdot\mu_{\text{truth}}^2 + \mu_{\text{truth}}\cdot\delta.\end{aligned} \tag{B.4}$$

Note that we assumed that $\delta > 0$. However, this is without loss of generality because the inequality trivially holds for $\delta = 0$. Using the following decomposition of μ_{truth},

$$\mu_{\text{truth}} = \int_0^{v'} (v-q)\cdot h(q)\cdot dq + \delta,$$

we obtain

$$\begin{aligned}\Gamma &= \frac{1}{\delta}\cdot\left(\int_0^{v'} (v-q)\cdot h(q)\cdot dq\right)\cdot\left(\int_{v'}^v (v-p)^2\cdot h(p)\cdot dp\right)\\ &\quad + \int_{v'}^v (v-p)^2\cdot h(p)\cdot dp\\ &= \frac{1}{\delta}\cdot\int_0^{v'} (v-q)\cdot\left(\int_{v'}^v (v-p)^2\cdot h(p)\cdot dp\right)\cdot h(q)\cdot dq\\ &\quad + \int_{v'}^v (v-p)^2\cdot h(p)\cdot dp\\ &\leq \frac{1}{\delta}\cdot\int_0^{v'} (v-q)\cdot\left(\int_{v'}^v (v-p)\cdot(v-q)\cdot h(p)\cdot dp\right)\cdot dq\\ &\quad + \int_{v'}^v (v-p)^2\cdot h(p)\cdot dp\\ &= \int_0^{v'} (v-q)^2\cdot h(q)\cdot dq + \int_{v'}^v (v-p)^2\cdot h(p)\cdot dp\\ &= \int_0^\infty [(v-p)^+]^2\cdot h(p)\cdot dp.\end{aligned} \tag{B.5}$$

The inequality follows from the fact that $(v-p)^2 \le (v-p)\cdot(v-q)$ for $0 \le q \le v' \le p \le v$. Replacing (B.5) in (B.4), we obtain

$$\begin{aligned}\Delta\cdot\frac{\mu_{\text{truth}}}{\delta} &\le \int_0^\infty ((v-p)^+)^2\cdot h(p)\cdot dp - 2\cdot\mu_{\text{truth}}^2 + \mu_{\text{truth}}\cdot\delta\\ &\le \int_0^\infty ((v-p)^+)^2\cdot h(p)\cdot dp - 2\cdot\mu_{\text{truth}}^2 + \mu_{\text{truth}}^2 \quad (\text{because } \delta\le\mu_{\text{truth}})\\ &= \mathsf{Var}_{\text{truth}},\end{aligned}$$

where the last equality follows from the variance definition $\mathsf{Var}(X) = E[X^2] - E[X]^2$. Note that the inequality we proved implies that when $\Delta \ge 0$, the relative variance reduction induced by misreporting is always dominated by the relative mean reduction.

Step 3 (Completing the proof). We first complete the proof in the first case (that is, when $U_{\text{truth}} > 0$) and show that $U_{\text{lie}} \le U_{\text{truth}}$. Using the previous steps, we can write

$$U_{\text{truth}} - U_{\text{lie}} = \delta - \beta\cdot\Delta.$$

Note that if $\Delta \le 0$, the result immediately follows. We thus assume that $\Delta > 0$. As a result, we have

$$\begin{aligned}U_{\text{truth}} - U_{\text{lie}} &\ge \delta - \mu_{\text{truth}}\cdot\frac{\Delta}{\mathsf{Var}_{\text{truth}}} \quad \left(\text{because } U_{\text{truth}}\ge 0 \Rightarrow \beta \le \frac{\mu_{\text{truth}}}{\mathsf{Var}_{\text{truth}}}\right)\\ &\ge 0. \quad (\text{by (B.3)})\end{aligned}$$

We next present a similar analysis to show that when $U_{\text{truth}} \le 0$, then $U_{\text{lie}} \le 0$. As before, we assume that $\Delta \ge 0$ as the result immediately follows otherwise. Because $\mathsf{Var}_{\text{truth}} - \Delta = \mathsf{Var}_{\text{lie}} \ge 0$, we have

$$\begin{aligned}U_{\text{lie}} &= \mu_{\text{truth}} - \delta - \beta\cdot(\mathsf{Var}_{\text{truth}} - \Delta)\\ &\le \mu_{\text{truth}} - \delta - \frac{\mu_{\text{truth}}}{\mathsf{Var}_{\text{truth}}}\cdot(\mathsf{Var}_{\text{truth}} - \Delta)\\ &\qquad \left(\text{because } U_{\text{truth}} \le 0 \Rightarrow \beta \ge \frac{\mu_{\text{truth}}}{\mathsf{Var}_{\text{truth}}}\right)\\ &= \mu_{\text{truth}}\cdot\frac{\Delta}{\mathsf{Var}_{\text{truth}}} - \delta\\ &\le 0. \quad (\text{by (B.3)})\end{aligned}$$

This concludes the proof.

B.2.2. Buyer Overbids. We now present the proof for the case of overbidding (i.e., $v' > v$). The steps are similar to the underbidding case. First, we derive closed-form expressions for the difference in the mean and variance incurred by misreporting. Second, we show an inequality relating these variations. Third, we use this inequality to complete the proof.

Step 1 (Rewriting the utilities when misreporting). The utility of the buyer when reporting v' can be written as $U_{\text{lie}} = \mu_{\text{lie}} - \beta\cdot\mathsf{Var}'$, where μ_{lie} and $\mathsf{Var}_{\text{lie}}$ are given by

$$\mu_{\text{lie}} = \mu_{\text{truth}} + \int_v^{v'} (v-p)\cdot h(p)\cdot dp = \mu_{\text{truth}} - \int_{v'}^{v} (v-p)\cdot h(p)\cdot dp$$

$$\begin{aligned}\mathsf{Var}_{\text{lie}} &= \int_0^{v'} [(v-p) - \mu_{\text{lie}}^2\cdot h(p)]\cdot dp + \int_{v'}^\infty (0-\mu_{\text{lie}})^2\cdot h(p)\cdot dp\\ &= \int_0^\infty [(v-p)^+ - \mu_{\text{lie}}^2]\cdot h(p)\cdot dp + \int_v^{v'} [((v-p)-\mu_{\text{lie}})^2\\ &\quad -(0-\mu_{\text{lie}})^2]\cdot h(p)\cdot dp\\ &= \int_0^\infty [(v-p)^+ - \mu_{\text{lie}}]^2\cdot h(p)\cdot dp + \int_{v'}^{v} [(0-\mu_{\text{lie}})^2\\ &\quad -((v-p)-\mu_{\text{lie}})^2]\cdot h(p)\cdot dp.\end{aligned}$$

Note that these are the same expressions we found for the underbidding case. Therefore, by replicating the analysis, we obtain $\mu_{\text{lie}} = \mu_{\text{truth}} - \delta$ and $\mathsf{Var}_{\text{lie}} = \mathsf{Var} - \Delta$, where

$$\begin{aligned}\delta &= \int_{v'}^{v} (v-p)\cdot h(p)\cdot dp \text{ and } \Delta\\ &= \int_{v'}^{v} (v-p)^2\cdot h(p)\cdot dp - 2\cdot\delta\cdot\mu_{\text{truth}} + \delta^2.\end{aligned}$$

We again have $\delta \ge 0$, but Δ can be of any sign. Indeed, recall that in this case, $v' > v$, so that

$$\delta = \int_{v'}^{v} (v-p)\cdot h(p)\cdot dp = -\int_{v}^{v'} (v-p)\cdot h(p)\cdot dp \ge 0,$$

because for $p\in[v,v']$, $(v-p)\le 0$.

