Geometry of Lines in $\mathbb{P}^3$

Max Weinstein

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1 Introduction

In modern terms, a collection of linear subspaces of a certain dimension relative to a larger vector space they live in is called a *Grassmanian.* From this perspective, the collection of all lines in $\mathbb{P}^3$ is the Grassmanian of 1-dimensional linear subspaces of 3-dimensional projective space, or alternatively the Grassmanian of 2-dimensional linear subspaces of 4-dimensional affine space.

Before Herman Grassman, however, there was Julius Plücker and his student Felix Klein, who studied the geometry of projective spaces and eventually led mathematics as a whole to abandon the primacy of Euclidean geometry in favor of a more egalitarian approach.

2 Lines in Lower Dimensional Spaces

In dimensions 1 and 2, the set of all lines is itself a projective space. All lines in $\mathbb{P}^1$ trivially constitute $\mathbb{P}^1$ and all lines in $\mathbb{P}^2$ are dual to the points in $\mathbb{P}^2$, and therefore dually constitute the projective plane. Lines through a single point in $\mathbb{P}^2$ form a pencil of lines, which is equivalent to a $\mathbb{P}^1$. (This case is due to the fact that any pencil of lines in $\mathbb{P}^2$ is uniquely determined by a choice of 2 lines, their unique intersection point being the center of the pencil, and any other line in this same pencil being a linear combination of the first 2: hence 2 coordinates suffice to determine any line in the pencil without any relations, hence a 1 dimensional projective space).

3 Lines in $\mathbb{P}^3$

Unlike lines in lower dimensional spaces, lines in $\mathbb{P}^3$ have non-trivial geometric structure (they are not equivalent to some $\mathbb{P}^n$), and this can be seen fairly easily with the help of Plücker coordinates.

Any line is determined uniquely by 2 points, so let $P^1 = [x_0, x_1, x_2, x_3]$ and $P^2 = [y_0, y_1, y_2, y_3]$ be two points in $\mathbb{P}^3$, then set $l_{ij} = P^1_i P^2_j - P^1_j P^2_i = x_i y_j - x_j y_i$. Notice first that $l_{ij} = -lji$ and $l_{ii} = 0$, hence of the 16 possible combinations (4 choices for i times 4 choices for j) we are only left with 6 independent quantities determined by the two points: $[l_{23}, l_{31}, l_{12}, l_{01}, l_{02}, l_{03}]$. Our line in $\mathbb{P}^3$ is thus represented by a 6 coordinate "point," which, lives in a projective space of dimension 5.

To see this is well defined correspondence, let $\overline{P^1} = sP^1 + tP^2$ and $\overline{P^2} = s'P^1 + t'P^2$ be two different points on the same line. Then

$$\overline{l_{ij}} = \overline{P^1_i P^2_j} - \overline{P^1_j P^2_i} = (sP^1_i + tP^2_i)(s'P^1_j + t'P^2_j) - (sP^1_j + tP^2_j)(s'P^1_i + t'P^2_i) = (st' - s't)l_{ij}$$

And hence $\bar{l}$ is just a scalar multiple of l, so they are equivalent in $\mathbb{P}^5$, and every line in $\mathbb{P}^3$ has a unique (up to equivalence) representation in $\mathbb{P}^5$.

In lower dimensions, a correspondence like the one above can be shown to be bijective, and as we've already seen lines in dimensions 2 and 1 correspond exactly to projective spaces and so have somewhat trivial geometry. The correspondence of 3 dimensional lines to points in $\mathbb{P}^5$, however, is not bijective.

It is a fact that the dimension of the grassmanian $\mathrm{Gr}(k,n)$ of k-dimensional linear subspaces of n-dimensional space is $k(n-k)$, so in our case lines in $\mathbb{P}^3$ form the grassmanian $\mathrm{Gr}(2,4)$ which has dimension $2(4-2)=4$, 1 dimension lower than $\mathbb{P}^5$. This tells us that there is additional structure to the correspondence above, in other words we can expect to find some relation amongst the lines that accounts for 1 missing dimension. This relation will define what is sometimes referred to as the Klein Quadric.

4 The Klein Quadric

We can find our relation in the following way:

If $P^1 = [x_0, x_1, x_2, x_3]$ and $P^2 = [y_0, y_1, y_2, y_3]$ are two points determining a line in $\mathbb{P}^3$ as before, then we can set up the following matrix:

$$\begin{bmatrix} x_0 & x_1 & x_2 & x_3 \\ y_0 & y_1 & y_2 & y_3 \\ x_0 & x_1 & x_2 & x_3 \\ y_0 & y_1 & y_2 & y_3 \end{bmatrix}$$

By choosing linearly dependent rows, we know the determinant of this matrix is 0, so we get the following equation:

$$x_0 \begin{vmatrix} y_1 & y_2 & y_3 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{vmatrix} - x_1 \begin{vmatrix} y_0 & y_2 & y_3 \\ x_0 & x_2 & x_3 \\ y_0 & y_2 & y_3 \end{vmatrix} + x_2 \begin{vmatrix} y_0 & y_1 & y_3 \\ x_0 & x_1 & x_3 \\ y_0 & y_1 & y_3 \end{vmatrix} - x_3 \begin{vmatrix} y_0 & y_1 & y_2 \\ x_0 & x_1 & x_2 \\ y_0 & y_1 & y_2 \end{vmatrix} = 0$$

$$x_0\Big(y_1(l_{23})-y_2(l_{13})+y_3(l_{12})\Big)-x_1\Big(y_1(l_{23})-y_2(l_{13})+y_3(l_{12})\Big)+x_2\Big(y_1(l_{23})-y_2(l_{13})+y_3(l_{12})\Big)-x_3\Big(y_1(l_{23})-y_2(l_{13})+y_3(l_{12})\Big) = 0$$

$$2(l_{01}l_{23} + l_{02}l_{13} + l_{03}l_{12}) = 0$$

Which as a homogeneous equation is equivalent to

$$l_{01}l_{23} + l_{02}l_{13} + l_{03}l_{12} = 0$$

This is a homogeneous equation of degree 2, and hence a quadric, in $\mathbb{P}^5$ that all lines in $\mathbb{P}^3$ satisfy.

5 Properties of the Klein Quadric

The fascinating properties of this quadric, Q, arise from the fact that the Plücker correspondence of lines in $\mathbb{P}^3$ to points in $\mathbb{P}^5$ is incidence preserving.

From this we get that two intersecting lines in $\mathbb{P}^3$ are mapped to points that lie on a line contained entirely in Q.

All lines in $\mathbb{P}^3$ through a single point are thus mapped to a set of points in Q that are all mutually colinear, in otherwords a plane contained in Q.

Similarly, if you fix a plane in $\mathbb{P}^3$, all lines in that plane all intersect and are hence mapped again to a set of mutually colinear points, another type of plane in Q

Thus we get two families of planes living in Q, those arising from points and those from planes (these are sometimes called "latins" and "greeks")

Two planes from the same family have the interesting property that they meet in Q at exactly 1 point! Take two points in $\mathbb{P}^3$ (that give us two planes in one family in Q), these span a unique line that corresponds to a single point in Q contained in both planes. Similarly take two planes in $\mathbb{P}^3$. These intersect on a unique line, corresponding to a unique point in Q!