

Addendum

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Proof of Remark 4.11.(ii) in
*Surface Diagrams for
 Frobenius Algebras and Frobenius-Schur Indicators
 in Grothendieck-Verdier Categories*

Let $\mathcal{C} = (\mathcal{C}, \otimes, 1, K)$ and $\mathcal{D} = (\mathcal{D}, \otimes, 1, K)$ be GV-categories. As stated in [DS25, Prop. 2.50], their duality functors are strong Frobenius LD-functors $\mathcal{C} \rightarrow \mathcal{C}^{\text{lop}}$ and $\mathcal{D} \rightarrow \mathcal{D}^{\text{lop}}$. For the definition of Frobenius LD-functors, refer to [DS25, Def. 2.13]. From now on, by abuse of notation, we denote both duality functors by D .

Let $F: \mathcal{C} \rightarrow \mathcal{D}$ be a Frobenius LD-functor. We consider its multiplication morphism

$$\varphi^2: \otimes \circ (F \times F) \rightarrow F \circ \otimes,$$

unit morphism $\varphi^0: 1 \rightarrow F(1)$, comultiplication morphism $\nu^2: F \circ \wp \rightarrow \wp \circ (F \times F)$, and counit morphism $\nu^0: F(K) \rightarrow K$.

Proposition 0.1. *The duality transformation of F*

$$\xi^F: F \circ D \xrightarrow{\simeq} D \circ F,$$

as defined in [DS25, Def. 4.10], is a morphism of Frobenius LD-functors.

Proof. We will prove that the duality transformation ξ^F is an \otimes -monoidal natural transformation. The proof that ξ^F is also \wp -monoidal follows by applying the 2-functor $(-)^{\text{cop}}$, as defined in [DS25, Def. 2.24], to the surface diagrams used in the \otimes -monoidal proof.

As a preliminary remark, observe that, as in the rigid case, two morphisms $f, g \in \text{Hom}_{\mathcal{C}}(A, D(B))$ are equal if and only if we have

$$\text{ev}_B \circ (f \otimes B) = \text{ev}_B \circ (g \otimes B). \quad (0.1)$$

This fact can be proven surface-diagrammatically using the snake equation (S1). We will utilize this fact in the remainder of the proof.

First, we show that the following diagram commutes for all objects $X, Y \in \mathcal{C}$:

$$\begin{array}{ccccc} FD(X) \otimes FD(Y) & \xrightarrow{\varphi_{DX, DY}^2} & F(DX \otimes DY) & \xrightarrow{\simeq} & FD(Y \wp X) \\ \xi_X^F \otimes \xi_X^F \downarrow & & & & \downarrow \xi_{Y \wp X}^F \\ DF(X) \otimes DF(Y) & \xrightarrow{\simeq} & D(F(Y) \wp F(X)) & \xrightarrow{D(\nu_{Y, X}^2)} & DF(Y \wp X). \end{array} \quad (0.2)$$

