Saha ionization equation

Sunday, April 4, 2021 4:46 PM

$$F = -T\ln\left[\sum_n \exp(-E_n/T)\right] = -T\ln Z$$

Free energy

$$Z = \sum_n \exp(-E_n/T)$$

$$Z = \sum \exp(-E_n/T)$$

energy of particle

$$= \frac{1}{N!}\left[\sum \exp(-\varepsilon_n/T)\right]^N$$

$$\sum \exp(-\varepsilon_n/T) = \frac{V}{(2\pi\hbar)^3}\int d\vec{p}\,\exp\left(-\frac{p^2}{2mT}\right) = \frac{V(2\pi mT)^{3/2}}{(2\pi\hbar)^3}$$

$$= \frac{1}{N!}\left[\frac{V(2\pi mT)^{3/2}}{(2\pi\hbar)^3}\, g\cdot\exp(-\varepsilon_0/T)\right]^N$$

stat factor

potential energy

$$F = -\sum_s T_s \ln Z_s$$

$$= -\sum_s T_s\left[-\ln(N!) + N\ln(C)\right]$$

species

For simplicity $C = \frac{V(2\pi mT)^{3/2}}{(2\pi\hbar)^3}\, g\exp(-\varepsilon_0 T)$

Stirling approximation:

$$\ln(N!) = N\ln N - N + O(\ln N)$$

$$F = -\sum_s T_s\{-N_s\ln N_s + N_s + N_s\ln(C)\}$$

$$= \sum_s -T_s N_s - T_s N_s \ln\left[\frac{C}{N_s}\right]$$

For thermal Equilibrium $\delta F = 0$

$$\Downarrow$$

$$\frac{\partial F}{\partial N_s} = 0$$

$$= -T_s - T_s\ln\left(\frac{C}{N_s}\right) - T_s N_s \frac{-C}{N_s^2}\frac{N_s}{C}$$

$$= -T_s\ln\left(\frac{C}{N_s}\right)$$

$$\frac{\partial F}{\partial N_i} + \frac{\partial F}{\partial N_e} - \frac{\partial F}{\partial N_a} = 0, \quad T_i = T_e = T_a$$

ion electron atom

$$0 = T\ln\left(\frac{C_i C_e N_a}{N_i N_e C_a}\right)$$

exp both sides, $n_s = \frac{N_s}{V}$

$$1 = \left(\frac{m_i m_e}{m_a}\right)^{3/2}\frac{g_i g_e}{g_a}\exp\left(-\frac{\varepsilon_i + \varepsilon_e - \varepsilon_a}{T}\right)\frac{n_a}{n_i n_e}\frac{(2\pi T)^{3/2}}{(2\pi\hbar)^3}$$

$$\frac{n_e n_i}{n_a} = \left(\frac{m_i m_e}{m_a}\right)^{3/2}\frac{g_i g_e}{g_a}\exp\left(-\frac{\varepsilon_i + \varepsilon_e - \varepsilon_a}{T}\right)\frac{(2\pi T)^{3/2}}{(2\pi\hbar)^3}$$

$I = \varepsilon_i + \varepsilon_e - \varepsilon_a$ ionization energy

$$\boxed{\frac{n_e n_i}{n_a} = \left(\frac{m_i m_e}{m_a}\right)^{3/2}\frac{g_i g_e}{g_a}\exp(-I/T)\frac{(2\pi T)^{3/2}}{(2\pi\hbar)^3}}$$

density mass stat factor ionization energy temperature

Saha Equation