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1 2023-07

1.1 Steepest Edge

2023-07-07 10:22 linear programming problem

$$\begin{aligned} \min \quad & \mathbf{c}'\mathbf{x} \\ s.t. \quad & \mathbf{A}\mathbf{x} = \mathbf{b} \\ & \mathbf{x} \geqslant 0 \end{aligned}$$

where $\mathbf{A}$ is an $m \times n$ matrix of rank m with $m < n$

$$\mathbf{A} = [\boldsymbol{a}_1\ \boldsymbol{a}_2\ \cdots\ \boldsymbol{a}_n] \to [\mathbf{B}\ \mathbf{N}]$$

Define the basic index set to be an integer set $\mathsf{B} = \{i_1, i_2, \cdots, i_m\}$ with $1 \leqslant i_1 < i_2 < \cdots < i_m \leqslant n$, such that $\mathbf{B} = [\boldsymbol{a}_{i_1}\ \boldsymbol{a}_{i_2}\ \cdots\ \boldsymbol{a}_{i_m}]$ is full rank. The matrix $\mathbf{N}$ is the block of $\mathbf{A}$ after $\mathbf{B}$ is taken out, and non-basic index set $\mathsf{N} = \{1, 2, \cdots, n\} \backslash \mathsf{B}$. Let $\mathbf{h}'_N = \mathbf{c}'_N - \mathbf{c}'_B \mathbf{B}^{-1}\mathbf{N}$, and

$$g_i = \frac{h_i^2}{1 + \|\mathbf{B}^{-1}\boldsymbol{a}_i\|^2} \qquad h_i < 0, i \in \mathsf{N}$$

then

$$k = \text{argmax}\{g_i : h_i < 0, i \in \mathsf{N}\}$$

is the entering index detected by the steepest edge rule.

See Goldfarb and Reid (1977) and Forrest and Goldfarb (1992) for more.

1.1.1 Zörnig (2006)

Zörnig (2006) Example 6.1, Problem (6.3) $\mathsf{B} = \{5, 6, 7\}$: entering index 1, leaving index 5

$\mathbf{c}$	$-\frac{7}{100}$	$-\frac{3}{50}$	$\frac{11}{50}$	$\frac{133}{12500}$	0	0	0
	$\blacktriangleleft\ \frac{33}{25}$	$\frac{1}{2}$	$-\frac{53}{25}$	$-\frac{9}{20}$	1	0	0
	-44	$-\frac{24}{5}$	6	$\frac{33}{100}$	0	1	0
	$\frac{1319}{1000}$	-40	$\frac{2123}{10}$	$\frac{1173}{100}$	0	0	1
$\mathbf{h}$	$-\frac{7}{100}$	$-\frac{3}{50}$	$\frac{11}{50}$	$\frac{133}{12\,500}$			
$\mathbf{g}$	$\checkmark\ \frac{4900}{1940\,482\,161}$	$\frac{3}{1353\,575}$					
i	1	2	3	4	5	6	7

for

$$\mathbf{B} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \mathbf{B}^{-1}\mathbf{N} = \begin{bmatrix} \frac{33}{25} & \frac{1}{2} & -\frac{53}{25} & -\frac{9}{20} \\ -44 & -\frac{24}{5} & 6 & \frac{33}{100} \\ \frac{1319}{1000} & -40 & \frac{2123}{10} & \frac{1173}{100} \end{bmatrix}$$

$$\mathbf{h}'_N = \mathbf{c}'_N - \mathbf{c}'_B\mathbf{B}^{-1}\mathbf{N} = \begin{bmatrix} -\frac{7}{100} & -\frac{3}{50} & \frac{11}{50} & \frac{133}{12\,500} \end{bmatrix}$$

$$g_i = \frac{h_i^2}{1 + \|\mathbf{B}^{-1}\boldsymbol{a}_i\|^2} \qquad h_i < 0$$

$$g_1 = \frac{\left(-\frac{7}{100}\right)^2}{1+\frac{1939\,482\,161}{1000\,000}} = \frac{4900}{1940\,482\,161} = 2.525\,1 \times 10^{-6}$$

$$g_2 = \frac{\left(-\frac{3}{50}\right)^2}{1+\frac{162\,329}{100}} = \frac{3}{1353\,575} = 2.216\,4 \times 10^{-6}$$