Step 2 (Showing a useful inequality). We next show the following inequality that relates δ and Δ:

$$\frac{\Delta}{\mathsf{Var}_{\text{truth}}} \le \frac{\delta}{\mu_{\text{truth}}}. \tag{B.6}$$

We again reproduce the steps in a similar fashion. We begin by computing the following:

$$\begin{aligned}\Delta\cdot\frac{\mu_{\text{truth}}}{\delta} &= \frac{\mu_{\text{truth}}}{\delta}\cdot\left[\int_{v'}^{v} (v-p)^2\cdot h(p)\cdot dp - 2\cdot\delta\cdot\mu_{\text{truth}} + \delta^2\right]\\ &= \underbrace{\frac{\mu_{\text{truth}}}{\delta}\cdot\left(\int_{v'}^{v} (v-p)^2\cdot h(p)\cdot dp\right)}_{=\Gamma} - 2\cdot\mu_{\text{truth}}^2 + \mu_{\text{truth}}\cdot\delta.\end{aligned} \tag{B.7}$$

Recall that $\mu_{\text{truth}} = \int_0^v (v-q)\cdot h(q)\cdot dq$, and hence,

$$\begin{aligned}\Gamma &= \frac{1}{\delta}\cdot\left(\int_0^v (v-q)\cdot h(q)\cdot dq\right)\cdot\left(\int_{v'}^{v} (v-p)^2\cdot h(p)\cdot dp\right)\\ &= \frac{1}{\delta}\cdot\int_0^v (v-q)\cdot\left(\int_{v'}^{v} (v-p)^2\cdot h(p)\cdot dp\right)\cdot h(q)\cdot dq\\ &\le \frac{1}{\delta}\cdot\int_0^v (v-q)\cdot\left(\int_{v'}^{v} (v-p)\cdot(v-q)\cdot h(p)\cdot dp\right)\cdot dq\\ &= \int_0^v (v-q)^2\cdot h(q)\cdot dq = \int_0^\infty [(v-p)^+]^2\cdot h(p)\cdot dp.\end{aligned} \tag{B.8}$$

The inequality follows from the fact that $0 \le q \le v \le p \le v'$ and therefore, $(v-p) \le 0 \le (v-q)$. Using $v \le v'$ and by swapping the order in the integral, we obtain

$$\int_{v'}^{v} (v-p)^2\cdot h(p)\cdot dp \le 0 \le \int_{v'}^{v} (v-p)\cdot(v-q)\cdot h(p)\cdot dp.$$

Finally, the last steps are the same as in the underbidding case.

Step 3 (Completing the proof). In this last step, the algebra is exactly the same as the underbidding case and hence, omitted for conciseness. □

Appendix C. Market-Maker Equilibrium Monotonicity: Proof of Theorem 2

Theorem 2 (Market-Maker Equilibrium Monotonicity). *Under both the CARA and mean-variance models in any two-period* *equilibrium induced by a Market-Maker price p_M, if there exists a value $\tilde{v}$ such that buyers with valuation $\tilde{v}$ purchase the Market-Maker contract in P1 (rather than waiting to participate in the auction in P2), all buyers with valuations $v \geq \tilde{v}$ will also purchase the Market-Maker contract in P1.*

Proof. We provide a separate analysis for each utility model. For a given v and a Market-Maker price p_M, let $U_M(v)$ ($U_A(v)$) be the utility earned by a buyer with valuation v from choosing the Market-Maker contract (the auction). As discussed in Remark 1, we assume that these expressions are given by Equation (1) (Equation (2)); namely, we do not worry about whether at most I buyers select the Market-Maker contract. Let $H(\cdot)$ denote the cumulative distribution function (cdf) and $h(\cdot)$ the probability mass function of the auction clearing price under the equilibrium induced (if an equilibrium exists) by the Market-Maker price of p_M. For ease of exposition, we assume that the distribution of valuations is continuous (only for this proof).

C.1. Mean-Variance Model

We next show that the difference in utilities between opting for the Market-Maker contract and participating in the auction is increasing with v. To this end, we first rewrite the utility derived from the auction:

$$U_A(v) = \int_0^v ((v-p) - \beta \cdot \mathsf{Var}_p[(v-p)^+])h(p) \cdot dp.$$

We now compute the derivative of each term separately. For the first term, we have

$$\frac{\partial}{\partial v}\left[\int_0^v (v-p) \cdot h(p) \cdot dp\right] = \int_0^v h(p) \cdot dp = H(v).$$

The derivative of the second term can be written as

$$\begin{aligned}\frac{\partial}{\partial v}\{\mathsf{Var}_p[(v-p)^+]\} &= \frac{\partial}{\partial v}(\mathbf{E}_p[\{(v-p)^+\}^2] - \mathbf{E}_p[(v-p)^+]^2)\\ &= \frac{\partial}{\partial v}\left[\int_0^v (v-p)^2 \cdot h(p) \cdot dp\right] - 2 \cdot \mathbf{E}_p[(v-p)^+] \cdot H(v)\\ &= 2 \cdot \int_0^v (v-p) \cdot h(p) \cdot dp - 2 \cdot \mathbf{E}_p[(v-p)^+] \cdot H(v)\\ &= 2 \cdot \mathbf{E}_p[(v-p)^+] - 2 \cdot \mathbf{E}_p[(v-p)^+] \cdot H(v)\\ &= 2 \cdot \mathbf{E}_p[(v-p)^+] \cdot \overline{H}(v).\end{aligned}$$

Here, $\overline{H}(v) = 1 - H(v)$ denotes the complementary cdf of the auction clearing price. Note that the variance is an increasing function of v. Putting both terms together, we obtain

$$\frac{\partial}{\partial v}U_A(v) = H(v) - 2 \cdot \beta \cdot \mathbf{E}_p[(v-p)^+] \cdot \overline{H}(v).$$

Because $U_M(v)$ is a linear function of v for $v \geq p_M$, namely $v - p_M$ (note that the theorem statement is only for values of p_M that are high enough so that the Market-Maker option is *not* sold out; therefore, any buyer with $v \geq p_M$ can afford the Market-Maker contract and earns a utility of $v - p_M$), its derivative with respect to v is equal to one. Therefore,

$$\frac{\partial}{\partial v}[U_M(v) - U_A(v)] = \overline{H}(v) \cdot (1 + 2 \cdot \beta \cdot \mathbf{E}_p[(v-p)^+]) \geq 0.$$

As a result, the difference in utility functions is increasing with v. Thus, if there exists a value $\tilde{v}$ such that buyers with valuation $\tilde{v}$ prefer the Market-Maker relative to the auction, then all buyers with valuation $v \geq \tilde{v}$ will also prefer the Market-Maker—hence, concluding the proof.

