Bases in standard categories

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Contents

1 Introduction

In this talk, we construct certain combinatorial bases of the endomorphism algebra of a tilting object in a standard category over a field. The talk closely follows [BT22, Section 3]. Consequently, for the most part we employ the notational conventions of [BT22]. The quoted article [BT22] generalises the construction of [AST18] from the context of quantum enveloping algebras to a broader categorical framework. More precisely, the category of modules over a quantum enveloping algebra is replaced by an arbitrary so-called standard category over a field.

For completeness, we recall the definition of a standard category over a field.

Definition 1.1
Let K be a field. Let Λ be a partially ordered set. A category $\mathscr{C}$ is called a standard category over K with respect to Λ if it satisfies the following four assumptions:

1. *(Basic properties) The category $\mathscr{C}$ is an essentially small, locally finite, K-linear, abelian category. By locally finite we mean on the one hand that all objects in $\mathscr{C}$ have finite length. On the other hand, the hom-space $\mathrm{Hom}_{\mathscr{C}}(X, Y)$ is required to be a finite-dimensional K-vector space for all objects $X, Y \in \mathscr{C}$.*

2. *(Simple objects) There exists a set $\{L(\lambda)\}_{\lambda\in\Lambda}$ of simple objects in $\mathscr{C}$ such that the function*
$$\begin{aligned} \Lambda &\longrightarrow \{\text{isomorphism classes of simple objects in } \mathscr{C}\} \\ \lambda &\longmapsto \text{isomorphism class of } L(\lambda) \end{aligned}$$
is a bijection.

3. *(Ext-vanishing condition) Each $L(\lambda)$ has a costandard object $\nabla(\lambda)$ and a standard object $\Delta(\lambda)$ such that the following Ext-vanishing condition holds for every $i \in \{0, 1, 2\}$ and all $\lambda, \mu \in \Lambda$:*
$$\mathrm{Ext}^i_{\mathscr{C}}(\Delta(\lambda), \nabla(\mu)) \cong \begin{cases} K & \text{if } i = 0 \text{ and } \lambda = \mu \\ 0 & \text{else} \end{cases}$$

4. *(Indecomposable tilting objects)*
   (a) *For any $\lambda \in \Lambda$, there is an indecomposable tilting object $T(\lambda)$ such that:*
      i. *$T(\lambda)$ is of highest weight λ,*
      ii. *There is a monomorphism $\Delta(\lambda) \hookrightarrow T(\lambda)$,*
      iii. *There is an epimorphism $T(\lambda) \twoheadrightarrow \nabla(\lambda)$.*

   (b) *The function*
$$\begin{aligned} \Lambda &\longrightarrow \{\text{isomorphism classes of indecomposable tilting objects in } \mathscr{C}\} \\ \lambda &\longmapsto \text{isomorphism class of } T(\lambda) \end{aligned}$$
   is a bijection.

Remark 1.2. (Terminology)
We will mostly call a 'standard category over K with respect to Λ' simply a 'standard category over K'.

Recall Proposition 3.2.5 from the previous talk by Kian.

Proposition 1.3. (Donkin's Ext-criteria)
Let $\mathscr{C}$ be a standard category over K. Then the following two equivalences hold:

(i) $M \in \mathscr{C}^{\Delta}$ if and only if $\operatorname{Ext}^1_{\mathscr{C}}(M, \nabla(\lambda) = 0$ for all $\lambda \in \Lambda$.

(ii) $N \in \mathscr{C}^{\nabla}$ if and only if $\operatorname{Ext}^1_{\mathscr{C}}(\Delta(\lambda), N) = 0$ for all $\lambda \in \Lambda$.

Corollary 1.4.
Let $\mathscr{C}$ be a standard category over K. Let $M \in \mathscr{C}^{\Delta}$ and $N \in \mathscr{C}^{\nabla}$. Then $\operatorname{Ext}^1_{\mathscr{C}}(M, N) = 0$.

Proof.
We proceed by induction on the length of a costandard filtration of N. The base case follows immediately from Proposition 1.3. For the induction step, let

$$0 = N_0 \subsetneq \cdots \subsetneq N_{n-1} \subsetneq N_n = N \tag{1}$$

be a costandard filtration of $N \in \mathscr{C}$. Let $\lambda \in \Lambda$ such that $N_n/N_{n-1} \cong \nabla(\lambda)$. Applying the functor $\operatorname{Hom}_{\mathscr{C}}(M, -)$ to the short exact sequence

$$0 \to N_{n-1} \to N \to \nabla(\lambda) \to 0 \tag{2}$$

yields an exact sequence

$$0 \to \cdots \to \operatorname{Ext}^1_{\mathscr{C}}(M, N_{n-1}) \to \operatorname{Ext}^1_{\mathscr{C}}(M, N) \to \operatorname{Ext}^1_{\mathscr{C}}(M, \nabla(\lambda)) \to \cdots . \tag{3}$$

By the induction hypothesis we know that $\operatorname{Ext}^1_{\mathscr{C}}(M, N_{n-1}) = 0$ holds. Moreover, by Proposition 1.3 we have $\operatorname{Ext}^1_{\mathscr{C}}(M, \nabla(\lambda)) = 0$. Since the sequence 3 is exact, it follows that $\operatorname{Ext}^1_{\mathscr{C}}(M, N) = 0$. By induction the claim follows. □

2 Tools: Lifts and extensions

For the remainder of this talk, we fix a field K and a standard category $\mathscr{C}$ over K. Additionally, we fix objects $M \in \mathscr{C}^{\Delta}$ and $N \in \mathscr{C}^{\nabla}$.

For every $\lambda \in \Lambda$, we choose an epimorphism $\pi^{\lambda} \colon T(\lambda) \twoheadrightarrow \nabla(\lambda)$ and a monomorphism $\iota^{\lambda} \colon \Delta(\lambda) \hookrightarrow T(\lambda)$. These exist by Definition 1.1.4. For every $\lambda \in \Lambda$, define $c^{\lambda} := \pi^{\lambda} \circ \iota^{\lambda}$.

Informally, we will construct bases of $\operatorname{Hom}_{\mathscr{C}}(M, N)$ by 'factorizing through the indecomposable tilting objects $\{T(\lambda)\}_{\lambda \in \Lambda}$ of $\mathscr{C}$.' Each factoring morphism will be realized either as a lift along π^{λ} or as an extension along ι^{λ}.

The following lemma will be the main tool in the subsequent construction of bases of $\operatorname{Hom}_{\mathscr{C}}(M, N)$.

Lemma 2.1. (Lifting lemma)
Let $\lambda \in \Lambda$.

(i) Any morphism $f \colon M \to \nabla(\lambda)$ lifts to a morphism $\tilde{f} \colon M \to T(\lambda)$ along $\pi^{\lambda} \colon T(\lambda) \twoheadrightarrow \nabla(\lambda)$. As a commutative diagram:

$$\begin{tikzcd} M \arrow[r, dashed, "\exists \tilde{f}"] \arrow[dr, "f"'] & T(\lambda) \arrow[d, two heads, "\pi^{\lambda}"] \\ & \nabla(\lambda) \end{tikzcd}$$

(ii) Dually, any morphism $g\colon \Delta(\lambda) \to N$ extends to a morphism $\widetilde{g}\colon T(\lambda) \to N$ along $\iota^\lambda\colon \Delta(\lambda) \hookrightarrow T(\lambda)$. As a commutative diagram:

$$\begin{tikzcd}
\Delta(\lambda) \arrow[d, hook, "\iota^\lambda"'] \arrow[dr, "g"] & \\
T(\lambda) \arrow[r, dashed, "\exists \widetilde{g}"'] & N
\end{tikzcd}$$

Proof.