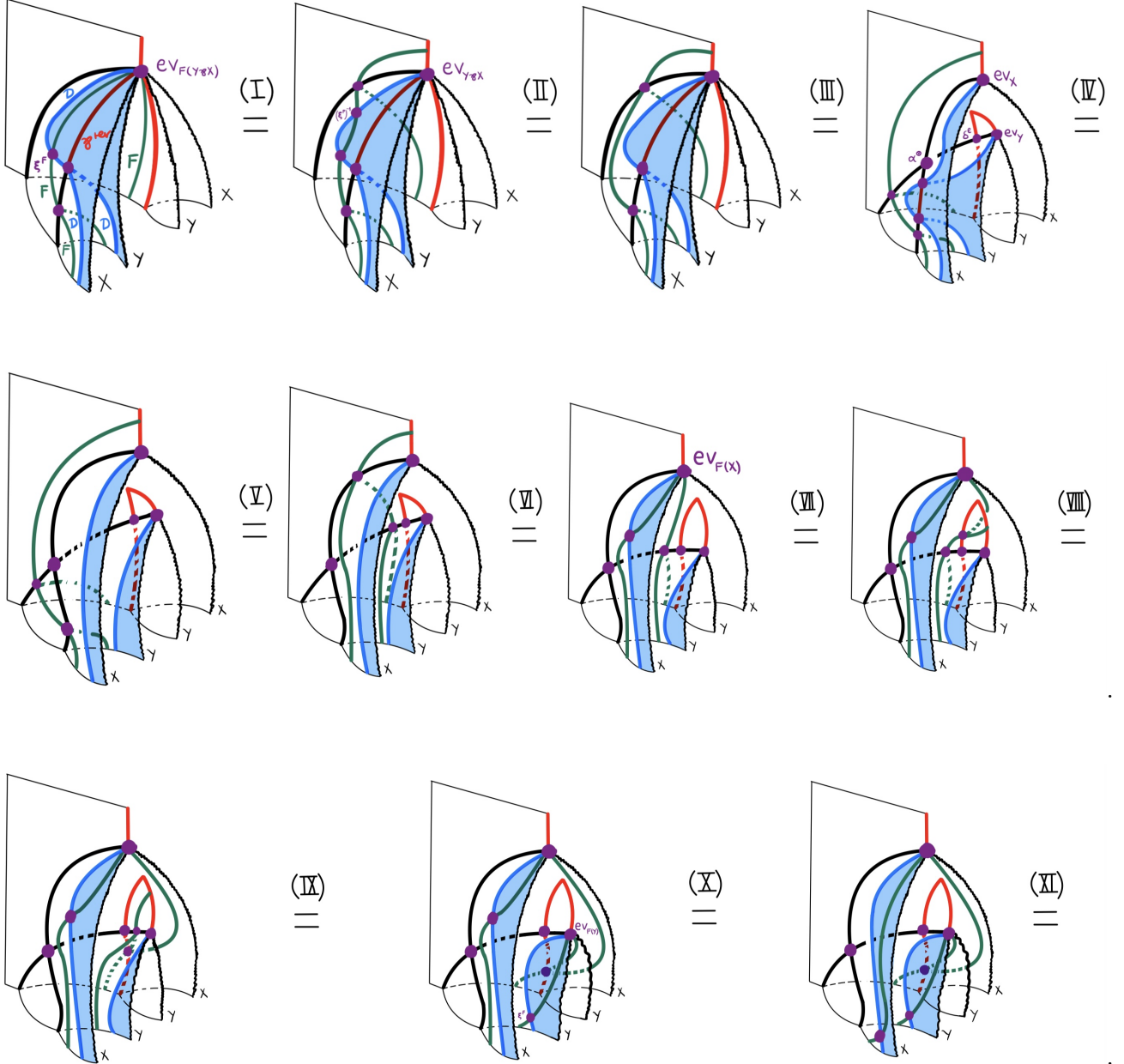
Here, the unlabeled isomorphisms are structure morphisms of the Frobenius LD-functor D .

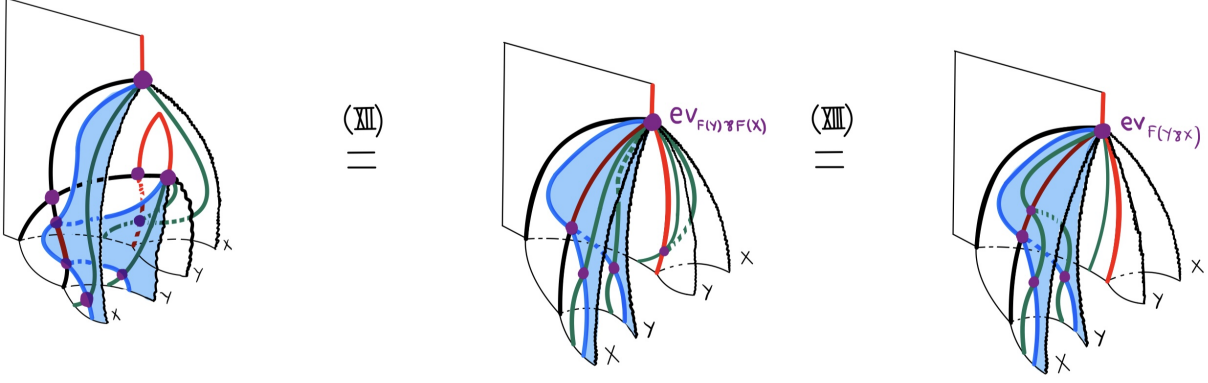
To prove commutativity, we use surface diagrams. In the following computation, the first surface diagram represents the morphism

