Critical properties of the double-frequency sine-Gordon model with applications

M. Fabrizio[a], A. O. Gogolin[b] and A. A. Nersesyan[c,d]

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[a]International School for Advanced Studies and INFM, Via Beirut 4, 34014 Trieste, Italy

[b]Department of Mathematics, Imperial College, 180 Queen's Gate, London SW7 2BZ, United Kingdom

[c] The Abdus Salam International Centre for Theoretical Physics, P.O.Box 586, 34100, Trieste, Italy

[d]The Andronikashvili Institute of Physics, Tamarashvili 6, 380077, Tbilisi, Georgia

Abstract

We study the properties of the double-frequency sine–Gordon model in the vicinity of the Ising quantum phase transition displayed by this model. Using a mapping onto a generalised lattice quantum Ashkin-Teller model, we obtain critical and nearly-off-critical correlation functions of various operators. We discuss applications of the double-sine-Gordon model to one-dimensional physical systems, like spin chains in a staggered external field and interacting electrons in a staggered potential.

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1 Introduction

The problem of determining the asymptotic behaviour of a conformal field theory (CFT) under the action of a relevant operator is well studied and understood, also in view of its relation to the physics of many quantum one-dimensional (1D) and classical two-dimensional (2D) models. A less studied case concerns a CFT perturbed by two relevant operators. Namely, consider a CFT subjected to such relevant perturbation, gO_{Δ_g}, with scaling dimension $\Delta_g < 2$, that turns it into a fully massive quantum field theory (QFT). Then add another relevant perturbation, $\lambda O_{\Delta_\lambda}$, which, if acting alone, would also make our QFT fully massive. Although this is not as common situation as the previous one, we will show that it displays interesting features which may be realized in physical systems.

Without loss of generality, we assume that $\Delta_\lambda < \Delta_g < 2$, i.e. that the second perturbation is more relevant then the first one. Moreover, we shall require, for the time being, that the operator product expansion (OPE) of these two operators is closed in the sense that it does not produce other relevant operators. The most general case will be discussed in a separate section. Then, the naïve expectation, which prevailed until recently, is that no qualitative changes occur in this case with respect to the standard situation of a CFT perturbed by a single relevant operator: the low-energy behaviour of the theory would be governed by the most relevant operator and, at any rate, the theory would remain fully massive. This expectation is however not a general rule if the two operators *exclude* each other, that is, if the field configurations which minimise one perturbation term do not minimise the other. In this case, the interplay between the two competing relevant operators can produce a novel quantum phase transition between two massive QFT's through a critical (massless) point. It is intuitively clear that such a scenario is only conceivable if the ratio of the bare coupling constants, $|g/\lambda|$, is large enough.

Recently Delfino and Mussardo (DM)[1] considered the double-frequency sine-Gordon (DSG) model, which is a Gaussian model of a scalar field Φ, perturbed by two relevant vertex operators with the ratio of their scaling dimensions $\Delta_g/\Delta_\lambda = 4$. The Hamiltonian density of the DSG model reads:

$$\begin{aligned} \mathcal{H}_{DSG}[\Phi] &= \mathcal{H}_0[\Phi] + \mathcal{U}[\Phi] \\ \mathcal{U}[\Phi] &= -g\cos\beta\Phi(x) - \lambda\sin[(\beta/2)\Phi(x)]. \end{aligned} \tag{1}$$

where

$$\mathcal{H}_0[\Phi] = v_0\left[(\partial_x\phi_R)^2 + (\partial_x\phi_L)^2\right], \tag{2}$$

$\phi_{R,L}$ being the chiral components of the Bose field, $\Phi = \phi_R + \phi_L$. DM have shown that there exists a quantum critical line $\lambda = \lambda_c(g)$ where the DSG model displays an Ising criticality with central charge $c = 1/2$.

The purpose of this paper is to investigate in detail the critical properties of this transition and to discuss physical applications to spin chains and other quantum one-dimensional (1D) systems. While we shall concentrate on the DSG model in what follows, it should be noted that the phenomenon is more general - similar quantum phase transitions do happen in more complicated models; those shall be addressed elsewhere [2].

Apart from the practical interest in physical realizations of the DSG model, this problem is of relevance also from a pure theoretical point of view. Indeed, we have already mentioned that the Ising critical point separates two *strong-coupling*, massive phases; hence, by definition, its analytical description is outside the range of applicability of perturbation theory. In this paper we propose a nonperturbative scheme, which is essentially based on the mapping of the DSG model onto another equivalent model – a generalised quantum Ashkin-Teller model on a 1D lattice, where the Ising critical point becomes accessible.

The paper is organised as follows. In section 2 we briefly discuss the DSG model at the semi-classical level and outline our approach to tackle the Ising criticality in the vicinity of the decoupling point $\beta^2 = 4\pi$. In sections 3–5 we present a complete description of the critical properties of the DSG model at the Ising transition. In section 3 we introduce a quantum lattice version of a generalised AT model which, apart from the conventional marginal inter-chain coupling, also includes a 'magnetic filed' type of interaction, $h\sigma_1\sigma_2$. We show that, in the continuum limit, this model is equivalent to the DSG model at $\beta^2 \sim 4\pi$. In section 4 we employ a new (σ-τ) representation of the deformed AT model which enables us to correctly identify those degrees of freedom that become critical. In section 5 we use the results of the quantum-spin-chain mappings to determine the physical properties of the DSG model close to the criticality.

In the next two sections we discuss applications of the DSG model to physical systems. In particular, we discuss Ising transitions in a spontaneously dimerized S=1/2 antiferromagnetic chain in a staggered magnetic field (section 6), and in the 1D Hubbard model with a staggered potential (section 7). In section 8 we show that at $\beta^2 < 2\pi$ the second-order Ising transition may transform to a first-order one. Section 9 contains discussion of the results and conclusions.

The paper is supplied with four Appendices which provide some details concerning bosonization of products of order and disorder operators for a system of two identical Ising models, the relationship between the quantum AT model and lattice fermions, a mean-field treatment of the σ-τ model, and calculation of the correlation functions.

2 The model and its quasi-classical analysis

In this paper we will be dealing with the particular version of the DSG model (1) corresponding to the case $g > 0$. We will also assume that $\lambda > 0$, although its sign is unimportant due to the obvious symmetry $\Phi \to -\Phi$.

For $2\pi < \beta^2 < 8\pi$ the Gaussian model is perturbed by two relevant vertex operators closed under the OPE [1]. The existence of the Ising phase transition can be qualitatively understood via a quasi-classical analysis: inspecting the profile of the potential $\mathcal{U}(\Phi)$ as a function of the ratio $x = \lambda/4g$. At $x = 0$ one has a periodic potential of a single-frequency sine-Gordon (SG) model. For $x \neq 0$ the period of the potential is doubled in such a way that in the region $0 < x < 1$ it can be viewed as a sequence of double-well potentials with the local structure $A\Phi^2 + B\Phi^4$, where $A < 0, B > 0$. Precisely at $x = 1$ each local double-well potential transforms to a Φ^4 one ($A = 0$), and (in the Ginzburg–Landau sense) this is a signature of the Ising criticality with the central charge $c = 1/2$. It should be stressed that the double-well structure of the potential $\mathcal{U}(\Phi)$ only occurs for $g > 0$. Hence the Ising critical point only exist for positive g. The case of $g < 0$ is qualitatively different: here the λ-perturbation removes the degeneracy between neighbouring minima of $\cos\beta\Phi$ and thus leads to soliton confinement (similarly to the analysis in Ref[3]). The spectrum in this case always remains massive.

Thus, for model (1) with $g > 0$, a plausible scenario is that the relevant perturbations naturally act on the two constituent Ising models of the starting Gaussian model, leaving one of them massless on the critical line $\lambda = \lambda_c(g)$. (Quasi-classically $\lambda_c(g) = 4g$. A better estimation which takes into account quantum fluctuations yields a power law $\lambda_c(g) \sim g^\nu$, where $\nu = (32\pi - \beta^2)\,/\,(16\pi - \beta^2)$). Indeed, DM have argued that the Ising transition is a universal property of the DSG model (1) as long as $\beta^2 < 8\pi$. This makes it possible to consider the vicinity of the point $\beta^2 = 4\pi$ where the description of the transition greatly simplifies. Namely, it is well known[4, 5] that $\beta^2 = 4\pi$ is the decoupling (or Luther-Emery) point of the ordinary SG model ($\lambda = 0$), at which the latter is equivalent to a theory of free massive fermions. The scaling dimension $1/4$ of the λ-term in (1) indicates that this point is special for the DSG model as well, suggesting an Ising model interpretation. In this paper we shall be working in the vicinity of the decoupling point. Rescaling the field Φ, the DSG model can be written in an equivalent form:

$$\begin{aligned}\mathcal{H}_{DSG} &= v\left[(\partial_x\phi_R)^2+(\partial_x\phi_L)^2\right]-\gamma\partial_x\phi_R(x)\partial_x\phi_L(x)\\ &- g\cos\sqrt{4\pi}\Phi(x)-\lambda\sin\sqrt{\pi}\Phi(x),\end{aligned} \tag{3}$$

where $v=(v_0/2)(\beta/\sqrt{4\pi}+\sqrt{4\pi}/\beta)$ and $\gamma=(v_0/\sqrt{4\pi}\beta)(\beta^2-4\pi)$. It was originally noted by DM that this form of the DSG model can be related to a deformed 2D Ashkin–Teller (AT) model, which can be viewed as the standard AT model, i.e. a model of two marginally coupled 2D Ising models with order parameters σ_1 and σ_2, extended to include a magnetic field type coupling, $\lambda\sigma_1\sigma_2$. DM then argued that, in the strong-coupling limit ($\lambda\to\infty$), the system effectively reduces to a single Ising model which may become critical if the temperature is properly tuned to its critical value. This argument can be extended to finite values of λ, provided that the two Ising copies are originally in a disordered phase, and thus explains the existence of a critical line in the t-λ plane ($t=(T-T_c)/T_c>0$) along which the deformed AT model flows from the ultraviolet $c=1$ fixed point to the infrared $c=1/2$ Ising fixed point.

We shall elaborate on the DM argument by explicitly constructing the mappings between the DSG model and various spin models, with the aim to fully describe the Ising transition and, in particular, calculate correlation functions of the operators in the original DSG model. The novel points of our analysis are as follows. First, we concentrate on quantum lattice spin chain models rather than on their classical counterparts (in the transfer matrix sense), as this formulation profits from the use of the powerful apparatus of various spin operator transformations like the duality transformation. Secondly, we identify the correct degrees of freedom in the deformed quantum AT model that become critical. Since the original Ising models enter symmetrically, this is a nontrivial step which is accomplished via a 'change of basis' transformation. Let us denote the critical degrees of freedom by σ and the remaining gaped degrees of freedom by τ (see the main text for precise definitions). The strategy to calculate the correlation functions is then to express the DSG-operators in terms of the lattice σ and τ operators. At the Ising transition σ and τ operators asymptotically decouple, with the σ-operators being critical and (some of) the τ operators acquiring finite average values. This allows us to trace the relation between the original DSG-operators and those from the operator content of the underlying critical Ising model, which ultimately accomplishes a complete description of the critical properties at the Ising transition.

3 Relation between DSG model and deformed quantum Ashkin-Teller model

3.1 Quantum Ising spin chain

We start with recollecting some basic facts about the quantum Ising (QI) spin chain. The Hamiltonian of the QI chain describes a 1D Ising model in a transverse magnetic field [6]:

$$H_{QI}[\sigma] = -\sum_n \left(J\sigma_n^z \sigma_{n+1}^z + \Delta \sigma_n^x \right), \tag{4}$$

where σ_n^α are the Pauli matrices associated with the lattice sites $\{n\}$. The Hamiltonian H_{QI} defines the transfer matrix of the classical 2D Ising model [7].

An important tool in studying 1D spin lattice models is the Kramers-Wannier duality transformation that we shall make extensive use of in the sequel. Consider a dual lattice consisting of sites $\{n + 1/2\}$, defined as the centres of the links $< n, n+1 >$ of the original lattice, and assign spin operators $\mu_{n+1/2}$ to the dual lattice sites. The duality transformation then relates the dual spins to the original ones as follows

$$\mu_{n+1/2}^z = \prod_{j=1}^{n} \sigma_j^x, \quad \mu_{n+1/2}^x = \sigma_n^z \sigma_{n+1}^z, \tag{5}$$

the inverse transformation being

$$\sigma_n^z = \prod_{j=0}^{n-1} \mu_{j+1/2}^x, \quad \sigma_n^x = \mu_{n-1/2}^z \mu_{n+1/2}^z. \tag{6}$$

In Hamiltonian (4), the parameters J and Δ are interchanged under the duality transformation, so that $J = \Delta$ is the self-duality point where the model displays an Ising criticality.

The operators σ_n^z and $\mu_{m+1/2}^z$, conventionally referred to as order and disorder operators, are mutually nonlocal and play the role of 'string operators' with respect to each other. In particular, they commute for $m < n$ but anticommute otherwise. These commutation properties make the duality construction a convenient starting point for introducing lattice fermions. It is immediate to check that two objects

$$\eta_n = \sigma_n^z \mu_{n-1/2}^z = \mu_{n-1/2}^z \sigma_n^z, \tag{7}$$

$$\zeta_n = \mathrm{i}\sigma_n^z \mu_{n+1/2}^z = -\mathrm{i}\mu_{n+1/2}^z \sigma_n^z, \tag{8}$$

satisfy the anticommutation relations for the real (Majorana) fermions on the lattice, with the normalisation $\eta_n^2 = \zeta_n^2 = 1$. Relations (7) and (8) are nothing but the inverse of the Jordan–Wigner transformation, with the 'direct' transformation being of the form

$$\sigma_n^x = i\zeta_n\eta_n, \quad \sigma_n^z = \eta_n \prod_{j=1}^{n-1} (i\zeta_j\eta_j)\,. \tag{9}$$

In terms of the Majorana fermions the QI Hamiltonian becomes

$$H_{QI} = \mathrm{i}\sum_n \left[J\zeta_n(\eta_{n+1} - \eta_n) - (\Delta - J)\zeta_n\eta_n\right]. \tag{10}$$

The next step is to take a continuum limit. To this end one introduces a lattice spacing a_0, treats $x = na_0$ as a continuum variable, and replaces η_n and ζ_n by slowly varying Majorana fields, $\eta(x)$ and $\zeta(x)$:

$$\eta_n \to \sqrt{2a_0}\eta(x), \quad \zeta_n \to \sqrt{2a_0}\zeta(x).$$

Notice that the factor $\sqrt{2}$ ensures the correct continuum anticommutation relations: $\{\eta(x), \eta(y)\} = \{\zeta(x), \zeta(y)\} = \delta(x - y)$. The Hamiltonian density of (10) then is

$$\mathcal{H}_{QI} = \mathrm{i}v\zeta\partial_x\eta - \mathrm{i}m\zeta\eta,$$

with $v = 2Ja_0$ and $m = 2(\Delta - J)$. Performing a chiral rotation of the Majorana spinor

$$\xi_R = \frac{-\eta + \zeta}{\sqrt{2}}, \quad \xi_L = \frac{\eta + \zeta}{\sqrt{2}}, \tag{11}$$

or, inversely,

$$\eta = \frac{-\xi_R + \xi_L}{\sqrt{2}}, \quad \zeta = \frac{\xi_R + \xi_L}{\sqrt{2}}, \tag{12}$$

transforms this Hamiltonian to a standard form:

$$\mathcal{H}_M^{(m)} = \frac{\mathrm{i}v}{2}(-\xi_R\partial_x\xi_R + \xi_L\partial_x\xi_L) - \mathrm{i}m\xi_R\xi_L. \tag{13}$$

The relation between the QI model and the massive Majorana QFT is therefore summarised as follows

$$\lim_{a_0\to 0} H_{QI} = \int \mathrm{d}x\ \mathcal{H}_M^{(m)}(x) \tag{14}$$

3.2 Bosonization of the deformed quantum Ashkin–Teller model

The standard quantum Ashkin-Teller (QAT) model is defined as a model of two identical QI spin chains, described by Hamiltonians $H_{QI}[\sigma_1]$ and $H_{QI}[\sigma_2]$ [see

Eq.(4)], which are coupled via a self-dual inter-chain interaction:

$$\begin{aligned} H_{QAT} &= H_{QI}[\sigma_1] + H_{QI}[\sigma_2] + H'_{AT}[\sigma_1, \sigma_2], \qquad (15) \\ H'_{AT}[\sigma_1, \sigma_2] &= K \sum_n \left(\sigma^z_{1,n}\sigma^z_{1,n+1}\sigma^z_{2,n}\sigma^z_{2,n+1} + \sigma^x_{1,n}\sigma^x_{2,n} \right). \qquad (16) \end{aligned}$$

We will be interested in a deformed version of this model which, apart from H'_{AT}, includes a 'magnetic field' type of coupling between the chains:

$$H_{DQAT} = H_{QAT} - h \sum_n \sigma^z_{1,n}\sigma^z_{2,n} \qquad (17)$$

We wish to establish a relationship between the deformed quantum Ashkin–Teller (DQAT) model (17) and the DSG model. These two models can be mapped onto each other using the Zuber-Itsykson trick[8]. The idea is to associate the two QI chains with two copies of Majorana fermions,

$$\begin{aligned} (\sigma_1, \mu_1) &\Rightarrow \left(\eta^1, \zeta^1\right) \Rightarrow \left(\xi^1_R, \xi^1_L\right), \\ (\sigma_2, \mu_2) &\Rightarrow \left(\eta^2, \zeta^2\right) \Rightarrow \left(\xi^2_R, \xi^2_L\right), \end{aligned}$$

combine ξ^1 and ξ^2 into a single Dirac field and then bosonize the latter using the standard rules of Abelian bosonization.