We only prove the first part of the lemma; the second one is shown dually.

Consider the short exact sequence

$$0 \to \ker(\pi^\lambda) \to T(\lambda) \xrightarrow{\pi^\lambda} \nabla(\lambda) \to 0. \tag{4}$$

We claim that the object $\ker(\pi^\lambda) \in \mathscr{C}$ admits a costandard filtration. By Donkin's Ext-criteria 1.3 this amounts to proving that $\operatorname{Ext}^1_{\mathscr{C}}(\Delta(\mu), \ker(\pi^\lambda)) = 0$ for all $\mu \in \Lambda$. The subsequent proof of this claim was suggested by Paul Wedrich. Let $\mu \in \Lambda$. Applying the functor $\operatorname{Hom}_{\mathscr{C}}(\Delta(\mu), -)$ to the short exact sequence 4 yields an exact sequence

$$\operatorname{Hom}_{\mathscr{C}}(\Delta(\mu), T(\lambda)) \xrightarrow{\pi^\lambda_*} \operatorname{Hom}_{\mathscr{C}}(\Delta(\mu), \nabla(\lambda)) \to \operatorname{Ext}^1_{\mathscr{C}}(\Delta(\mu), \ker(\pi^\lambda)) \to \operatorname{Ext}^1_{\mathscr{C}}(\Delta(\mu), T(\lambda)). \tag{5}$$

Note that by Donkin's Ext-criteria 1.3, we have $\operatorname{Ext}^1_{\mathscr{C}}(\Delta(\mu), T(\lambda)) = 0$. We make a case distinction. If $\mu \neq \lambda$, then by Definition 1.1.3, we have $\operatorname{Hom}_{\mathscr{C}}(\Delta(\mu), \nabla(\lambda)) = 0$. Since sequence 5 is exact, we conclude $\operatorname{Ext}^1_{\mathscr{C}}(\Delta(\mu), \ker(\pi^\lambda)) = 0$. If on the other hand $\mu = \lambda$, then by Proposition 3.2.7.3 from Kian's talk, the morphism

$$\pi^\lambda_*\colon \operatorname{Hom}_{\mathscr{C}}(\Delta(\lambda), T(\lambda)) \to \operatorname{Hom}_{\mathscr{C}}(\Delta(\lambda), \nabla(\lambda)) \tag{6}$$

is surjective. By the exactness of sequence 5 and since $\operatorname{Ext}^1_{\mathscr{C}}(\Delta(\lambda), T(\lambda)) = 0$ holds, this implies $\operatorname{Ext}^1_{\mathscr{C}}(\Delta(\lambda), \ker(\pi^\lambda)) = 0$. Thus, the object $\ker(\pi^\lambda) \in \mathscr{C}$ admits a costandard filtration.

Since by assumption, $M \in \mathscr{C}$ admits a standard filtration, it follows from Corollary 1.4 that $\operatorname{Ext}^1_{\mathscr{C}}(M, \ker(\pi^\lambda)) = 0$. Consequently, by applying the left exact functor $\operatorname{Hom}_{\mathscr{C}}(M, -)$ to sequence 4 we obtain the exact sequence

$$0 \to \operatorname{Hom}_{\mathscr{C}}(M, \ker(\pi^\lambda)) \to \operatorname{Hom}_{\mathscr{C}}(M, T(\lambda)) \xrightarrow{\pi^\lambda_*} \operatorname{Hom}_{\mathscr{C}}(M, \nabla(\lambda)) \to \operatorname{Ext}^1_{\mathscr{C}}(M, \ker(\pi^\lambda)) = 0.$$

Thus, the map

$$\pi^\lambda_*\colon \operatorname{Hom}_{\mathscr{C}}(\Delta(M, T(\lambda)) \to \operatorname{Hom}_{\mathscr{C}}(M, \nabla(\lambda))$$

is surjective. □

Remark 2.2.

In general, the lifts and extensions of the lifting lemma 2.1 are not unique. In other words, post-composition with π^λ (pre-composition with ι^λ) is not injective in general. Moreover, the proof of the lifting lemma 2.1 gives a criterion for the injectivity of the map π^λ_*. Namely, the map π^λ_* is injective if and only if $\operatorname{Hom}_{\mathscr{C}}(M, \ker(\pi^\lambda)) = 0$.

3 Basic construction

By employing the lifting lemma 2.1, we will now construct bases of the finite-dimensional K-vector space $\mathrm{Hom}_{\mathscr{C}}(M, N)$.

Construction 3.1.
The construction consists of six steps.

(1) For every $\lambda \in \Lambda$, choose a basis F_M^λ of the finite-dimensional K-vector space $\mathrm{Hom}_{\mathscr{C}}(M, \nabla(\lambda))$.

(2) Next, for every $\lambda \in \Lambda$ and every basis element $f \in F_M^\lambda$ choose a lift $\widetilde{f} \in \mathrm{Hom}_{\mathscr{C}}(M, T(\lambda))$ along π^λ. Such a lift exists by the lifting lemma 2.1. Denote the set of lifts by

$$\widetilde{F}_M^\lambda := \{\widetilde{f} \mid f \in F_M^\lambda\}.$$

(3) Dually, for every $\lambda \in \Lambda$, choose a basis G_M^λ of the finite-dimensional K-vector space $\mathrm{Hom}_{\mathscr{C}}(\Delta(\lambda), N)$.

(4) For every $\lambda \in \Lambda$ and every basis element $g \in G_N^\lambda$ pick an extension $\widetilde{g} \in \mathrm{Hom}_{\mathscr{C}}(T(\lambda), N)$ along ι^λ. Denote the set of extensions by

$$\widetilde{G}_N^\lambda := \{\widetilde{g} \mid g \in G_N^\lambda\}.$$

(5) Next, for every $\lambda \in \Lambda$, define the subset

$$\widetilde{G}_N^\lambda \widetilde{F}_M^\lambda := \left\{\widetilde{g} \circ \widetilde{f} \mid \widetilde{g} \in \widetilde{G}_N^\lambda \text{ and } \widetilde{f} \in \widetilde{F}_M^\lambda\right\} \subseteq \mathrm{Hom}_{\mathscr{C}}(M, N).$$

Here, we employ the following convention: If one of the sets $\widetilde{G}_N^\lambda$ or $\widetilde{F}_M^\lambda$ is empty, then we let the set $\widetilde{G}_N^\lambda \widetilde{F}_M^\lambda$ be empty.

One may illustrate the set $\widetilde{G}_N^\lambda \widetilde{F}_M^\lambda$ by the subsequent commutative diagram:

$$\begin{tikzcd}
 & \Delta(\lambda) \arrow[d, hook, "\iota^\lambda"'] \arrow[dr, "g"] & \\
M \arrow[r, dashed, "\widetilde{f}"] \arrow[dr, "f"'] & T(\lambda) \arrow[r, dashed, "\widetilde{g}"'] \arrow[d, two heads, "\pi^\lambda"] & N \\
 & \nabla(\lambda) &
\end{tikzcd}$$

(6) Finally, define the following set

$$\widetilde{G}_N \widetilde{F}_M := \bigcup_{\lambda \in \Lambda} \widetilde{G}_N^\lambda \widetilde{F}_M^\lambda \subseteq \mathrm{Hom}_{\mathscr{C}}(M, N).$$

4 Basis theorem(s)

The next theorem is the main result of this section.