$\mathsf{B} = \{1, 6, 7\}$: entering index 2, leaving index 1

$\mathbf{c}$	$-\frac{7}{100}$	$-\frac{3}{50}$	$\frac{11}{50}$	$\frac{133}{12500}$	0	0	0
	1	$\blacktriangleleft\ \frac{25}{66}$	$-\frac{53}{33}$	$-\frac{15}{44}$	$\frac{25}{33}$	0	0
	0	$\frac{178}{15}$	$-\frac{194}{3}$	$-\frac{1467}{100}$	$\frac{100}{3}$	1	0
	0	$-\frac{106\,919}{2640}$	$\frac{7075\,807}{33\,000}$	$\frac{107\,181}{8800}$	$-\frac{1319}{1320}$	0	1
$\mathbf{h}$		$-\frac{221}{6600}$	$\frac{71}{660}$	$-\frac{7273}{550\,000}$	$\frac{7}{132}$		
$\mathbf{g}$		$\checkmark\ \frac{195\,364}{310\,527\,143\,625}$		$\frac{211\,586\,116}{441\,249\,749\,640\,625}$			
i	1	2	3	4	5	6	7

for

$$\mathbf{B}^{-1}\mathbf{N} = \begin{bmatrix} \frac{33}{25} & 0 & 0 \\ -44 & 1 & 0 \\ \frac{1319}{1000} & 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} \frac{1}{2} & -\frac{53}{25} & -\frac{9}{20} & 1 \\ -\frac{24}{5} & 6 & \frac{33}{100} & 0 \\ -40 & \frac{2123}{10} & \frac{1173}{100} & 0 \end{bmatrix} = \begin{bmatrix} \frac{25}{66} & -\frac{53}{33} & -\frac{15}{44} & \frac{25}{33} \\ \frac{178}{15} & -\frac{194}{3} & -\frac{1467}{100} & \frac{100}{3} \\ -\frac{106\,919}{2640} & \frac{7075\,807}{33\,000} & \frac{107\,181}{8800} & -\frac{1319}{1320} \end{bmatrix}$$

$$\mathbf{c}'_B\mathbf{B}^{-1}\mathbf{N} = \begin{bmatrix} -\frac{7}{100} & 0 & 0 \end{bmatrix} \mathbf{B}^{-1}\mathbf{N} = \begin{bmatrix} -\frac{7}{264} & \frac{371}{3300} & \frac{21}{880} & -\frac{7}{132} \end{bmatrix}$$

$$\mathbf{h}'_N = \mathbf{c}'_N - \mathbf{c}'_B\mathbf{B}^{-1}\mathbf{N} = \begin{bmatrix} -\frac{3}{50} & \frac{11}{50} & \frac{133}{12\,500} & 0 \end{bmatrix} - \mathbf{c}'_B\mathbf{B}^{-1}\mathbf{N} = \begin{bmatrix} -\frac{221}{6600} & \frac{71}{660} & -\frac{7273}{550\,000} & \frac{7}{132} \end{bmatrix}$$

$$g_2 = \frac{\left(-\frac{221}{6600}\right)^2}{1+\frac{827\,607\,743}{464\,640}} = \frac{195\,364}{310\,527\,143\,625} = 6.291\,4 \times 10^{-7}$$
$$g_4 = \frac{\left(-\frac{7273}{550\,000}\right)^2}{1+\frac{28\,162\,543\,977}{77\,440\,000}} = \frac{211\,586\,116}{441\,249\,749\,640\,625} = 4.795\,2 \times 10^{-7}$$