First we notice that, in the case of two QI spin chains, the Jordan–Wigner transformation (7)–(9) should be slightly modified. This follows from the requirement that the spin operators belonging to different chains should commute. While this is automatically true for the disorder operators $\mu^\alpha_{a,n+1/2}$ ($\alpha = x, y, z;\ a = 1, 2$) because of their bosonic character, to ensure commutation between the order parameters σ^z_{1n} and σ^z_{2n} one has to introduce two anticommuting (Klein) factors

$$\{\kappa_1, \kappa_2\} = 0, \quad \kappa_1^2 = \kappa_2^2 = 1, \qquad (18)$$

and replace (7), (8) by

$$\begin{aligned} \eta_{1,n} &= \kappa_1 \sigma^z_{1,n}\mu^z_{1,n-1/2}, \quad \zeta_{1,n} = i\kappa_1 \sigma^z_{1,n}\mu^z_{1,n+1/2}, \\ \eta_{2,n} &= \kappa_2 \sigma^z_{2,n}\mu^z_{2,n-1/2}, \quad \zeta_{2,n} = i\kappa_2 \sigma^z_{2,n}\mu^z_{2,n+1/2}. \qquad (19) \end{aligned}$$

With the spin variables $\sigma_{1,2}$ and $\mu_{1,2}$ subject to the duality relations (5), (6), definitions (19) ensure the correct statistics for the Majorana fermions $\eta_{a,n}$ and $\zeta_{a,n}$.

In terms of the lattice Majorana fermions, the standard QAT Hamiltonian becomes

$$\begin{aligned} H_{QAT} &= \mathrm{i} \sum_{a=1,2} \sum_n \left[J\zeta_{a,n}(\eta_{a,n+1} - \eta_{a,n}) - (\Delta - J)\zeta_{a,n}\eta_{a,n} \right] \\ &+ K \sum_n \zeta_{1,n}\zeta_{2,n} \left[\eta_{1,n+1}\eta_{2,n+1} + \eta_{1,n}\eta_{2,n} \right], \qquad (20) \end{aligned}$$

and admits a straightforward passage to the continuum limit. The corresponding Hamiltonian density reduces to a theory of two interacting massive Majorana fermions

$$\mathcal{H}_{QAT}(x) = \sum_{a=1,2} \mathcal{H}_M^{(m)}[\xi_a(x)] + 8Ka_0\xi_{1R}(x)\xi_{2R}(x)\xi_{1L}(x)\xi_{2L}(x), \tag{21}$$

which is obviously equivalent to the massive Thirring model for a single Dirac fermion. Using the standard rules of Abelian bosonization which are briefly summarised in Appendix A.1, one maps the QAT model onto a $\beta^2 = 4\pi$ quantum SG model with a marginal perturbation [4]:

$$\mathcal{H}_{QAT} \Rightarrow \mathcal{H}_0[\Phi] - 8Ka_0\partial_x\phi_R\partial_x\phi_L - (m/a_0)\cos\sqrt{4\pi}\Phi. \tag{22}$$

Notice that, the bosonization of the QAT model can be also achieved by first mapping the model onto a single chain of spinless fermions, and then bosonizing the latter. This alternative route is traced in Appendix B.

Turning to the DQAT model, we observe that the h-term in (17) is nonlocal in terms of lattice Majorana operators. However, it is known that, for two identical Ising models close to criticality, products of two Ising operators belonging to different chains, $\sigma_1\sigma_2$, $\mu_1\mu_2$, $\sigma_1\mu_2$, $\mu_1\sigma_2$, all with scaling dimension 1/4, can be expressed *locally* in terms of bosonic vertex operators with the same scaling dimension (see, for instance, [9] and references therein). In Appendix A.2, we rederive this correspondence starting from the lattice theory of two quantum Ising chains and paying special attention to the Klein factors:

$$\begin{aligned}
\mu^z_{1,n+1/2}\mu^z_{2,n+1/2} &= \ :\cos\sqrt{\pi}\Phi(x):, &(23)\\
\sigma^z_{1n}\sigma^z_{2n} &= -\mathrm{i}\kappa_1\kappa_2 :\sin\sqrt{\pi}\Phi(x):, &(24)\\
\sigma^z_{1n}\mu^z_{2,n+1/2} &= -\mathrm{i}\kappa_1 :\cos\sqrt{\pi}\Theta(x):, &(25)\\
\mu^z_{1,n+1/2}\sigma^z_{2n} &= \mathrm{i}\kappa_2 :\sin\sqrt{\pi}\Theta(x): . &(26)
\end{aligned}$$

Here $\Theta = -\phi_R + \phi_L$ is a scalar field dual to Φ.

Thus, Eq. (24) establishes the continuum bosonized version of the "magnetic field" coupling term in (17):

$$h\sum_n \sigma^z_{1n}\sigma^z_{2n} = \mathrm{i}\left(\frac{h}{a_0}\right)\kappa_1\kappa_2 \int \mathrm{d}x \ :\sin\sqrt{\pi}\Phi(x): . \tag{27}$$

Notice that algebra (18) of the Klein factors allows one to identify them with Pauli matrices,

$$\kappa_1 = \tau_1, \quad \kappa_2 = \tau_2, \quad \kappa_1\kappa_2 = \mathrm{i}\tau_3.$$

Therefore, in the continuum limit, the DQAT model can be represented in a diagonal 2×2 matrix form:

$$\begin{aligned}
\lim_{a_0\to 0} H_{DQAT} &= \int \mathrm{d}x\ \hat{\mathcal{H}}(x), \\
\hat{\mathcal{H}}(x) &= \begin{pmatrix} \mathcal{H}_+(x) & 0 \\ 0 & \mathcal{H}_-(x) \end{pmatrix},
\end{aligned} \tag{28}$$

with the two Hamiltonians

$$\begin{aligned}
\mathcal{H}_\pm(x) &= \mathcal{H}_0[\Phi(x)] - 8Ka_0\partial_x\phi_R(x)\partial_x\phi_L(x) \\
&- \left(\frac{m}{a_0}\right)\cos\sqrt{4\pi}\Phi(x) \mp \left(\frac{h}{a_0}\right)\sin\sqrt{\pi}\Phi(x)\,,
\end{aligned} \tag{29}$$

having the structure of the DSG model (3) and differing only in the sign of the coupling constant h. The identification of the parameters is as follows: $\gamma = 8Ka_0$, $g = m/a_0$, and $\lambda = h/a_0$ (it is understood that when $a_0 \to 0$, $K \to \infty$, $m, h \to 0$ so as to keep the DSG parameters finite).

The 2×2 matrix form of $\hat{\mathcal{H}}$ reflects the symmetry of the DQAT model with respect to the interchange of the two constituent quantum chains, ($\mathcal{P}_{12}$), under which the DSG Hamiltonians $\mathcal{H}_\pm$ transform to each other. Formally, this symmetry appears as "unphysical" because the Hamiltonian $\hat{\mathcal{H}}$ commutes with τ_3. This allows one to set

$$\kappa_1\kappa_2 = \mathrm{i}, \tag{30}$$

and study the Ising transition by projecting the matrix DQAT model onto the subspace of a single DSG Hamiltonian $\mathcal{H}_+$ (the choice $\kappa_1\kappa_2 = -\mathrm{i}$, $\mathcal{H} \to \mathcal{H}_-$ would be as good as the above one). However, while $\kappa_1\kappa_2$ is a conserved quantity, each κ_i is not, implying that correlations functions of operators which contain the Klein factors may involve transitions between the two different sectors $\tau_3 = \pm 1$.

We shall see below that some physical models are automatically mapped onto a single DSG model with a fixed sign of h (section 5), while some others reduce to the matrix form (28) (section 6). In the latter case, the effective $\mathcal{P}_{12}$ symmetry corresponds to a *physical* discrete Z_2 symmetry of the model in question, which is spontaneously broken in the ground state, a typical example of this kind being a spontaneously dimerized phase. In such situations, the Hamiltonians $H_\pm$, when considered separately, describe only excitations above each of the two degenerate vacua. On the other hand, the two-fold degeneracy of the ground state necessarily implies the existence of topological kinks which interpolate between neighbouring degenerate minima of two *different* potentials $\mathcal{U}_\pm$, corresponding to $\mathcal{H}_\pm$. However, in the entire parameter space of the model (28), including the Ising transition point, the sets of minima of the potentials $\mathcal{U}_\pm$ always have a finite relative shift. This means that the topological Z_2 kinks always remain massive and thus appear as irrelevant excitations as long as Ising criticality is concerned.

Thus, when working with a single Hamiltonian $\mathcal{H}_+$, all operators representing strongly fluctuating fields at the Ising critical point are automatically taken into account. All "off-diagonal" operators proportional to κ_1 or κ_2, such as those given by (25) and (26), can be dropped as they represent short-ranged fields at the critical point.

4 Deformed quantum Ashkin-Teller model in the (σ, τ) representation and the Ising transition

The original two-chain representation of the Hamiltonian (17) is not the most appropriate one to describe the role of the coupling term (27), but some general statements can already be drawn at this stage. First of all it is clear that, if the chains are originally in the ordered phase, the role of the h-term would be exhausted by removing the degeneracy between the four ground states specified by the signs of the order parameters $\sigma_1 = \langle \sigma^z_{1n} \rangle$ and $\sigma_2 = \langle \sigma^z_{2n} \rangle$. The lowest-energy sector, determined by the condition $h\sigma_1\sigma_2 > 0$, will stay massive anyway. (For two ordered Ising copies, the h-term gives rise to an effective longitudinal magnetic field applied to both Ising systems and thus keeping them massive.) Therefore, new effects can be only expected if the two chains are originally disordered.

We can give an heuristic argument in favour of an Ising critical point that appears at a finite h, if at $h = 0$ the chains are disordered. For the sake of clarity, let us make a duality transformation and replace the original model by a pair of two ordered Ising copies well below T_c, coupled by the interaction $h\mu_1\mu_2$. Consider a single quantum Ising model (4) at $J \gg \Delta$. In the leading approximation, its ground state is fully polarised. The disorder operator $\mu^z_{n+1/2}$ creates a domain wall by flipping all the spins within the interval $1 \leq j \leq n$. At short distances, the domain walls behave like hard-core bosons which, in the dilute limit, are equivalent to spinless fermions with the dispersion

$$\epsilon(k) = 2J - 2\Delta \cos ka_0.$$

The coupling term $h\mu_1\mu_2$ creates or destroys a pair of domains walls, one for each chain. Therefore, in the dilute limit, the effective Hamiltonian for the fermions describing domain walls is of the form:

$$H = \sum_k \sum_{i=1,2} \epsilon(k) c^\dagger_{i,k} c_{i,k} + h_{\mathit{eff}} \sum_k \left(c^\dagger_{1,-k} + c_{1,k}\right)\left(c^\dagger_{2,k} + c_{2,-k}\right) \tag{31}$$

where $h_{\mathit{eff}} \propto \Delta h/J$. The combinations of the creation and annihilation operators that appear in the coupling term reflect the fact that $\left[\mu^z_{n+1/2}\right]^2 = 1$. It is clear that the model (31) is a theory of four Majorana fermions, two of which are not affected by the coupling term. The remaining part of Hamiltonian can be straightforwardly diagonalized and reduced to two Majorana fields with the masses

$$m_\pm = 2J \pm \text{const.}\frac{\Delta h}{J}.$$

The Ising criticality is reached when one of the masses vanishes, which occurs when $h \sim J^2/\Delta$. In the dual representation, the above estimation of the critical field should be replaced by $h \sim \Delta^2/J$. Strictly speaking, the domain walls do not posses a fermionic statistics but contain a Jordan-Wigner type phase, which can only be neglected in the extreme dilute limit.

