Math, Problem Set #2, Inner Prouct Spaces

Zachery

Due Wednesday, July 5 at 8:00am

Homework: 1, 2, 3, 8, 9, 10, 11, 16, 17, 23, 24, 26, 28, 29, 30, 37, 38, 39, 40, 44, 45, 46, 47, 48, 50 at the end of Chapter 3 of Humpherys et al. (2017)

3.1) i)

$$
\begin{aligned}
&\frac{1}{4}\left(||x+y||^2 - ||x-y||^2\right) \\
=&\frac{1}{4}\left(||x||^2 + ||y||^2 + 2||x||\,||y||\,cos(\theta) - [||x||^2 + ||y||^2 - 2||x||\,||y||\,cos(\theta)]\right) \\
=&\frac{1}{4}\left(4||x||\,||y||\,cos(\theta)\right) \quad by\, definition\, of\, cos(\theta) \\
=&||x||\,||y||\,\frac{\langle x, y\rangle}{||x||\,||y||} \\
=&\langle x, y\rangle
\end{aligned}
$$

ii)

$$
\begin{aligned}
&\frac{1}{2}\left(||x+y||^2 + ||x-y||^2\right) \\
=&\frac{1}{2}\left(||x||^2\,||y||^2 + 2||x||\,||y||\,cos(\theta) + ||x||^2 + ||y||^2 - 2||x||\,||y||\,cos(\theta)\right) \\
=&\frac{1}{2}2\left(||x||^2 + ||y||^2\right) \\
=&||x||^2 + ||y||^2
\end{aligned}
$$

3.2)

$$
\begin{aligned}
&\frac{1}{4}\left(||x+y||^2-||x-y||^2+i||x-iy||^2-i||x+iy||^2\right)\\
&\frac{1}{4}\left(||x+y||^2-||x-y||^2-i(||x+iy||^2-||x-iy||^2)\right)\\
=&\frac{1}{4}(||x||^2+\langle x,y\rangle+\langle y,x\rangle+||y||^2\\
&-||x||^2+\langle x,y\rangle+\langle y,x\rangle-||y||^2\\
&-i(||x||^2+\langle x,iy\rangle+\langle iy,x\rangle+||y||^2\\
&-||x||^2+\langle x,iy\rangle+\langle iy,x\rangle-||y||^2))\\
=&\frac{1}{4}(2\langle x,y\rangle+2\langle y,x\rangle\\
&-i(2i\langle x,y\rangle-2i\langle y,x\rangle)\\
=&\frac{1}{4}(2\langle x,y\rangle+2\langle y,x\rangle\\
&+2\langle x,y\rangle-2\langle y,x\rangle)\\
=&\frac{1}{4}(4\langle x,y\rangle)=\langle x,y\rangle
\end{aligned}
$$

3.3)

$$\theta = cos^{-1}\left(\frac{\langle f,g\rangle}{||f||\,||g||}\right)$$

for i) and ii) we have

$$\langle f,g\rangle=\int_o^1 x^6\,dx=\frac{1}{7}$$

and for i) we further have:

$$
\begin{aligned}
||g|| =&||x|| = \sqrt{\langle g,g\rangle}=\left(\frac{1}{3}\right)^{\frac{1}{2}}\\
||f|| =&= ||x^5||\sqrt{\langle f,f\rangle}=\left(\frac{1}{11}\right)^{\frac{1}{2}}\\
\theta =& \cos^{-1}\left(\frac{\sqrt{33}}{7}\right)
\end{aligned}
$$

and for ii)

$$||g|| = ||x^2|| = \sqrt{\langle g, g\rangle} = \left(\frac{1}{5}\right)^{\frac{1}{2}}$$
$$||f|| = = ||x^4|| = \sqrt{\langle f, f\rangle} = \left(\frac{1}{9}\right)^{\frac{1}{2}}$$
$$\theta = \cos^{-1}\left(\frac{\sqrt{45}}{7}\right)$$

