Math, Problem Set #1, Probability Theory

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Due Monday, June 26 at 8:00am

1. **Exercises from chapter.** Do the following exercises in Chapter 3 of **?**: 3.6, 3.8, 3.11, 3.12 (watch this movie clip), 3.16, 3.33, 3.36.

   Solutions:
   3.6
   Assumptions:

   (a) $\Omega = \cup_{i \in I} B_i$
   (b) $B_i \cap B_j = 0$

   Proof:

   $$P(A) = P(\Omega \cap A) = P(\cup_{i \in I} B_i \cap A) = P(\cup_{i \in I}(B_i \cap A)) = \sum_{i \in I} P(A \cap B_i)$$

   3.8
   First, if the events E_k are independent, than there complements E_k^c are independet, too.

   $$\begin{aligned} 1 - \prod_{k=1}^{n}(1 - P(E_k)) =& 1 - \prod_{k=1}^{n} (P(E))^c \\ =& 1 - P(\cap_{k=1}^{n} E_k^c) \end{aligned}$$

   and by De Morgans law we get

   $$\begin{aligned} =& 1 - P((\cup_{k=1}^{n} E_k)^c) \\ =& P(\cap_{k=1}^{n} E_k)) \end{aligned}$$

   3.11 TODO
   3.12
   In the first round the unconditional probability of picking the door with the car out of the three door is is just $P(car) = \frac{1}{3}$. After having observed one goat the conditional probability of choosing the door with the car is $P(car|seen\,one\,goat) = \frac{1}{2}$ because one chooses randomly out of two possible choices.
   When there are 10 doors $P(car) = \frac{1}{10}$ and $P(car|seen\,8\,goats) = \frac{1}{2}$ because, again, you are randomly choosing one out of two possible choices conditional on the 8 goats you have already observed.

3.16

$$\begin{aligned}Var[X] =&E\left[(X-\mu)^2\right] = E\left[X^2 - 2X\mu + \mu^2\right] \\ =&E[X^2] - 2E[X]\mu + \mu^2 = E[X^2] - 2\mu^2 + \mu^2 \\ =&E[X^2] - \mu^2\end{aligned}$$

3.33

Since $B \sim Binom(n, p)$, $E[B] = np$ and $Var[B] = \sigma^2 = p(1-p)$

$$P\left(|\frac{B}{n} - p| \geq \varepsilon\right) = P\left(|B - np| \geq n\varepsilon\right)$$

And by the Chebyshev's Inequality we get

$$\begin{aligned}P\left(|B-np| \geq n\varepsilon\right) \leq &\frac{np(1-p)}{n^2\varepsilon^2} \\ \leq &\frac{p(1-p}{n\varepsilon^2}\end{aligned}$$

3.36

Since $X_i \sim Bernoulli(p)\, for\, i \in (1, 6242)$ with $E[x] = p = 0.801$ and $Var[X] = \sigma^2 = p(1-p) = 0.199 \times 0.801$, let the number of students actually enrolling be $S = \sum^6 242_i = 1X_i$, then the variable $\frac{S-6242p}{\sigma\sqrt{6242}} \sim N(0,1)$. Then the probability that more than 5500 students will enroll is given by $1 - P(x < \frac{5500-6242p}{\sigma\sqrt{6242}}) = 0$. This probability is practically zero.

2. Construct examples of events A, B, and C, each of probability strictly between 0 and 1, such that
   (a) $P(A \cap B) = P(A)P(B)$, $P(A \cap C) = P(A)P(C)$, $P(B \cap C) = P(B)P(C)$, but $P(A \cap B \cap C) \neq P(A)P(B)P(C)$.
   (b) $P(A \cap B) = P(A)P(B)$, $P(A \cap C) = P(A)P(C)$, $P(A \cap B \cap C) = P(A)P(B)P(C)$, but $P(B \cap C) \neq P(B)P(C)$. (Hint: You can let Ω be a set of eight equally likely points.)

Solutions:

(a) let $\Omega = 1, 2, 3, 4, 5, 6, 7, 8$ with $P(x = x_i) = \frac{1}{8} \, \forall \, x_i \in \Omega$ and with the events

$$A = \{1, 2, 3, 4\}$$
$$B = \{1, 2, 5, 6\}$$
$$C = \{3, 4, 5, 6\}$$

Such that $P(A) = P(B) = P(C) = \frac{4}{8}$ and the intersections $A \cap B$, $A \cap B$, $B \cap C$ each have two elements leading to $P(A \cap B) = P(A \cap C) == P(B \cap C) = P(A)P(C) = P(A)P(B) = P(B)P(C) = \frac{2}{8}$. Furthermore, $P(A \cap B \cap C) = P(\emptyset) = 0 \neq P(A)P(B)P(C) = \frac{1}{8}$. $\emptyset$

(b) for the same Ω as in (a) be

$$A = \{1, 2, 3, 4\}$$
$$B = \{3, 4, 6, 7\}$$
$$C = \{1, 2, 4, 5\}$$

Such that $P(A) = P(B) = P(C) = \frac{4}{8}$ and the intersections $A \cap B$ and $A \cap B$ each have two elements leading to $P(A \cap B) = P(A \cap C) = P(A)P(C) = P(A)P(B) = \frac{2}{8}$. Furthermore, $P(B \cap C) = P(x = 4) = \frac{1}{8} \neq P(B)P(C) = \frac{2}{8}$ and $P(A \cap B \cap C) = P(4) = \frac{1}{8} = P(A)P(B)P(C)$.