$$(FD(X) \otimes FD(Y)) \otimes F(Y \mathfrak{Y} X) \longrightarrow K,$$

which is built from the clockwise morphism in diagram (0.2), and the evaluation

$$\text{ev}_{F(Y \mathfrak{Y} X)}: DF(Y \mathfrak{Y} Y) \otimes F(Y \mathfrak{Y} X) \stackrel{\text{def}}{=} DF(X \mathfrak{Y}^{\text{rev}} Y) \otimes F(Y \mathfrak{Y} X) \longrightarrow K.$$





Equations (I), (VI) and (IX) hold by the defining equation of the inverse duality transformation $(\xi^F)^{-1}$; see [DS25, Remark 4.11.(i)]. Equation (II) follows from the invertibility of the duality transformation. For Equation (III), consider the composite

$$D(Y \wp X) \otimes (Y \wp X) \cong (DX \otimes DY) \otimes (Y \wp X) \xrightarrow{\overline{\text{ev}}} K, \quad (0.3)$$

where we define

$$\overline{\text{ev}} := \text{ev}_X \circ (DX \otimes l_X^\wp) \circ (DX \otimes (\text{ev}_Y \wp X)) \circ (DX \otimes \delta_{DY, Y, X}^l) \circ \alpha_{DX, DY, Y \wp X}^\otimes.$$

One can show that this composite is a side inverse, in the sense of [DS25, Def. 2.26], to the composite

$$1 \xrightarrow{\overline{\text{coev}}} (Y \wp X) \wp (DX \otimes DY) \cong (Y \wp X) \wp D(Y \wp X),$$

where we set

$$\overline{\text{coev}} := (\alpha_{Y, X, DX \otimes DY}^\wp)^{-1} \circ (Y \wp \delta_{X, DX, DY}^r) \circ (Y \wp (\text{coev}_X \otimes DY)) \circ (Y \wp (l_{DY}^\otimes)^{-1}) \circ \text{coev}_Y.$$

Thus, we define the evaluation $\text{ev}_{Y \wp X}: D(Y \wp X) \otimes (Y \wp X) \rightarrow K$ as the composite in Equation (0.3). With this choice, Equation (III) holds. Equations (IV) and (XI) follow since the Frobenius LD-structure on the duality functor D is strong. Equation (V) is an instance of the associativity of the lax \otimes -monoidal functor F . Equation (VII) follows from the counitality axiom of the \wp -monoidal functor F . Equation (VIII) holds by the Frobenius relation (F1) of the Frobenius LD-functor F . Equation (X) amounts to pushing down the duality transformation ξ^F , and is valid by the axioms of a strict monoidal 2-category. Equation (XII) follows analogously to Equation (III). Finally, Equation (XIII) holds by [DS25, Prop. 2.46].

By Equation (0.1), the surface-diagrammatic computation shows that diagram (0.2) commutes.

Next, we show that the following diagram commutes:

$$\begin{array}{ccc} 1 & \xrightarrow{\varphi^0} & F(1) \xrightarrow{\simeq} FD(K) \\ \simeq \downarrow & & \downarrow \xi_K^F \\ D(K) & \xrightarrow{D(\nu^0)} & DF(K). \end{array} \quad (0.4)$$