3.8) i)

$$\begin{aligned}
\langle cos^2(t)\rangle = \frac{1}{\pi}\int_{-\pi}^{\pi} cos^2(t)dt =& \frac{1}{\pi}\int_{-\pi}^{\pi}\frac{1+cos(2t)}{2}dt = \frac{1}{2\pi}\left(\int_{-\pi}^{\pi} 1dt + \int_{-\pi}^{\pi} cos(t)dt\right)\\
=& \frac{1}{2\pi}\left[t\right]_{-\pi}^{\pi} = \frac{1}{2\pi}[\pi+\pi] = 1\\
\langle cos^2(2t)\rangle = \frac{1}{\pi}\int_{-\pi}^{\pi} cos^2(2t)dt =& \frac{1}{\pi}\int_{-\pi}^{\pi}\frac{1+cos(4t)}{2}dt \quad and\, because\, cos(4t) = cos(2t)\\
=& \frac{1}{2\pi}\left(\int_{-\pi}^{\pi} 1dt + \int_{-\pi}^{\pi} cos(t)dt\right)\\
=& \frac{1}{2\pi}\left[t\right]_{-\pi}^{\pi} = \frac{1}{2\pi}[\pi+\pi] = 1\\
\langle sin^2(t)\rangle = \frac{1}{\pi}\int_{-\pi}^{\pi} sin^2(t)dt =& \frac{1}{\pi}\int_{-\pi}^{\pi}\frac{1-cos(2t)}{2}dt = \frac{1}{2\pi}\left(\int_{-\pi}^{\pi} 1\,dt - \int_{-\pi}^{\pi} cos(t)dt\right)\\
=& \frac{1}{2\pi}\left[t\right]_{-\pi}^{\pi} = \frac{1}{2\pi}[\pi+\pi] = 1\\
\langle sin^2(2t)\rangle = \frac{1}{\pi}\int_{-\pi}^{\pi} sin^2(2t)dt =& \frac{1}{\pi}\int_{-\pi}^{\pi}\frac{1-cos(4t)}{2}dt \quad and\, because\, sin(4t) = cos(2t)\\
=& \frac{1}{2\pi}\left(\int_{-\pi}^{\pi} 1dt + \int_{-\pi}^{\pi} cos(t)dt\right)\\
=& \frac{1}{2\pi}\left[t\right]_{-\pi}^{\pi} = \frac{1}{2\pi}[\pi+\pi] = 1
\end{aligned}$$

$$
\begin{aligned}
\langle cos(t,)sin(t)\rangle &= \int_{-\pi}^{\pi} cos(t)sin(t)dt = 0 \\
\langle cos(t), cos(2t)\rangle &= \int_{-\pi}^{\pi} cos(t)sin(2t)dt = 0 \\
\langle cos(t), sin(2t)\rangle &= \int_{-\pi}^{\pi} cos(t)sin(2t)dt = 0 \\
\langle sin(t), cos(2t)\rangle &= \int_{-\pi}^{\pi} sin(t)cos(2t)dt = 0 \\
\langle sin(t), sin(2t)\rangle &= \int_{-\pi}^{\pi} sin(t)cos(2t)dt = 0 \\
\langle cos(2t), sin(2t)\rangle &= \int_{-\pi}^{\pi} cos(2t)sin(2t)dt = 0
\end{aligned}
$$

ii)

$$
||t|| = \langle t, t\rangle^{0.5} = \left(\frac{1}{\pi}\int_{-\pi}^{\pi} t^2\, dt\right)^{0.5} = \left(\frac{1}{\pi} \times \frac{2\pi^3}{3}\right)^{0.5} = \sqrt{\frac{2}{3}}\pi
$$

iii)

$$
\begin{aligned}
proj_X(cos(3t)) =& \sum_{i=1}^{4} \langle x_i, cos(3t)\rangle x_i \\
=& \langle cos(t), cos(3t)\rangle cos(t) + \langle sin(t), cos(3t)\rangle sin(t) \\
& + \langle cos(2t), cos(3t)\rangle cos(2t) + \langle sin(2t), cos(3t)\rangle sin(2t) \\
=& 0 + 0 + 0 + 0 = 0
\end{aligned}
$$

iv)

$$
\begin{aligned}
proj_X(t) =& \sum_{i=1}^{4} \langle x_i, t\rangle x_i \\
=& \frac{1}{\pi}\int_{-\pi}^{\pi} t\, cos(t)\, dt\, cos(t) + \frac{1}{\pi}\int_{-\pi}^{\pi} t\, sin(t)\, dt\, sin(t) \\
& + \frac{1}{\pi}\int_{-\pi}^{\pi} t\, cos(2t)\, dt\, cos(2t) + \frac{1}{\pi}\int_{-\pi}^{\pi} t\, sin(2t)\, dt\, sin(2t) \\
=& 0 + \frac{2\pi}{\pi} sin(t) + 0 + -\frac{\pi}{\pi} sin(2t) \\
=& 2sin(t) - sin(2t)
\end{aligned}
$$