$\mathsf{B} = \{2, 6, 7\}$: entering index 4, leaving index ◄

$\mathbf{c}$	$-\frac{7}{100}$	$-\frac{3}{50}$	$\frac{11}{50}$	$\frac{133}{12500}$	0	0	0
	$\frac{66}{25}$	1	$-\frac{106}{25}$	$-\frac{9}{10}$	2	0	0
	$-\frac{3916}{125}$	0	$-\frac{1794}{125}$	$-\frac{399}{100}$	$\frac{48}{5}$	1	0
	$\frac{106\,919}{1000}$	0	$\frac{427}{10}$	$-\frac{2427}{100}$	80	0	1
$\mathbf{h}$	$\frac{221}{2500}$		$-\frac{43}{1250}$	$-\frac{271}{6250}$	$\frac{3}{25}$		
$\mathbf{g}$			$\frac{1849}{3200\,386\,725}$	✓ $\frac{146\,882}{47\,403\,359\,375}$			
i	1	2	3	4	5	6	7

for

$$\mathbf{B}^{-1}\mathbf{N} = \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ -\frac{24}{5} & 1 & 0 \\ -40 & 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} \frac{33}{25} & -\frac{53}{25} & -\frac{9}{20} & 1 \\ -44 & 6 & \frac{33}{100} & 0 \\ \frac{1319}{1000} & \frac{2123}{10} & \frac{1173}{100} & 0 \end{bmatrix} = \begin{bmatrix} \frac{66}{25} & -\frac{106}{25} & -\frac{9}{10} & 2 \\ -\frac{3916}{125} & -\frac{1794}{125} & -\frac{399}{100} & \frac{48}{5} \\ \frac{106\,919}{1000} & \frac{427}{10} & -\frac{2427}{100} & 80 \end{bmatrix}$$

$$\mathbf{c}_B'\mathbf{B}^{-1}\mathbf{N} = \begin{bmatrix} -\frac{3}{50} & 0 & 0 \end{bmatrix} \mathbf{B}^{-1}\mathbf{N} = \begin{bmatrix} -\frac{99}{625} & \frac{159}{625} & \frac{27}{500} & -\frac{3}{25} \end{bmatrix}$$

$$\mathbf{h}_N' = \mathbf{c}_N' - \mathbf{c}_B'\mathbf{B}^{-1}\mathbf{N} = \begin{bmatrix} -\frac{7}{100} & \frac{11}{50} & \frac{133}{12500} & 0 \end{bmatrix} - \mathbf{c}_B'\mathbf{B}^{-1}\mathbf{N} = \begin{bmatrix} \frac{221}{2500} & -\frac{43}{1250} & -\frac{271}{6250} & \frac{3}{25} \end{bmatrix}$$

$$g_3 = \frac{\left(-\frac{43}{1250}\right)^2}{1+\frac{127\,952\,969}{62\,500}} = \frac{1849}{3200\,386\,725} = 5.777\,4 \times 10^{-7}$$
$$g_4 = \frac{\left(-\frac{271}{6250}\right)^2}{1+\frac{605\,763}{1000}} = \frac{146\,882}{47\,403\,359\,375} = 3.098\,6 \times 10^{-6}$$

The entering index 4, since $\mathbf{B}^{-1}\boldsymbol{a}_4 < 0$, the LP is unbounded

Zörnig (2006) Example 6.1, Problem (6.4) $\mathsf{B} = \{7, 8, 9\}$: entering index 5, leaving index 7