Theorem 4.1. (Basis theorem)
The set $\widetilde{G}_N\widetilde{F}_M$ is a basis of the K-vector space $\mathrm{Hom}_{\mathscr{C}}(M,N)$.

Before we prove the basis theorem, we first show a weaker version. To be able to formulate and prove the weaker version we need a few additional definitions and results.

Definition 4.2
Let $\phi \in \mathrm{Hom}_{\mathscr{C}}(M,N)$ be a morphism. Let $\mu \in \Lambda$. We call $\phi_\mu := [\mathrm{im}(\phi)\colon L(\mu)] \in \mathbb{N}$ the $\underline{\mu\text{-multiplicity}}$ of ϕ.

Lemma 4.3. (Subadditivity of the μ-multiplicity)
Let $\phi, \psi \in \mathrm{Hom}_{\mathscr{C}}(M,N)$ and $\mu \in \Lambda$. Then we have $(\phi + \psi)_\mu \leqslant \phi_\mu + \psi_\mu$.

The proof of Lemma 4.3 employs a corollary of the following proposition.

Proposition 4.4.
Let $\lambda \in \Lambda$. Let

$$0 \to X \to Z \to Y \to 0$$

be a short exact sequence in $\mathscr{C}$. Then the following equality holds

$$[Z\colon L(\lambda)] = [X\colon L(\lambda)] + [Y\colon L(\lambda)].$$

Proof.
The proposition immediately follows from the proof of [P70, Corollary 4.5.2].

More specifically, let

$$0 = Z_0 \subsetneq \cdots \subsetneq Z_i = X \subsetneq \cdots \subsetneq Z_n = Z$$

be a composition series of Z through X. Such as composition series exists by [P70, Proposition 4.5.1]. Note that for all $k \in \{i, \ldots, n-1\}$ we have

$$(Z_{k+1}/X)/(Z_k/X) \cong Z_{k+1}/Z_k \tag{7}$$

by one of the isomorphism theorems. In particular, the object $(Z_{k+1}/X)/(Z_k/X)$ is simple for all $k \in \{i, \ldots, n-1\}$. Thus, $0 = Z_i/X \subsetneq \cdots \subsetneq Z_n/X \cong Y$ is a composition series of Y. Now, equation 7 immediately yields $[Z\colon L(\lambda)] = [X\colon L(\lambda)] + [Y\colon L(\lambda)]$.

□

Corollary 4.5.
Let $\lambda \in \Lambda$. Let $X, Y \in \mathscr{C}$ be objects. Then the following hold:

(i) If there exists a monomorphism $X \hookrightarrow Y$, then we have $[Y\colon L(\lambda)] \geqslant [X\colon L(\lambda)]$.

(ii) If there exists an epimorphism $Y \twoheadrightarrow X$, then we have $[Y\colon L(\lambda)] \geqslant [X\colon L(\lambda)]$.

(iii) We have $[X \oplus Y\colon L(\lambda)] = [X\colon L(\lambda)] + [Y\colon L(\lambda)]$.

Proof of Lemma 4.3.
Let $\phi, \psi \in \operatorname{Hom}_{\mathscr{C}}(M, N)$ and $\mu \in \Lambda$. Firstly, note that $\operatorname{im}(\phi + \psi) \subseteq \operatorname{im}(\phi) \cup \operatorname{im}(\psi)$ by [P70, Lemma 4.3.6.]. Here, $\operatorname{im}(\phi) \cup \operatorname{im}(\psi)$ denotes the join in the preorder of subobjects of N. The join exists in any abelian category. Additionally, the two canonical morphisms

$$\operatorname{im}(\phi) \to \operatorname{im}(\phi) \cup \operatorname{im}(\psi)$$
$$\operatorname{im}(\psi) \to \operatorname{im}(\phi) \cup \operatorname{im}(\psi)$$

induce an epimorphism

$$\operatorname{im}(\phi) \oplus \operatorname{im}(\psi) \twoheadrightarrow \operatorname{im}(\phi) \cup \operatorname{im}(\psi).$$

Thus, by Corollary 4.5 we have for all $\mu \in \Lambda$:

$$\begin{aligned}(\phi+\psi)_\mu \overset{\text{def}}{=} [\operatorname{im}(\phi+\psi) \colon L(\mu)] \leqslant [\operatorname{im}(\phi) \cup \operatorname{im}(\psi) \colon L(\mu)] &\leqslant [\operatorname{im}(\phi) \oplus \operatorname{im}(\psi) \colon L(\mu)] \\ &= [\operatorname{im}(\phi) \colon L(\mu)] + [\operatorname{im}(\psi) \colon L(\mu)] \\ &\overset{\text{def}}{=} \phi_\mu + \psi_\mu.\end{aligned}$$

□

Next, given $\lambda \in \Lambda$, define two sets

$$\operatorname{Hom}_{\mathscr{C}}(M, N)^{\leqslant \lambda} := \{\phi \in \operatorname{Hom}_{\mathscr{C}}(M, N) \mid \phi_\mu \neq 0 \text{ implies } \mu \leqslant \lambda\},$$
$$\operatorname{Hom}_{\mathscr{C}}(M, N)^{< \lambda} := \{\phi \in \operatorname{Hom}_{\mathscr{C}}(M, N) \mid \phi_\mu \neq 0 \text{ implies } \mu < \lambda\}.$$

Remark 4.6. (A filtration-like structure)
Let $\mu, \lambda \in \Lambda$. If $\mu \leqslant \lambda$ holds, then we have the subsequent inclusions:

$$\operatorname{Hom}_{\mathscr{C}}(M, N)^{\leqslant \mu} \subseteq \operatorname{Hom}_{\mathscr{C}}(M, N)^{\leqslant \lambda},$$
$$\operatorname{Hom}_{\mathscr{C}}(M, N)^{< \mu} \subseteq \operatorname{Hom}_{\mathscr{C}}(M, N)^{< \lambda}.$$

Proposition 4.7.
Let $\lambda \in \Lambda$.

(i) The set $\operatorname{Hom}_{\mathscr{C}}(M, N)^{\leqslant \lambda}$ is a K-vector subspace of $\operatorname{Hom}_{\mathscr{C}}(M, N)$.

(ii) Similarly, the set $\operatorname{Hom}_{\mathscr{C}}(M, N)^{< \lambda}$ is a K-vector subspace of $\operatorname{Hom}_{\mathscr{C}}(M, N)$.

Proof.
We only show statement (i). Statement (ii) is proved analogously.

Clearly, we have $0_{MN} \in \operatorname{Hom}_{\mathscr{C}}(M, N)^{\leqslant \lambda}$. Next, we convince ourselves that the set $\operatorname{Hom}_{\mathscr{C}}(M, N)^{\leqslant \lambda}$ is closed under scalar multiplication. To do so, let $k \in K$ and $\operatorname{Hom}_{\mathscr{C}}(M, N)^{\leqslant \lambda}$. Since the category $\mathscr{C}$ is K-linear, we have

$$\operatorname{im}(k \cdot \phi) \cong \begin{cases} \operatorname{im}(\phi) & \text{if } k \neq 0 \\ 0 & \text{else} \end{cases} \tag{8}$$

Hence, $k \cdot \phi \in \operatorname{Hom}_{\mathscr{C}}(M, N)^{\leqslant \lambda}$.

