Econ, Problem Set #2, Dynamic Programming

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Proofs accompanying the Jupyter Notebooks

Notebook 1), Exercise 2)

For $x, y \in \mathbf{R}_+$ show that $\| \max\{\beta,\, x\} - \max\{\beta,\, y\} \| \leq \|x - y\|$ is a contraction mapping.

$$
\begin{aligned}
& \| \max\{\beta,\, x\} - \max\{\beta,\, y\} \| \\
=& \| \, c(1-\beta) + \beta \sum_{k=1}^{K} \max\{w_k,\, x\}\; p_k \\
& \quad - c(1-\beta) - \beta \sum_{k=1}^{K} \max\{w_k,\, y\}\; p_k \| \\
=& \| \, \beta \sum_{k=1}^{K} \max\{w_k,\, x\}\; p_k - \beta \sum_{k=1}^{K} \max\{w_k,\, y\}\; p_k \| \\
=& \| \, \beta \left(\sum_{k=1}^{K} \max\{w_k,\, x\}\; p_k - \max\{w_k,\, y\}\; p_k \right) \| \\
\leq & \beta \sum_{k=1}^{K} \| \max\{w_k,\, x\}\; p_k - \max\{w_k,\, y\}\; p_k \| \\
\leq & \beta \sum_{k=1}^{K} p_k \|x - y\| \\
\leq & \beta \|x - y\| \sum_{k=1}^{K} p_k \\
\| \max\{\beta,\, x\} - \max\{\beta,\, y\} \| \leq & \beta \|x - y\|
\end{aligned}
$$

Notebook 2), Exercise 1)

Show $\forall y \in \mathbf{R}_+$ and $1 > b > 0$ that $\|Uw(y) - Uw'(y)\|_{sup} \leq \|w(y) - w'(y)\|_{sup}$:

$$\begin{aligned}
\|Uw(y) - Uw'(y)\| =& \|u(\sigma(y)) + \beta \int w(f(y-\sigma(y))z)\phi(dz) \\
& - u(\sigma(y)) - \beta \int w'(f(y-\sigma(y))z)\phi(dz)\| \\
=& \beta \| \int w(f(y-\sigma(y))z)\phi(dz) - \int w'(f(y-\sigma(y))z)\phi(dz)\| \\
=& \beta \| \int \left[w(f(y-\sigma(y))z) - w'(f(y-\sigma(y))z)\right]\phi(dz)\| \\
\leq& \beta \| \int \left[w(f(y-\sigma(y))z) - w'(f(y-\sigma(y))z)\right]\phi(dz)\|_{sup} \\
\leq& \beta \| \left[w(f(y-\sigma(y))z) - w'(f(y-\sigma(y))z)\right] \|_{sup} \underbrace{\int \phi(dz)}_{=1} \\
\leq& \beta \|w(y) - w'(y)\|_{sup}
\end{aligned}$$

so $\quad \|Uw(y) - Uw'(y)\|_{sup} \leq \beta \|w(y) - w'(y)\|_{sup}$