$\mathbf{c}$	36	$-\frac{3}{5}$	20	$\frac{1}{4}$	$-\frac{1}{20}$	$-\frac{1}{20}$	0	0	0
	2	$\frac{1}{5}$	-5	$-\frac{9}{10}$	$\blacktriangleleft\, 1$	$\frac{23}{1000}$	1	0	0
	-41	$-\frac{6}{5}$	12	$\frac{1}{5}$	$-\frac{14}{5}$	$-\frac{1}{500}$	0	1	0
	165000	2600	9600	125	-100	-300	0	0	1
$\mathbf{h}$	36	$-\frac{3}{5}$	20	$\frac{1}{4}$	$-\frac{1}{20}$	$-\frac{1}{20}$			
$\mathbf{g}$		$\frac{3}{56\,333\,354}$			$\checkmark\ \frac{1}{4003\,936}$	$\frac{2500}{90\,001\,000\,533}$			
i	1	2	3	4	5	6	7	8	9

for

$$\mathbf{B}^{-1}\mathbf{N} = \begin{bmatrix} 2 & \frac{1}{5} & -5 & -\frac{9}{10} & 1 & \frac{23}{1000} \\ -41 & -\frac{6}{5} & 12 & \frac{1}{5} & -\frac{14}{5} & -\frac{1}{500} \\ 165000 & 2600 & 9600 & 125 & -100 & -300 \end{bmatrix}$$

$\mathbf{c}'_B\mathbf{B}^{-1}\mathbf{N} = 0$

$$\mathbf{h}'_N = \mathbf{c}'_N - \mathbf{c}'_B\mathbf{B}^{-1}\mathbf{N} = \begin{bmatrix} 36 & -\frac{3}{5} & 20 & \frac{1}{4} & -\frac{1}{20} & -\frac{1}{20} \end{bmatrix}$$

$$g_2 = \frac{\left(-\frac{3}{5}\right)^2}{1+\frac{169\,000\,037}{25}} = \frac{3}{56\,333\,354} = 5.325\,4 \times 10^{-8}$$

$$g_5 = \frac{\left(-\frac{1}{20}\right)^2}{1+\frac{250\,221}{25}} = \frac{1}{4003\,936} = 2.497\,5 \times 10^{-7}$$

$$g_6 = \frac{\left(-\frac{1}{20}\right)^2}{1+\frac{90\,000\,000\,533}{1000\,000}} = \frac{2500}{90\,001\,000\,533} = 2.777\,7 \times 10^{-8}$$

$\mathsf{B} = \{5, 8, 9\}$: entering index 2, leaving index 5

$\mathbf{c}$	36	$-\frac{3}{5}$	20	$\frac{1}{4}$	$-\frac{1}{20}$	$-\frac{1}{20}$	0	0	0
	2	$\blacktriangleleft\, \frac{1}{5}$	-5	$-\frac{9}{10}$	1	$\frac{23}{1000}$	1	0	0
	$-\frac{177}{5}$	$-\frac{16}{25}$	-2	$-\frac{58}{25}$	0	$\frac{39}{625}$	$\frac{14}{5}$	1	0
	$165\,200$	2620	9100	35	0	$-\frac{2977}{10}$	100	0	1
$\mathbf{h}$	$\frac{361}{10}$	$-\frac{59}{100}$	$\frac{79}{4}$	$\frac{41}{200}$		$-\frac{977}{20\,000}$	$\frac{1}{20}$		
$\mathbf{g}$		$\checkmark\ \frac{3481}{68\,644\,014\,496}$				$\frac{954\,529}{35\,450\,517\,769\,104}$			
i	1	2	3	4	5	6	7	8	9

for

$$\mathbf{B}^{-1}\mathbf{N} = \begin{bmatrix} 1 & 0 & 0 \\ -\frac{14}{5} & 1 & 0 \\ -100 & 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 2 & \frac{1}{5} & -5 & -\frac{9}{10} & \frac{23}{1000} & 1 \\ -41 & -\frac{6}{5} & 12 & \frac{1}{5} & -\frac{1}{500} & 0 \\ 165000 & 2600 & 9600 & 125 & -300 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & \frac{1}{5} & -5 & -\frac{9}{10} & \frac{23}{1000} & 1 \\ -\frac{177}{5} & -\frac{16}{25} & -2 & -\frac{58}{25} & \frac{39}{625} & \frac{14}{5} \\ 165\,200 & 2620 & 9100 & 35 & -\frac{2977}{10} & 100 \end{bmatrix}$$

