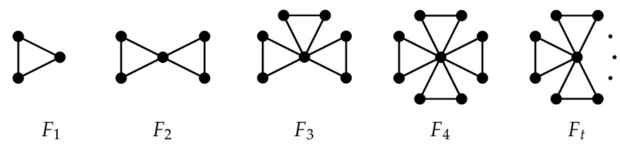
1 Friendship Graphs

A **friendship graph** F_n is a planar, undirected graph with $2_n + 1$ vertices and $3n$ edges. The friendship graph can be constructed by joining n copies of the cycle C_3 with a common vertex, which becomes the **universal vertex** for the graph.



Theorem 1.1 (Friendship Theorem)**.** *Let G be a finite graph such that every pair of vertices have exactly one common neighbor. Then G has as universal vertex, i.e., a vertex adjacent to every other vertex.*

Proof. Suppose to the contrary that each pair of vertices in G has exactly one common neighbor but no universal vertex. Following from the common neighbor property as stated in 1.1, we observe the following:

Observation 1.2. *G has no 4-cycles, as any pair of vertices on a C_4 would have two common neighbors.*

From this observation we make the following claim:

Claim 1.3. *G is a regular graph.*

Begin with non-adjacent vertices $u, v \in G$ via no universal vertex. There must be exactly one common vertex between u and v, i.e., this common vertex must be contained in $N(u) = \{w_1, w_2, \ldots, w_k\}$. Say without loss of generality this common vertex is w_2. There must also be exactly one common vertex between u and w_2. Say without loss of generality this common vertex is w_1. We take notice of the following:

1. v is not adjacent to any w_i (where $i \neq 2$)

2. No pair of w_i, w_j have a common neighbor other than u

3. v, w_i have a common neighbor (where $i \geq 2$)

From these observations we conclude $d(u) \geq d(u)$. We can argue similarly that $d(v) \geq d(u)$. Combining both conclusions we have $d(u) = d(v) = k$. From here we make the following observation:

Observation 1.4. *Every vertex x (except w_2) is **not** adjacent to u or v. Hence, $d(x) = k$.*

Since w_2 is not universal (as stated in the aim of the proof), there is some vertex y to which w_2 is not adjacent. But from our observation 1.4, $d(y) = k$. Thus, $d(w_2) = k$, which proves Claim 1.3.

The regularity of G allows the conclusion $\sum_1^k d(w_i) = k^2$. Notice that this sum counts every vertex of G, as every vertex in G has a common neighbor with u, so this common neighbor must exist as w_i (and is not repeated). Further, u is counted k times for each of its k edges. Thus we can deduce $|V(G)| = n = k^2 - k + 1$.

We can then make the following observation:

Observation 1.5. *If $k = 1$, then $n = 1$. If $k = 2$, then $n = 3$. Thus, $k > 2$.*

Let us consider the adjacency matrix A of G and its square, A^2

$$A = \begin{bmatrix} 0 & a_{01} & \dots & a_{0j} \\ a_{10} & 0 & \dots & \dots \\ \vdots & \vdots & \ddots & \dots \\ a_{i0} & \dots & \dots & 0 \end{bmatrix}$$

$$a_{ij} = \begin{cases} 1 & \text{if } v_i v_j \text{ is an edge} \\ 0 & \text{otherwise} \end{cases}$$

We can leverage A toward a contradiction by considering the following property of A^2:

Theorem 1.6. *A^2 counts the number of walks of length 2 in G.*

$$A^2 = \begin{bmatrix} k & 1 & \dots & 1 & 1 \\ 1 & k & \dots & \dots & 1 \\ \vdots & \vdots & \ddots & \dots & \dots \\ 1 & \dots & \dots & k & 1 \\ 1 & 1 & \dots & 1 & k \end{bmatrix}$$

$$a_{ij}^2 = \begin{cases} k & \text{if } v_i = v_j \\ 1 & \text{otherwise} \end{cases}$$

When $v_i \neq v_j$ we know each $v_i v_j$ pair has exactly one walk of length 2. And when $v_i = v_j$, the number of walks is simply the degree of the vertex, as a length-two walk simply travels along one out-edge return on the same edge for each of its k out-edges.

Written another way, $A^2 = J + (k-1)I$, where J is the all-ones matrix and I is the identity matrix. Computing the eigenvalues of A^2 gives:

$$\lambda = \begin{pmatrix} k - 1 + n = k^2 \\ k - 1 \\ \vdots \end{pmatrix} \begin{array}{l} \leftarrow \text{multiplicity of } 1 \\ \leftarrow \text{multiplicity of } n-1 \\ \end{array}$$

Note that $k - 1 + n$ is indeed k^2 as we concluded from 1.4. We can simply calculate the element-wise square root of A^2's eigenvalues of to determine the eigenvalues of A:

$$\lambda = \begin{pmatrix} \sqrt{k^2} = k \\ \pm\sqrt{k-1} \\ \vdots \end{pmatrix} \begin{array}{l} \leftarrow \text{multiplicity of } 1 \\ \leftarrow \text{multiplicity of } n-1 \\ \end{array}$$

We unpack the $\pm\sqrt{k-1}$ and restate the values and multiplicities the following way:

$$\lambda = \begin{pmatrix} \sqrt{k^2} = k \\ +\sqrt{k-1} \\ \vdots \\ -\sqrt{k-1} \\ \vdots \end{pmatrix} \begin{array}{l} \leftarrow \text{multiplicity of } 1 \\ \leftarrow \text{multiplicity of } r \\ \\ \leftarrow \text{multiplicity of } s \\ \end{array}$$

Summing the eigenvalues of A gives $k + r\sqrt{k-1} - s\sqrt{k-1}$. This value is equivalent to summing the diagonal of A, which is a property of eigenvalues we can now exploit. Recall A's diagonal is all 0's, which gives:

$$k + r\sqrt{k-1} - s\sqrt{k-1} = 0.$$

Refactoring above and multiplying k throughout results in the following:

$$(r - s)(k - 1) = k^2.$$

This implies that $(k - 1)$ divides k^2, which only happens if $k \leq 2$ which contradicts Observation 1.5. $\square$