the s l o w fourier transform algorithm

a silly talk about the einstein-wiener-khinchin theorem

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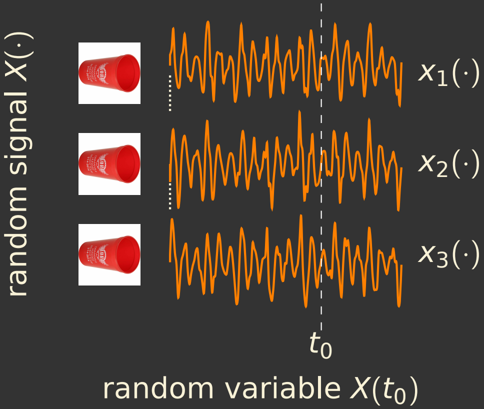
random signals (trigger warning: american-style presentation)

- A *random signal* is a function that maps each outcome of a probabilistic experiment to a real signal among an ensemble of signals.
- Permit me to call an *entire* random signal $X(t)$ if continuous, $X[n]$ if discrete. Sometimes I will more sensibly call it $X(\cdot), X[\cdot]$, or simply X.
- A signal's value at a particular time t_i (or n_i) can be seen as a real random variable—call it $X(t_i)$ (or $X[n_i]$)—taking its value according to the distribution of values among the ensembles' signals at t_i (or n_i).
- At each point in time, we model $X(t_i)$ (or $X[n_i]$) as a continuous random variable with PDF $f_{X(t_i)}(\cdot)$ (or $f_{X[n_i]}(\cdot)$). (For certain random signals, a PMF over a discrete set of signal values at each time is more appropriate.)

“you’re giving three-card monte a bad name!”



first & second moments of random signals X, Y

We can consider joint densities of several of these random variables, be they in the same random signal $f_{X(t_1),X(t_2),\ldots,X(t_k)}(x_1, x_2, \ldots, x_k)$ or in different signals $f_{X(t_1),\ldots,X(t_k),Y(t'_1),\ldots,Y(t'_l)}(x_1, \ldots, x_k, y_1, \ldots y_l)$.

- *Mean:* $\mu_X(t_i) = \mathbb{E}[X(t_i)]$,
- *Autocorrelation:* $R_{XX}(t_i, t_j) = \mathbb{E}[X(t_i)X(t_j)]$,
- *Autocovariance:*

$$\begin{aligned} C_{XX}(t_i, t_j) &= \mathbb{E}[(X(t_i) - \mu_X(t_i))(X(t_j) - \mu_X(t_j))] \\ &= R_{XX}(t_i, t_j) - \mu_X(t_i)\mu_X(t_j) \end{aligned}$$

- *Cross-correlation:* $R_{XY}(t_i, t_j) = \mathbb{E}[X(t_i)Y(t_j)]$
- *Cross-covariance:*

$$\begin{aligned} C_{XY}(t_i, t_j) &= \mathbb{E}[(X(t_i) - \mu_X(t_i))(Y(t_j) - \mu_Y(t_j))] \\ &= R_{XY}(t_i, t_j) - \mu_X(t_i)\mu_Y(t_j) \end{aligned}$$

model simplification

Figure: Grab-and-Go for Kids on the Move!

We don't have time to find a spoon, let alone to specify the joint PDF for every possible combination of samples $X(t_1), \dots, X(t_k)$ and $Y(t'_1), \dots, Y(t'_l)$.

We may assume two random signals X and Y are independent:

$$f_{X(t_1),\dots,X(t_k),Y(t'_1),\dots,Y(t'_l)}(x_1, \dots, x_k, y_1, \dots, y_l) =$$
$$f_{X(t_1),\dots,X(t_k)}(x_1, \dots, x_k) \cdot f_{Y(t'_1),\dots,Y(t'_l)}(y_1, \dots, y_l)$$

Now characterizing each random signal is as easy as specifying the PDFs for every possible subset of samples of each signal!

Natural signals of interest, even those assumed to be uncorrelated noise, exhibit strong intertemporal correlation when observation involves filtering.

strict-sense stationarity (sss)

If X is SSS, all joint statistics for $X(t_1), \ldots, X(t_k)$ depend on the relative values of the sampling instances $t_1, \ldots, t_k$, not on the sampling instances themselves.

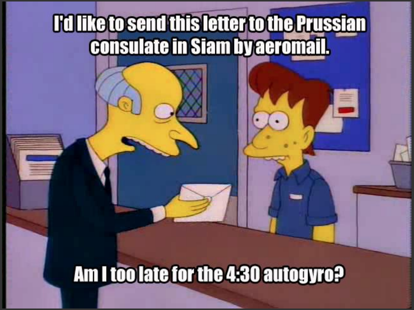


Figure: The US Postal Service is not SSS.