$$\mathbf{c}_B'\mathbf{B}^{-1}\mathbf{N} = \begin{bmatrix} -\frac{1}{20} & 0 & 0 \end{bmatrix}\mathbf{B}^{-1}\mathbf{N} = \begin{bmatrix} -\frac{1}{10} & -\frac{1}{100} & \frac{1}{4} & \frac{9}{200} & -\frac{23}{20\,000} & -\frac{1}{20} \end{bmatrix}$$

$$\mathbf{h}_N' = \mathbf{c}_N' - \mathbf{c}_B'\mathbf{B}^{-1}\mathbf{N} = \begin{bmatrix} 36 & -\frac{3}{5} & 20 & \frac{1}{4} & -\frac{1}{20} & 0 \end{bmatrix} - \mathbf{c}_B'\mathbf{B}^{-1}\mathbf{N}$$

$$= \begin{bmatrix} \frac{361}{10} & -\frac{59}{100} & \frac{79}{4} & \frac{41}{200} & -\frac{977}{20\,000} & \frac{1}{20} \end{bmatrix}$$

$$g_2 = \frac{\left(-\frac{59}{100}\right)^2}{1+\frac{4290\,250\,281}{625}} = \frac{3481}{68\,644\,014\,496} = 5.071\,1\times 10^{-8}$$

$$g_6 = \frac{\left(-\frac{977}{20\,000}\right)^2}{1+\frac{2215\,632\,360\,569}{25\,000\,000}} = \frac{954\,529}{35\,450\,517\,769\,104} = 2.692\,6\times 10^{-8}$$

$\mathsf{B} = \{2, 8, 9\}$: entering index 4, leaving index 9

$\mathbf{c}$	36	$-\frac{3}{5}$	20	$\frac{1}{4}$	$-\frac{1}{20}$	$-\frac{1}{20}$	0	0	0
	10	1	-25	$-\frac{9}{2}$	5	$\frac{23}{200}$	5	0	0
	-29	0	-18	$-\frac{26}{5}$	$\frac{16}{5}$	$\frac{17}{125}$	6	1	0
	$139\,000$	0	$74\,600$	$\blacktriangleleft\ 11\,825$	$-13\,100$	-599	$-13\,000$	0	1
$\mathbf{h}$	42		5	$-\frac{49}{20}$	$\frac{59}{20}$	$\frac{19}{1000}$	3		
$\mathbf{g}$				$\checkmark$					
i	1	2	3	4	5	6	7	8	9

for

$$\mathbf{B}^{-1}\mathbf{N} = \begin{bmatrix} \frac{1}{5} & 0 & 0 \\ -\frac{6}{5} & 1 & 0 \\ 2600 & 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 2 & -5 & -\frac{9}{10} & 1 & \frac{23}{1000} & 1 \\ -41 & 12 & \frac{1}{5} & -\frac{14}{5} & -\frac{1}{500} & 0 \\ 165000 & 9600 & 125 & -100 & -300 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 10 & -25 & -\frac{9}{2} & 5 & \frac{23}{200} & 5 \\ -29 & -18 & -\frac{26}{5} & \frac{16}{5} & \frac{17}{125} & 6 \\ 139\,000 & 74\,600 & 11\,825 & -13\,100 & -599 & -13\,000 \end{bmatrix}$$

$$\mathbf{c}_B'\mathbf{B}^{-1}\mathbf{N} = \begin{bmatrix} -\frac{3}{5} & 0 & 0 \end{bmatrix}\mathbf{B}^{-1}\mathbf{N} = \begin{bmatrix} -6 & 15 & \frac{27}{10} & -3 & -\frac{69}{1000} & -3 \end{bmatrix}$$