Although the above argument provides a qualitative explanation of the transition, a satisfying description of the critical region can only be reached if we are able to properly identify the degrees of freedom which get critical and those which do not (and, of course, take into account the correct stastistics of the domain walls). For this purpose, let us introduce a new Ising variable

$$\tau^z_n = \sigma^z_{1,n}\sigma^z_{2,n}. \tag{32}$$

Namely, let us switch from the original two-chain representation, with basic spin operators σ^z_{1n} and σ^z_{2n}, to a new one where the basic variables are τ^z_n and $\sigma^z_n = \sigma^z_{1n}$ (due to the $\mathcal{P}_{12}$ symmetry of the model, the choice τ^z_n and $\sigma^z_n = \sigma^z_{2n}$ would be equivalent). In the original (σ_1-σ_2) representation, the local (i.e. at a fixed lattice site n) Hilbert space of the two-chain model is spanned by the basis vectors $|\sigma_1, \sigma_2\rangle$ which are eigenstates of σ^z_1 and σ^z_2:

$$\sigma^z_{1,2}|\sigma_1, \sigma_2\rangle = \sigma_{1,2}|\sigma_1, \sigma_2\rangle, \quad \sigma_{1,2} = \pm 1.$$

The new local basis $|\sigma, \tau\rangle$ is defined as

$$\sigma^z|\sigma, \tau\rangle = \sigma|\sigma, \tau\rangle, \quad \tau^z|\sigma, \tau\rangle = \tau|\sigma, \tau\rangle,$$

where $\sigma = \sigma_1,\ \tau = \sigma_1\sigma_2$. Comparing matrix elements of the operators σ^α_{1n} and σ^α_{2n} in the two bases, we find the following correspondence:

$$\begin{aligned} \sigma^z_{1n} &= \sigma^z_n, \qquad & \sigma^z_{2n} &= \sigma^z_n\tau^z_n, \\ \sigma^x_{1n} &= \sigma^x_n\tau^x_n, \qquad & \sigma^x_{2n} &= \tau^x_n. \end{aligned} \tag{33}$$

We also need to define the variables μ_n and ν_n dual to σ_n and τ_n, respectively. The pairs (σ_n, μ_n) and (τ_n, ν_n) should obey the duality relations (5) and (6). Using these relations together with (33), one finds out how the dual spins transform under the change of basis:

$$\begin{aligned} \mu^z_{1,n+1/2} &= \mu^z_{n+1/2}\nu^z_{n+1/2}, \qquad & \mu^z_{2,n+1/2} &= \nu^z_{n+1/2}, \\ \mu^x_{1,n+1/2} &= \mu^x_{n+1/2}, \qquad & \mu^x_{2,n+1/2} &= \mu^x_{n+1/2}\nu^x_{n+1/2}. \end{aligned} \tag{34}$$

In the σ-τ representation the DQAT model (17) transforms to another two-chain model which we call the σ-τ model:

$$H[\sigma, \tau] = H_\sigma + H_\tau + H_{\sigma\tau}. \tag{35}$$

Here H_σ is a QI Hamiltonian similar to (4) but with different parameters:

$$H_\sigma = \sum_n \left(-J\sigma^z_n\sigma^z_{n+1} + K\sigma^x_n \right). \tag{36}$$

H_τ is also of the QI type model but the magnetic field is nonzero both in the transverse and longitudinal directions:

$$H_\tau = K\sum_n \tau^z_n\tau^z_{n+1} - \sum_n \left(h\tau^z_n + \Delta\tau^x_n \right). \tag{37}$$

Finally, the coupling term is of the Ashkin-Teller type:

$$H_{\sigma\tau} = -\sum_n \left(J\sigma^z_n\sigma^z_{n+1}\tau^z_n\tau^z_{n+1} + \Delta\sigma^x_n\tau^x_n \right). \tag{38}$$

A mean-field approach to (35) is outlined in Appendix C, here we shall concentrate on the large-h limit. For large h, the τ-degrees of freedom freeze in a configuration where $\langle\tau^z_n\rangle \simeq 1$, and $\langle\tau^x_n\rangle \simeq \Delta/h$. The σ-degrees of freedom are then described by an effective Ising model

$$H[\sigma] \to -\sum_n \left(2J\sigma^z_n\sigma^z_{n+1} + \left(\frac{\Delta^2}{h} - K \right) \sigma^x_n \right), \tag{39}$$

which can indeed become critical when $\Delta \sim \sqrt{Jh}$. Although this simple picture holds only when $h \gg \Delta \gg J, K$, if a universal behaviour is to be expected, then we are lead to conclude that, both at strong and weak h, the Ising transition essentially corresponds the situation when the σ degrees of freedom go massless, while the τ degrees of freedom remain frozen in a disordered configuration with both $\langle \tau^z \rangle$ and $\langle \tau^x \rangle$ nonzero. Notice that, with the above strong-coupling description, we are unable to determine the critical line and even estimate the strength of the *irrelevant* operators close to this line. This means that, even though the universal properties of the DSG model at the Ising transition, including the singular parts of physical quantities and critical exponents of the correlation functions, will be captured correctly, the nonuniversal parts, such as prefactors and subleading corrections to main asymptotics, are not to be trusted.

Let us conclude this section by noticing two important facts which will prove useful in the next Section devoted to the correlation functions.

(i) Quantum critical points are associated with gapless phases which are realized under certain conditions imposed on the parameters of the model. As soon as any of those parameters is shifted away from the criticality constraint, the system becomes off-critical. The corresponding perturbations to the critical Hamiltonian are therefore *relevant* operators with conformal dimensions determined by the universality class of the critical model. Notice that there is no magnetic field coupled to σ^z in the exact lattice Hamiltonian (35). Therefore a departure from criticality by changing any one of the couplings (h, J, Δ_J, or K) will give rise to an Ising mass term.

(ii) From Eq.(39), if we reasonably take $|K| < 2J$, we arrive at the conclusion that, for $h > h_c$, the σ degrees of freedom are ordered, while they are disordered otherwise.

5 Correlation functions

In this Section we use the results of the above quantum-spin-chain type mappings to determine the physical properties of the DSG model close to the criticality.

5.1 DSG operators in the (σ, τ) representation: UV-IR transmutation

In preceding sections we have shown that (i) the DQAT model can be mapped onto the DSG model which is a Gaussian free theory of the field $\Phi(x)$ in the ultraviolet (UV) limit, and that (ii) the DQAT can also be mapped onto the (σ, τ) model which, at the Ising transition, essentially reduces to a single critical Ising model of the order field $\sigma(x)$ and disorder field $\mu(x)$ in the infrared (IR) limit. Our aim here is to find out how the operators of the DSG model, originally defined in the vicinity of the UV fixed point, "transmute" when going from the UV limit to the IR limit.

5.1.1 Current operators

We start with holomorphic, or current, operators that are made up of additive (analytic and anti-analytic) chiral parts. Of physical interest are the vector and axial current densities which, in terms of the bosonic field of the DSG model, are defined as

$$\begin{aligned} J(x) &= J_R(x) + J_L(x) = \frac{1}{\sqrt{\pi}} \partial_x \Phi(x), && (40) \\ J_5(x) &= J_R(x) - J_L(x) = -\frac{1}{\sqrt{\pi}} \partial_x \Theta(x) = -\frac{1}{\sqrt{\pi}} \partial_t \Phi(x). && (41) \end{aligned}$$

In physical situations, $J(x)$ determines the smooth part of the charge or spin density $[J(x) \equiv \rho(x)]$, while $J_5(x)$ describes the corresponding charge or spin current $[J_5(x) \equiv j(x)]$. As follows from (114), (115), these can be expressed in terms of the Majorana fields ξ^1 and ξ^2:

$$\begin{aligned} J(x) &= \mathrm{i} \left[\xi_{1R}(x) \xi_{2R}(x) + \xi_{1L}(x) \xi_{2L}(x) \right], && (42) \\ J_5(x) &= \mathrm{i} \left[\xi_{1R}(x) \xi_{2R}(x) - \xi_{1L}(x) \xi_{2L}(x) \right]. && (43) \end{aligned}$$

Making the inverse chiral rotation from (ξ^a_R, ξ^a_L) to (η^a, ζ^a), $(a = 1, 2)$, we can define a local lattice operator

$$J_n = \frac{\mathrm{i}}{2} \left(\eta_{1n} \eta_{2n} + \zeta_{1n} \zeta_{2n} \right), \qquad (44)$$

which reproduces (42) in the continuum limit. Using the inverse Jordan-Wigner relations (19) and transformations (33),(34), we obtain:

$$\begin{aligned} J_n &= -\frac{\mathrm{i}}{2} \kappa_1 \kappa_2 \sigma^z_{1n} \sigma^z_{2n} \left(\mu^z_{1,n+1/2} \mu^z_{2,n+1/2} - \mu^z_{1,n-1/2} \mu^z_{2,n-1/2} \right) \\ &= \frac{1}{2} \tau^z_n \left(\mu^z_{n+1/2} - \mu^z_{n-1/2} \right) && (45) \end{aligned}$$

(here we have implemented our Klein factor convention (30)). Using the fact that the τ-field is noncritical and has a nonzero expectation value, we pass to the continuum limit and thus find the expression for the current density at the infrared fixed point:

$$J(x) \to C\partial_x \mu(x), \tag{46}$$

where $C \sim \langle \tau^z \rangle$ is a nonuniversal number, and $\mu(x)$ is the Ising disorder field at the criticality. Thus, the UV-IR transmutation of the current density is given by

$$J(x) = \begin{cases} \frac{1}{\sqrt{\pi}}\partial_x \Phi(x) & \text{UV}, \\ C\partial_x \mu(x) & \text{IR}. \end{cases} \tag{47}$$

Turning to the axial current J_5, we notice that the latter is related to the vector current J via the continuity equation:

$$\partial_t J(x,t) + \partial_x J_5(x,t) = 0. \tag{48}$$

As a result, the IR form of $J_5(x)$ can be immediately recovered. Indeed, the Ising disorder field μ is a scalar field (with zero conformal spin). This means that holomorphic properties of the vector and axial current operators are lost at the IR fixed point. Then the result (46), together with the requirements of Lorentz invariance and continuity equation (48), leads to the correspondence

$$J_5(x) = \begin{cases} -\frac{1}{\sqrt{\pi}}\partial_t \Phi(x) & \text{UV}, \\ -C\partial_t \mu(x) & \text{IR}, \end{cases} \tag{49}$$

with the same prefactor C as in Eq.(47), provided that the velocity is set $v = 1$.

As an important consistency check, we still need to explicitly determine the lattice version of the current operator. One might naïvely conclude that, since under the inverse chiral rotation (12) the axial current density transforms to

$$J_5(x) = -\mathrm{i}\left[\zeta_1(x)\eta_2(x) + \eta_1(x)\zeta_2(x)\right], \tag{50}$$

it would be sufficient to replace the r.h.s. of (50) by its local (single-site) counterpart which one might identify with J_n^5. This route is misleading because, in the lattice formulation, the particle current is always defined on a *link* $(n, n+1)$ and, therefore, should be determined from the equation of motion

$$\mathrm{i}\partial_t J_n = \left[J_n, H_{DQAT}\right]. \tag{51}$$

which, in the continuum limit, is supposed to reduce to Eq.(48). It can easily be checked that the (density) operator, J_n, commutes with all the interaction terms in the lattice DQAT Hamiltonian (17), since all of them, including the h-term, are made up of the density operators. Therefore the above commutator is only contributed to by the Majorana kinetic energy term in (17) and so is easily computed:

$$\mathrm{i}\partial_t J_n = Q_{n,n+1} - Q_{n-1,n}, \tag{52}$$

where

$$Q_{n,n+1} = -\frac{J}{4}\left(\eta_{1,n+1}\zeta_{2n} + \zeta_{1n}\eta_{2,n+1}\right). \tag{53}$$

In terms of the lattice spins this reads

$$J^5_{n,n+1} = -\frac{J}{2}(\tau^z_n + \tau^z_{n+1})\mu^y_{n+1/2}. \tag{54}$$

In the IR continuum limit this becomes $\sim \partial_t \mu$ since $\partial_t \mu \sim i[\mu^z, H_{QI}] \sim \mu^y$, as immediately follows from the dual version of the QI spin chain Hamiltonian. Thus we arrive at the IR representation of the axial current given by (49).

5.1.2 Vertex operators

The analysis of the vertex operators

$$V_\beta[\Phi] = e^{i\beta\Phi}$$

is in fact simpler than that of the holomorphic operators. There are two cases of physical interest: $\beta = \pm\sqrt{\pi}$ and $\beta = \pm\sqrt{4\pi}$.

From (24), (30) and (32) it follows that

$$\sin\sqrt{\pi}\Phi \sim \sigma_1\sigma_2 \sim \tau \sim I, \tag{55}$$

(I being the identity operator), as the τ model is always off-critical. This is a reasonable result since the above operator is directly present in the DSG Hamiltonian (3). However, as we mentioned in the previous Section, the operator $\sin\sqrt{\pi}\Phi$ corresponds to the departure from the criticality of the σ-model. So, this operator, though having a finite average value, should also possess an extra term which, at the critical point, represents a strongly fluctuating field (with a power-law decaying correlation function). A more correct version of the formula (55) is therefore as follows:

$$\sin\sqrt{\pi}\Phi \sim I + \varepsilon, \tag{56}$$

where ε is the energy density (or a Majorana mass bilinear) operator. This follows from the first observation **(i)** at the end of Section 4, which states that moving h from its critical value results in the stress-energy tensor renormalisation

Table 1: UV-IR transmutation of DSG-operators

UV limit	$\partial_x \Phi$	$V_{\sqrt{\pi}} + V_{-\sqrt{\pi}}$	$V_{\sqrt{\pi}} - V_{-\sqrt{\pi}}$, $V_{\pm\sqrt{4\pi}}$
IR limit	$\partial_x \mu$	μ	$I + \epsilon$

and, more importantly, in the appearance of the Majorana mass, i.e. the Ising energy-density operator.

Furthermore, (23) and the lattice fusion rule (34) give

$$\cos\sqrt{\pi}\Phi \sim \mu_1\mu_2 \sim \mu. \tag{57}$$

Thus, the operator $\cos\sqrt{\pi}\Phi$ is the most divergent operator of the DSG model, with a nonzero expectation value at $h < h_c$ and vanishing upon approaching the Ising critical point as

$$\langle\mu\rangle \sim (h_c - h)^{1/8}\,. \tag{58}$$

Finally, the behaviour of $\beta = \pm\sqrt{4\pi}$ operators is determined in full analogy to the above discussion:

$$V_{\pm\sqrt{4\pi}}[\Phi] \sim I + \epsilon. \tag{59}$$

Our results on UV-IR DSG operators transmutations are summarised in table I.

5.2 Correlation functions

Having identified the operators at the IR fixed point, we can now determine leading asymptotics of the correlation functions. Most of the operators of physical interest can be expressed in terms of the Ising disorder operator μ at the Ising transition point and in its close vicinity.

For physical applications of the DSG model, it is important to investigate the dynamical susceptibility defined as the frequency-momentum Fourier transform of the retarded auto-correlation function of the Ising disorder parameter: function

$$D^{(R)}(\omega, p) = -i\int_{-\infty}^{\infty} dx \int_0^{\infty} dt e^{-ipx+i\omega t}\langle[\mu(x,t), \mu(0,0)]\rangle. \tag{60}$$

It is known that, at criticality, $D(r) = 1/r^{1/4}$ where $\mathbf{r} = (\tau, x)$ ($\tau =$ it, and v is set to 1). Furthermore, away from criticality, in the *ordered* phase, $D(r) =$

$(A_1/\pi)K_0(mr)$ where A_1 is related to the Glaisher constant. The crossover between these two regimes, $r \ll \xi = 1/m$ and $r > \xi$, respectively, is complicated and is described in terms of the Painlevé theory [10]. If, following [10], we introduce,

$$\zeta = r\frac{d\ln D}{dr},$$

then the function ζ can be shown to satisfy

$$(r\zeta'')^2 = 4(r\zeta' - \zeta^2)(r\zeta' - \zeta) + (\zeta')^2,$$

which is related to Painlevé V equation and can, in turn, be shown to produce the correct conformal and massive limits. As the exact expressions are uncomfortable for calculations, we shall estimate the Fourier transforms from the asymptotes of the correlation function $D(r)$. Therefore, first we summarise known results on the limiting behaviour of this function.

At criticality

$$D_0(r) = \frac{1}{r^{1/4}}. \tag{61}$$

It is understood that this asymptotics is still valid away from criticality in the region $r \ll \xi$.

In the ordered phase

$$D_>(r) = \frac{A_1}{\pi}K_0(mr), \tag{62}$$

the large-r limit of which is

$$D_>(r) = \frac{A_1\sqrt{\pi}}{\sqrt{2m}}\frac{1}{r^{1/2}}e^{-mr}. \tag{63}$$

In the disordered phase, thanks to the work by McCoy, Wu, Tracy, and collaborators [10], we know that

$$\begin{aligned} D_<(r) &= \frac{A_1}{\pi^2}\left\{m^2r^2\left[K_1^2(mr) - K_0^2(mr)\right] - mrK_0(mr)K_1(mr)\right. \\ &+ \left.\frac{1}{2}K_0^2(mr)\right\}, \end{aligned} \tag{64}$$

which is understood to be the connected part of the correlator (i.e. without the constant piece). The large-r asymptotics of this function can be obtained from the relevant expansions of the modified Bessel functions quoted e.g. in [9]:

$$D_<(r) = \frac{A_1}{8\pi m^2}\frac{1}{r^2}e^{-2mr}. \tag{65}$$

The retarded function (60) can be found by calculating the Fourier transforms of the above asymptotic forms with a subsequent analytic continuation to real frequencies. This procedure is outlined in Appendix D. The results are as follows.