Next, we argue that $\operatorname{Hom}_{\mathscr{C}}(M, N)^{\leqslant \lambda}$ is closed under addition. Let $\phi, \psi \in \operatorname{Hom}_{\mathscr{C}}(M, N)^{\leqslant \lambda}$. Let $\mu \in \Lambda$ such that $\mu \nleqslant \lambda$. By the subadditivity of the μ-multiplicity (i.e. Lemma 4.3), we then have:

$$0 \leqslant (\phi + \psi)_\mu \leqslant \phi_\mu + \psi_\mu = 0 + 0 = 0. \tag{9}$$

□

Lemma 4.8.
Let $\lambda \in \Lambda$. If $\phi, \psi \in \mathrm{Hom}_{\mathscr{C}}(M, N)$ and $\phi_\lambda = 0$, then $(\phi + \psi)_\lambda = \psi_\lambda$.

Proof.
Consider the map $q := \mathrm{coker}(\phi)\colon N \to N/\mathrm{im}(\phi)$. Set $\overline{\phi} = q \circ \phi$ and $\overline{\phi + \psi} = q \circ (\phi + \psi)$. Since in an abelian category composition is bilinear, we have $\overline{\phi + \psi} = \overline{\psi}$. Moreover, we claim that $(\phi + \psi)_\lambda = (\overline{\phi + \psi})_\lambda$ and $\psi_\lambda = \overline{\psi}_\lambda$. We only show $\psi_\lambda = \overline{\psi}_\lambda$. The other equality is proven analogously. By Corollary 4.5 we know that

$$0 \leqslant [\mathrm{im}(\psi) \cap \mathrm{im}(\phi)\colon L(\lambda)] \leqslant [\mathrm{im}(\phi)\colon L(\lambda)] \overset{\text{def}}{=} \phi_\lambda = 0. \tag{10}$$

Consider the following short exact sequence

$$0 \to \mathrm{im}(\phi) \cap \mathrm{im}(\psi) \xrightarrow{\text{inc}} \mathrm{im}(\psi) \xrightarrow{q_{\upharpoonright \mathrm{im}(\psi)}} \mathrm{im}(q_{\upharpoonright \mathrm{im}(\psi)}) = \mathrm{im}(q \circ \psi) \to 0.$$

Here, the morphism $q_{\upharpoonright \mathrm{im}(\psi)}\colon \mathrm{im}(\psi) \to N/\,\mathrm{im}(\phi)$ is defined as the composition of morphisms $q_{\upharpoonright \mathrm{im}(\psi)} := q \circ m$, where $m\colon \mathrm{im}(\psi) \hookrightarrow N$ denotes the canonical monomorphism. With Proposition 4.4 and equation 10 we conclude $\psi_\lambda = \overline{\psi}_\lambda$.

In total, we find

$$(\phi + \psi)_\lambda = (\overline{\phi + \psi})_\lambda = \overline{\psi}_\lambda = \psi_\lambda.$$

□

We are now finally able to prove the weaker version of the basis theorem 4.1.

Proposition 4.9. (Dependent basis theorem)
For every $\mu \in \Lambda$, choose a basis F_M^μ and a corresponding set of lifts $\widetilde{F}_M^\mu$. Then for all $\mu \in \Lambda$, there exists a basis G_N^μ and a set of extensions $\widetilde{G}_N^\mu$ satisfying the following properties:

(i) The set $\widetilde{G}_N \widetilde{F}_M$ is a basis of the K-vector space $\mathrm{Hom}_{\mathscr{C}}(M, N)$.

(ii) Let $\mu \in \Lambda$. If ϕ is a non-zero morphism in the K-span $\mathrm{span}_K\left(\widetilde{G}_N^\mu \widetilde{F}_M^\mu\right) \subseteq \mathrm{Hom}_{\mathscr{C}}(M, N)$ of $\widetilde{G}_N^\mu \widetilde{F}_M^\mu$, then $\phi_\mu \neq 0$.

Additionally, any choice of G_N^μ and $\widetilde{G}_N^\mu$ that satisfies properties (i) and (ii) also satisfies the following two properties:

(iii) For all $\lambda \in \Lambda$, the set $\bigcup_{\mu \leqslant \lambda} \widetilde{G}_N^\mu \widetilde{F}_M^\mu$ is a K-vector space basis of $\mathrm{Hom}_{\mathscr{C}}(M, N)^{\leqslant \lambda}$.
Similarly, for all $\lambda \in \Lambda$, the set $\bigcup_{\mu < \lambda} \widetilde{G}_N^\mu \widetilde{F}_M^\mu$ is a K-vector space basis of $\mathrm{Hom}_{\mathscr{C}}(M, N)^{< \lambda}$.

(iv) The sets $\widetilde{G}_N^\mu \widetilde{F}_M^\mu$ are pairwise disjoint for the various $\mu \in \Lambda$.

Proof.
Regarding (i) and (ii): We will show statements (i) and (ii) simultaneously by induction on the length l of a costandard filtration of N.

Base case: If $l = 1$ holds, then there exists an element $\lambda \in \Lambda$ such that $N \cong \nabla(\lambda)$.
We set

$$G_N^\mu := \begin{cases} \{c^\lambda\} & \text{if } \lambda = \mu \\ \varnothing & \text{else} \end{cases}$$

Here, the morphism c^λ is defined as in Section 2. If $\mu = \lambda$ holds, then $G_N^\mu = \{c^\lambda\}$ is a basis of $\mathrm{Hom}_{\mathscr{C}}(\Delta(\mu), \nabla(\lambda)) \cong K$ since c^λ is non-zero. If on the other hand $\mu \neq \lambda$ holds, then $\mathrm{Hom}_{\mathscr{C}}(\Delta(\mu), \nabla(\lambda)) \cong \mathrm{Ext}^0_{\mathscr{C}}(\Delta(\mu), \nabla(\lambda)) = 0$ by Definition 1.1.3. Thus, G_N^μ is a basis of $\mathrm{Hom}_{\mathscr{C}}(\Delta(\mu), N)$ for all $\mu \in \Lambda$. Next, define

$$\widetilde{G}_N^\mu := \begin{cases} \{\pi^\lambda\} & \text{if } \lambda = \mu \\ \varnothing & \text{else} \end{cases}$$

By definition of c^λ (c.f. Section 2), the morphism π^λ extends c^λ along ι^λ.

Now, observe that the following chain of equalities holds

$$\widetilde{G}_N \widetilde{F}_M = \widetilde{G}_N^\lambda \widetilde{F}_M^\lambda = F_M^\lambda. \tag{11}$$

Thus, the set $\widetilde{G}_N \widetilde{F}_M = F_M^\lambda$ defines a basis of the K-vector space $\mathrm{Hom}_{\mathscr{C}}(M, N) = \mathrm{Hom}_{\mathscr{C}}(M, \nabla(\lambda))$.