Thus, for arbitrary time shift τ, we have that
$f_{X(t_1),\ldots,X(t_k)}(x_1, \ldots, x_k) = f_{X(t_1+\tau),\ldots,X(t_k+\tau)}(x_1, \ldots, x_k)$.

wide-sense stationarity

A random signal X is WSS if its first two moments are stationary:

- $\mu_X(t_i) = \mu_X$ for all t_i
- $R_{XX}(t_i, t_j) = R_{XX}(t_i + \alpha, t_j + \alpha)$ for all t_i, t_j, α. By setting $\alpha = t - t_j$ for arbitrary t, we can write $R_{XX}(\tau) = R_{XX}(t + \tau, t) = \mathbb{E}[X(t + \tau)X(t)]$.
- It follows that C_{XX} too depends only on the difference τ of its arguments.

Two random signals X and Y are jointly WSS if they are individually WSS and if $R_{XY}(t_i, t_j)$ depends only on the time lag $\tau = t_i - t_j$, ie if we we can write $R_{XY}(\tau) = \mathbb{E}[X(t + \tau)Y(t)]$.

properties of wss signals

- *Symmetry:*

$$R_{XX}(\tau) = R_{XX}(-\tau),\ R_{XY}(\tau) = R_{YX}(-\tau),$$
$$C_{XX}(\tau) = C_{XX}(-\tau),\ C_{XY}(\tau) = C_{YX}(-\tau).$$

It follows then that $R_{XX}(\tau)$ and $C_{XX}(\tau)$ have real, even Fourier transforms at all frequencies. We will soon see the transforms are nonnegative.

- *Second auto-moments attain max at $\tau = 0$:*

$$-C_{XX}(0) \leq C_{XX}(\tau) \leq C_{XX}(0)$$
$$2\mu_X^2 - R_{XX}(0) \leq R_{XX}(\tau) \leq R_{XX}(0)$$

Follows from the fact that a correlation coefficient's magnitude never exceeds 1.

"ergodicity"

- Time-averages of statistics of ensemble member approach those of the ensemble
- *Ergodic in the mean:* time averages of X converge in squared mean to μ_X, ie

$$\langle X \rangle_T = \frac{1}{2T} \int_{-T}^{T} X(t)dt \overset{L^2}{\to} \mu_X$$

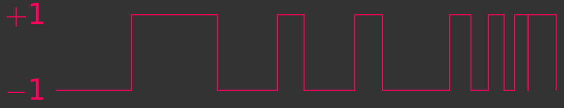
- Any WSS process with finite variance at each instant and with a covariance function that approaches 0 for large τ is ergodic in the mean
- EE approach: assume ergodicity unless it's very unreasonable. Then obtain first moment $\mathbb{E}[X(t)] = \frac{1}{2T} \int_{-T}^{T} x(t)dt$ and autocorrelation $\mathbb{E}[X(t+\tau)X(t)] = \frac{1}{2T} \int_{-T}^{T} x(t)dt$ on x, a realization of X, for some large T.

example: random telegraph signal

- Random initial polarity PMF: $X(0) = \pm 1$ w.p. 0.5
- Polarity changes at Poisson times:

$$\text{P}(k \text{ sign changes in interval of length } T) = \frac{(\lambda T)^k e^{-\lambda T}}{k!}$$

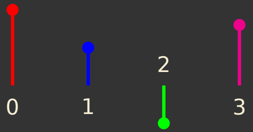
- Probability of an even number of sign changes in an interval of length T: $\frac{1}{2}(1 + e^{-2\lambda T})$. Probability of an odd number of sign changes: $\frac{1}{2}(1 - e^{-2\lambda T})$.
- First moment is stationary: $\mu_X(t) = 0$ because $\text{P}(X(t) = +1) = \frac{1}{2}(1 + e^{-2\lambda T}) \cdot \frac{1}{2} + \frac{1}{2}(1 - e^{-2\lambda T}) \cdot \frac{1}{2} = \frac{1}{2}$.
- Second moment is stationary: $R_{XX}(\tau) = e^{-2\lambda|\tau|}$ as $\mathbb{E}[X(t_i)X(t_j)] = +1 \cdot \text{P}(X(t_i) = X(t_j)) - 1 \cdot \text{P}(X(t_i) \neq X(t_j)) = e^{-2\lambda|t_j - t_i|}$.



passing a wss process through an lti filter

$$x[n] \longrightarrow \boxed{h[n]} \longrightarrow y[n]$$

Recall: for deterministic signals, linear, time-invariant systems are completely characterized by their impulse-response function h. $y = h * x$.