3. Prove that Benford's Law is, in fact, a well-defined discrete probability distribution.

Solutions: Benford's law holds for the sample space

1. since $2 \geq (1 + \frac{1}{d}) \geq (1 + \frac{1}{9}) \, \forall \, d \in \Omega$ it follows that $0 \geq log_{10}(1 + \frac{1}{d}) \geq 1$ holds.

2.

$$\sum_{d=1}^{9} log_{10}\left(1 + \frac{1}{d}\right) = log_{10}\left(\prod_{d=1}^{9} 1 + \frac{1}{d}\right)$$
$$= log_{10}(2 \times 1.5 \times 1.33 \times 1.25 \times 1.2 \times 1.167 \times 1.143 \times 1.125 \times 1.111 \times)$$
$$= log_{10}(10) = 1$$

3. for any events d_i and d_j with $i \neq j$, the intersection $d_i \cap d_j = \emptyset$ (i.e. they are pairwise disjoint) such that $P(d_i \cup d_j) = P(d_i) + P(d_j)$

4. A person tosses a fair coin until a tail appears for the first time. If the tail appears on the nth flip, the person wins 2^n dollars. Let the random variable X denote the player's winnings.

(a) (St. Petersburg paradox) Show that $E[X] = +\infty$.

(b) Suppose the agent has log utility. Calculate $E[\ln X]$.

Let X be the payoff from the St. Petersburg game, than its expected value can be calculated by

$$E[X] = \sum_{k=1}^{\infty} \left(\frac{1}{2}\right)^k 2^k = 1 + 1 + 1 + 1 + \ ... = +\infty$$

When the payoff of one game is weighted with the utility function $u(x) = ln(x)$, the expected utility is given by

$$E[u(X)] = \sum_{k=1}^{\infty} \left(\frac{1}{2}\right)^k ln(2^k) = ln(2) \sum_{k=1}^{\infty} \frac{k}{2^k} = ln(2)S$$

where

$$S = \sum_{k=1}^{\infty} \frac{k}{2^k} = S - \frac{1}{2}S + \frac{1}{2}S = S - \sum_{k=1}^{\infty} \frac{1}{2}\frac{k}{2^k} + \frac{1}{2}S = S - \sum_{k=1}^{\infty} \frac{k}{2^{k+1}} + \frac{1}{2}S$$
$$= \sum_{k=1}^{\infty} \frac{k}{2^k} - \sum_{k=2}^{\infty} \frac{k-1}{2^k} + \frac{1}{2}S = \sum_{k=1}^{\infty} \frac{1}{2^k} + \frac{1}{2}S = 1 + \frac{1}{2}S \Rightarrow S = 2$$

so,

$$E[X] = 2ln(2)$$

5. (Siegel's paradox) Suppose the exchange rate between USD and CHF is 1:1. Both a U.S. investor and a Swiss investor believe that a year from now the exchange rate will be either 1.25 : 1 or 1 : 1.25, with each scenario having a probability of 0.5. Both investors want to maximize their wealth in their respective home currency (a year from now) by investing in a risk-free asset; the risk-free interest rates in the U.S. and in Switzerland are the same. Where should the two investors invest?

6. Consider a probability measure space with $\Omega = [0, 1]$.

   (a) Construct a random variable X such that $E[X] < \infty$ but $E[X^2] = \infty$.

   (b) Construct random variables X and Y such that $P(X > Y) > \frac{1}{2}$ but $E[X] < E[Y]$.

   (c) Construct random variables X, Y, and Z such that $P(X > Y)P(Y > Z)P(X > Z) > 0$ and $E(X) = E(Y) = E(Z) = 0$.

7. Let the random variables X and Z be independent with $X \sim N(0, 1)$ and $P(Z = 1) = P(Z = -1) = \frac{1}{2}$. Define $Y = XZ$ as the product of X and Z. Prove or disprove each of the following statements.

(a) $Y \sim N(0,1)$.

(b) $P(|X| = |Y|) = 1$.

(c) X and Y are not independent.

(d) $Cov[X,Y] = 0$.

(e) If X and Y are normally distributed random variables with $Cov[X,Y] = 0$, then X and Y must be dependent.

8. Let the random variables X_i, $i = 1, 2, \ldots, n$, be i.i.d. having the uniform distribution on $[0,1]$, denoted $X_i \sim U[0,1]$. Consider the random variables $m = \min\{X_1, X_2, \ldots, X_n\}$ and $M = \max\{X_1, X_2, \ldots, X_n\}$. For both random variables m and M, derive their respective cumulative distribution (cdf), probability density function (pdf), and expected value.

9. You want to simulate a dynamic economy (e.g., an OLG model) with two possible states in each period, a "good" state and a "bad" state. In each period, the probability of both shocks is $\frac{1}{2}$. Across periods the shocks are independent. Answer the following questions using the Central Limit Theorem and the Chebyshev Inequality.

   (a) What is the probability that the number of good states over 1000 periods differs from 500 by at most 2%?

   (b) Over how many periods do you need to simulate the economy to have a probability of at least 0.99 that the proportion of good states differs from $\frac{1}{2}$ by less than 1%?

10. If $E[X] < 0$ and $\theta \neq 0$ is such that $E[e^{\theta X}] = 1$, prove that $\theta > 0$.