To show statement (ii) in the case $l = 1$, pick $\mu \in \Lambda$ and let $\phi \in \mathrm{span}_K\left(\widetilde{G}_N^\mu \widetilde{F}_M^\mu\right)$ be a non-zero morphism. Since $\mathrm{span}_K\left(\widetilde{G}_N^\mu \widetilde{F}_M^\mu\right) = 0$ for $\mu \neq \lambda$ by definition of $\widetilde{G}_N^\mu$ and by the convention mentioned in step (5) of Construction 3.1, we conclude $\mu = \lambda$. Thus, we have with equation 11

$$\phi \in \mathrm{span}_K\left(F_M^\lambda\right) = \mathrm{Hom}_{\mathscr{C}}(M, \nabla(\lambda)).$$

In particular, the image $\mathrm{im}(\phi)$ is a non-zero subobject of $\nabla(\lambda)$. Hence, the socle $L(\lambda) \cong \mathrm{Soc}\,\nabla(\lambda)$ is a subobject of $\mathrm{im}(\phi)$. Thus, by [P70, Proposition 4.5.1] there exists a composition series

$$0 = A_0 \subsetneq A_1 \subsetneq \cdots \subsetneq A_n = \mathrm{im}(\phi)$$

in which $L(\lambda)$ appears as an element. Since $L(\lambda)$ is simple, we know $A_1 = L(\lambda)$ and thus $0 \neq [\mathrm{im}(\phi) \colon L(\lambda)] \overset{\mathrm{def}}{=} \phi_\lambda$. This shows statement (ii) in the case that $l = 1$.

Induction step: For the induction step let $l \in \mathbb{N} \backslash \{1\}$. Let $N \in \mathscr{C}$ be an object with costandard filtration

$$0 = N_0 \subsetneq \cdots N_{l-1} \subsetneq N_l = N. \tag{12}$$

By the induction hypothesis, there exists a basis $G_{N_{l-1}}$ of $\mathrm{Hom}_{\mathscr{C}}(\Delta(\lambda), N_{l-1})$ and a set of extensions $\widetilde{G}_{N_{l-1}}$ such that the set $\widetilde{G}_{N_{l-1}} \widetilde{F}_M$ is a basis of $\mathrm{Hom}_{\mathscr{C}}(M, N_{l-1})$ satisfying (ii). From said data we now construct a basis of $\mathrm{Hom}_{\mathscr{C}}(M, N)$.

To do so, find $\lambda \in \Lambda$ with $N_l / N_{l-1} \cong \nabla(\lambda)$. Consider the short exact sequence

$$0 \to N_{l-1} \xrightarrow{\mathrm{inc}} N \xrightarrow{\mathrm{proj}} \nabla(\lambda) \to 0. \tag{13}$$

Let $\mu \in \Lambda$. Applying the left exact functor $\mathrm{Hom}_{\mathscr{C}}(\Delta(\mu), -)$ to sequence 13 and using Corollary 1.4 yields the following short exact sequence in the category of finite-dimensional K-vector spaces:

$$0 \to \mathrm{Hom}_{\mathscr{C}}(\Delta(\mu), N_{l-1}) \xrightarrow{\mathrm{inc}_*} \mathrm{Hom}_{\mathscr{C}}(\Delta(\mu), N) \xrightarrow{\mathrm{proj}_*} \mathrm{Hom}_{\mathscr{C}}(\Delta(\mu), \nabla(\lambda)) \to 0. \tag{14}$$

Analogously, by applying the left exact functor $\mathrm{Hom}_{\mathscr{C}}(M, -)$ to sequence 13 we obtain the following short exact sequence in the category of finite-dimensional K-vector spaces:

$$0 \to \mathrm{Hom}_{\mathscr{C}}(M, N_{l-1}) \xrightarrow{\mathrm{inc}_*} \mathrm{Hom}_{\mathscr{C}}(M, N) \xrightarrow{\mathrm{proj}_*} \mathrm{Hom}_{\mathscr{C}}(M, \nabla(\lambda)) \to 0. \tag{15}$$

Next, we define the set G_N^μ depending on $\mu \in \Lambda$.

In the case that $\mu \neq \lambda$, set

$$G_N^\mu := \mathrm{inc}_*\big(G_{N_{l-1}}^\mu\big) \tag{16}$$

$$\widetilde{G}_N^\mu := \mathrm{inc}_*\big(\widetilde{G}_{N_{l-1}}^\mu\big). \tag{17}$$

Since by Definition 1.1.3 we have $\mathrm{Hom}_{\mathscr{C}}(\Delta(\mu), \nabla(\lambda)) = 0$ if $\mu \neq \lambda$, the morphism inc_* in sequence 14 is a K-linear isomorphism. Thus, the set G_N^μ is a basis of $\mathrm{Hom}_{\mathscr{C}}(\Delta(\mu), N)$. Moreover, by construction the set $\widetilde{G}_N^\mu$ is a set of extensions of G_N^μ along ι^μ.

In the case that $\mu = \lambda$, choose a morphism $g^\lambda \colon \Delta(\lambda) \to N$ such that $\mathrm{proj} \circ g^\lambda = c^\lambda$. Here, the morphism c^λ is defined as in Section 2. Such a g^λ exists, since the sequence 14 is exact. Then the set

$$G_N^\mu := \mathrm{inc}_*\big(G_{N_{l-1}}^\lambda\big) \cup \{g^\lambda\}$$

defines a basis of $\mathrm{Hom}_{\mathscr{C}}(\Delta(\lambda), N)$, since $\{c^\lambda\}$ is a basis of $\mathrm{Hom}_{\mathscr{C}}(\Delta(\lambda), \nabla(\lambda)) \cong K$ and any short exact sequence of vector spaces (in particular sequence 14) splits. Next, choose by the lifting lemma 2.1 any extension $\widetilde{g}^\lambda \colon T(\lambda) \to N$ of g^λ along ι^λ and set

$$\widetilde{G}_N^\lambda := \mathrm{inc}_*\big(\widetilde{G}_{N_{l-1}}^\lambda\big) \cup \{\widetilde{g}^\lambda\}. \tag{18}$$

By construction the set $\widetilde{G}_N^\lambda$ is a set of extensions of G_N^λ along ι^λ. We omit the proof that with definitions 17 and 18 the set $\widetilde{G}_N \widetilde{F}_M = \bigcup_{\lambda \in \Lambda} \widetilde{G}_N^\lambda \widetilde{F}_M^\lambda$ is a basis of $\mathrm{Hom}_{\mathscr{C}}(M, N)$. A proof is given in [AST18, Proposition 3.3.4]. That this basis satisfies (ii) is shown in detail in [BT22, p. 18].

To prove statements (iii) and (iv) choose a basis G_N and a set of extensions $\widetilde{G}_N$ satisfying (i) and (ii).