C.2. CARA Model

Unlike the proof for the mean-variance model, where we analyzed the derivative of the difference in utilities, for the CARA model we will use the ratio of the derivatives. As before, we start by computing the derivatives of the utility functions. For $v \geq p_M$, we have

$$\frac{\partial}{\partial v}U_M(v) = \alpha \cdot e^{-\alpha\cdot(v-p_M)} \geq 0 \quad \text{and}$$
$$\frac{\partial^2}{\partial^2 v}U_M(v) = -\alpha^2 \cdot e^{-\alpha\cdot(v-p_M)} \leq 0.$$

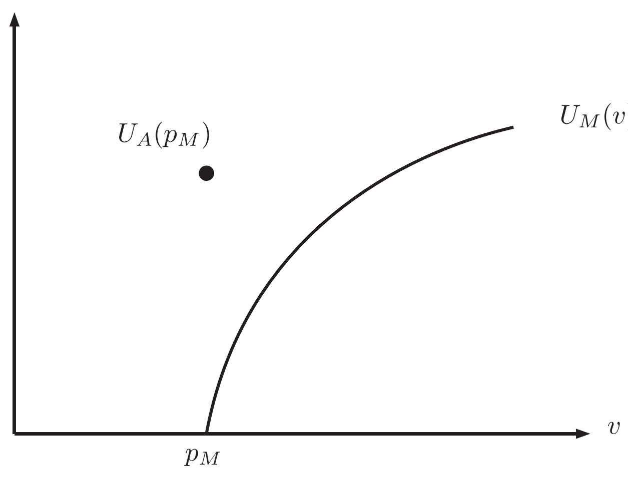
One can write $\frac{\partial}{\partial v}U_A(v) = \alpha \cdot \int_0^v e^{-\alpha\cdot(v-p)} \cdot h(p) \cdot dp \geq 0$. This shows that $U_M(v)$ and $U_A(v)$ are both increasing functions of v. In addition, $U_M(v)$ is concave. Observe that for $v \leq p_M$, $U_M(v) = 0$. Consequently, $U_A(p_M) \geq U_M(p_M) = 0$. To prove the claim, it suffices to show that the functions $U_A(v)$ and $U_M(v)$ intersect at most once. Figure A.1 depicts the situation. We next consider two cases. For both cases, the analysis will use the ratio of derivatives. By using the expressions of the derivatives, we have

$$\frac{\partial U_A}{\partial v}(v) \Big/ \frac{\partial U_M}{\partial v}(v) = \int_0^v e^{-\alpha\cdot(p_M-p)} \cdot h(p) \cdot dp. \tag{C.1}$$

This implies that the ratio in (C.1) is increasing with v. Let $v_{\max}$ denote the maximal value of the valuation v (if the support of v is unbounded, one can take $v_{\max} = \infty$ and use a limit argument). We next separate the analysis depending on the ordering of $U_A(v_{\max})$ and $U_M(v_{\max})$.

Case C.1. We first consider $U_A(v_{\max}) \leq U_M(v_{\max})$. In this case, because $U_A(p_M) \geq U_M(p_M) = 0$, the functions $U_A(\cdot)$ and $U_M(\cdot)$ have to intersect at least once. In other words, there must exist at least one value $\bar{v} > p_M$ such that $U_M(\bar{v}) = U_A(\bar{v})$. In addition, it is not possible to have an even number of crossing points (by crossing point, we mean that the ordering of the function switches) as this would contradict the fact that $U_A(v_{\max}) \leq U_M(v_{\max})$. We assume by contradiction that the number of crossing points is at least three. In this case, note that when the curves intersect for the second time, say at $\hat{v}$,

Figure A.1. Comparing $U_A(v)$ and $U_M(v)$



we must have $\partial U_A/\partial v(\hat{v}) \geq \partial U_M/\partial v(\hat{v})$: that is, the ratio of derivatives must be greater than one. Because this ratio is increasing by Equation (C.1), this means that this ratio remains above one for all $v \geq \hat{v}$. Consequently, we cannot have $U_A(v) > U_M(v)$ for any $v \geq \hat{v}$. As a result, there exists a single value $\overline{v}$ such that for all $v \geq \overline{v}$, $U_M(v) \geq U_A(v)$.

Case C.2. We now consider $U_A(v_{\max}) > U_M(v_{\max})$. By using the expressions derived at the beginning of the proof, we can write for all $v \geq p_M$:

$$\frac{\partial}{\partial v}U_M(v) = \alpha \cdot (1 - U_M(v)) \quad \text{and} \quad \frac{\partial}{\partial v}U_A(v) = \alpha \cdot (1 - U_A(v)).$$

Consequently,

$$\frac{\partial U_A}{\partial v}(v) \bigg/ \frac{\partial U_M}{\partial v}(v)\bigg|_{v=v_{\max}} = \frac{1 - U_A(v_{\max})}{1 - U_M(v_{\max})}. \tag{C.2}$$

Our assumption for this case implies that the ratio in (C.2) is smaller than one. Because the ratio of derivatives is increasing with v, it remains strictly smaller than one for all $p_M \leq v \leq v_{\max}$. This implies that the functions $U_A(\cdot)$ and $U_M(\cdot)$ never intersect. Indeed, recall that at $v = p_M$, $U_A(p_M) \geq U_M(p_M)$, and thus, if $U_M(\cdot)$ was to intersect $U_A(\cdot)$, its derivative must be greater than the derivative of $U_A(\cdot)$ at some point (because both functions are increasing). As a result, there is no value of v such that $U_M(v) = U_A(v)$—concluding the proof.

Appendix D. Market-Maker Price Characterization: Proof of Proposition 1

Proposition 1 (Market-Maker Price Characterization). *Let $p_{\max}$ be the highest auction clearing price (in the support of the distribution of clearing prices) in a single-period (P2) equilibrium where there is no Market-Maker contract. Let p_M^* be the Market-Maker price such that $\overline{v}(p_M^*) = p_{max}$ (see Corollary 1). Then, we have the following.*

1. *A Market-Maker price will result in an equilibrium where at most, I buyers opt for the Market-Maker contract only if it is at least p_M^*.*
2. *A Market-Maker price will result in at least one buyer opting for the Market-Maker contract only if it is at most p_M^*.*

Moreover, a closed-form expression for p_M^* is given by

$$p_M^* = \begin{cases} \mu_A + \beta \cdot \sigma_A^2 & (\textit{Mean-variance model}), \\ \dfrac{1}{\alpha} \cdot \log(\mathbf{E}_p[e^{\alpha \cdot p}]) & (\textit{CARA model}). \end{cases} \tag{3}$$

Here, μ_A and σ_A are the mean and standard deviation of the auction clearing price, respectively, in the single-period (P2) equilibrium, where the Market-Maker contract is not offered. Likewise, the expectation $\mathbf{E}_p$ in the CARA model is over the distribution of auction clearing prices in the single-period equilibrium, where the Market-Maker contract is not offered.

Proof. We start by deriving the closed-form expression for p_M^*, which is defined as the Market-Marker price such that $\overline{v}(p_M^*) = p_{\max}$. When selecting p_M^* for the Market-Maker contract, recall that buyers with valuation higher or equal than $\overline{v}(p_M^*) = p_{\max}$ will opt for the Market-Maker contract (Corollary 1). Because such buyers have a valuation higher than the highest possible clearing price, they will always be allocated in the auction. Therefore, for $v \geq p_{\max}$, we have $(v-p)^+ = v - p$ for any auction clearing price p. In addition, for a given value of $v \geq p_{\max}$, the variance of $(v-p)$ is simply the variance of p, which is σ_A^2. Putting everything together, this means that the utility of buyers with valuation $v \geq p_{\max}$ if they choose the auction is

$$U_A(v) = \begin{cases} v - \mu_A - \beta \cdot \sigma_A^2 & (\textit{Mean-variance model}), \\ 1 - e^{-\alpha \cdot v} \cdot \mathbf{E}_p[e^{\alpha \cdot p}] & (\textit{CARA model}). \end{cases} \tag{D.1}$$

If they decide to opt for the Market-Maker contract at price p_M, their utility is given by

$$U_M(v) = \begin{cases} v - p_M & (\text{Mean-variance model}), \\ 1 - e^{-\alpha \cdot v} \cdot e^{\alpha \cdot p_M} & (\text{CARA model}). \end{cases} \tag{D.2}$$