- **Criticality.** Using (186) and (182), one obtains

$$\begin{aligned} D_0^{(R)}(\omega,p) &= 2^{1/4}\pi\frac{\Gamma(7/8)}{\Gamma(1/8)}\left\{\frac{\theta(p^2-\omega^2)+\cos(7\pi/8)\theta(\omega^2-p^2)}{|\omega^2-p^2|^{7/8}}\right. \\ &- \left.\frac{2i\sin(7\pi/8)\theta(\omega^2-p^2)}{|\omega^2-p^2|^{7/8}}\right\}. \end{aligned} \tag{66}$$

- **Ordered Phase.** Here we have a particle pole (181):

$$D_>^{(R)}(\omega,p) = \frac{\pi A_1}{-\omega^2+\epsilon_p^2} - \frac{i\pi^2 A_1}{2\epsilon_p}\left[\delta(\omega+\epsilon_p)+\delta(\omega-\epsilon_p)\right]. \tag{67}$$

- **Disordered phase.** Specialising to $\alpha = -1/2$ in (186) we have:

$$f(\omega) = \sqrt{|\omega^2-\epsilon_p^2|}\theta(\epsilon_p^2-\omega^2) + 2i\sqrt{|\omega^2-\epsilon_p^2|}\theta(\omega^2-\epsilon_p^2). \tag{68}$$

  Therefore, as long as $\omega^2 < p^2+4m^2$, the argument of the log-function in (185) is real and positive, so the correlation function remains real without dissipation. Above the two-particle threshold ($\omega^2 > p^2+4m^2$), the correlation function has a branch cut and the dissipative part will appear as follows:

$$\begin{aligned} & D_<^{(R)}(\omega,p) = \\ & \frac{A_1}{8m^2}\left\{\ln\left(\frac{4m}{\sqrt{|-\omega^2+p^2+4m^2|}+2m}\right)\theta(p^2+4m^2-\omega^2)\right. \\ & \left.-i\,\mathrm{Arctg}\frac{\sqrt{\omega^2-p^2-4m^2}}{m}\theta(\omega^2-p^2-4m^2)\right\}. \end{aligned} \tag{69}$$

The excitations are therefore incoherent in this regime.

6 Dimerized Heisenberg chain in a staggered magnetic field

The DSG model exhibiting a nontrivial flow towards Ising criticality can be realized as an effective continuum theory for a number of quantum 1D models of strongly correlated electrons, in particular quantum spin chains and ladders. In the context of spin systems, an effective DSG model can emerge within the

Abelian bosonization scheme when staggered fields breaking translational invariance, such as an explicit dimerization (bond alternation)[3] or a staggered magnetic field, are added to an originally translationally invariant model with a gaped ground state[1].

Perhaps the simplest example of this kind is given by the spin-1/2 Heisenberg chain with nearest-neighbour (J_1) and next-nearest-neighbour (J_2) antiferromagnetic exchange interactions

$$H_{J_1-J_2} = \sum_n \left(J_1 \mathbf{S}_n \cdot \mathbf{S}_{n+1} + J_2 \mathbf{S}_n \cdot \mathbf{S}_{n+2}\right) \tag{70}$$

This model has been extensively studied during past years. If frustrating interaction J_2 is small enough, the model maintains the critical properties of the unfrustrated Heisenberg chain ($J_2 = 0$). At $J_2 \geq J_{2c} \simeq 0.24 J_1$, frustration gets relevant and drives the model to a massive phase characterised by spontaneously broken parity[13, 14]. The ground state is dimerized and doubly degenerate, and there exist massive elementary excitations - topological Z_2 kinks carrying the spin 1/2. At a special (Majumdar–Ghosh) point, $J_2 = 0.5 J_1$, the picture is particularly simple because the two Z_2-degenerate ground states are given by matrix products of singlet dimers formed either on the lattice links $< 2n, 2n+1 >$ or $< 2n-1, 2n >$.

Under assumption that $J_1 \gg J_2$, the continuum limit of the $J_1 - J_2$ model can be considered, and the resulting quantum field theory is that of a critical $SU(2)_1$ Wess–Zumino–Novikov–Witten (WZNW) model perturbed a marginal current-current interaction[14]:

$$H_{J_1-J_2} = \frac{2\pi}{3} \left(: \mathbf{J}_R \cdot \mathbf{J}_R : + : \mathbf{J}_L \cdot \mathbf{J}_L :\right) + \gamma \mathbf{J}_R \cdot \mathbf{J}_L, \tag{71}$$

where $\gamma \sim J_{2c} - J_2 > 0$. At $\gamma < 0$, the perturbation is marginally irrelevant. However, at $\gamma > 0$ the effective interaction flows to strong coupling, and the

[1]While alternation of the nearest-neighbour exchange constants can originate from the spin-phonon coupling, the case of a nonuniform magnetic field with a period $2a_0$ (a_0 being the lattice constant) used to be regarded as unrealistic, not achievable in experimental conditions. Fortunately, the status of the staggered magnetic field has recently changed from exotic to legitimate. It has been shown that, in some quasi-1D antiferromagnetic compounds, such field can be realized as an intrinsic one. An effective staggered field can originate from the Neel ordering of one magnetic sublattice and being experienced by loosely connected magnetic chains which form another sublattice and remain disordered down to very low temperatures [11]. A sign-alternating component of the magnetic field can be also effectively generated due to a staggered anisotropy of the gyromagnetic tensor, as it is the case for the Copper-Benzoate organic molecule [12].

system ends up in a spontaneously dimerized phase with a dynamically generated spectral gap[14]: $m_{dim} \sim \Lambda \exp(-2\pi/\gamma)$, where $\Lambda \sim J_1$ is the UV cutoff.

Using Abelian bosonization, we can rewrite (71) as a sine-Gordon (SG) model:

$$H_{J_1-J_2} = \frac{1}{2}\left[(\partial_x\Phi)^2 + (\partial_x\Theta)^2\right] + \frac{\gamma}{2\pi}\partial_x\Phi_R\partial_x\Phi_L - \frac{\gamma}{(2\pi\alpha)^2}\cos\sqrt{8\pi}\Phi. \tag{72}$$

The "hidden" SU(2) symmetry of this bosonic Hamiltonian is encoded in the robust structure of the last two terms in Eq.(72), parametrised by a single coupling constant γ. This fact enforces the SG model (72) to occur either on the weak-coupling SU(2) separatrix of the Kosterliz-Thouless phase diagram ($\gamma < 0$), or on the strong-coupling SU(2) separatrix ($\gamma > 0$). In the latter case, quantum solitons with the mass m_{dim} and topological charge $Q = 1$ are identified with the Z_2 dimerization kinks carrying the spin $S = Q/2 = 1/2$. In the remainder of this section we will be dealing with the massive, spontaneously dimerized phase.

Consider the following deformation of the model: $H = H_{J_1-J_2} + H'$, where

$$H' = \sum_{a=0}^{3} \lambda_a Tr\left(\tau_a \hat{g}\right). \tag{73}$$

Here $\hat{g}$ is the 2×2 WZNW matrix field with conformal dimensions (1/4, 1/4), and τ_a are the Pauli matrices including the unit matrix $\tau_0 = I$. The scalar and vector parts of $\hat{g}$,

$$\mathbf{n}_s \sim Tr\left(\vec{\tau}\hat{g}\right) \sim \left(\cos\sqrt{2\pi}\Theta, \sin\sqrt{2\pi}\Theta, -\sin\sqrt{2\pi}\Phi\right), \tag{74}$$

$$\epsilon_s \sim Tr\left(\hat{g}\right) \sim \cos\sqrt{2\pi}\Phi, \tag{75}$$

constitute the staggered magnetisation and dimerization field of the S=1/2 Heisenberg chain. Representing

$$H' = \lambda\epsilon_s + \mathbf{h}_s \cdot \mathbf{n}_s, \tag{76}$$

let us consider the two terms in (76) separately.

The role of weak explicit (spin-Peierls) dimerization in the spontaneously dimerized $J_1 - J_2$ spin-1/2 chain has already been addressed by Affleck [3]. The effective double-frequency-sine-Gordon potential appearing in this case is different from the one studied in this paper (c.f. Eq.(1)):

$$\mathcal{U}_{dimer} = -\frac{\gamma}{(2\pi\alpha)^2}\cos\sqrt{8\pi}\Phi + \lambda\cos\sqrt{2\pi}\Phi. \tag{77}$$

The λ-term in (77) removes the degeneracy between the neighbouring minima of the unperturbed potential $-\cos\sqrt{8\pi}\Phi$ (i.e. between the two degenerate dimerized ground states) and thus leads to confinement of the solitons. The main physical effect is the spinon-magnon transmutation: deconfined spinons of the frustrated Heisenberg chain, carrying the spin S = 1/2, form bound states with S = 0 and S = 1, the latter representing coherent triplet magnon excitations.

Let us concentrate on the case of the staggered magnetic field, $\mathbf{h}_s$:

$$H' = \mathbf{h}_s \cdot \mathbf{n}_s. \tag{78}$$

Choosing $\mathbf{h}_s = h_s \hat{z}$, we arrive at a bosonic model

$$\begin{aligned} H &= H_0[\Phi] + \frac{\gamma}{2\pi}\partial_x \Phi_R \partial_x \Phi_L - \frac{\gamma}{(2\pi\alpha)^2}\cos\sqrt{8\pi}\Phi \\ &- h_s \sin\sqrt{2\pi}\Phi, \end{aligned} \tag{79}$$

in which we recognise the DSG model with the structure (1). From the analysis of the preceding section we conclude that the spontaneously dimerized chain in a staggered magnetic field has two phases separated by a quantum critical point at $h_s = h_s^*$. At $h_s < h_s^*$ a "mixed" phase is realized, with coexisting dimerization $\langle \epsilon_s \rangle \neq 0$ and staggered magnetization $\langle \mathbf{n}_s \rangle \neq 0$. Notice that, as opposed to the case of uniform magnetic field that couples to the (conserved) total magnetisation, the dependence $\mathbf{n}_s = \mathbf{n}_s(h_s)$ shows no threshold in h_s. Dimerization vanishes at the Ising critical point $h_s = h_s^*$ and remains zero in the "pure Neel" phase, $h_s > h_s^*$. The critical field h_s^* can be estimated by comparing the dimerization gap m_{dim} with the gap that would open up at $\gamma \leq 0$: $m_h \sim h_s^{2/3}$. So the critical staggered field is exponentially small: $h_s^* \sim (m_{dim})^{3/2}$.

Since the spin SU(2) symmetry is broken by the (staggered) magnetic field, the total spin is not conserved, but the spin projection S^z is. The latter circumstance allows one to identify the spin S^z of elementary excitations as the topological quantum number of the kinks interpolating between the nearest degenerate minima of the potential $\mathcal{U}(\Phi)$ in (79). According to the structure of $\mathcal{U}(\Phi)$, there will be "short" and "long" kinks, carrying the spin $S^z_{\pm} = \frac{1}{2} \mp \delta$, where $\delta = \delta(h_s)$ smoothly increases from $\delta = 0$ at $h_s = 0$ to $\delta = 1/2$ at $h_s \geq h_s^*$. Therefore, in the mixed phase, the original massive S = 1/2 spinon splits into two topological excitations carrying fractional spins $S^z_{\pm}$. These spins become $S^z = 1$ and $S^z = 0$ at the Ising transition, and it is just the singlet kink which loses its topological charge and becomes massless at $h_s = h_s^*$. The existence of the fractional-spin excitations in the mixed phase ($h_s < h_s^*$) is nothing but the spin version of the charge fractionization of topological excitations found earlier in 1D

Table 2: UV-IR transmutation of operators

field	UV	IR
uniform spin density	$J^z \sim \partial_x \Phi$	$J^z \sim \partial_x \mu$
uniform spin current	$\mathcal{J} \sim \partial_t \Phi$	$\mathcal{J} \sim \partial_t \mu$
dimerization	$\epsilon_s \sim \cos\sqrt{2\pi}\Phi$	$\epsilon_s \sim \mu$
staggered magnetisation	$n^z \sim \sin\sqrt{2\pi}\Phi$	$n^z \sim I + \varepsilon$

commensurate Peierls insulators with broken charge conjugation symmetry (e.g. cis-polyacetylene)[15], and also in a recent study of a 1D Mott insulator with alternating single-site energy [16].

To estimate the behaviour of physical quantities at the transition, let us consider an anisotropic $(\gamma_\|, \gamma_\perp)$ version of the model in which

$$\begin{aligned} H &= H_0[\Phi] + \frac{\gamma_\|}{2\pi}\partial_x\Phi_R\partial_x\Phi_L - \frac{\gamma_\perp}{(2\pi\alpha)^2}\cos\sqrt{8\pi}\Phi \\ &- h_s \sin\sqrt{2\pi}\Phi, \end{aligned} \tag{80}$$

The $\gamma_\|$-term in (80) can be eliminated by an appropriate rescaling of the field, $\Phi \to \sqrt{K}\Phi$:

$$H \to H_0[\Phi] - \frac{\gamma_\perp}{(2\pi\alpha)^2}\cos\sqrt{8\pi K}\Phi + h_s \sin\sqrt{2\pi K}\Phi, \tag{81}$$

As already mentioned, universality arguments lead to the conclusion that the anisotropic model (81) also incorporates the Ising criticality at some value of h_s. Choosing $K(\gamma_\|) = 1/2$, we reduce the perturbation to the form (3)

$$H' = -\frac{\gamma_\perp}{(2\pi\alpha)^2}\cos\sqrt{4\pi}\Phi + h_s \sin\sqrt{\pi}\Phi, \tag{82}$$

discussed in detail in previous sections. Using the (σ-τ) representation, we derive the relations in Table 6 describing the UV-IR transmutation of the physical fields.