Regarding (iii): Let $\lambda \in \Lambda$. By assumption (ii), it suffices to prove that $\bigcup_{\mu \leqslant \lambda} \widetilde{G}_N^\mu \widetilde{F}_M^\mu$ spans $\mathrm{Hom}_{\mathscr{C}}(M, N)^{\leqslant \lambda}$. We first convince ourselves that if $\mu \leqslant \lambda$ holds, then we have the inclusion

$$\widetilde{G}_N^\mu \widetilde{F}_M^\mu \subseteq \mathrm{Hom}_{\mathscr{C}}(M, N)^{\leqslant \lambda}. \tag{19}$$

To see this, let $\widetilde{g}_i^\mu \circ \widetilde{f}_j^\mu \in \widetilde{G}_N^\mu \widetilde{F}_M^\mu$. Set $c_{ij}^\mu := \widetilde{g}_i^\mu \circ \widetilde{f}_j^\mu$. If $(c_{ij}^\mu)_\nu \overset{\text{def}}{=} [\mathrm{im}(c_{ij}^\mu) \colon L(\nu)] \neq 0$ for $\nu \in \Lambda$, then $[\mathrm{im}(\widetilde{g}_i^\mu) \colon L(\nu)] \neq 0$. This follows from Corollary 4.5 since by [P70, Lemma 4.3.6 a)] we have $\mathrm{im}(\widetilde{g}_i^\mu \circ \widetilde{f}_j^\mu) \subseteq \mathrm{im}(\widetilde{g}_i^\mu)$. Thus, again by Corollary 4.5:

$$[T(\mu) \colon L(\nu)] \geqslant [T(\mu)/\ker(\widetilde{g}_i^\mu) \colon L(\nu)] = [\mathrm{im}(\widetilde{g}_i^\mu) \colon L(\nu)] > 0. \tag{20}$$

Since $T(\mu)$ is of highest weight μ by Definition 1.1.4, we conclude $\mu \geqslant \nu$. With the transitivity of the partial order, we hence have $\lambda \geqslant \nu$. This shows inclusion 19. From inclusion 19 we conclude

$$\mathrm{span}_K \Big(\bigcup_{\mu \leqslant \lambda} \widetilde{G}_N^\mu \widetilde{F}_M^\mu \Big) \subseteq \mathrm{Hom}_{\mathscr{C}}(M, N)^{\leqslant \lambda}. \tag{21}$$

Now, to show that the two sets appearing in 21 are actually equal, pick a non-zero morphism $\phi \in \mathrm{Hom}_{\mathscr{C}}(M, N)^{\leqslant \lambda}$. By assumption (i), we can write

$$\phi = \sum_{\mu \in \Lambda} a_{ij}^\mu c_{ij}^\mu$$

with $a_{ij}^\mu \in K$ and $c_{ij}^\mu \in \widetilde{G}_N^\mu \widetilde{F}_M^\mu$ for $\mu \in \Lambda$. Pick a maximal (not necessarily unique) $\mu \in \Lambda$ such that $a_{ij}^\mu \neq 0$ for some i and j. Define

$$\phi^\mu := \sum_{i,j} a_{ij}^\mu c_{ij}^\mu \quad \text{and} \quad \phi^{\neq\mu} := \sum_{\substack{\nu\neq\mu \\ i,j}} a_{ij}^\nu c_{ij}^\nu.$$

Since the set of c_{ij}^μ's is K-linearly independent, we know that $\phi^\mu \in \operatorname{span}_K \left(\widetilde{G}_N^\mu \widetilde{F}_M^\mu\right)$ is non-zero. With (ii) we conclude that the μ-multiplicity $(\phi^\mu)_\mu$ is non-zero. By the maximality of μ we have $(\phi^{\neq\mu})_\mu = 0$. Now, Lemma 4.8 yields the chain of equalities $\phi_\mu = (\phi^\mu + \phi^{\neq\mu})_\mu = (\phi^\mu)_\mu \neq 0$. By definition of $\operatorname{Hom}_{\mathscr{C}}(M,N)^{\leqslant\lambda}$, we conclude $\mu \leqslant \lambda$. Since this inequality holds for any maximal μ, this implies $\phi \in \operatorname{span}_K \left(\bigcup_{\mu\leqslant\lambda} \widetilde{G}_N^\mu \widetilde{F}_M^\mu\right)$.

Regarding (iv): Let $\phi \in \widetilde{G}_N^\lambda \widetilde{F}_M^\lambda \cap \widetilde{G}_N^\mu \widetilde{F}_M^\mu$. By definition of a basis we have $\phi \neq 0$. With property (ii) we conclude $\phi_\mu \neq 0$. Additionally, by property (iii) we know that $\phi \in \operatorname{Hom}_{\mathscr{C}}(M,N)^{\leqslant\lambda}$. Thus, $\mu \leqslant \lambda$. By exchanging μ and λ one shows $\lambda \leqslant \mu$. With the antisymmetry of a partial order, we conclude $\lambda = \mu$.

□

Remark 4.10.
One may exchange the roles of F_M and G_N in the basis theorem 4.9.

With Lemma 3.2.4. from Kian's talk, we obtain the following result.

Corollary 4.11. (Counting dimension via multiplicities in (co)standard filtrations)
We have

$$\dim_K \left(\operatorname{Hom}_{\mathscr{C}}(M,N)\right) = \sum_{\lambda\in\Lambda} (M\colon \nabla(\lambda)) \cdot (N\colon \Delta(\lambda)).$$

The dependent basis theorem now allows us to prove the original basis theorem by a change-of-basis argument.

Proof of Theorem 4.1.
A detailed proof is given in [BT22, Proof of Theorem 3.2.]. The proof amounts to showing that a certain linear transformation is an isomorphism. □

The proof of the basis theorem 4.1 yields the subsequent proposition.

Proposition 4.12.
Any choice for $F_M^\lambda, G_N^\lambda, \widetilde{F}_M^\lambda, \widetilde{G}_N^\lambda$ in Construction 3.1 satisfies property (ii) of the dependent basis theorem 4.9.

Proposition 4.12 together with the dependent basis theorem 4.9 implies the following result.

Corollary 4.13.
Any choice for $F_M^\lambda, G_N^\lambda, \widetilde{F}_M^\lambda, \widetilde{G}_N^\lambda$ in Construction 3.1 satisfies properties (iii) and (iv) of the dependent basis theorem 4.9.

5 Combinatorial bases

We fix a tilting object $T \in \mathscr{C}$. The basis theorem 4.1 immediately gives us a way to construct bases of the endomorphism algebra $\mathrm{End}_{\mathscr{C}}(T)$ of T. In this section, we equip these bases with extra (combinatorial) structure.

5.1 Standard bases

It turns out that the bases of the endomorphism algebra $\mathrm{End}_{\mathscr{C}}(T)$ constructed in the basis theorem 4.1 already come with a canonical combinatorial structure, making them so-called *standard bases.*

Definition 5.1 *(Du-Rui)*
Let E be a finite-dimensional K-algebra. A standard basis of E consists of:

(i) A K-basis B of E,

(ii) A poset Λ and a function $f\colon \mathsf{B} \to \Lambda$. We write $\mathsf{B}^\lambda := f^{-1}(\lambda)$,

(iii) For any $\lambda \in \Lambda$, indexing sets I^λ and J^λ such that:

(a) We can write $\mathsf{B}^\lambda = \{c_{ij}^\lambda \mid i \in \mathsf{I}^\lambda \text{ and } j \in \mathsf{J}^\lambda\}$,

(b) For any $\phi \in E$ and any $c_{ij}^\lambda \in \mathsf{B}^\lambda$, we have:

$$\phi \cdot c_{ij}^\lambda \equiv \sum_{k \in \mathsf{I}^\lambda} r_k^\lambda(\phi, i) c_{kj}^\lambda \mod E^{<\lambda}, \tag{22}$$

$$c_{ij}^\lambda \cdot \phi \equiv \sum_{l \in \mathsf{J}^\lambda} r_l^\lambda(j, \phi) c_{il}^\lambda \mod E^{<\lambda}, \tag{23}$$

where $r_k^\lambda(\phi, i)$, $r_l^\lambda(j, \phi) \in K$ are independent of j and i, respectively.
Here, we let $E^{<\lambda} := \mathrm{span}_K(\bigcup_{\mu<\lambda} \mathsf{B}^\mu) \subseteq E$.