As a result, for the Market-Maker contract to be (weakly) preferred relative to the auction for buyers with valuation $v \geq p_{\max}$, the Market-Maker price p_M must be at most p_M^* given by

$$p_M^* = \begin{cases} \mu_A + \beta \cdot \sigma_A^2 & (\text{Mean-variance model}), \\ \dfrac{1}{\alpha} \cdot \mathbf{E}_p[e^{\alpha \cdot p}] & (\text{CARA model}). \end{cases}$$

When setting a Market-Maker price below p_M^*, all buyers with valuation above $p_{\max}$ prefer the Market-Maker contract relative to the auction. If we decrease the Market-Maker price strictly below p_M^*, buyers with valuation exactly $p_{\max}$ will strictly prefer the Market-Maker price: that is, $U_A(p_{\max}) < U_M(p_{\max})$. By continuity of the utility functions, this implies that for some $\epsilon > 0$, a buyer with $v = p_{\max} - \epsilon$ also strictly prefers the Market-Maker contract. This implies that $\overline{v}(p_M^*) \leq p_{\max} - \epsilon$, which contradicts the definition of p_M^* (recall that p_M^* is defined by $\overline{v}(p_M^*) = p_{\max}$).

D.1. Uniqueness of p_M^*

We now show that p_M^* is the only feasible price for which at least one buyer selects the Market-Maker contract. First, note that for any price strictly below p_M^*, more than I buyers will choose the Market-Maker when $p_{\max}$ is the auction clearing price. If the Market-Maker price p_M is strictly below p_M^*, this implies that $\overline{v}(p_M) < p_{\max}$, and consequently, more than I buyers will choose the Market-Maker contract when $N_{\max}$ realizes. In this case, the Market-Maker will default, and hence, this is not a feasible price. We next show that the Market-Maker price cannot exceed p_M^*. Observe that when $p_M = p_M^*$ by comparing (D.1) and (D.2), we have $U_A(v) = U_M(v)$ for all $v \geq p_{\max}$. This implies that (i) buyers with valuations above $p_{\max}$ are indifferent between the Market-Maker contract and the auction and that (ii) for any $p_M > p_M^*$, all buyers strictly prefer participating in the auction relative to opting for the Market-Maker contract. As a result, p_M^* is the unique feasible price for which at least one buyer selects the Market-Maker contract. □

Appendix E. Market-Maker Price as a Function of Risk: Proof of Proposition 2

Proposition 2 (Market-Maker Price as a Function of Risk). *The revenue-optimal Market-Maker price p_M^* is an increasing function* *of the risk aversion parameter (α in the CARA model and β in the mean-variance model). Further, in the mean-variance model, p^*_M is an increasing function of the variance of the auction clearing price (σ^2_A).*

Proof. Note that for the mean-variance model, the result directly follows from the expression in (3). We next prove the result for the CARA model. The first-order derivative with α is given by

$$(p^*_M)'(\alpha) = -\frac{1}{\alpha^2}\cdot\log\left(\mathbf{E}_p[e^{\alpha\cdot p}]\right) + \frac{1}{\alpha}\cdot\frac{\mathbf{E}_p[p\cdot e^{\alpha\cdot p}]}{\mathbf{E}_p[e^{\alpha\cdot p}]} = -\frac{1}{\alpha^2}\cdot h(\alpha),$$

where we denote $h(\alpha) = \log\left(\mathbf{E}_p[e^{\alpha\cdot p}]\right) - \alpha\cdot\frac{\mathbf{E}_p[p\cdot e^{\alpha\cdot p}]}{\mathbf{E}_p[e^{\alpha\cdot p}]}$.

Note that $h(0) = 0$, and therefore, it suffices to show that $h'(\alpha) \le 0$ for all $\alpha \ge 0$. The first derivative of $h(\alpha)$ is given by

$$\begin{aligned} h'(\alpha) &= \frac{\mathbf{E}_p[p\cdot e^{\alpha\cdot p}]}{\mathbf{E}_p[e^{\alpha\cdot p}]} - \frac{\mathbf{E}_p[p\cdot e^{\alpha\cdot p}]}{\mathbf{E}_p[e^{\alpha\cdot p}]} \\ &\quad - \alpha\cdot\frac{\mathbf{E}_p[p^2\cdot e^{\alpha\cdot p}]\cdot\mathbf{E}_p[e^{\alpha\cdot p}] - \mathbf{E}_p[p\cdot e^{\alpha\cdot p}]^2}{\mathbf{E}_p[e^{\alpha\cdot p}]^2} \\ &= -\alpha\cdot\frac{\mathbf{E}_p[p^2\cdot e^{\alpha\cdot p}]\cdot\mathbf{E}_p[e^{\alpha\cdot p}] - \mathbf{E}_p[p\cdot e^{\alpha\cdot p}]^2}{\mathbf{E}_p[e^{\alpha\cdot p}]^2} \le 0, \end{aligned}$$

where the last inequality follows from the Cauchy–Schwarz inequality. Because $h'(\alpha) \le 0$ for all $\alpha \ge 0$ and $h(0) = 0$, we have $h(\alpha) \le 0$ for all $\alpha \ge 0$. As a result, $(p^*_M)'(\alpha) \ge 0$ for all $\alpha \ge 0$. □

Appendix F. Relaxing Assumption A.1

In the paper, we assume that at the beginning of P2, the bidders know the auction clearing price with no uncertainty (Assumption A.1). We next examine how our results would be affected if instead, there is some level of uncertainty at the beginning of P2. For simplicity, we assume that the bidders are unit demand (i.e., $k_j = 1$ for all buyers). However, the results extend for all k_j's using Assumption 2.

Specifically, we assume that at the beginning of P2, the remaining uncertainty is captured by a distribution $\ell(\cdot)$ that represents the belief of buyers over the $(I+1)$ th highest value v_{I+1}. We note that the distribution $\ell(\cdot)$ not only is a decision-making tool for bidders in P2 but also, models the utility they actually experience. When there is no uncertainty, the distribution $\ell(\cdot)$ will boil down to a point mass at v_{I+1}. We assume that at the beginning of P2, the buyers do not have an exact knowledge of v_{I+1} but only know that it belongs to some interval $[v^-, v^+]$, such that $v^+ - v^- \le \delta$ for a small $\delta > 0$. Under this assumption, we extend Theorem A.1 as follows.

Theorem F.1. *In a repeated first- or second-price auction, every bidding δ equilibrium where no bidder overbids will result in buyers with value at least $v^- - O(\delta)$ being allocated.*

Proof. First, we show the existence of a δ equilibrium for both the repeated first-price and second-price auctions. Consider the following bidding strategy for buyers. Buyers j with a value strictly higher than v^+ bid v^+ in every repeated auction until they get allocated and bid zero afterward. Buyers with value at most v^+ bid their true value. We claim that in this bidding strategy profile, no buyer can benefit by more than δ from a unilateral deviation. Indeed, buyers with value at most v^+ cannot afford to pay more than v^+ per item and hence, cannot earn a utility higher than δ by bidding more than v^+. Bidding any value in $[v^-, v^+]$ gives them at most δ utility. The remaining buyers, who bid v^+, get allocated and pay at most v^+ in either a first- or second-price auction. These buyers can increase their utility by at most δ by reducing their bids, and they would not benefit from raising their bids.

