# Ising Model Summary

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### Overview

The **Ising model** is a mathematical model of ferromagnetism in statistical mechanics, consisting of discrete variables representing magnetic dipole moments of atomic spins that can be in one of two states (+1 or -1).

### Basic Definitions

| Symbol | Meaning |
|---|---|
| $s_i$ | Spin at site $i$ ($\pm 1$) |

| Symbol | Meaning |
|---|---|
| $J$ | Coupling constant (interaction strength) |
| $h$ | External magnetic field |
| $k_B$ | Boltzmann constant |
| $T$ | Temperature |
| $\beta$ | Inverse temperature $= 1/(k_B T)$ |
| $N$ | Number of spins |

## Hamiltonian

### General Form

| Equation | Description |
|---|---|
| $H = -J\sum_{\langle i,j\rangle} s_i s_j - h\sum_i s_i$ | Full Hamiltonian with nearest-neighbor interactions and external field |
| $H = -J\sum_{\langle i,j\rangle} s_i s_j$ | Zero-field Hamiltonian |

- $\langle i,j\rangle$ denotes nearest-neighbor pairs
- $J > 0$: ferromagnetic (parallel spins favored)
- $J < 0$: antiferromagnetic (antiparallel spins favored)

## Statistical Mechanics Recap

### Partition Function and Energy

| Quantity | Formula | Description |
|---|---|---|
| **Partition function** | $Z = \sum_{\{s\}} e^{-\beta H}$ | Sum over all possible spin configurations |
| **Free energy** | $F = -k_B T \ln Z$ | Thermodynamic potential |
| **Energy** | $\langle E\rangle = -\frac{\partial \ln Z}{\partial \beta}$ | Average energy |
| **Probability** | $P(\{s\}) = \frac{e^{-\beta H(\{s\})}}{Z}$ | Boltzmann distribution |

## Order Parameters and Observables

| Observable | Formula | Physical Meaning |
|---|---|---|
| **Magnetization per spin** | $m = \frac{1}{N}\sum_i s_i$ | Average spin per site |
| **Susceptibility** | $\chi = \beta N(\langle m^2\rangle - \langle m\rangle^2)$ | Response to external field |
| **Specific heat** | $c = k_B\beta^2 N(\langle e^2\rangle - \langle e\rangle^2)$ | Heat capacity fluctuations |
| **Correlation function** | $G(r) = \langle s_i s_j\rangle - \langle s_i\rangle\langle s_j\rangle$ | Spin correlations at distance $r$ |

## Thermodynamic Quantities and Response Functions

| Quantity | Thermodynamic Definition | Response Function (Fluctuation-Dissipation) |
|---|---|---|
| **Magnetization per spin** | $m = -\frac{1}{N}\frac{\partial F}{\partial h}\Big\vert_T$ | $\chi = -\frac{\partial^2 F}{\partial h^2}\Big\vert_T = \beta N(\langle m^2\rangle - \langle m\rangle^2)$ |
| **Energy per spin** | $e = \frac{\partial(\beta F)}{\partial \beta}\Big\vert_h$ | $c = -\frac{1}{N}k_B\beta^2\frac{\partial^2 F}{\partial \beta^2}\Big\vert_h = k_B\beta^2 N(\langle e^2\rangle - \langle e\rangle^2)$ |

## Critical Exponents and Scaling

### Critical Scaling Laws

Near the critical point, physical quantities diverge or vanish as power laws in $t = (T - T_c)/T_c$:

| Exponent | Quantity | Behavior | Physical Meaning |
|---|---|---|---|
| $\alpha$ | Specific heat | $c \sim \|t\|^{-\alpha}$ | Heat capacity divergence |
| $\beta$ | Magnetization | $m \sim (-t)^\beta$ for $t < 0$ | Order parameter |
| $\gamma$ | Susceptibility | $\chi \sim \|t\|^{-\gamma}$ | Response to field |
| $\delta$ | Critical isotherm | $h \sim m^\delta$ at $T = T_c$ | Field-magnetization relation |
| $\nu$ | Correlation length | $\xi \sim \|t\|^{-\nu}$ | Spatial correlations |
| $\eta$ | Correlation function | $G(r) \sim r^{-(d-2+\eta)}$ at $T_c$ | Decay of correlations |

### Relations Between Exponents

Critical exponents are not independent

| Relation | Equation |
|---|---|
| **Rushbrooke** | $\alpha + 2\beta + \gamma = 2$ |
| **Widom** | $\gamma = \beta(\delta - 1)$ |
| **Fisher** | $\gamma = \nu(2 - \eta)$ |
| **Hyperscaling** | $d\nu = 2 - \alpha$ (valid for $d < 4$) |

**Note:** Only 2 exponents are independent; others follow from these relations.

### Universal Values by Dimension

| Dimension | $\alpha$ | $\beta$ | $\gamma$ | $\delta$ | $\nu$ | $\eta$ |
|---|---|---|---|---|---|---|
| **1D** | - | - | - | - | - | No transition |
| **2D (Exact)** | 0 (log) | 1/8 | 7/4 | 15 | 1 | 1/4 |
| **3D (Numerical)** | 0.110 | 0.326 | 1.237 | 4.789 | 0.630 | 0.036 |
| **Mean Field ($d \geq 4$)** | 0 | 1/2 | 1 | 3 | 1/2 | 0 |

## Exact Results (Onsager Solution, 2D)

| Property | Value/Formula | Notes |
|---|---|---|
| **Critical temperature** | $T_c = \frac{2J}{k_B \ln(1+\sqrt{2})} \approx 2.269 J/k_B$ | Phase transition point |
| **Critical exponents** | $\alpha = 0$ (log), $\beta = 1/8$, $\gamma = 7/4$, $\nu = 1$ | Exact values for 2D |
| **Spontaneous magnetization** | $m(T) \sim (T_c - T)^\beta$ for $T < T_c$ | Order parameter near $T_c$ |
| **Susceptibility** | $\chi(T) \sim \|T - T_c\|^{-\gamma}$ | Diverges at $T_c$ |

## Monte Carlo Simulation

### Metropolis Algorithm

| Step | Action | Formula |
|---|---|---|
| 1 | Select random spin $s_i$ | - |
| 2 | Calculate energy change | $\Delta E = 2s_i(J \sum_{\text{neighbors}} s_j + h)$ |
| 3 | Accept flip if | $\Delta E < 0$ or $\text{rand}(0,1) < e^{-\beta \Delta E}$ |
| 4 | Update configuration | Flip spin if accepted |

**Note:** One Monte Carlo step (MCS) = $N$ attempted flips

## Finite Size Effects

| Effect | Scaling | Notes |
|---|---|---|
| **Rounded transition** | Width $\sim L^{-1/\nu}$ | Transition smears out |
| **Shifted $T_c$** | $T_c(L) - T_c(\infty) \sim L^{-1/\nu}$ | Apparent $T_c$ moves |
| **Finite magnetization** | $m(T_c, L) \sim L^{-\beta/\nu}$ | Non-zero at pseudo-$T_c$ |

$L$ = linear system size, $\nu$ = correlation length exponent