Theorem 5.2.
Any K-basis $\widetilde{G}_T\widetilde{F}_T$ of $\mathrm{End}_{\mathscr{C}}(T)$ arising from Construction 3.1 carries a canonical structure of a standard basis.

Proof.
Firstly, note that by Corollary 4.13 the set B is fibered over the poset Λ, i.e.

$$\widetilde{G}_T\widetilde{F}_T = \coprod_{\lambda \in \Lambda} \widetilde{G}_T^\lambda \widetilde{F}_T^\lambda. \tag{24}$$

For any $\lambda \in \Lambda$, we can thus set

$$\mathsf{B}^\lambda := \widetilde{G}_T^\lambda \widetilde{F}_T^\lambda. \tag{25}$$

Secondly, for any $\lambda \in \Lambda$, define

$$\mathsf{I}^\lambda := \{n \in \mathbb{N} \mid n \leqslant (T\colon \Delta(\lambda))\} \tag{26}$$

$$\mathsf{J}^\lambda := \{n \in \mathbb{N} \mid n \leqslant (T\colon \nabla(\lambda))\}. \tag{27}$$

By Lemma 3.2.4. from Kian's talk, we can index

$$\widetilde{G}_T^\lambda = \{\widetilde{g}_i^\lambda \mid i \in \mathsf{I}^\lambda\} \tag{28}$$

$$\widetilde{F}_T^\lambda = \{\widetilde{f}_j^\lambda \mid j \in \mathsf{J}^\lambda\}. \tag{29}$$

We can thus write

$$\mathsf{B}^\lambda = \{\widetilde{g}_i^\lambda \circ \widetilde{f}_j^\lambda \mid i \in \mathsf{I}^\lambda \text{ and } j \in \mathsf{J}^\lambda\}. \tag{30}$$

Next, we prove equation 22. Let $\phi \in \operatorname{End}_{\mathscr{C}}(T)$. Choose $\widetilde{g}_i^\lambda \circ \widetilde{f}_j^\lambda \in \widetilde{G}_T^\lambda \widetilde{F}_T^\lambda$. Set $c_{ij}^\lambda := \widetilde{g}_i^\lambda \circ \widetilde{f}_j^\lambda$. Since G_T^λ is a basis of $\operatorname{Hom}(\Delta(\lambda), T)$, we find $r_k^\lambda(\phi, i) \in K$ such that

$$\phi \circ g_i^\lambda = \sum_{k \in \mathsf{I}^\lambda} r_k^\lambda(\phi, i) g_k^\lambda. \tag{31}$$

Since for each $k \in \mathsf{I}^\lambda$ the morphism $\widetilde{g}_k^\lambda$ extends g_k^λ along ι^λ, equation 31 yields

$$\phi \circ \widetilde{g}_i^\lambda \circ \iota^\lambda = \sum_{k \in \mathsf{I}^\lambda} r_k^\lambda(\phi, i)(\widetilde{g}_k^\lambda \circ \iota^\lambda). \tag{32}$$

Since composition in an abelian category is bilinear, this amounts to

$$\Big(\phi \circ \widetilde{g}_i^\lambda - \sum_{k \in \mathsf{I}^\lambda} r_k^\lambda(\phi, i)\widetilde{g}_k^\lambda\Big) \circ \iota^\lambda = 0. \tag{33}$$

Note that $\operatorname{Hom}_{\mathscr{C}}(T(\lambda), T) = \operatorname{Hom}_{\mathscr{C}}(T(\lambda), T)^{\leqslant \lambda}$. The inclusion $\supseteq$ is clear. For the other inclusion, let $\psi \colon T(\lambda) \to T$ with $\psi_\mu \neq 0$. Then

$$[T(\lambda) \colon L(\mu)] \geqslant [T(\lambda)/\ker(\psi) \colon L(\mu)] = [\operatorname{im}(\psi) \colon L(\mu)] > 0.$$

Since by assumption $T(\lambda)$ is of highest weight λ, we conclude $\mu \leqslant \lambda$. The inclusion $\subseteq$ follows.

In particular, we see that $\psi := \phi \circ \widetilde{g}_i^\lambda - \sum_{k \in \mathsf{I}^\lambda} r_k^\lambda(\phi, i)\widetilde{g}_k^\lambda \in \operatorname{Hom}_{\mathscr{C}}(T(\lambda), T)^{\leqslant \lambda}$. We even know that $\psi \in \operatorname{Hom}_{\mathscr{C}}(T(\lambda), T)^{< \lambda}$. To see this, note that by the universal property of the kernel and by 33 there exists a monomorphism $\Delta(\lambda) \hookrightarrow \ker(\lambda)$. Thus, by Corollary 4.5 we have

$$[\ker(\psi) \colon L(\lambda)] \geqslant [\Delta(\lambda) \colon L(\lambda)]. \tag{34}$$

Since both $T(\lambda)$ and $\Delta(\lambda)$ are of highest weight λ, we conclude with one of the isomorphism theorems:

$$\begin{aligned} 0 = [T(\lambda) \colon L(\lambda)] - [\Delta(\lambda) \colon L(\lambda)] &\overset{34}{\geqslant} [T(\lambda) \colon L(\lambda)] - [\ker(\psi) \colon L(\lambda)] \\ &\overset{4.4}{\geqslant} [T(\lambda)/\ker(\psi) \colon L(\lambda)] \\ &= [\operatorname{im}(\psi) \colon L(\lambda)]. \end{aligned}$$

Next, note that $\operatorname{im}(\psi \circ \widetilde{f}_j^\lambda) \subseteq \operatorname{im}(\psi)$ by [P70, Lemma 4.3.6 a)]. Thus, by Proposition 4.4 we know

$$\phi \circ \widetilde{g}_i^\lambda \circ \widetilde{f}_j^\lambda - \sum_{k \in \mathsf{I}^\lambda} r_k^\lambda(\phi, i)\widetilde{g}_k^\lambda \circ \widetilde{f}_j^\lambda = \psi \circ \widetilde{f}_j^\lambda \in \operatorname{Hom}_{\mathscr{C}}(T, T)^{< \lambda}. \tag{35}$$

Since by Corollary 4.13 the following two sets

$$\mathrm{Hom}_{\mathscr{C}}(T,T)^{<\lambda} = \mathrm{span}_K\Big(\bigcup_{\mu<\lambda} \widetilde{G}_T^{\lambda}\widetilde{F}_T^{\lambda}\Big) \tag{36}$$

are equal, we therefore have shown

$$\phi \circ c_{ij}^{\lambda} \equiv \sum_{k\in \mathsf{I}^{\lambda}} r_k^{\lambda}(\phi, i) c_{kj}^{\lambda} \mod E^{<\lambda}. \tag{37}$$

Equation 23 is proved analogously. □

5.2 Cellular bases

We recall the definition of a cellular basis as presented in Muskan's talk.

Definition 5.3 *(Graham-Lehrer)*
Let E be a finite-dimensional K-algebra. A cellular basis of E consists of a standard basis B together with an involutive antialgebra morphism $*\colon E \to E$ such that the following two equalities hold:

(i) $\mathsf{I}^{\lambda} = \mathsf{J}^{\lambda}$ *for every* $\lambda \in \Lambda$,

(ii) $c_{ij}^{*} = c_{ji}$ *for every* $(i,j) \in \mathsf{I}^{\lambda} \times \mathsf{J}^{\lambda}$.