Second, we show that in both repeated first- and second-price auctions, any equilibrium allocation outcome will never have a bidder with value $v \le v^- - O(\delta)$ getting allocated. Indeed, suppose on the contrary that a bidder with value $v < v^- - O(\delta)$ gets allocated. Such a bidder could not have paid more than $v^- - O(\delta)$ in equilibrium because this would lead to a negative utility. Consequently, this means that one of the bidders with value $v \ge v^-$ who did not get allocated (and hence, earns zero utility in the contrary equilibrium) could have raised his or her bid to v^- and got at least δ utility. It would allow this bidder to be allocated and earn positive utility, leading to a contradiction. □

At the beginning of P1, values are drawn from a joint distribution $F(\cdot)$, which is not necessarily independent across bidders. This distribution $F(\cdot)$ along with other sources of randomness creates a distribution in P1 on the $(I+1)$ th highest value v_{I+1}, which we call $g(\cdot)$. If there was no uncertainty in P2, then the distribution of the auction clearing price in P1 would be the same as $g(\cdot)$. However, because we have relaxed Assumption A.1 and hence, there is some level of uncertainty in P2 (this uncertainty is captured by distribution $\ell(\cdot)$ described at the beginning of Appendix F), it follows that the distribution of the clearing price in P1 is not the same as $g(\cdot)$. Let the distribution $\tilde{g}(\cdot)$ capture the clearing prices in P1 when bidders make a decision between the Market-Maker contract and the sequence of repeated auctions. Importantly, we assume that the Market-Maker has only access to $g(\cdot)$ and sets the price according to that distribution. Otherwise, because Theorem 5 holds under any equilibrium distribution in P2, one can simply compute the optimal Market-Maker price in Equation (3) by using $\tilde{g}(\cdot)$ instead of $g(\cdot)$.

We now extend Corollary 2, which is a central result enabling other results. The goal is to understand how relaxing Assumption A.1 affects which buyers select the Market-Maker contract. To analyze what happens in P1, we focus on the *canonical* δ equilibrium described. More precisely, we focus on the δ equilibrium where every bidder with value $v \ge v^+$ bids v^+, whereas other bidders bid their true value. We note that when δ goes to zero, this coincides with the equilibrium in Theorem A.1. We begin with the repeated second-price auction setting. Under this equilibrium, note that exactly the top I bidders get allocated and pay v_{I+1} in the repeated second-price auction. This is exactly the same outcome as in a single-shot multi-unit auction, and thus, all our results continue to hold in this case. In particular, the same set of buyers who selected Market-Maker will continue to do so even after relaxing Assumption A.1.

Under the first-price price auction setting, things get slightly different. Indeed, bidders with value higher than v^+ will bid v^+, where $v_{I+1} \leq v^+ \leq v_{I+1} + \delta$. This means that when these bidders opt for the auction, they end up paying a slightly higher price because their payment of v^+ (which is their bid) is slightly higher than v_{I+1}.[4] This means that the auction is slightly less attractive for some buyers, and they could potentially switch to the Market-Maker contract. We now have to argue that there are not too many buyers who will switch to the Market-Maker contract because of this. Note that as per Corollary 2, buyers with value at least p_{max} are indifferent between Market-Maker and auction when there is no uncertainty. Regardless of how they broke ties when there was no uncertainty, even if all such buyers are now picking the Market-Maker option, this still does not cause any more such buyers choosing Market-Maker than when all the indifferent buyers picked Market-Maker when there was no uncertainty. As for buyers with value $v < p_{max}$, note that they strictly preferred the auction when there was no uncertainty. With uncertainty, their utility from the auction goes down by at most $O(\delta)$. Given that the separation between auction and Market-Maker utilities increases as v decreases (see the proof of Theorem 5), this means that for $v < p_{max} - r(\delta)$ for some $r(\cdot)$ that is monotonically increasing and has $r(0) = 0$, despite reduced auction utility, such buyers will still strictly prefer the auction. Thus, buyers in an interval of size at most $O(r(\delta))$ will switch to the Market-Maker. As a result, we can extend Corollary 2 as follows.

Corollary F.1. *When offering a Market-Maker contract at price p_M^*, buyers with valuations $v < p_{\max} - O(r(\delta))$ strictly prefer to participate in the auction (i.e., wait until P2), whereas buyers with valuations $v \geq p_{\max}$ strictly prefer to choose the Market-Maker contract (i.e., buy in P1).*

Corollary F.1 conveys that when there is uncertainty in P2 (i.e., when $\delta > 0$), only buyers with values in a small interval of size $O(r(\delta))$ will make a different decision relative to their decision in the absence of uncertainty. When the distribution $F(\cdot)$ is Lipschitz continuous, this means that the mass of such buyers is small. Therefore, this implies that the revenue from the Market-Maker contract in the presence of uncertainty differs by at most $O(r(\delta))$ from the Market-Maker revenue without uncertainty. Additionally, the probability of defaulting is also at most $O(r(\delta))$. As a result, this ultimately shows that our results continue to hold even when there is some uncertainty at the beginning of P2.

Appendix G. Heterogeneous Risk Aversion: Proof of Theorem 4

Theorem 4 (Heterogeneous Risk Aversion). *Under heterogeneous risk aversion, the equilibrium Market-Maker price is not necessarily unique. Instead, there exists a range of equilibrium prices that depends on the range of risk parameters. There always exists at least one price, $p^*_{M,k}$, that strictly increases* (i) *the seller's revenue and* (ii) *the welfare. In addition, each buyer's utility stays the same, and the efficiency remains optimal.*

Proof. Let $\alpha_1 < \cdots < \alpha_k$ $(\beta_1 < \cdots < \beta_k)$ be the risk aversion parameters for the CARA (mean-variance) model. Recall that by Proposition 2, α_k and β_k correspond to the most risk-averse population of buyers. Consider setting the Market-Maker price assuming that the entire population has a risk aversion parameter α_k (or β_k); that is, we set the price $p^*_{M,k}$ by using Equation (3) with $\alpha = \alpha_k$ (or $\beta = \beta_k$). Under this Market-Maker price, buyers from population k with valuation at least p_{max} are indifferent between the Market-Maker contract and the auction. As before, we assume that such indifferent buyers will opt for the Market-Maker contract. However, the less risk-averse buyers (i.e., all the buyers from populations $\ell < k$) strictly prefer participating in the auction when the Market-Maker price is set at $p^*_{M,k}$. Thus, each buyer's utility stays the same, and the efficiency remains optimal. At this price, the auction clearing price is unaffected, and the revenue strictly increases, just as in the homogeneous setting. This concludes the Pareto improvement proof. □

We next show the existence of a range of prices. When setting the Market-Maker price at $p^*_{M,k}$, all buyers from population k with value above $p_{\max}$ opt for the Market-Maker. Note that when $p^*_{M,k}$ realizes, this represents $\rho_k \cdot I$ buyers. Consequently, because at most $\rho_k \cdot I$ buyers select the Market-Maker, there is room to lower the Market-Maker price to attract additional buyers. As the Market-Maker price decreases below $p^*_{M,k}$, some of the less risk-averse buyers will start switching to the Market-Maker contract. This will occur until the point in which we cannot decrease the Market-Maker price any further: that is, when at most I buyers select the Market-Maker contract. This shows that there is a range of feasible price for the Market-Maker.

Appendix H. Proof of Theorem 5

Theorem 5 (Impact on Sum of Buyers' Utilities). *Consider a heterogeneous population of buyers. If the valuation function has an increasing failure rate (i.e., $f(x)/[1-F(x)]$ is increasing)*[5] *and the support of the clearing price distribution consists of two points, then under the mean-variance model, the sum of buyers' utilities increases as the Market-Maker price decreases (as long as the Market-Maker does not default).*

Proof. For simplicity, we present the proof for the case with two types of buyers. The proof naturally extends for $k > 2$ types. As discussed at the end of Section 6, we focus on the mean-variance model as the utilities of buyers with different risk aversion parameters under the CARA model are not comparable quantities.