We have the following correspondence:

$$\begin{aligned} h_s < h_s^* :& \quad \textit{disordered phase} \ : \langle\mu\rangle \neq 0; \\ h_s > h_s^* :& \quad \textit{ordered phase} \ : \langle\mu\rangle = 0. \end{aligned}$$

We see that dimerization is finite at $h_s < h_s^*$ and vanishes as

$$\langle\epsilon_s\rangle \sim (h_s^* - h_s)^{1/8},$$

on approaching the critical point. The staggered magnetisation, on the other hand, is always finite in both phases. Its behaviour at the transition is determined by the subleading correction to the identity operator (see Table 6):

$$\begin{aligned} \langle\langle n^z \rangle\rangle &\equiv \langle n^z \rangle_{h_s} - \langle n^z \rangle_{h_s^*} \\ &\sim (h_s - h_s^*) \ln \frac{h_s^*}{|h_s - h_s^*|}. \end{aligned} \tag{83}$$

The logarithmic divergence of the staggered magnetic susceptibility at the transition is similar to that of the specific heat of the Ising model:

$$\chi_{stag} \sim \ln \frac{h_s^*}{|h_s - h_s^*|}. \tag{84}$$

Next we consider the dynamical magnetic susceptibility. In analogy to (60), it is defined by

$$\chi(\omega, p) = -i \int_{-\infty}^{\infty} dx \int_0^{\infty} dt e^{-ipx + i\omega t} \langle [S^z(x,t), S^z(0,0)] \rangle \tag{85}$$

Using the continuum limit decomposition of the spin operators and the above glossary, we conclude that while the uniform magnetic susceptibility is readily given by

$$\chi(\omega, q \sim 0) \sim q^2 D^{(R)}(\omega, q) \tag{86}$$

(the function $D^{(R)}$ has been extensively discussed in Section (5)), the staggered susceptibility is in turn related to the correlation function of the Ising energy-density operator. The latter object is a Majorana bilinear

$$\varepsilon(x) \sim \xi_R(x) \xi_L(x).$$

Therefore, calculating the staggered magnetic susceptibility reduces to a simple task of computing the polarisation loop diagram for free, massive Majorana fermions. The result is

$$\chi(\omega, q = \pi) \sim \ln\left(1 - \frac{\omega^2}{4m^2}\right) - 2\sqrt{\frac{4m^2 - \omega^2}{\omega^2}} \text{arctg}\left(\frac{\omega^2}{4m^2 - \omega^2}\right), \tag{87}$$

inside the gap ($|\omega| < 2m$) and

$$\chi(\omega, \pi) \sim \ln\left(\frac{\omega^2}{4m^2} - 1\right) - i\text{sign}\omega \left\{\pi + 2\sqrt{\frac{\omega^2 - 4m^2}{\omega^2}} \text{arctg}\left(\frac{\omega^2}{\omega^2 - 4m^2}\right)\right\}, \tag{88}$$

for $|\omega| > 2m$.

We notice that for $h > h_s$ it obviously is the staggered field operator ($\beta^2 = 2\pi$) which alone dictates the physics of our effective DSG model (not only is it the most relevant operator, but it also has a large amplitude). It is therefore instructive to compare the results for the dynamical magnetic susceptibility of the DSG model we have found via Ising-type mappings with those for the SG model with a $\beta^2 = 2\pi$ operator only, which were obtained in the paper [17] by means of the form-factor technique. As for $h > h_s$ we enter the ordered phase ($\langle\mu\rangle$=0), the uniform susceptibility (86,67) is of a coherent nature, very much in agreement with [17] ('magnon' contribution). The staggered susceptibility (87,88), on the other hand, is incoherent in excellent agreement with the kink-antikink continuum observed in [17] (the breather contribution is missed in our approach; it might be recovered as a bound state of Majorana fermions, but we shall not push our analysis beyond this point). No analogous comparison can be made for $h < h_s$ (no form-factor calculations are, to our knowledge, currently available for the $\beta^2 = 8\pi$ SG model, and, even if they were, a comparison would have been of dubious validity, as the addition of the $\beta^2 = 2\pi$-operator qualitatively changes the spectrum from the start).

7 Ising transition in the 1D Hubbard model with alternating chemical potential

The DSG model finds a number of interesting applications in the theory of 1D strongly correlated electron systems. In this section we shall consider a particular example of this kind which has been recently discussed in Ref.[16] – a 1D repulsive Hubbard model at 1/2-filling with a sign-alternating single-site energy (i.e. staggered chemical potential). The Hamiltonian of this model reads:

$$H = -t \sum_{i,\sigma} \left(c^\dagger_{i\sigma} c_{i+1,\sigma} + h.c. \right) + U \sum_i n_{i\uparrow} n_{i\downarrow} + \Delta \sum_{i,\sigma} (-1)^i n_{i\sigma}, \qquad (89)$$

where $c_{i\sigma}$ is the annihilation operator of an electron with the spin projection σ, residing at the lattice site i, and $n_{i\sigma} = c^\dagger_{i\sigma} c_{i\sigma}$. The model (89) was originally proposed in the context of quasi-1D organic materials [18] and is also believed to be prototypical for ferroelectric perovskites [19].

In spite of its apparent simplicity, the model (89) reveals nontrivial physics. At $U = 0$ it describes a band insulator (BI) with a spectral gap for all excitations. At $\Delta \ll t$, the low-energy spectrum of the BI is that of free massive Dirac fermions.

When the Hubbard interaction is switched on, the finite fermionic mass Δ makes the theory free of infrared divergences, so that the BI phase remains stable for small enough U. On the other hand, at $\Delta = 0$, the Hamiltonian (89) coincides with the standard (translationally invariant) Hubbard model which is exactly solvable [20] and is well known to describe a Mott insulator (MI) at any positive value of U, if the electron concentration $n = (1/N)\sum_{i,\sigma} n_{i\sigma} = 1$ (the case of a 1/2-filled energy band). The MI state has a finite mass gap m_c in the charge sector induced by commensurability of the electron density with the underlying lattice. At energies well below m_c, local charge fluctuations are suppressed, and the low-energy spin dynamics of the model coincides with that of the spin-1/2 Heisenberg antiferromagnetic chain, the latter possessing a gapless spectrum. At a finite U, the charge-gaped MI phase is stable against site alternation, provided that Δ is small enough [18].

Thus, the issue of interest is the nature of the crossover between the BI and MI regimes which is expected to occur in the strong-coupling region where the single-particle mass gap Δ becomes comparable with the MI charge gap m_c. Starting out from the MI phase and decreasing U at a fixed Δ, one has to identify the mechanism for the mass generation in the spin sector. On the other hand, it is clear that the charge degrees of freedom should also be involved in the BI-MI crossover. Indeed, dividing the lattice into two sublattices, A and B, with the single-site electron energies Δ and $-\Delta$, respectively, and considering electronic states of a diatomic (AB) unit cell in the limit $t \ll U, \Delta$, one finds a region $U \sim 2\Delta$ where two charge configurations, A^1B^1 and A^0B^2, become almost degenerate. This is the so-called mixed-valence regime where those excitations responsible for the charge redistribution among the two unit-cell configurations become soft. This means that, apart from the spin transition, a charge transition associated with vanishing of the charge gap at some value of U is also expected to occur.

In Ref.[16] we have shown that the MI-to-BI crossover, taking place on decreasing U at a fixed Δ, is realized as a sequence of two continuous transitions: a Berezinskii-Kosterlitz-Thouless (BKT) transition at $U = U_{c2}$ where a spin gap is dynamically generated, and an Ising critical point at $U = U_{c1} < U_{c2}$ where the charge gap vanishes. Assuming that $U, \Delta \ll t$, below we shall consider the effective low-energy field theory for the lattice model (89). We shall then briefly comment on the spin transition and mostly concentrate on the Ising transition in the charge sector of the model which will be described in terms of a DSG model.

The standard bosonization procedure (see e.g. [9]) allows one to represent the

Hamiltonian density as

$$\mathcal{H}_{eff} = \mathcal{H}_c + \mathcal{H}_s + \mathcal{H}_{cs}.$$

Here the spin sector is described by the $SU(2)_1$ WZNW model with a marginally irrelevant current-current perturbation originating from the electron backscattering processes ($g \sim Ua_0 > 0$):

$$\mathcal{H}_s = \frac{2\pi v_s}{3} \left(: \mathbf{J}_R \cdot \mathbf{J}_R : + : \mathbf{J}_L \cdot \mathbf{J}_L : \right) - 2g \mathbf{J}_R \cdot \mathbf{J}_L \tag{90}$$

where $\mathbf{J}_{R,L}$ are chiral components of the vector spin current satisfying the $SU_1(2)$ Kac-Moody algebra. This Hamiltonian accounts for the universal properties of the spin-1/2 antiferromagnetic Heisenberg chain in the scaling limit [21]. The charge degrees of freedom are represented by a sine-Gordon model for a scalar field Φ_c:

$$\mathcal{H}_c = \frac{v_c}{2} \left[\Pi_c^2 + (\partial_x \Phi_c)^2 \right] - \frac{m_0}{\pi\alpha} \cos \sqrt{8\pi K_c} \Phi_c \tag{91}$$

where $m_0 \sim g$. The cosine perturbation is caused by the electron Umklapp processes (see e.g.[9]). For a wide class of Hamiltonians with finite-range interactions (of which the Hubbard model is a member) the parameter $K_c < 1$. This means that the model (91) is in a strong-coupling regime, and the dynamically generated mass determines the gap m_c in the charge sector. The charge and spin sectors are coupled by the Δ-term:

$$\mathcal{H}_{cs} = -\frac{2\Delta}{\pi\alpha} \epsilon_s \sin \sqrt{2\pi K_c} \Phi_c \tag{92}$$

where ϵ_s is the spin dimerization field of the S=1/2 Heisenberg chain, defined in (75).

Let us assume that we are in the MI regime with a gaped charge sector. Since the charge field Φ_c is locked in one of degenerate minima of the periodic potential in (91),

$$(\Phi_c)_m = \sqrt{\frac{\pi}{2K_c}} m, \quad m \in Z_\infty$$

it follows immediately that, in the MI phase, the operator $\sin \sqrt{2\pi K_c} \Phi_c$ appearing in (92) does not represent a strongly fluctuating field; it is rather short-ranged at distances $\xi_c \sim v_c / m_c$, and this explains the stability of the MI phase against a small Δ-perturbation. Assuming that the charge gap is nonzero, we integrate out the massive charge degrees of freedom to obtain the effective action in the spin sector. According to the unbroken SU(2) symmetry of the model, the effective Hamiltonian in the spin sector retains its form (90), but the current-current coupling constant undergoes an additive renormalization. In the second order in Δ we obtain:

$$g \to g_{\mathit{eff}} = g - C\,(\Delta/m_c)^2\, v_c$$

where $C \sim 1$ is a nonuniversal numerical constant. The spin sector now resembles the $J_1 - J_2$ frustrated spin-1/2 chain (see section 5) with an effecive next-nearest-neighbour interaction J_2 generated by the staggered chemical potential Δ. As long as $g_{\mathit{eff}} > 0$, the spin excitation spectrum remains gapless. However, decreasing U at a fixed Δ (or increasing Δ at a fixed U) eventually reverts the above inequality to $g_{\mathit{eff}} < 0$. In this case the current-current perturbation becomes marginally relevant, and a continuous (BKT) transition takes place to a spontaneously dimerized insulating (SDI) phase with broken (site) parity and a finite mass gap in the spin sector.

Thus, the condition $g_{\mathit{eff}} = 0$ determines the spin transition point U_{c2}. Using the exact result for the charge gap in the small-U Hubbard model[20],

$$m_c \sim \sqrt{Ut}e^{-2\pi t/U},$$

we find that

$$U_{c2} = \frac{2\pi t}{\ln(t/U)}\left[1 + O\left(\frac{\ln\ln(t/\Delta)}{\ln(t/\Delta)}\right)\right] \tag{93}$$

The dimerization order parameter that becomes nonzero at $U < U_{c2}$ is defined as

$$D = \sum_{i,\sigma}(-1)^i\left(c^\dagger_{i\sigma}c_{i+1,\sigma} + h.c.\right) \tag{94}$$

and in the continuum limit its density is given by

$$D(x) \sim \cos\sqrt{2\pi K_c}\Phi_c(x) Tr\hat{g}_s(x) \tag{95}$$

It is instructive to compare the parity properties of the SDI and BI phases. For models defined on a 1D lattice, there are two parity transformations – the site parity (P_S) and link parity (P_L) [22]. The difference between P_S and P_L survives the continuum limit and shows up in two inequivalent parity transformations that keep the massless Dirac equation

$$(\partial_t + \sigma_3\partial_x)\,\psi(x) = 0, \quad \psi = \begin{pmatrix} R \\ L \end{pmatrix}$$

invariant. These are

$$P_S: \qquad \psi(x) \to \sigma_1\psi(-x) \tag{96}$$

$$P_L: \qquad \psi(x) \to \sigma_2\psi(-x) \tag{97}$$

Using bosonization rules for spin-1/2 Dirac fermions (see e.g. [9]), one easily finds that both P_S and P_L lead to

$$\Phi_c(x) \to -\Phi_c(-x),$$

whereas the WZNW field $\hat{g}_s$ transforms differently:

$$P_S: \qquad \hat{g}_s(x) \to -\hat{g}_s(-x) \tag{98}$$

$$P_L: \qquad \hat{g}_s(x) \to \hat{g}_s(-x) \tag{99}$$

Comparing (92) and (95), we see that the Δ-perturbation breaks P_L but is invariant under P_S (in fact, P_S is the symmetry of the Hamiltonian (89)), while the dimerization operator D breaks P_S but preserves P_L. This is easily understood by noticing that these two terms have the structure of two different fermionic mass bilinears, $\psi^\dagger\sigma_1\psi$ and $\psi^\dagger\sigma_2\psi$, with opposite transformation properties with respect to P_S and P_L. Thus, the parity properties of the SDI and BI phases are different, and this is a strong indication that the passage from SDI to BI should be associated with a significant redistribution of the charge density (charge transition).

In what follows, we will not be dealing with estimation of the transition point U_{c1}. Referring the reader to Ref.[16] where the mechanism of the charge transition is discussed in the context of excitonic instability of the BI phase, here we simply claim that for the model (89), within the leading logarithmic accuracy, $U_{c2}/U_{c1} - 1 = const/\ln(t/\Delta)$, where the positive constant is of the order of unity. Let us instead focus on a nonperturbative description of the charge transition to the BI phase. Suppose we are in the SDI phase with both charge and spin sectors gaped. Adopting an Abelian bosonic representation for the $SU(2)_1$ WZNW model with a marginally *relevant* perturbation,

$$\begin{aligned} \mathcal{H}_s &= \frac{v_s}{2}\left[\Pi_s^2 + (\partial_x\Phi_s)^2\right] \\ &- \frac{\lambda}{\pi}\partial_x\Phi_{sR}\partial_x\Phi_{sL} + \frac{\lambda}{2(\pi\alpha)^2}\cos\sqrt{8\pi}\Phi_s, \end{aligned} \tag{100}$$

we can regard the effective Hamiltonian given by (100), (91) and (92) as a phenomenological Landau-Ginzburg energy functional, in the sense that all the couplings $(m_0, K_c, \lambda, \Delta)$ and velocities (v_s, v_c) are effective ones obtained by integrating out high-energy degrees of freedom. The effective potential is given by:

$$\mathcal{U}(\Phi_c, \Phi_s) = -\mu_c\cos\sqrt{8\pi K_c}\Phi_c - \mu_s\cos\sqrt{8\pi}\Phi_s - \delta\sin\sqrt{2\pi K_c}\Phi_c\cos\sqrt{2\pi}\Phi_s \tag{101}$$

with

$$\mu_c = \frac{m_0}{\pi\alpha} > 0, \qquad \mu_s = \frac{\lambda}{2(\pi\alpha)^2} > 0, \qquad \delta = \frac{\Delta}{\pi\alpha}.$$

The robust ingredients of the potential (101) are the vertex operators and the signs of the corresponding amplitudes. It can be shown that, as long as the parameter K_c is confined within the interval $1/2 < K_c < 1$, this potential, with all terms being strongly relevant perturbations, is indeed the most representative one for the model under discussion because no new relevant operators are generated in the course of renormalization. This is no longer true if $K_c < 1/2$, the regime which can be realized for an extended model with extra finite-range interactions, e.g. $V\sum_i n_i n_{i+1}$. In this case, new relevant vertex operators will be generated upon renormalization, and the continuous Ising transition in the charge sector can transform to a first-order one. We will comment on that in the end of this section.