$$\mathbf{h}_N' = \mathbf{c}_N' - \mathbf{c}_B'\mathbf{B}^{-1}\mathbf{N} = \begin{bmatrix} 36 & 20 & \frac{1}{4} & -\frac{1}{20} & -\frac{1}{20} & 0 \end{bmatrix} - \mathbf{c}_B'\mathbf{B}^{-1}\mathbf{N}$$

$$= \begin{bmatrix} 42 & 5 & -\frac{49}{20} & \frac{59}{20} & \frac{19}{1000} & 3 \end{bmatrix}$$

$g_4 = \checkmark$, unique choice

$\mathsf{B} = \{2, 4, 8\}$: entering index 6, leaving index ◀

$\mathbf{c}$	36	$-\frac{3}{5}$	20	$\frac{1}{4}$	$-\frac{1}{20}$	$-\frac{1}{20}$	0	0	0
	$\frac{29\,750}{473}$	1	$\frac{1603}{473}$	0	$\frac{7}{473}$	$-\frac{2137}{18\,920}$	$\frac{25}{473}$	0	$\frac{9}{23\,650}$
	$\frac{5560}{473}$	0	$\frac{2984}{473}$	1	$-\frac{524}{473}$	$-\frac{599}{11\,825}$	$-\frac{520}{473}$	0	$\frac{1}{11\,825}$
	$\frac{15\,195}{473}$	0	$\frac{35\,014}{2365}$	0	$-\frac{6056}{2365}$	$-\frac{7533}{59\,125}$	$\frac{134}{473}$	1	$\frac{26}{59\,125}$
$\mathbf{h}$	$\frac{33\,488}{473}$		$\frac{48\,379}{2365}$		$\frac{2231}{9460}$	$-\frac{9943}{94\,600}$	$\frac{145}{473}$		$\frac{49}{236\,500}$
$\mathbf{g}$						✓			
i	1	2	3	4	5	6	7	8	9

for

$$\mathbf{B}^{-1}\mathbf{N} = \begin{bmatrix} \frac{1}{5} & -\frac{9}{10} & 0 \\ -\frac{6}{5} & \frac{1}{5} & 1 \\ 2600 & 125 & 0 \end{bmatrix}^{-1} \begin{bmatrix} 2 & -5 & 1 & \frac{23}{1000} & 1 & 0 \\ -41 & 12 & -\frac{14}{5} & -\frac{1}{500} & 0 & 0 \\ 165000 & 9600 & -100 & -300 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{29\,750}{473} & \frac{1603}{473} & \frac{7}{473} & -\frac{2137}{18\,920} & \frac{25}{473} & \frac{9}{23\,650} \\ \frac{5560}{473} & \frac{2984}{473} & -\frac{524}{473} & -\frac{599}{11\,825} & -\frac{520}{473} & \frac{1}{11\,825} \\ \frac{15\,195}{473} & \frac{35\,014}{2365} & -\frac{6056}{2365} & -\frac{7533}{59\,125} & \frac{134}{473} & \frac{26}{59\,125} \end{bmatrix}$$

$$\mathbf{c}'_B\mathbf{B}^{-1}\mathbf{N} = \begin{bmatrix} -\frac{3}{5} & \frac{1}{4} & 0 \end{bmatrix} \mathbf{B}^{-1}\mathbf{N} = \begin{bmatrix} -\frac{16\,460}{473} & -\frac{1079}{2365} & -\frac{676}{2365} & \frac{5213}{94\,600} & -\frac{145}{473} & -\frac{49}{236\,500} \end{bmatrix}$$

$$\mathbf{h}'_N = \mathbf{c}'_N - \mathbf{c}'_B\mathbf{B}^{-1}\mathbf{N} = \begin{bmatrix} 36 & 20 & -\frac{1}{20} & -\frac{1}{20} & 0 & 0 \end{bmatrix} - \mathbf{c}'_B\mathbf{B}^{-1}\mathbf{N}$$

$$= \begin{bmatrix} \frac{33\,488}{473} & \frac{48\,379}{2365} & \frac{2231}{9460} & -\frac{9943}{94\,600} & \frac{145}{473} & \frac{49}{236\,500} \end{bmatrix}$$