Without loss of generality, we assume that $\beta_2 > \beta_1$, and we refer to the buyers with risk aversion parameter β_1 (β_2) as type 1 (type 2). Moreover, we assume that each type of buyer represents a fraction $0 < \rho_i < 1$ of the population. Additionally, we assume that the clearing price can only take two values and is equal to p_L with probability $0 < q_L < 1$ and to p_H otherwise, where $p_H > p_L$. We define

$$p(\beta) = \mu_A + \beta \cdot \sigma_A^2,$$

where $\mu_A = q_L \cdot p_L + (1 - q_L) \cdot p_H$ is the average clearing price and $\sigma_A = (p_H - p_L) \cdot \sqrt{q_L \cdot (1 - q_L)}$ is the standard deviation of the clearing price. Without the Market-Maker contract, because the auction clearing price can only take two values, we can compute all the quantities of interest in closed form. In particular, when all the buyers participate in the auction, we have

$$U_{A,i}(v) = \begin{cases} s(v,\beta) & \text{if } p_L \le v \le p_H, \\ v - p(\beta_i) & \text{if } v > p_H, \\ 0 & \text{otherwise.} \end{cases}$$

Here, $U_{A,i}(\cdot)$ denotes the utility of a buyer of type I, and for all v and β, we have

$$h(v,\beta) = -\beta \cdot q_L^2 \cdot (1-q_L) \cdot (v-p_L)^2 + (1-q_L) \cdot (v-p_L).$$

We start from the situation described in Theorem A.1 (i.e., the Market-Maker price is set to $p^*_{M,2} = p(\beta_2)$). Under this Market-Maker price, type 2 buyers with valuations $v \ge p_H$ are indifferent between the Market-Maker contract and participating in the auction, whereas the other buyers strictly prefer the auction. In this case, as we explained before, there is no change in allocation. We next consider decreasing the Market-Maker price below $p(\beta_2)$. Note that when $p^*_{M,2} = p(\beta_2)$, at most $\rho_2 \cdot I < I$ buyers will opt for the Market-Maker contract. As a result, decreasing the Market-Maker price is feasible and does not lead to defaulting (as it was the case in the homogeneous setting).

For any Market-Maker price $p_M < p(\beta_2)$, some type 2 buyers will prefer the Market-Maker relative to the auction. For a given p_M, there exists a unique threshold on the type 2 buyers' valuation, denoted by $(p_H - \delta)$, that induces a switch to the Market-Maker. We denote the Market-Maker price that induces this threshold by $p_2(\delta)$. We parametrize the analysis as a function of δ. In other words, the type 2 buyers with valuations higher than or equal to $(p_H - \delta)$ strictly prefer the Market-Maker contract relative to the auction. In addition, the type 2 buyers with valuation exactly equal to $(p_H - \delta)$ are indifferent between the two options. For a given value of δ, we have

$$U_2(\delta, v) = \begin{cases} h(v,\beta_2) & \text{if } p_L \le v \le p_H - \delta, \\ v - p_2(\delta) & \text{if } v > p_H - \delta, \\ 0 & \text{otherwise.} \end{cases}$$

The indifference experienced by the type 2 buyers with valuation $(p_H - \delta)$ yields the following continuity constraint:

$$h(p_H - \delta, \beta_2) = p_H - \delta - p_2(\delta). \tag{H.1}$$

Recall that when p_H realizes, the number of buyers (of both types 1 and 2) with valuations higher or equal than p_H is exactly I. Because some type 2 buyers with valuations strictly less than p_H now opt for the Market-Maker contract, some type 1 buyers with valuations p_H and higher will be unallocated when p_H realizes. This follows from the following facts. (i) For a small-enough value of δ, type 1 buyers still strictly prefer participating in the auction, and (ii) because there is a smaller number of items to allocate in the auction, some type 1 buyers will be unallocated when p_H realizes. In particular, there is a threshold denoted by $(p_H + t(\delta))$ such that the type 1 buyers with valuations strictly above this threshold are always allocated in the auction. Therefore, the type 1 buyers earn the following utility:

$$U_1(\delta, v) = \begin{cases} h(v,\beta_1) & \text{if } p_L \le v \le p_H + t(\delta), \\ v - p_1(\delta) & \text{if } v > p_H + t(\delta), \\ 0 & \text{otherwise.} \end{cases}$$

Note that in this case, no type 1 buyer opts for the Market-Maker contract. Note also that the average auction clearing price denoted by $p_1(\delta)$ has increased. As before, the continuity of the utility function at $(p_H + t(\delta))$ implies

$$h(p_H + t(\delta), \beta_1) = p_H + t(\delta) - p_1(\delta). \tag{H.2}$$

Finally, we relate $t(\delta)$ to δ by using the fact that the number of type 2 buyers who switch to the Market-Maker is equal to the number of type 1 buyers who lose their allocation when p_H realizes. This conservation condition can be written as follows:

$$\rho_1 \cdot [F(p_H + t(\delta)) - F(p_H)] = \rho_2 \cdot [F(p_H) - F(p_H - \delta)]. \tag{H.3}$$

We next complete the proof by showing the monotonicity of the sum of buyers' utilities as a function of δ. First, we consider the sum of utilities of the type 2 buyers:

$$\begin{aligned} U_2(\delta) &= \int_0^{v_{\max}} U_2(\delta, v) \cdot f(v) \cdot dv \\ &= \int_{p_L}^{p_H-\delta} h(v,\beta_2) \cdot f(v) \cdot dv + \int_{p_H-\delta}^{v_{\max}} [v - p_2(\delta)] \cdot f(v) \cdot dv. \end{aligned}$$

The first-order derivative is given by

$$\begin{aligned} U_2'(\delta) &= -h(p_H - \delta) \cdot f(p_H - \delta) + (p_H - \delta - p_2(\delta)) \cdot f(p_H - \delta) \\ &\quad - \int_{p_H-\delta}^{v_{\max}} p_2'(\delta) \cdot f(v) \cdot dv. \end{aligned}$$

By using Equation (H.1), we obtain

$$U_2'(\delta) = -\int_{p_H-\delta}^{v_{\max}} p_2'(\delta) \cdot f(v) \cdot dv. \tag{H.4}$$

Next, we consider the sum of utilities of the type 1 buyers:

$$\begin{aligned} U_1(\delta) &= \int_0^{v_{\max}} U_1(\delta, v) \cdot f(v) \cdot dv \\ &= \int_{p_L}^{p_H+t(\delta)} h(v,\beta_1) \cdot f(v) \cdot dv + \int_{p_H+t(\delta)}^{v_{\max}} [v - p_1(\delta)] \cdot f(v) \cdot dv. \end{aligned}$$

Taking the first-order derivative yields

$$\begin{aligned} U_1'(\delta) &= t'(\delta) \cdot h(p_H + t(\delta), \beta_1) \cdot f(p_H + t(\delta)) \\ &\quad - (p_H + t(\delta) - p_1(t(\delta))) \cdot f(p_H + t(\delta)) \cdot g'(\delta) \\ &\quad - \int_{p_H+t(\delta)}^{1} p_1'(t(\delta)) \cdot t'(\delta) \cdot f(v) \cdot dv. \end{aligned}$$

Using Equation (H.2), we obtain

$$U_1'(\delta) = -\int_{p_H+t(\delta)}^{v_{\max}} p_1'(\delta) \cdot f(v) \cdot dv. \tag{H.5}$$