A simple analysis of the saddle points of the potential $\mathcal{U}(\varepsilon_c, \varepsilon_s)$[16] shows that the location of its minima in the spin sector, $\varepsilon_s = \sqrt{\pi/2}n$, and hence the spin quantum numbers of the topological excitations, are the same as in the BI phase ($U < U_{c1}$). So the spin part of the spectrum in the SDI phase smoothly transforms to that of the BI phase. Therefore, being interested in the redistribution of the charge degrees of freedom in the vicinity of U_{c1}, in (101) we can replace $\cos\sqrt{2\pi}\Phi_s$ by its vacuum expectation value, $\langle\cos\sqrt{2\pi}\Phi_s\rangle = \pm c_0$. The two signs here reflect the Z_2 degeneracy of the dimerized ground state (spontaneously broken site parity; see Eq. (98)). Thus, the Hamiltonian of effective model describing the charge degrees of freedom can be represented in the following 2×2 matrix form:

$$H_{c;\mathit{eff}} = \begin{pmatrix} H_c^{(+)} & 0 \\ 0 & H_c^{(-)} \end{pmatrix} \tag{102}$$

where

$$\begin{aligned} H_c^{(\pm)} &= \frac{v_c}{2}\left[(\partial_x\Phi_c)^2 + (\partial_x\Theta_c)^2\right] \\ &- \mu_c \cos\sqrt{8\pi K_c}\Phi_c \mp h \sin\sqrt{2\pi K_c}\Phi_c \end{aligned} \tag{103}$$

(here $h = \delta c_0$). Notice that under P_S $H_c^{(+)} \leftrightarrow H_c^{(-)}$.

We have arrived at the matrix version of DSG model similar to (28), (29). To make contact with the DQAT model discussed in detail in sections 2 and 3, we shall consider $H_{c;\mathit{eff}}$ in the vicinity of the point $K_c = 1/2$. Setting $K_c =$

$1/2\,(1+\gamma_0)$ and rescaling the fields

$$\Phi_c = \frac{1}{\sqrt{2K_c}}\,\Phi, \quad \Theta_c = \sqrt{2K_c}\,\Theta$$

transforms $H_c^{(\pm)}$ in (103) to a form similar to $\mathcal{H}_\pm$ of Eq.(29).

With this correspondence and all the results of the previous sections, we are now able to describe in detail the Ising transition in the charge sector. As already explained in section 2.2, this can be done by considering only the Hamiltonian $H_c^{(+)}$.

When studying the physical properties of our model (89) at the charge transition point, it is important to remember that the disordered ($h < h_c$) and ordered ($h > h_c$) phases of the effective Ising model correspond to the SDI and BI phases of the electronic model (89). First we consider the dimerization operator (95):

$$\mathcal{D} \sim \cos\sqrt{2\pi K_c}\Phi_c \cos\sqrt{2\pi}\Phi_s.$$

With the spin field Φ_s locked in the SDI phase, $\langle\cos\sqrt{2\pi}\Phi_s\rangle = +c_0$ and $K_c = 1/2$, this transforms to the charge polarisation field

$$\mathcal{D} \sim \cos\sqrt{\pi}\Phi_c.$$

In full agreement with the physical picture, from (57) it follows that the operator $\mathcal{D}$, being the order parameter of the SDI phase, is the most strongly fluctuating field at the Ising trasntion:

$$\mathcal{D} \sim \mu \tag{104}$$

Its average value is nonzero in the SDI phase and vanishes as $\langle\mathcal{D}\rangle \sim (h_c - h)^{1/8}$ on approaching the charge transition point (remaining zero in the whole BI phase).

Using (55), we find that the average value of the Δ-perturbation, which, at $K_c = 1/2$, is given by $\mathcal{O}_\Delta \sim \sin\pi\Phi_c$ is nonsingular across the transition and remains finite in both phases.

The most interesting feature of the Ising transition is the UV-IR transmutation of the charge density $\rho_c(x)$ and current $J_c(x)$. At $K_c = 1/2$, the UV limit of our model represents a metallic state with central charge $c_{UV} = 2$, described in terms of two massless Gaussian fields Φ_c and Φ_s. In this limit

$$\rho_c(x) = \frac{1}{\sqrt{\pi}}\partial_x\Phi_c(x), \quad J_c(x) = -\frac{1}{\sqrt{\pi}}\partial_x\Theta_c(x).$$

According to (47),(49), at the Ising criticality ($c_{IR} = 1/2$),

$$\rho_c(x) = C\partial_x\mu(x), \quad J_c(x) = -C\partial_t\mu(x) \tag{105}$$

With $\mu(x)$ representing the charge polarization field, Eqs.(105) identify ρ_c and J_c as the bound charge and polarisation-current densities, respectively. Such an identification is typical for insulators rather than metals. This could have been anticipated from the fact that a true metallic state with charge-carrying gapless excitations cannot be described by a single massless real (Majorana) field.

An insulating (semi-metallic) behaviour of the model at the quantum critical point, reached at the charge transition, becomes manifest when one estimates the optical conductivity:

$$\sigma(\omega) \sim -\omega \Im m D^{(R)}(\omega, 0). \tag{106}$$

Here $D^{(R)}(\omega, q)$ is the retarded correlation function defined in (60). Using the result (66), we find that, at zero temperature, $\sigma(\omega)$ displays a universal power-law behaviour:

$$\sigma_0(\omega) \sim \omega^{-3/4} \tag{107}$$

Although the optical conductivity is divergent in the zero-frequency limit, there is no Drude-peak contribution ($\sim \delta(\omega)$) typical of true metals.

It is also possible to estimate the optical conductivity at finite temperatures (keeping in mind the $\sigma - \tau$ representation of the DQAT model, one should assume that T is much smaller than the mass gap in the decoupled τ degrees of freedom). This can be done using conformal mapping from a cylinder

$$ontoacomplexplane$$

C.$Startingwiththe$T=0$asymptoticsofthecorrelationfunction \langle \mu(z_1, \bar{z}_1)\mu(z_2, \bar{z}_2)\rangle \propto \frac{1}{|z_1 - z_2|^{1/4}}, andusingtheabovementionedconformalmapping, oneobtains \langle \mu(x,\tau)\mu(0,0)\rangle \propto \left[\frac{\pi T}{\sinh \pi T(x - \mathrm{i}\tau)\sinh \pi T(x + \mathrm{i}\tau)}\right]^{1/8}$.(108)It can be shown that[23]

$$-\Im m \chi(\omega) \sim \frac{1}{T^{7/4}} \Im m \ \left[\rho\left(\frac{\omega}{4\pi T}\right)\right]^2, \tag{109}$$

where

$$\rho(x) = \frac{\Gamma\left(\frac{1}{16} - \mathrm{i}x\right)}{\Gamma\left(\frac{15}{16} - \mathrm{i}x\right)}. \tag{110}$$

The final result for the temperature dependent optical conductivity at the Ising critical point is given by the following *universal* formula:

$$\sigma(\omega, T) \propto \frac{\omega}{T^{7/4}} \Im m \ \left[\rho\left(\frac{\omega}{4\pi T}\right)\right]^2 \tag{111}$$

At finite $\omega \ll T$, the frequency dependence of $\sigma(\omega, T)$ is classical but with a quantum, temperature dependent prefactor:

$$\sigma(\omega, T) \sim \omega^2 / T^{11/4}.$$

At $\omega \sim T$, $\sigma(\omega, T)$ reaches its maximum and then crosses over to its quantum-critical high-frequency ($\omega \gg T$) asymptotics (107).

8 First order transition at $\beta^2 < 2\pi$

Until now we assumed that $2\pi < \beta^2 < 8\pi$, so that no other relevant operators were generated upon renormalization. Indeed, already at $\beta^2 < 32\pi/9$, the operator $\sin[(3\beta/2)\,\Phi(x)]$, which is generated by the OPE of the two operators present in (1), becomes relevant. However, such an operator does not modify the qualitative behaviour of the model, as one can easily realize by a quasi-classical analysis. On the contrary, the operator $\cos(2\beta\Phi(x))$, which is also generated by the OPE, and which becomes relevant at $\beta^2 < 2\pi$, can modify the properties of the model in a relevant manner. Indeed, inspecting the quasi-classical potential

$$\mathcal{U}[\Phi] = -g\, \cos\beta\Phi - \lambda\, \sin\left(\frac{\beta}{2}\Phi\right) - V\, \cos 2\beta\Phi,$$

one finds that the Ising transition is turned by a sufficiently large V into a first order one.

The capability of an apparently subleading operator to change a continuous transition to a first order one is indeed common to a variety of models. For instance, it is known that the critical line with non universal exponents separating the Charge Density Wave (CDW) and the Spin Density Wave (SDW) phases of the extended (U-V) Hubbard model, becomes a first order line at sufficiently strong coupling[24]. In fact, in the extended Hubbard model, the charge Luttinger liquid exponent K_c can be lower than $1/2$, the point at which the second harmonics of the Umklapp scattering starts to be relevant. Therefore, at sufficiently strong interaction, it can indeed turn the CDW–SDW transition line into a first order one. A similar situation occurs with the charge transition in the electronic model, considered in section 7, when the latter is generalised to include a sufficiently strong nearest-neighbour repulsion [18].

9 Conclusions

In this paper, we have proposed a nonperturbative description of the Ising criticality in the DSG model (1). Using the equivalence between the DSG model and a deformed quantum Ashkin-Teller model, valid in the vicinity of the decoupling point, $\beta^2 = 4\pi$, we were able to identify the effective Ising degrees of freedom that asymptotically decouple from the rest of the spectrum and become critical in the infrared limit. This identification allowed us to describe the UV-IR "transmutation" of all physical fields of the DSG model and calculate the correlation functions at and close to the transition. We have also demonstrated the efficiency of our approach to describe Ising transitions in some physical realizations of the DSG model.

We believe that our quantum-Ising-chain approach can be generalised to the case when the number of the constituent Ising models, coupled by the interaction $h\prod_j \sigma_j$, is larger than 2. Such situation can indeed be realized in certain SU(2)-invariant spin-ladders models which can be driven to criticality under the action of external staggered fields. In such systems, the quantum critical points may be not only of the Ising type but also correspond to $\mathrm{SU}(2)_k$ WZNW universality class. Such examples will be considered elsewhere [2].

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A Bosonization

A.1 Bosonization of Fermi fields, currents and mass bilinears

We build up a Dirac field out of two Majorana fields, ξ_1 and ξ_2, and bosonize it:

$$\psi = \begin{pmatrix} R \\ L \end{pmatrix} = \left[\frac{\xi_1 + \mathrm{i}\xi_2}{\sqrt{2}}\right]_{R,L} \Rightarrow \frac{1}{\sqrt{2\pi\alpha}} \exp\left(\pm \mathrm{i}\sqrt{4\pi}\phi_{R,L}\right) \tag{112}$$

To ensure anticommutation between the right and left components of the Fermi field, it is assumed that

$$[\phi_R, \phi_L] = \frac{\mathrm{i}}{4}. \tag{113}$$

Notice that α is an ultraviolet cutoff in the *bosonic* theory which actually appears as a short-distance regulator in the normal-mode expansion of bosonic fields. For this reason it does not need to coincide with the original lattice constant a_0 which appears in the continuum representation of *fermionic* fields. These two cutoffs are, however, related, as shown in Appendix A.2.

The chiral components of the U(1) current are defined as

$$J_R = :R^\dagger R:= \mathrm{i}\xi_{1R}\xi_{2R} = \frac{1}{\sqrt{\pi}}\partial_x \phi_R \tag{114}$$

$$J_L = :L^\dagger L:= \mathrm{i}\xi_{1L}\xi_{2L} = \frac{1}{\sqrt{\pi}}\partial_x \phi_L \tag{115}$$

Using (112), one also finds that

$$\begin{aligned} R^\dagger L &= -\frac{\mathrm{i}}{2\pi\alpha} e^{-\mathrm{i}\sqrt{4\pi}\Phi} \\ &= \frac{1}{2}\left(\xi_{1R} - \mathrm{i}\xi_{2R}\right)\left(\xi_{1L} + \mathrm{i}\xi_{2L}\right), \end{aligned} \tag{116}$$

implying that

$$\cos\sqrt{4\pi}\Phi = \mathrm{i}\pi\alpha\left(\xi_{1R}\xi_{1L} + \xi_{2R}\xi_{2L}\right) \tag{117}$$

$$\sin\sqrt{4\pi}\Phi = -\mathrm{i}\pi\alpha\left(\xi_{1R}\xi_{2L} + \xi_{1L}\xi_{2R}\right) \tag{118}$$

The DQAT Hamiltonian (17) is symmetric under interchange ($\mathcal{P}_{12}$) of the two chains. Let us find the transformation properties of all the fields under $\mathcal{P}_{12}$. ¿From (112) it follows that interchanging the two chains leads to transformations

$$\phi_R \to \frac{\sqrt{\pi}}{4} - \phi_R, \quad \phi_L \to -\frac{\sqrt{\pi}}{4} - \phi_L \tag{119}$$

The chiral currents $J_{R,L}$, and therefore the total ($J = J_R + J_L$) and axial ($J_5 = J_R - J_L$) currents change their signs:

$$J_{R(L)} \to -J_{R(L)}, \quad J \to -J, \quad J_5 \to -J_5 \tag{120}$$

Under $\mathcal{P}_{12}$ the scalar field $\Phi = \phi_R + \phi_L$ and its dual counterpart $\Theta = -\phi_R + \phi_L$ transform as follows:

$$\Phi \to -\Phi, \quad \Theta \to -\frac{\sqrt{\pi}}{2} - \Theta \tag{121}$$

Notice that the symmetry properties of r.h.sides of (117)and (118) are consistent with (121).

A.2 Bosonization of products of order and disorder operators

Starting with the Jordan–Wigner transformation for two QI spin chains, which can be summarised as follows

$$\sigma^x_{an} = \mathrm{i}\zeta_{an}\eta_{an}, \qquad \mu^x_{a,n+1/2} = -\mathrm{i}\zeta_{an}\eta_{a,n+1} \tag{122}$$

$$\sigma^z_{an} = \mathrm{i}\kappa_a \left(\prod_{j=1}^{n} \sigma^x_{aj}\right) \zeta_{an}, \qquad \mu^z_{a,n+1/2} = \prod_{j=1}^{n} \sigma^x_{aj} \tag{123}$$

$$\sigma^y_{an} = \mathrm{i}\kappa_a \left(\prod_{j=1}^{n} \sigma^x_{aj}\right) \eta_{an}, \qquad \mu^y_{a,n+1/2} = \left(\prod_{j=1}^{n} \sigma^x_{aj}\right), \eta_{a,n+1}\zeta_{an} \tag{124}$$

here we derive bosonized expressions for products of Ising fields belonging to different chains.