$g_6 = $ ✓, unique choice

The entering index 6, since $\mathbf{B}^{-1}\boldsymbol{a}_6 < 0$, the LP is unbounded

1.1.2 Hall and McKinnon (2004)

Hall and McKinnon (2004) Table 2: $\mathsf{B} = \{5, 6, 7\}$: entering index 1, leaving index 5

$$
\begin{array}{c|ccccccc}
\mathbf{c} & -1 & -\frac{7}{4} & \frac{49}{4} & \frac{1}{2} & 0 & 0 & 0 \\
\hline
 & \blacktriangleleft \frac{2}{5} & \frac{1}{5} & -\frac{7}{5} & -\frac{1}{5} & 1 & 0 & 0 \\
 & -\frac{39}{5} & -\frac{7}{5} & \frac{39}{5} & \frac{2}{5} & 0 & 1 & 0 \\
 & 0 & -20 & 156 & 8 & 0 & 0 & 1 \\
\hline
\mathbf{h} & -1 & -\frac{7}{4} & \frac{49}{4} & \frac{1}{2} & & & \\
\mathbf{g} & \checkmark \frac{1}{62} & \frac{49}{6448} & & & & & \\
i & 1 & 2 & 3 & 4 & 5 & 6 & 7
\end{array}
$$

for

$$
\mathbf{B}^{-1}\mathbf{N} = \begin{bmatrix} \frac{2}{5} & \frac{1}{5} & -\frac{7}{5} & -\frac{1}{5} \\ -\frac{39}{5} & -\frac{7}{5} & \frac{39}{5} & \frac{2}{5} \\ 0 & -20 & 156 & 8 \end{bmatrix}
$$

$\mathbf{c}_B'\mathbf{B}^{-1}\mathbf{N} = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix} \mathbf{B}^{-1}\mathbf{N} = 0$

$\mathbf{h}_N' = \mathbf{c}_N' - \mathbf{c}_B'\mathbf{B}^{-1}\mathbf{N} = \begin{bmatrix} -1 & -\frac{7}{4} & \frac{49}{4} & \frac{1}{2} \end{bmatrix}$

$g_1 = \frac{(-1)^2}{1+61} = \frac{1}{62} = 1.6129 \times 10^{-2}$

$g_2 = \frac{\left(-\frac{7}{4}\right)^2}{1+402} = \frac{49}{6448} = 7.5993 \times 10^{-3}$

$\mathsf{B} = \{1, 6, 7\}$: entering index 2, leaving index 1

$$
\begin{array}{c|ccccccc}
\mathbf{c} & -1 & -\frac{7}{4} & \frac{49}{4} & \frac{1}{2} & 0 & 0 & 0 \\
\hline
 & 1 & \blacktriangleleft \frac{1}{2} & -\frac{7}{2} & -\frac{1}{2} & \frac{5}{2} & 0 & 0 \\
 & 0 & \frac{5}{2} & -\frac{39}{2} & -\frac{7}{2} & \frac{39}{2} & 1 & 0 \\
 & 0 & -20 & 156 & 8 & 0 & 0 & 1 \\
\hline
\mathbf{h} & & -\frac{5}{4} & \frac{35}{4} & 0 & \frac{5}{2} & & \\
\mathbf{g} & & \checkmark & & & & & \\
i & 1 & 2 & 3 & 4 & 5 & 6 & 7
\end{array}
$$