We now get the closed-form expression for $p_1'(t(\delta)) \cdot t'(\delta)$, $p_2'(\delta)$ and $t'(\delta)$. First, using (H.1), we have

$$p_2'(\delta) = -1 + \ell(p_H - \delta, \beta_2),$$

where for all v and β, we have

$$\ell(v,\beta) = \frac{\partial}{\partial v} h(v,\beta) = -2 \cdot \beta \cdot q_L^2 \cdot (1-q_L) \cdot (v-p_L) + (1-q_L). \tag{H.6}$$

Similarly, using (H.2), we obtain

$$t'(\delta)\cdot p_1'(t(\delta)) = t'(\delta)(1-\ell(p_H+t(\delta),\beta_1)).$$

Finally, using (H.3), we have

$$t'(\delta) = \tau\cdot\frac{f(p_H-\delta)}{f(p_H+t(\delta))},$$

where $\tau = \rho_2/\rho_1$. Using these expressions combined with (H.4) and (H.5), we obtain that the first derivative of the sum of buyers' utilities is given by

$$\begin{aligned} U'(\delta) &= \rho_1\cdot U_1'(\delta)+\rho_2\cdot U_2'(\delta)\\ &= -\rho_1\cdot t'(\delta)\cdot p_1'(t(\delta))\cdot(1-F(p_H+t(\delta)))\\ &\quad -\rho_2\cdot p_2'(\delta)\cdot(1-F(p_H-\delta))\\ &= \rho_2\cdot f(p_H-\delta)\cdot\Bigg[[1-\ell(p_H-\delta,\beta_2)]\cdot\frac{(1-F(p_H-\delta))}{f(p_H-\delta)}\\ &\quad -(1-\ell(p_H+t(\delta),\beta_1))\cdot\frac{1-F(p_H+t(\delta))}{f(p_H+t(\delta))}\Bigg].\end{aligned}$$

The increasing hazard rate assumption implies that

$$\frac{f(p_H+t(\delta))}{1-F(p_H+t(\delta))}\ge\frac{f(p_H-\delta)}{1-F(p_H-\delta)}.$$

To conclude the proof, it therefore remains to show that

$$\ell(p_H+t(\delta),\beta_1)\ge\ell(p_H-\delta,\beta_2).$$

Using (H.6), this is equivalent to showing

$$\begin{aligned}&-2\cdot\beta_1\cdot q_L^2\cdot(1-q_L)\cdot(p_H+t(\delta)-p_L)+(1-q_L)\\ &\quad\ge -2\cdot\beta_2\cdot q_L^2\cdot(1-q_L)\cdot(p_H-\delta-p_L)+(1-q_L),\end{aligned}$$

which is in turn equivalent to

$$\beta_2\cdot(p_H-\delta-p_L)\ge\beta_1\cdot(p_H+t(\delta)-p_L). \tag{H.7}$$

Note that the Market-Maker price associated with δ is feasible (i.e., does not default) as long as no type 2 buyers switch to the Market-Maker: that is, as long as

$$h(p_H+t(\delta),\beta_1)\ge p_H+t(\delta)-p_2(\delta).$$

Using (H.1) to replace $p_2(\delta)$, this implies that

$$h(p_H+t(\delta),\beta_1)-h(p_H-\delta,\beta_2)\ge t(\delta)+\delta.$$

Using the expression of $h(v,\beta)$, the assumption that the Market-Maker does not default therefore implies that

$$\begin{aligned}&\beta_2\cdot q_L^2\cdot(1-q_L)\cdot(p_H-\delta-p_L)^2\ge\beta_1\cdot q_L^2\cdot(1-q_L)\\ &\quad\cdot(p_H+t(\delta)-p_L)^2+q_L\cdot t(\delta)+(2-q_L)\cdot\delta.\end{aligned}$$

Collecting the terms, we obtain the following inequality:

$$\beta_2\cdot(p_H-\delta-p_L)^2\ge\beta_1\cdot(p_H+t(\delta)-p_L)^2.$$

Multiplying the left-hand side by $\beta_2/\beta_1\ge 1$, we obtain

$$\beta_2^2\cdot(p_H-\delta-p_L)^2\ge\beta_1^2\cdot(p_H+t(\delta)-p_L)^2,$$

which implies (H.7) and concludes the proof.

Appendix I. Additional Figures

Figure A.2. (Color online) Expected Revenue Improvement for the Setting with Homogeneous Risk Aversion (Parameters: N Is Either Discrete Two Point in $\{I, 1.4I\}$ or Normal with $(1.2I, 0.1I)$, F Is Uniform in $[0,1]$, and $I = 1{,}000$)

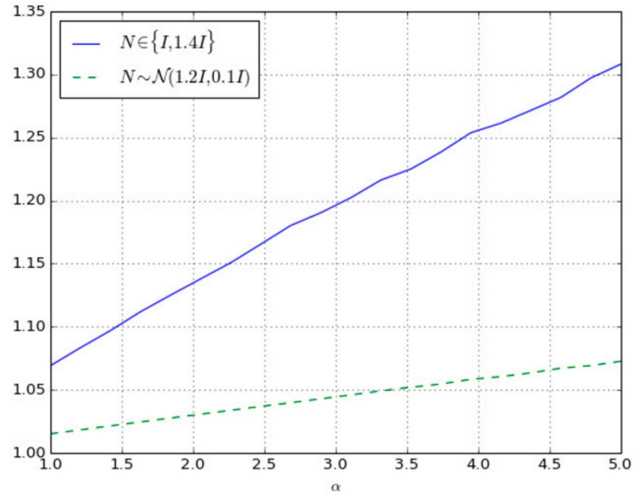
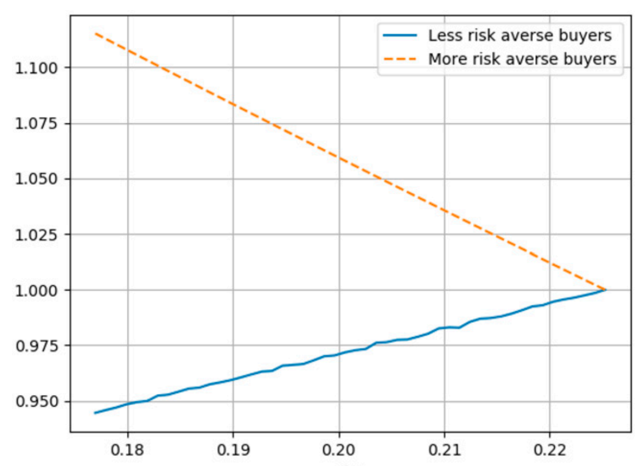


Figure A.3. (Color online) Relative Improvement in the Buyers' Utilities Obtained by Adding the Market-Maker Contract for the Setting with Heterogeneous Risk Aversion (Same Parameters as in Figure 2)



Endnotes

[1] We leave the analysis of the joint equilibrium in the presence of repeated first- or second-price auctions, rather than the reduced-form single-shot multiunit auction that we consider, as an open question. The main technical challenge here is that the repeated auction process with persistent values has to be analyzed by

understanding the perfect Bayesian equilibrium (PBE) for that setting (see analyses of related settings in Immorlica et al. 2017 and Devanur et al. 2019). Although PBE is already an involved notion and in fact, the analysis of PBEs in the related settings cited is quite intricate, in our setting one would have to analyze the joint equilibrium between the Market-Maker and the auction, making it even more challenging.

[2] We note that most of our results and insights also hold for the mean-standard deviation model (i.e., when the variance term is replaced by the standard deviation).

[3] This is a common assumption that is satisfied by several popular distributions, such as Gamma, Weibull, modified extreme value, and truncated normal distribution (see, e.g., Lariviere 2006).