We will now turn some of the bases constructed in the basis theorem 4.1 into cellular bases. To do so, we need the following definitions.

Definition 5.4
A strong duality on $\mathscr{C}$ consists of:

(i) A contravariant K-linear endofunctor $D\colon \mathscr{C} \to \mathscr{C}$ on $\mathscr{C}$,

(ii) A natural isomorphism of K-linear functors

$$\xi\colon \mathrm{id}_{\mathscr{C}} \to D^2$$

that satisfies

$$\mathrm{id}_{D(X)} = D(\xi_X) \circ \xi_{D(X)}$$

for all objects $X \in \mathscr{C}$. As a commutative diagram:

$$\begin{tikzcd} D(X) \arrow[rr, "\mathrm{id}_{D(X)}"] \arrow[dr, "\xi_{D(X)}"'] & & D(X) \\ & D^3(X) \arrow[ur, "D(\xi_X)"'] & \end{tikzcd}$$

Definition 5.5
Let $D \colon \mathscr{C} \to \mathscr{C}$ be a strong duality on $\mathscr{C}$. A <u>*self-dual*</u> *object (X, ϕ) in $\mathscr{C}$ consists of an object $X \in \mathscr{C}$ and an isomorphism $\phi \colon D(X) \xrightarrow{\sim} X$.*

Construction 5.6.
Let (D, ξ) be a strong duality on $\mathscr{C}$. Let (X, ϕ) be a self-dual object in $\mathscr{C}$. Define the K-antialgebra morphism

$$\alpha_\phi^{-1} \in \mathrm{End}_{\mathscr{C}}(X)$$

by letting $\alpha_\phi^{-1}(\varphi) := \phi \circ D(\varphi) \circ \phi^{-1}$ for all $\varphi \in \mathrm{End}_{\mathscr{C}}(X)$. Define

$$a := \phi \circ D(\phi^{-1}) \circ \xi_X \in \mathrm{End}_{\mathscr{C}}(X).$$

Using the naturality of ξ, one sees that

$$\alpha_\phi^{-2}(\varphi) = a \circ \varphi \circ a^{-1}$$

for all $\varphi \in \mathrm{End}_{\mathscr{C}}(X)$. In particular, α_ϕ^{-1} has a two-sided inverse

$$\alpha_\phi \in \mathrm{End}(X).$$

Definition 5.7
Let (D, ξ) be a strong duality on $\mathscr{C}$. A self-dual object (X, ϕ) is called a <u>*fixed point*</u> *of D if α_ϕ from Construction 5.6 is an anti-involution, i.e. if*

$$a \overset{\text{def}}{=} \phi \circ D(\phi^{-1}) \circ \xi_X = \mathrm{id}_X \, .$$

Definition 5.8
An object $X \in \mathscr{C}$ is called <u>*fixed*</u> *by D if there exists an isomorphism $\phi \colon D(X) \xrightarrow{\sim} X$ such that the pair (X, ϕ) is a fixed point of D.*

A proof of the following lemma is given in [BT22, Lemma 3.13.].

Lemma 5.9.
Let D be a strong duality on $\mathscr{C}$ that exchanges standard and costandard objects, i.e.

$$D(\nabla(\lambda)) \cong \Delta(\lambda)$$

for all $\lambda \in \Lambda$. Then we have:

(i) Every simple object $L(\lambda)$ is fixed by D,

(ii) Every tilting object $T \in \mathscr{C}$ admits a self-dual structure, i.e. there exists an isomorphism $\phi \colon D(T) \xrightarrow{\sim} T$.

Definition 5.10
A <u>*standard duality*</u> *on $\mathscr{C}$ is a strong duality D on $\mathscr{C}$ such that:*

(i) The duality D exchanges standard and costandard objects, i.e. for all $\lambda \in \Lambda$ we have

$$D(\nabla(\lambda)) \cong \Delta(\lambda),$$

(ii) Every indecomposable tilting object $T(\lambda)$ is a fixed point, i.e. for every $\lambda \in \Lambda$ and every isomorphism $\phi\colon D(T(\lambda)) \xrightarrow{\sim} T(\lambda)$, the pair $(T(\lambda), \phi)$ is a fixed point of D.

Remark 5.11.
Lemma 5.9 ensures that condition (ii) in Definition 5.10 is not void (, i.e. that there exists an isomorphism $\phi\colon D(T(\lambda)) \xrightarrow{\sim} T(\lambda)$).

Construction 5.12.
Let D be a standard duality on $\mathscr{C}$. Let $T \in \mathscr{C}$ be a tilting object. We will sketch how some of the bases of $\mathrm{End}(T)$ constructed in the basis theorem carry a cellular structure.

For this purpose, choose for every $\lambda \in \Lambda$ a basis G_T^λ of $\mathrm{Hom}_{\mathscr{C}}(\Delta(\lambda), T)$. For any $\lambda \in \Lambda$, then pick a set of extensions $\widetilde{G}_T^\lambda$ of G_T^λ along the monomorphism ι^λ as in Construction 3.1.

Since the full subcategory $\mathscr{C}^t$ consisting of tilting objects in $\mathscr{C}$ is a Krull-Schmidt category (as shown in Corollary 3.2.6.3. of Kian's talk), the tilting object T is a finite direct sum of indecomposable tilting objects $T(\lambda)$. By property (ii) of a standard duality and since the functor D is required to be K-linear, we therefore know that the tilting object T is fixed by D. In other words, we can choose an isomorphism

$$\phi_T\colon D(T) \xrightarrow{\sim} T$$

such that

$$\phi_T \circ D(\phi_T^{-1}) \circ \xi_T = \mathrm{id}_T\,.$$

Using this isomorphism, one constructs for every $\lambda \in \Lambda$ a basis $D(G_T^\lambda)$ of $\mathrm{Hom}_{\mathscr{C}}(T, \nabla(\lambda))$ and a corresponding set of lifts $D(\widetilde{G}_T^\lambda)$. For details see [BT22, p. 24]. We then let

$$D(\widetilde{G}_T) := \coprod_{\lambda \in \Lambda} D(\widetilde{G}_T^\lambda).$$

Theorem 5.13.
Let D be a standard duality on $\mathscr{C}$. Let $T \in \mathscr{C}$ be a tilting object. Choose a basis G_T and a set of extensions $\widetilde{G}_T$. Then the standard basis $\widetilde{G}_T D(\widetilde{G}_T)$ of $\mathrm{End}_{\mathscr{C}}(T)$ together with the involutive antialgebra morphism α_{ϕ_T} (as defined in Construction 5.6) define a cellular basis.

Proof.
Note that the duality D is fully faithful and exchanges standard and costandard objects. Together with the fact that the tilting object T admits a self-duality, this implies for all $\lambda \in \Lambda$:

$$\mathrm{Hom}_{\mathscr{C}}(\Delta(\lambda), T) \cong \mathrm{Hom}_{\mathscr{C}}(D(T), D(\Delta(\lambda))) \cong \mathrm{Hom}_{\mathscr{C}}(T, \nabla(\lambda)). \tag{38}$$

With Lemma 3.2.4. from Kian's talk, we conclude $\mathsf{I}^\lambda = \mathsf{J}^\lambda$.

A detailed proof of the fact that the equality $\alpha_{\phi_T}(c_{ij}) = c_{ji}$ holds for all $(i,j) \in \mathsf{I}^\lambda \times \mathsf{J}^\lambda$ is given in [BT22, Theorem 3.16]. □

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