- $\underline{\mu^z_{1,n+1/2}\ \mu^z_{2,n+1/2}}$. Using the lattice definition (123), we have:

$$\mu^z_{1,n+1/2}\mu^z_{2,n+1/2} = \prod_{j=1}^{n} \sigma^x_{1j}\sigma^x_{2j} \tag{125}$$

There are many ways to exponentiate the product in (125), the naive assumption would be to use the identity

$$\sigma^x = \mp \mathrm{i} e^{\pm \mathrm{i}(\pi/2)\sigma^x}$$

and then replace in the exponential σ^x by $\mathrm{i}\eta\zeta$. This will bring us to a phase proportional to

$$\sum_{j=1}^{n} (\eta_{1n}\zeta_{1n} + \eta_{2n}\zeta_{2n})$$

which, in the continuum limit, reduces to

$$\int_0^x \mathrm{d}x' \, (\xi_{1R}\xi_{1L} + \xi_{2R}\xi_{2L}) \, .$$

According to (117), the integrand represents a bosonic cosine operator, which is not what we would like to have for practical purposes. To get a better representation, one has first to rearrange the four Majorana fields in the product $\sigma^x_{1j}\sigma^x_{2j}$:

$$\sigma^x_{1j}\sigma^x_{2j} = (\eta_{1j}\eta_{2j})\,(\zeta_{1j}\zeta_{2j}) \tag{126}$$

It is readily seen that a product of two local Majorana fields, defined on the lattice, can be represented as

$$\eta_1\eta_2 = \pm \exp\left(\pm\frac{\pi}{2}\eta_1\eta_2\right) \tag{127}$$

Therefore

$$\mu^z_{1,n+1/2}\mu^z_{2,n+1/2} = \prod_{j=1}^{n} \sigma^x_{1j}\sigma^x_{2j} = \exp\left[\pm\frac{\pi}{2}\sum_{j=1}^{n}(\eta_{1j}\eta_{2j} + \zeta_{1j}\zeta_{2j})\right] \tag{128}$$

Now we pass to the continuum limit by making chiral rotation (12) and using (114), (115):

$$\begin{aligned}
\mu^z_{1,n+1/2}\mu^z_{2,n+1/2} &\to \exp\left[\pm\pi\int_0^x \mathrm{d}y \;\; (\xi_{1R}\xi_{2R} + \xi_{1L}\xi_{2L})\right] \\
&= \exp\left[\mp \mathrm{i}\sqrt{\pi}\int_0^x \mathrm{d}y \;\partial_y\Phi(y)\right] \\
&= e^{\mp \mathrm{i}\sqrt{\pi}\Phi(x)}
\end{aligned} \tag{129}$$

The l.h.s. of this relation is symmetric under $\mathcal{P}_{12}$, so must the r.h.side. As follows from (121), under $\mathcal{P}_{12}$ the field Φ changes its sign. So the sign ambiguity in (129) is resolved by replacing the phase exponential by a cosine:

$$\mu^z_{1,n+1/2}\mu^z_{2,n+1/2} = \cos\sqrt{\pi}\Phi(x) \tag{130}$$

- $\underline{\sigma^z_{1,n+1/2}\;\sigma^z_{2,n+1/2}}$. Using (123), we have:

$$\begin{aligned}
\sigma^z_{1n}\sigma^z_{2n} &= -(\kappa_1\kappa_2)\,(\zeta_{1n}\zeta_{2n})\left(\prod_{j=1}^{n}\sigma^x_{1j}\sigma^x_{2j}\right) \\
&= -\kappa_1\kappa_2\;\zeta_{1n}\zeta_{2n}\left(\mu^z_{1,n+1/2}\mu^z_{2,n+1/2}\right)
\end{aligned} \tag{131}$$

With the representation (130) at hand, we only need to pass to the continuum limit in $\zeta_{1n}\zeta_{2n}$ and then make use of proper operator product expansions. We have:

$$
\begin{aligned}
\zeta_{1n}\zeta_{2n} &\rightarrow 2a_0\zeta_1(x)\zeta_2(x) \\
&= a_0\left[\xi_{1R}(x)+\xi_{1L}(x)\right]\left[\xi_{2R}(x)+\xi_{2L}(x)\right] \\
&= -\frac{\mathrm{i}a_0}{\sqrt{\pi}}\partial_x\Phi(x)+\frac{\mathrm{i}a_0}{\pi\alpha}\sin\sqrt{4\pi}\Phi(x)
\end{aligned} \tag{132}
$$

So

$$
\sigma^z_{1n}\sigma^z_{2n}\rightarrow \mathrm{i}\kappa_1\kappa_2 a_0\left[\frac{1}{\sqrt{\pi}}\partial_x\Phi(x)-\frac{1}{\pi\alpha}\sin\sqrt{4\pi}\Phi(x)\right]\cos\sqrt{\pi}\Phi(x+\alpha) \tag{133}
$$

Picking up the most relevant operators in the following OPE

$$
\partial_x\Phi(x):\cos\sqrt{\pi}\Phi(x\mp\alpha): \;=\; \pm\frac{1}{2\sqrt{\pi}\alpha}:\sin\sqrt{\pi}\Phi(x):+\cdots \tag{134}
$$

$$
:\sin\sqrt{4\pi}\Phi(x)::\cos\sqrt{\pi}\Phi(x): \;=\; \frac{1}{2}:\sin\sqrt{\pi}\Phi(x):+\cdots \tag{135}
$$

one finds that

$$
\sigma^z_{1n}\sigma^z_{2n}\rightarrow -\mathrm{i}\kappa_1\kappa_2\left(\frac{a_0}{2\pi\alpha}\right):\sin\sqrt{\pi}\Phi(x): \tag{136}
$$

Intending to keep duality among (136) and (130) we set

$$
\alpha=\frac{a_0}{\pi} \tag{137}
$$

and finally obtain:

$$
\sigma^z_{1n}\sigma^z_{2n}=-\mathrm{i}\kappa_1\kappa_2:\sin\sqrt{\pi}\Phi(x): \tag{138}
$$

Notice the important role of the Klein product $\kappa_1\kappa_2$. The l.h.s. of (138) is $\mathcal{P}_{12}$-symmetric while in the r.h.side $\sin\sqrt{\pi}\Phi(x)$ is antisymmetric (see (121)). So the r.h.s. of (138) is symmetric just due to the presence of $\kappa_1\kappa_2$. If we had replaced $\kappa_1\kappa_2$ by a constant, $\kappa_1\kappa_2\rightarrow \mathrm{i}$, we would obtain

$$
\sigma^z_{1n}\sigma^z_{2n}=:\sin\sqrt{\pi}\Phi(x):
$$

In this representation, the only way to ensure the $\mathcal{P}_{12}$-symmetry would be to impose a constraint that identifies Φ and $-\Phi$, in which case the scalar field Φ would transform to an orbifold.

- $\underline{\sigma^z_{1n}\mu^z_{2,n+1/2}}$. First we represent $\sigma^z_{1n}\mu^z_{2,n+1/2}$ as

$$
\sigma^z_{1n}\mu^z_{2,n+1/2}=\left(\sigma^z_{1n}\mu^z_{1,n+1/2}\right)\left(\mu^z_{1,n+1/2}\mu^z_{2,n+1/2}\right).
$$

The continuum limit for second product has already been found (see Eq.(130)). The first product reduces to a Majorana field: $\sigma^z_{1n}\mu^z_{1,n+1/2} = -\mathrm{i}\kappa_1\zeta_{1n}$. In the continuum limit, with relation (137) taken into account,

$$\begin{aligned} \zeta_{1n} &\to \sqrt{a_0}\,[\xi_{1R}(x) + \xi_{1L}(x)] \\ &= \cos\sqrt{4\pi}\Phi_R(x) + \cos\sqrt{4\pi}\Phi_L(x) \end{aligned} \tag{139}$$

So

$$\sigma^z_{1n}\mu^z_{2,n+1/2} = -\mathrm{i}\kappa_1\left[\cos\sqrt{4\pi}\Phi_R(x) + \cos\sqrt{4\pi}\Phi_L(x)\right]\cos\sqrt{\pi}\Phi(x+\alpha) \tag{140}$$

Making use of the following OPE

$$\begin{aligned} :\cos\sqrt{4\pi}\Phi_R(x)::\cos\sqrt{\pi}\Phi(x+\alpha): &= :\cos\sqrt{4\pi}\Phi_L(x)::\cos\sqrt{\pi}\Phi(x+\alpha): \\ &= \frac{1}{2}:\cos\sqrt{\pi}\Theta(x):+\cdots \end{aligned} \tag{141}$$

we arrive at the result:

$$\sigma^z_{1n}\mu^z_{2,n+1/2} = -\mathrm{i}\kappa_1 :\cos\sqrt{\pi}\Theta(x): \tag{142}$$

- $\underline{\mu^z_{1,n+1/2}\sigma^z_{2n}}$. Quite similarly one finds that

$$\mu^z_{1,n+1/2}\sigma^z_{2n} = \mathrm{i}\kappa_2 :\sin\sqrt{\pi}\Theta(x): \tag{143}$$

Using (121), we see that under $\mathcal{P}_{12}$ the r.h.sides of Eqs.(142) and (143) indeed transform to each other.

B More on lattice fermions

An alternative way to bosonize the QAT model, is to pass through a mapping onto spinless fermions, and bosonize the latter.

We start by the quantum Ising model (10). By means of the following unitary transformation

$$\zeta_n = -\frac{1}{\sqrt{2}}\left(\xi_{Rn} - \xi_{Ln}\right), \tag{144}$$

$$\eta_n = \frac{1}{\sqrt{2}}\left(\xi_{Rn} + \xi_{Ln}\right), \tag{145}$$

the Hamiltonian (10) transforms onto

$$\begin{aligned} H_{QI} &= \frac{J}{2}\left(-\frac{i}{2}\right)\sum_n \left[\xi_{Rn}\left(\xi_{Rn+1} - \xi_{Rn-1}\right) - \xi_{Ln}\left(\xi_{Ln+1} - \xi_{Ln-1}\right)\right] \\ &+ i\frac{J}{4}\sum_n \left[\xi_{Rn}\left(2\xi_{Ln} - \xi_{Ln+1} - \xi_{Ln-1}\right) - \xi_{Ln}\left(2\xi_{Rn} - \xi_{Rn+1} - \xi_{Rn-1}\right)\right] \\ &+ i\left(\Delta_J - J\right)\sum_n \xi_{Rn}\xi_{Ln}. \end{aligned} \tag{146}$$

We now consider the following quantum Ashkin–Teller Hamiltonian of two-coupled Ising chains

$$H_{QAT} = H_{QI}\left[\sigma_1\right] + H_{QI}\left[\sigma_2\right] + H'_{AT}\left[\sigma_1, \sigma_2\right], \tag{147}$$

where H_{QI}'s are Ising Hamiltonians [see Eq.(146)] for two spin species, σ_1 and σ_2, and

$$H'_{AT} = K\sum_n \left(\sigma^z_{1,n}\sigma^z_{1,n+1}\sigma^z_{2,n}\sigma^z_{2,n+1} + \sigma^x_{1,n}\sigma^x_{2,n}\right), \tag{148}$$

is the coupling term. For each spin species we introduce Majorana's fermions. In order to make σ_1 and σ_2 commute we multiply each Majorana's fermion for the chain 1 by another Majorana κ_1, and for chain 2 by κ_2. That is, if $a = 1, 2$, we define

$$\zeta_{a,n} = \kappa_a\left(i\sigma^z_{a,n}\mu^z_{a,n+\frac{1}{2}}\right), \quad \eta_{a,n} = \kappa_a\left(\sigma^z_{a,n}\mu^z_{a,n-\frac{1}{2}}\right), \tag{149}$$

as well as their R, L components

$$\zeta_{1,n} = -\frac{1}{\sqrt{2}}\left(\xi_{Rn} - \xi_{Ln}\right), \tag{150}$$

$$\eta_{1,n} = \frac{1}{\sqrt{2}}\left(\xi_{Rn} + \xi_{Ln}\right), \tag{151}$$

$$\zeta_{2,n} = -\frac{1}{\sqrt{2}}\left(\eta_{Rn} - \eta_{Ln}\right), \tag{152}$$

$$\eta_{2,n} = \frac{1}{\sqrt{2}}\left(\eta_{Rn} + \eta_{Ln}\right). \tag{153}$$

Let us concentrate for the moment onto the two Ising Hamiltonians, which are bilinear in the Majorana fermions. We notice the following general property $(p, q = R, L)$

$$\begin{aligned} \xi_{pn}\xi_{qm} + \eta_{pn}\eta_{qm} &= \frac{1}{2}\left[\left(\xi_{pn} - i\eta_{pn}\right)\left(\xi_{qm} + i\eta_{qm}\right) + \left(\xi_{pn} + i\eta_{pn}\right)\left(\xi_{qm} - i\eta_{qm}\right)\right] \\ = \quad 2\left[c^\dagger_{pn}c_{qm} + c_{pn}c^\dagger_{qm}\right] &= 2\left[c^\dagger_{pn}c_{qm} - c^\dagger_{qm}c_{pn} + \delta_{pq}\delta_{nm}\right], \end{aligned}$$

where we define the Fermi operators

$$c_{R(L)n} = \frac{1}{2}\left(\xi_{R(L)n} + i\eta_{R(L)n}\right). \tag{154}$$

Therefore, through (146), we find

$$\begin{aligned} &H_{QI}\left[\sigma_1\right] + H_{QI}\left[\sigma_2\right] = \\ = \quad &J\left(-\frac{i}{2}\right)\sum_n \left[c^\dagger_{Rn}\left(c_{Rn+1} - c_{Rn-1}\right) - c^\dagger_{Ln}\left(c_{Ln+1} - c_{Ln-1}\right) - H.c.\right] \\ &+i\frac{J}{2}\sum_n \left[c^\dagger_{Rn}\left(2c_{Ln} - c_{Ln+1} - c_{Ln-1}\right) - c^\dagger_{Ln}\left(2c_{Rn} - c_{Rn+1} - c_{Rn-1}\right) - H.c.\right] \\ &+2i\left(\Delta_J - J\right)\sum_n c^\dagger_{Rn}c_{Ln} - c^\dagger_{Ln}c_{Rn}. \end{aligned} \tag{155}$$

The analogy with a fermionic model on a lattice is already apparent. Let us make this analogy more firm. We consider the following spinless fermion model on a lattice of $2L$ sites

$$\begin{aligned} H &= -t\sum_{n=0}^{2L-1} c^\dagger_n c_{n+1} + H.c. \\ &= -t\sum_{n=0}^{L-1} c^\dagger_{2n+1}\left(c_{2n} + c_{2n+2}\right) + c^\dagger_{2n}\left(c_{2n-1} + c_{2n+1}\right). \end{aligned}$$

We make the following unitary transformation

$$\begin{aligned} c_{2n} &= \frac{(-1)^n}{\sqrt{2}}\left(c_{Rn} + c_{Ln}\right), \\ c_{2n+1} &= i\frac{(-1)^n}{\sqrt{2}}\left(c_{Rn} - c_{Ln}\right), \end{aligned}$$

where the right- and left-moving fermions are defined on a lattice of half the number of sites, i.e. L. The Hamiltonian becomes

$$\begin{aligned} H &= -i\frac{t}{2}\sum_n c^\dagger_{Rn}\left(c_{Rn+1} - c_{Rn-1}\right) - c^\dagger_{Ln}\left(c_{Ln+1} - c_{Ln-1}\right) \\ &+ i\frac{t}{2}\sum_n c^\dagger_{Rn}\left(2c_{Ln} - c_{Ln+1} - c_{Ln-1}\right) - c^\dagger_{Ln}\left(2c_{Rn} - c_{Rn+1} - c_{Rn-1}\right). \end{aligned} \tag{156}$$

Let us add to this Hamiltonian a staggered hopping

$$\delta H = -\Delta \sum_n c^\dagger_{2n} c_{2n+1} + H.c. \tag{157}$$

$$= i\Delta \sum_n c^\dagger_{Rn} c_{Ln} - H.c.\,. \tag{158}$$

We notice that (156) plus (158) coincide with (155) if $2J = t$ and $2(\Delta_J - J) = \Delta$.