$$x[n] = x[0]\delta[n-0] + x[1]\delta[n-1] + x[2]\delta[n-2] + x[3]\delta[n-3]$$

If x is the sum of scaled and shifted lollipops, and h is an LTI system, then y is the sum of scaled and shifted *responses* to the unit lollipop.[1]

$$y[n] = x[0]h[n-0] + x[1]h[n-1] + x[2]h[n-2] + x[3]h[n-3]$$

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[1] The basic convolution relation holds for a wide class of practical functions, for instance, when x is bounded and h is absolutely summable/integrable.

passing a wss process through an lti filter

Preserves wise-sense stationarity! Y is jointly WSS with X.

$$\begin{aligned}\mathbb{E}[y(t)] &= \mathbb{E}\left[\int_{-\infty}^{\infty} h(v)x(t-v)dv\right] \\ &= \int_{-\infty}^{\infty} h(v)\mathbb{E}[x(t-v)]dv^2 \\ &= \mu_X \int_{-\infty}^{\infty} h(v)dv = H(j0)\mu_X,\end{aligned}$$

which does not depend on the time t.

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[2] Electrical engineers to mathematicians' quibbles: 

When working with WSS processes we generally do not require that every function in the random signal ensemble of the input X need be bounded, merely that $R_{XX}(\tau)$ be bounded, but with care this result can apply to nonbounded inputs (e.g., continuous white noise, where $R_{XX}(\tau) = \delta(\tau)$ and non-BIBO systems, such as the ideal LPF, when the frequency response function is well-defined.

passing a wss process through an lti filter

Deterministic relationship between autocorrelation of input and cross-correlation of output and input:

$$R_{XX}(\tau) \longrightarrow \boxed{h(\tau)} \longrightarrow R_{YX}(\tau)$$

$$\begin{aligned}
R_{YX}(\tau) = \mathbb{E}[y(t+\tau)x(t)] &= \mathbb{E}\left[\int_{-\infty}^{\infty}\big(h(v)x(t+\tau-v)\big)x(v)dv\right] \\
&= \int_{-\infty}^{\infty} h(v)\mathbb{E}[x(t+\tau-v)x(v)]dv \\
&= \int -\infty^{\infty} h(v)R_{XX}(\tau-v)dv \\
&= (h * R_{xx})(\tau).
\end{aligned}$$

Noting that $R_{XY}(\tau) = R_{YX}(-\tau) = h(-\tau) * R_{XX}(-\tau) = R_{XX}(\tau) * h(-\tau)$, we can conclude further that:

$$R_{YY}(\tau) = h(-\tau) * h(\tau) * R_{XX}(\tau) \tag{1}$$

passing a wss signal through an lti filter

Putting it together:

$$R_{XX}(\tau) \longrightarrow \boxed{h(-\tau)} \longrightarrow R_{XY}(\tau)$$

$$R_{XY}(\tau) \longrightarrow \boxed{h(\tau)} \longrightarrow R_{YY}(\tau)$$

Define the deterministic autocorrelation function:
$\overline{R_{hh}}(\tau) = h(\tau) * h(-\tau)$.

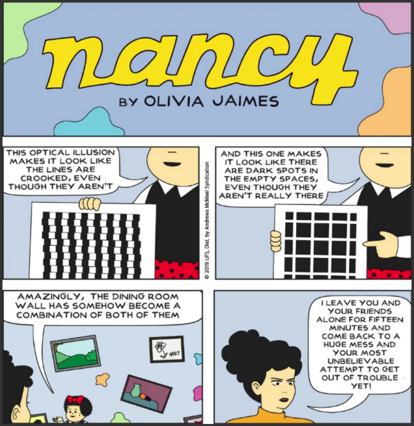
$$R_{XX}(\tau) \longrightarrow \boxed{\overline{R_{hh}}(\tau)} \longrightarrow R_{YY}(\tau)$$

In the frequency domain:

$$S_{YX}(j\omega) = S_{XX}(j\omega)H(j\omega) \; S_{XY}(j\omega) = S_{XX}(j\omega)H^*(j\omega)$$
$$S_{YY}(j\omega) = S_{YX}(j\omega)H^*(j\omega) \; S_{YY}(j\omega) = S_{XX}(j\omega)|H(j\omega)|^2.$$

where $S_{XX}(j\omega)$ be the Fourier transform of $R_{XX}(\tau)$ and so forth.

the fourier transform of a random signal: cleaning up the mess



california history (as taught in france)

California became its own state in 1850 so that marijuana-legalization advocates could define an idiotic marijuana culture and slowly build momentum for the 2016 legalization of the (then-legal) substance.

A young Arnold Schwarzenegger smoking marijuana, 1977.

but california is so much more

A young Arnold Schwarzenegger smoking marijuana, 1977.