for

$$
\mathbf{B}^{-1}\mathbf{N} = \begin{bmatrix} \frac{2}{5} & 0 & 0 \\ -\frac{39}{5} & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} \frac{1}{5} & -\frac{7}{5} & -\frac{1}{5} & 1 \\ -\frac{7}{5} & \frac{39}{5} & \frac{2}{5} & 0 \\ -20 & 156 & 8 & 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & -\frac{7}{2} & -\frac{1}{2} & \frac{5}{2} \\ \frac{5}{2} & -\frac{39}{2} & -\frac{7}{2} & \frac{39}{2} \\ -20 & 156 & 8 & 0 \end{bmatrix}
$$

$\mathbf{c}_B'\mathbf{B}^{-1}\mathbf{N} = \begin{bmatrix} -1 & 0 & 0 \end{bmatrix} \mathbf{B}^{-1}\mathbf{N} = \begin{bmatrix} -\frac{1}{2} & \frac{7}{2} & \frac{1}{2} & -\frac{5}{2} \end{bmatrix}$

$\mathbf{h}_N' = \mathbf{c}_N' - \mathbf{c}_B'\mathbf{B}^{-1}\mathbf{N} = \begin{bmatrix} -\frac{7}{4} & \frac{49}{4} & \frac{1}{2} & 0 \end{bmatrix} - \mathbf{c}_B'\mathbf{B}^{-1}\mathbf{N} = \begin{bmatrix} -\frac{5}{4} & \frac{35}{4} & 0 & \frac{5}{2} \end{bmatrix}$

$g_2 = \checkmark$, unique choice

$\mathsf{B} = \{2, 6, 7\}$: entering index 4, leaving index ◀

$\mathbf{c}$	-1	$-\frac{7}{4}$	$\frac{49}{4}$	$\frac{1}{2}$	0	0	0
	2	1	-7	-1	5	0	0
	-5	0	-2	-1	7	1	0
	40	0	16	-12	100	0	1
$\mathbf{h}$	$\frac{5}{2}$		0	$-\frac{5}{4}$	$\frac{35}{4}$		
$\mathbf{g}$				✓			
i	1	2	3	4	5	6	7

for

$$\mathbf{B}^{-1}\mathbf{N} = \begin{bmatrix} \frac{1}{5} & 0 & 0 \\ -\frac{7}{5} & 1 & 0 \\ -20 & 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} \frac{2}{5} & -\frac{7}{5} & -\frac{1}{5} & 1 \\ -\frac{39}{5} & \frac{39}{5} & \frac{2}{5} & 0 \\ 0 & 156 & 8 & 0 \end{bmatrix} = \begin{bmatrix} 2 & -7 & -1 & 5 \\ -5 & -2 & -1 & 7 \\ 40 & 16 & -12 & 100 \end{bmatrix}$$

$$\mathbf{c}_B'\mathbf{B}^{-1}\mathbf{N} = \begin{bmatrix} -\frac{7}{4} & 0 & 0 \end{bmatrix} \mathbf{B}^{-1}\mathbf{N} = \begin{bmatrix} -\frac{7}{2} & \frac{49}{4} & \frac{7}{4} & -\frac{35}{4} \end{bmatrix}$$

$$\mathbf{h}_N' = \mathbf{c}_N' - \mathbf{c}_B'\mathbf{B}^{-1}\mathbf{N} = \begin{bmatrix} -1 & \frac{49}{4} & \frac{1}{2} & 0 \end{bmatrix} - \mathbf{c}_B'\mathbf{B}^{-1}\mathbf{N} = \begin{bmatrix} \frac{5}{2} & 0 & -\frac{5}{4} & \frac{35}{4} \end{bmatrix}$$

$g_4 = \checkmark$, unique choice

The entering index 4, since $\mathbf{B}^{-1}\boldsymbol{a}_4 < 0$, the LP is unbounded

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