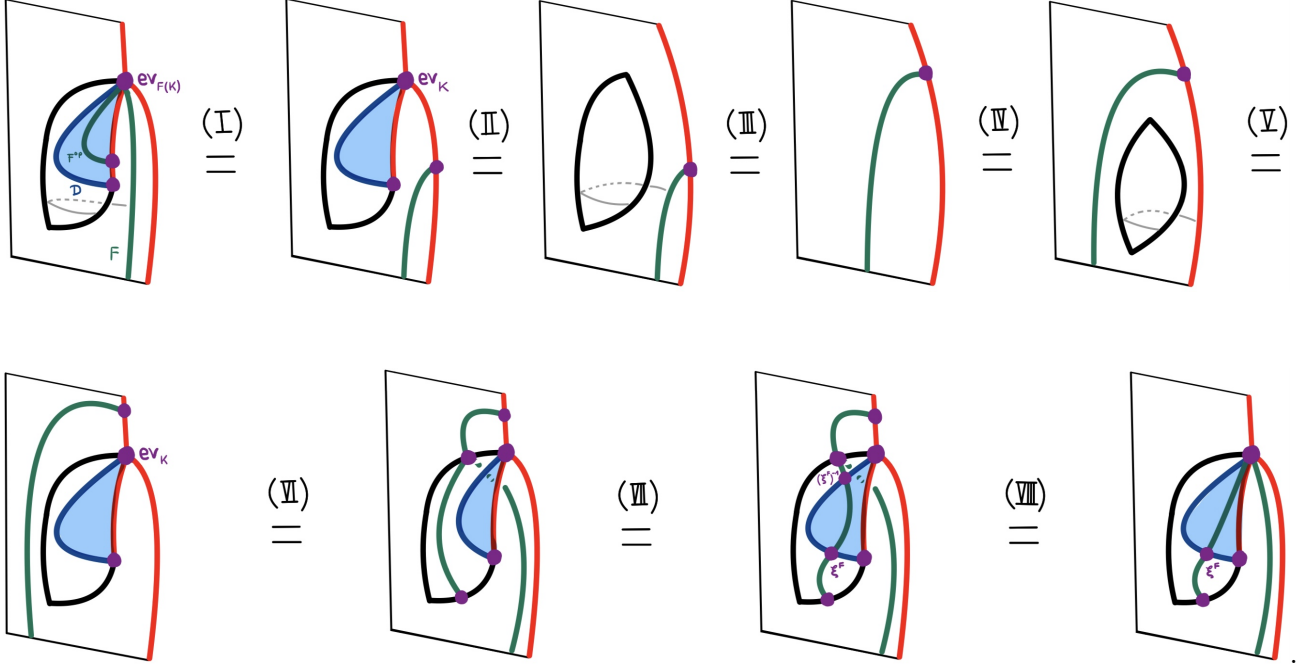
Again, the unlabelled isomorphisms are structure morphisms of the Frobenius LD-functor D .

To prove commutativity we use surface diagrams. In the following computation, the first surface diagram represents the morphism

$$F(K) \xrightarrow{\simeq} 1 \otimes F(K) \longrightarrow DF(K) \otimes F(K) \longrightarrow K,$$

which is built from the inverse left unitor, the counterclockwise morphism in diagram (0.2), and the evaluation

$$\text{ev}_{F(K)}: F(K) \otimes K \longrightarrow K.$$



Equation (I) holds by [DS25, Prop. 2.46]. By [DS25, Ex. 2.28], we can assume that we have $D(1) = K$, $D(K) = 1$, and $\text{ev}_K = l_K^\otimes$. With this choice, Equations (II) and (V) hold. Equation (III) and (IV) follow from the invertibility of the unitors. The unitality of the lax \otimes -monoidal functor F implies Equation (VI). Equation (VII) holds by the invertibility of the duality transformation ξ^F , while Equation (VIII) follows from the defining equation of the inverse duality transformation $(\xi^F)^{-1}$; see [DS25, Remark 4.11.(i)].

Finally, by Equation (0.1) and the invertibility of the left unitor, the above calculation implies that diagram (0.4) commutes.

This completes the proof that the duality transformation ξ^F is an \otimes -monoidal natural transformation. \square

References

- [DS25] M. Demirdilek and C. Schweigert. Surface Diagrams for Frobenius Algebras and Frobenius-Schur Indicators in Grothendieck-Verdier Categories, 2025. Preprint.