Now we focus on the coupling term (148), which, in terms of Majorana's fermions reads

$$H'_{AT} = K \sum_n \zeta_{1n}\zeta_{2n} \left(\eta_{1n+1}\eta_{2n+1} + \eta_{1n}\eta_{2n}\right).$$

Transforming into right- and left-moving Majorana fermions we find that

$$\begin{aligned}
\zeta_{1n}\zeta_{2n} &= \frac{1}{2}\left(\xi_{Rn} - \xi_{Ln}\right)\left(\eta_{Rn} - \eta_{Ln}\right) \\
&= \frac{1}{2}\left(\xi_{Rn}\eta_{Rn} + \xi_{Ln}\eta_{Ln} - \xi_{Rn}\eta_{Ln} - \xi_{Ln}\eta_{Rn}\right) \\
&= -i\left(\rho_{Rn} + \rho_{Ln} - 1 - \Delta_n\right),
\end{aligned}$$

where we used (154), and we defined $\rho_{pn} = c^\dagger_{pn} c_{pn}$ $(p = R, L)$ as well as $\Delta_n = c^\dagger_{Rn} c_{Ln} + H.c.$. We can equivalently show that

$$\eta_{1n}\eta_{2n} = -i\left(\rho_{Rn} + \rho_{Ln} - 1 + \Delta_n\right).$$

Therefore the coupling term can be written as

$$\begin{aligned}
H'_{AT} &= -K \sum_n \left(\rho_{Rn} + \rho_{Ln} - 1 - \Delta_n\right)\left(\rho_{Rn} + \rho_{Ln} - 1 + \Delta_n\right) \\
&- K \sum_n \left(\rho_{Rn} + \rho_{Ln} - 1 - \Delta_n\right)\left(\rho_{Rn+1} + \rho_{Ln+1} - 1 + \Delta_{n+1}\right).
\end{aligned} \tag{159}$$

On the other hand let us consider the simplest nearest-neighbour interaction for spinless fermions, namely

$$V \sum_n c^\dagger_{2n} c_{2n} c^\dagger_{2n+1} c_{2n+1} + c^\dagger_{2n+1} c_{2n+1} c^\dagger_{2n+2} c_{2n+2}.$$

This interaction term turns out to be equivalent to (159), apart from a chemical potential term, provided $4K = -V$. Notice that, if one considers an Ashkin-Teller coupling which is not self-dual, e.g.

$$H'_{AT} = K_1 \sum_n \sigma^z_{1,n}\sigma^z_{1,n+1}\sigma^z_{2,n}\sigma^z_{2,n+1} + K_2 \sum_n \sigma^x_{1,n}\sigma^x_{2,n},$$

this translates into a staggered interaction for the spinless fermion model. This interaction, even without an explicit dimerization, is able to gap the fermionic spectrum, thus showing the importance of self-duality, even at the level of the coupling term (147), to get a critical behaviour.

Therefore we have shown the equivalence between a model of two coupled Ising chains in a transverse field, given by the Hamiltonian (147), and a model of spinless fermion with nearest neighbour interaction and dimerized hopping.

An interesting point in this respect regards quantum numbers. The spinless fermion model has for instance conserved number of particle. In ρ is the density, in the reduced chain we must have that

$$\frac{1}{L}\sum_n \rho_{Rn} + \rho_{Ln} = 2\rho, \tag{160}$$

is conserved. On the other hand

$$\begin{aligned}\frac{1}{L}\sum_n (\rho_{Rn} + \rho_{Ln} - 1) &= 2\rho - 1 \\ \equiv -\frac{i}{2L}\sum_n \zeta_{1n}\zeta_{2n} &+ \eta_{1n}\eta_{2n},\end{aligned}$$

which, in terms of Ising variables, implies the conservation of a very non-local operator, which includes the spins and their dual counterparts.

B.1 Operator identities

We can build up several operator identities in the lattice representation.

(1) The simplest one is obtained by the identity

$$2\left(\rho_{Rn} + \rho_{Ln} - 1\right) = i\left(\zeta_{1n}\zeta_{2n} + \xi_{1n}\xi_{2n}\right), \tag{161}$$

which relates the density operator of the spinless fermions to a particular bilinear of Majorana's fermions.

(2) The other identity derives from the equality

$$2\left(c^\dagger_{Rn}c_{Ln} + H.c.\right) = i\left(\xi_{1n}\xi_{2n} - \zeta_{1n}\zeta_{2n}\right), \tag{162}$$

which relates the charge density wave operator of the spinless fermions to the Majorana's fermions.

(3) The last identity is obtained by the equality

$$-2i\left(c^\dagger_{Rn}c_{Ln} - H.c.\right) = i\left(\zeta_{1n}\xi_{1n} + \zeta_{2n}\xi_{2n}\right), \tag{163}$$

which provides a relation between the dimerization operator and the Majorana's fermions.

C Mean-field treatment of the (σ, τ)-model

In oder to devise an appropriate mean-field scheme, let us formally rewrite Hamiltonian (35) as

$$H[\sigma, \tau] = H_{mf}[\sigma] + H_{mf}[\tau] + H_{fluc}[\sigma, \tau] + \text{const.} \tag{164}$$

Here the two terms of the mean-field Hamiltonian are:

$$H_{mf}[\sigma] = -\sum_n \left(J_\sigma \sigma_n^z \sigma_{n+1}^z + \Delta_\sigma \sigma_n^x \right) \tag{165}$$

$$H_{mf}[\tau] = -\sum_n \left(J_\tau \tau_n^z \tau_{n+1}^z + h\tau_n^z + \Delta_\tau \tau_n^x \right) \tag{166}$$

where

$$\begin{aligned} J_\sigma = J\left(1 + \langle \tau_n^z \tau_{n+1}^z \rangle\right), \qquad J_\tau = J\langle \sigma_n^z \sigma_{n+1}^z \rangle - K \\ \Delta_\sigma = \Delta\langle \tau_n^x \rangle - K, \qquad \Delta_\tau = \Delta\left(1 + \langle \sigma_n^x \rangle\right) \end{aligned} \tag{167}$$

The coupling between the fluctuations, is given by

$$\begin{aligned} H_{fluc}[\sigma, \tau] = -\sum_n \Big[& J(\sigma_n^z \sigma_{n+1}^z - \langle \sigma_n^z \sigma_{n+1}^z \rangle)(\tau_n^z \tau_{n+1}^z - \langle \tau_n^z \tau_{n+1}^z \rangle) \\ + & \Delta(\sigma_n^x - \langle \sigma_n^x \rangle)(\tau_n^x - \langle \tau_n^x \rangle)\Big] \end{aligned} \tag{168}$$

In the leading approximation, fluctuations described by (168) are neglected, and the σ and τ degrees of freedom decouple. We notice that the parameter h appears only in $H_{mf}[\tau]$ where it plays the role of an effective *longitudinal* field. Therefore the model (166) is a quantum counterpart of a 2D Ising model in a nonzero external field which is known to be always noncritical. Thus, as expected, the τ-chain is always gapped. The condition for the σ-chain to be critical then reads:

$$J\left(1 + \langle \tau_n^z \tau_{n+1}^z \rangle\right) = \Delta\langle \tau_n^x \rangle - K \tag{169}$$

Despite the fact that the Ising model in the field is supposed to be integrable in the continuum limit (see references in [10]), the explicit expectation values of the τ operators appearing in (169) are not known. To make further analytic progress possible, we choose a special value of the field rescaling parameter K,

$$K = J\langle \sigma_n^z \sigma_{n+1}^z \rangle, \tag{170}$$

at which $J_\tau = 0$, and interaction between the neighbouring τ-spins in $H_{mf}[\tau]$ vanishes. This trivialises the τ-model and leads to

$$\langle \tau^z \rangle = \frac{h}{\sqrt{h^2 + \Delta_\tau^2}}, \quad \langle \tau^x \rangle = \frac{\Delta_\tau}{\sqrt{h^2 + \Delta_\tau^2}} \tag{171}$$

Self-consistency requires the knowledge of the averages related to the σ-model. Specialising to the critical point and making use of the known results [6],

$$\langle \sigma_n^z \sigma_{n+1}^z \rangle_{crit} = \langle \sigma_n^x \rangle_{crit} = \frac{2}{\pi}, \tag{172}$$

we obtain

$$K = \frac{2J}{\pi}, \quad \Delta_\tau = \Delta \left(1 + \frac{2}{\pi} \right). \tag{173}$$

Then the condition (169) reduces to

$$1 + \langle \tau^z \rangle^2 = (\Delta/J) \langle \tau^x \rangle - \frac{2}{\pi} \tag{174}$$

This equation determines the critical value of h at a given ratio Δ/J. Introducing the quantity

$$x = \frac{\sqrt{h^2 + C^2 \Delta^2}}{J}, \quad C = 1 + \frac{2}{\pi}$$

we obtain a quadratic equation

$$(C+1)x^2 - C(\Delta/J)^2 x + C^2 (\Delta/J)^2 = 0,$$

whose solution determines the critical value of h:

$$\left(\frac{h_c}{C\Delta} \right)^2 = \frac{1}{4(C+1)^2} \left[\left(\frac{\Delta}{J} \right) + \sqrt{ \left(\frac{\Delta}{J} \right)^2 + 4(C+1) } \right]^2 - 1 \tag{175}$$

The requirement that the r.h.side of (175) is positive yields a restriction upon Δ:

$$\frac{\Delta}{J} > 1 + \frac{2}{\pi} \tag{176}$$

Recall that we already assumed that at $h = 0$ the two QI chains of the QAT model are disordered: $\Delta > J$. The restriction (176) tells us that, for the Ising criticality to be reached at some critical value of h, the original disordered σ_1 and σ_2 chains should be far enough from their original critical point. In fact, as follows from (176), the value of the Majorana mass, when estimated as $m \sim 2(\Delta - J) \sim 4J/\pi$, turns out to be of the order of the ultraviolet cutoff. This estimation reflects the already mentioned fact that the DM transition is indeed not a weak-coupling one from the standpoint of the DQAT model.

One could, in principle, perturbatively calculate the fluctuation corrections, i.e. those originating from (168), to the criticality condition. As the model is only tractable under fine tuning (170) of the parameter K, that is not expected to generalise the qualitative picture obtained in the mean-field approximation.

D Correlation functions

Let us start calculating the Fourier transform of $D^{(R)}(\omega, p)$, given by (60), with the remark that using the Bessel functions (despite some convenient Fourier transforms) is *beyond the accuracy*. The analytic properties (and the issue of coherent particle poles) should not depend on fine details of the correlators at $r \sim \xi = 1/m$. Therefore what one ought to do is to study analytic properties of the integral

$$I_\alpha^m(q) = \int_0^\infty r^{\alpha-1/2} dr J_0(qr) e^{-mr} \tag{177}$$

in terms of which the Fourier transforms,

$$D(q) = \frac{1}{2}\int d^2\vec{r} e^{i\vec{q}\vec{r}} D(r) = \pi \int_0^\infty r dr J_0(qr) D(r) \ ,$$

$\vec{q} = (\omega_n, p)$, $q = \sqrt{\omega_n^2 + p^2}$, are given by

$$D_0(q) = \pi I_{5/4}^0(q), \quad D_>(q) = \frac{\pi^{3/2} A_1}{\sqrt{2m}} I_1^m(q), \quad D_<(q) = \frac{35 A_1}{256 m^2} I_{-1/2}^{2m}(q) \tag{178}$$

In the area of convergence, we have [25]:

$$I_\alpha^m(q) = \frac{\Gamma(\alpha + 1/2)}{m^{\alpha+1/2}} F(\alpha/2 + 1/4, \alpha/2 + 3/4; 1; -q^2/m^2) \tag{179}$$

where $F(a, b; c; z)$ stands for the hypergeometric function. The character of the singularity at $z = 1$ can be determined by using appropriate transformation formulas for the hypergeometric functions and the Γ-function doubling formula[25]:

$$I_\alpha^m(q)|_{q^2 \simeq -m^2} = \frac{(2m)^{\alpha-1/2}\Gamma(\alpha)}{\sqrt{\pi}} \frac{1}{(q^2 + m^2)^\alpha} \tag{180}$$

where only the leading, most divergent term is retained.

This calculation explains how the correlation function has a branch cut at the threshold, unless $\alpha = 1$ which, of course, corresponds to the disordered case:

$$D_>(q) = \frac{\pi A_1}{q^2 + m^2} \tag{181}$$

where the coefficient, as expected, is the same as one finds by Fourier transforming $K_0(mr)$

$$K_0(mr) = \frac{1}{2\pi} \int d^2\vec{q} \frac{e^{i\vec{q}\vec{r}}}{q^2 + m^2}$$

Although this calculation is instructive, formula (180) doesn't solve all our problems. So, the critical correlation function has nothing to do with the $m \to 0$ limit of this formula, as the main contribution to the integral comes from a different spatial domain. Therefore we should rather return to the general formula (179), set $m = 0$ there, and analytically continue beyond the convergence domain $(-1/2 < \alpha < 1/2)$. The result is

$$D_0(q) = 2^{1/4}\pi \frac{\Gamma(7/8)}{\Gamma(1/8)} \frac{1}{q^{7/4}} \tag{182}$$

The correlation function in the disordered phase is also outside the validity of (180) and, furthermore, the integral (179) is logarithmically divergent at the lower limit in this case. It should therefore be regularised. One way to do it is as follows [25]:

$$\begin{aligned} I^{2m}_{-1/2}(q) &= \int\limits_0^\infty \frac{dr}{r} J_0(qr) e^{-2mr} \to \int_0^\infty \frac{dr}{r} J_\nu(qr) e^{-2mr} \\ &= \frac{(\sqrt{q^2+4m^2}-2m)^\nu}{\nu q^\nu} \to \frac{1}{\nu} + \ln\left(\frac{\sqrt{q^2+4m^2}-2m}{q}\right) + \mathrm{O}(\nu) \end{aligned} \tag{183}$$

This regularisation is, however, not good enough. That is because the integral still diverges in the $q \to 0$ limit as

$$\ln\left(\frac{q}{4m}\right)$$

while the physical regularisation is *finite* in the $q \to 0$ limit and expands as:

$$I^{2m}_{-1/2}(q) \to \int\limits_\xi^\infty \frac{dr}{r} J_0(qr) e^{-2mr} = \int\limits_\xi^\infty \frac{dr}{r} e^{-2mr} + \sum_{k=1}^{+\infty} \frac{(2k)!}{2^{2k}(k!)^2} \left(-\frac{q^2}{4m^2}\right)^k \tag{184}$$

Since different limiting processes must lead to the same result, one can simply substruct the unphysical $q \to 0$ divergence from (183). This leads to the following result for the correlation function (up to an unessential additive constant):

$$D_<(q) = \frac{35 A_1}{256 m^2} \ln\left(\frac{4m}{\sqrt{q^2+4m^2}+2m}\right) \tag{185}$$

Turning to the problem of analytic continuation, we recall that, according to the standard approach [26]

$$D(i\omega_n, p) = D^{(R)}(\omega, p)$$

in the upper half-plane. Put another way, one has to substitute

$$i\omega_n \to \omega + i\delta$$

(that is provided the resulting function has no 'accidental' poles and is otherwise regular in the upper half-plane - there is no general solution to the analytic continuation problem.) The resulting function is, by construction, analytic in the upper half-plane and automatically satisfies the Kramers–Kronig relation.

We need to analytically continue formula (180), or equivalently the function

$$f(i\omega_n) = \frac{1}{(\omega_n^2 + p^2 + m^2)^\alpha}$$

It is easy to see that

$$\begin{aligned} f(\omega) &= \frac{\theta(\epsilon_p^2 - \omega^2) + \cos(\pi\alpha)\theta(\omega^2 - \epsilon_p^2)}{|\omega^2 - \epsilon_p^2|^\alpha} \\ &- \frac{2i\sin(\pi\alpha)\theta(\omega^2 - \epsilon_p^2)}{|\omega^2 - \epsilon_p^2|^\alpha} \end{aligned} \tag{186}$$

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