[4] This is consistent with the fact that, in general, a first-price auction under risk aversion will lead to a higher revenue relative to a second-price auction.

[5] This is a common assumption that is satisfied by several popular distributions, such as Gamma, Weibull, modified extreme value, and truncated normal distribution (see, e.g., Lariviere 2006).

References

Ang A (2014) *Asset Management: A Systematic Approach to Factor Investing* (Oxford University Press, Oxford, UK).

Araman VF, Popescu I (2010) Media revenue management with audience uncertainty: Balancing upfront and spot market sales. *Manufacturing Service Oper. Management* 12(2):190–212.

Arrow KJ (1971) The theory of risk aversion. Helsinki YJS, ed. *Essays in the Theory of Risk-Bearing* (Markham Publing Co., Chicago), 90–120.

Athey S, Calvano E, Gans J (2013) The impact of the internet on advertising markets for news media. NBER Working Paper No. 19419, National Bureau of Economic Research, Cambridge, MA.

Balseiro SR, Feldman J, Mirrokni VS, Muthukrishnan S (2011) Yield optimization of display advertising with ad exchange. *Proc. 12th ACM Conf. Electronic Commerce (EC-2011) (San Jose, CA)*, 27–28.

Boyacı T, Özer Ö (2010) Information acquisition for capacity planning via pricing and advance selling: When to stop and act? *Oper. Res.* 58(5):1328–1349.

Budish EB, Takeyama LN (2001) Buy prices in online auctions: Irrationality on the Internet? *Econom. Lett.* 72(3):325–333.

Cachon GP (2004) The allocation of inventory risk in a supply chain: Push, pull, and advance-purchase discount contracts. *Management Sci.* 50(2):222–238.

Caldentey R, Vulcano G (2007) Online auction and list price revenue management. *Management Sci.* 53(5):795–813.

Chawla S, Goldner K, Miller JB, Pountourakis E (2018) Revenue maximization with an uncertainty-averse buyer. *Proc. 29th Annual ACM-SIAM Sympos. Discrete Algorithms*, 2050–2068.

Chen B (2016) Risk-aware dynamic reserve prices of programmatic guarantee in display advertising. *2016 IEEE 16th Internat. Conf. Data Mining Workshops (ICDMW)* (IEEE), 511–518.

Chen B, Yuan S, Wang J (2014) A dynamic pricing model for unifying programmatic guarantee and real-time bidding in display advertising. *Proc. 8th Internat. Workshop Data Mining Online Advertising* (ACM), 1–9.

Cho SH, Tang CS (2013) Advance selling in a supply chain under uncertain supply and demand. *Manufacturing Service Oper. Management* 15(2):305–319.

Devanur NR, Peres Y, Sivan B (2019) Perfect Bayesian equilibria in repeated sales. *Games Econom. Behav.* 118:570–588.

Fu H, Hartline J, Hoy D (2013) Prior-independent auctions for risk-averse agents. *Proc. 14th ACM Conf. Electronic Commerce*, 471–488.

Gao J, Iyer K, Topaloglu H (2019) When fixed price meets priority auctions: Competing firms with different pricing and service rules. *Stochastic Systems* 9(1):47–80.

Ghosh A, McAfee P, Papineni K, Vassilvitskii S (2009) Bidding for representative allocations for display advertising. Leonardi S, ed. *Internet and Network Economics* (Springer, Berlin), 208–219.

Hidvegi Z, Wang W, Whinston AB (2006) Buy-price English auction. *J. Econom. Theory* 129(1):31–56.

Hoy D, Immorlica N, Lucier B (2016) On-demand or spot? Selling the cloud to risk-averse customers. *Internat. Conf. Web Internet Econom.* (Springer), 73–86.

Immorlica N, Lucier B, Pountourakis E, Taggart S (2017) Repeated sales with multiple strategic buyers. *Proc. 2017 ACM Conf. Econom. Comput. EC '17*, 167–168.

Iyer K, Johari R, Moallemi CC (2012) Information and the value of execution guarantees. *ACM Conf. Electronic Commerce, EC '12 (Valencia, Spain)*, 673.

Kirkegaard R, Overgaard PB (2008) Buy-out prices in auctions: Seller competition and multi-unit demands. *RAND J. Econom.* 39(3):770–789.

Lariviere MA (2006) A note on probability distributions with increasing generalized failure rates. *Oper. Res.* 54(3):602–604.

Levy H, Markowitz HM (1979) Approximating expected utility by a function of mean and variance. *Amer. Econom. Rev.* 69(3):308–317.

Markowitz H (2014) Mean-variance approximations to expected utility. *Eur. J. Oper. Res.* 234(2):346–355.

Maskin E, Riley J (1984) Optimal auctions with risk averse buyers. *Econometrica* 52(6):1473–1518.

Medium (2018) Managing Advertising Agreements. Accessed September 8, 2022, https://medium.com/@Clause/managing-advertising-agreements-c1147feb3cee.

Milgrom P (2004) *Putting Auction Theory to Work* (Cambridge University Press, Cambridge, UK).

Mirrokni V, Nazerzadeh H (2017) Deals or no deals: Contract design for online advertising. *Proc. 26th Internat. Conf. World Wide Web* (International World Wide Web Conferences Steering Committee), 7–14.

Nautz D, Wolfstetter E (1997) Bid shading and risk aversion in multi-unit auctions with many bidders. *Econom. Lett.* 56(2):195–200.

Nisan N, Roughgarden T, Tardos É, Vazirani VV, eds. (2007) *Algorithmic Game Theory* (Cambridge University Press, Cambridge, UK).

Paravisini D, Rappoport V, Ravina E (2016) Risk aversion and wealth: Evidence from person-to-person lending portfolios. *Management Sci.* 63(2):279–297.

Prasad A, Stecke KE, Zhao X (2011) Advance selling by a newsvendor retailer. *Production Oper. Management* 20(1):129–142.

Pratt JW (1992) Risk aversion in the small and in the large. Dionne G, Harrington SE, eds. *Foundations of Insurance Economics*, Heubner International Series on Risk, Insurance, and Economic Security, vol. 14 (Springer, Dordrecht, Netherlands), 83–98.

Rabin M (2000) Risk aversion and expected-utility theory: A calibration theorem. *Econometrica* 68(5):1281–1292.

Reynolds SS, Wooders J (2009) Auctions with a buy price. *Econom. Theory* 38(1):9–39.

Sayedi A (2018) Real-time bidding in online display advertising. *Marketing Sci.* 37(4):553–568.

Shugan SM, Xie J (2000) Advance pricing of services and other implications of separating purchase and consumption. *J. Service Res.* 2(3):227–239.

Shugan SM, Xie J (2005) Advance-selling as a competitive marketing tool. *Internat. J. Res. Marketing* 22(3):351–373.

Shunda N (2009) Auctions with a buy price: The case of reference-dependent preferences. *Games Econom. Behav.* 67(2):645–664.

Turner J (2012) The planning of guaranteed targeted display advertising. *Oper. Res.* 60(1):18–33.

von Neumann J, Morgenstern O (1953) *Theory of Games and Economic Behavior* (Princeton University Press, Princeton, NJ).

Wang J, Chen B (2012) Selling futures online advertising slots via option contracts. *Proc. 21st Internat. Conf. World Wide Web* (ACM), 627–628.

Yang J, Vee E, Vassilvitskii S, Tomlin J, Shanmugasundaram J, Anastasakos T, Kennedy O (2010) Inventory allocation for online graphical display advertising. Preprint, submitted August 10, https://arxiv.org/abs/1008.3551.