Symplectic Reduction and Geometric Invariant Theory

Maxence Mayrand

OVERVIEW

- M complex algebraic variety (non-singular) $\Rightarrow (M, I)$ complex manifold
- g Kähler metric on M $\Rightarrow (M, \omega)$ symplectic manifold
- K compact Lie group s.t. $G := K_{\mathbb{C}} \circlearrowright (M, I)$, $K \circlearrowright (M, \omega)$

3 types of quotients:

symplectic

$$\mu^{-1}(0)/K \cong M/\!\!/_L G = M^{L\text{-ss}}/G$$

algebraic (GIT)

$\boxed{\text{choice of linearization } L \text{ (line bundle)}}$

categorical quotient in AG

- homeomorphisms
- stratified
- Kähler

$\boxed{\text{choice of moment map } \mu}$

always true: $\mu^{-1}(0)/K \cong M/\!\!/_\mu G$

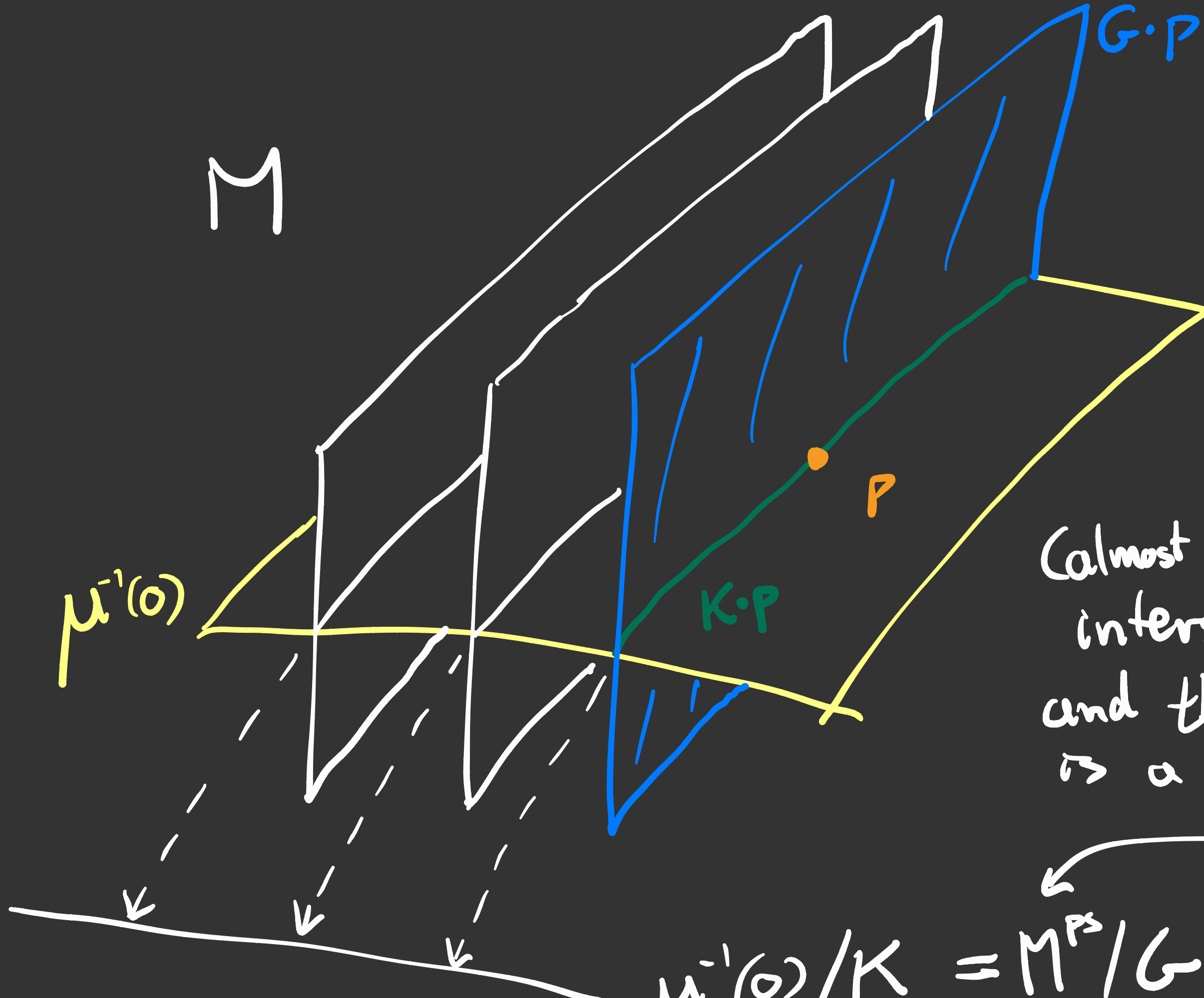
true $\Leftrightarrow M^{\mu\text{-ss}} = M^{L\text{-ss}}$: $M/\!\!/_\mu G \cong M/\!\!/_L G$

$\Leftrightarrow$
- compatibility between μ and L
- good asymptotic properties at ∞ of M (always true if M is compact)

$$M/\!\!/_\mu G = M^{\mu\text{-ss}}/G$$

complex analytic

categorical quotient in complex analytic spaces



$G\cdot p \supseteq K\cdot p$
since $K \subseteq G$

(almost all) G-orbits intersect $\mu^{-1}(0)$, and the intersection is a K-orbit, so

$$\mu^{-1}(0)/K = M^{ps}/G = M /\!\!/_{L}\, G$$

What is a quotient?

A **quotient** of a geometric space M (Riemanian, complex, symplectic, ...) by a group G of symmetries (isometries, biholomorphisms, symplectomorphisms, ...) should be a space N (in the same category) encoding a large class of G-invariant objects on M.

Example M smooth manifold, K compact Lie group $\circlearrowleft M$ freely

The orbit space M/K encodes the K-invariant subsets.

It also encodes the K-invariant smooth maps.

$$\begin{tikzcd} M \arrow[d] \arrow[dr, "\forall\ G\text{-invariant}"] & \\ M/K \arrow[r, dashed, "\exists!"] & P \end{tikzcd}$$

Definition. A categorical quotient for an action of a group G on an object M is a G-invariant morphism $M \longrightarrow N$ such that

$$\begin{tikzcd} M \arrow[d] \arrow[dr, "\forall\, G\text{-invariant morphism}"] & \\ N \arrow[r, dashed, "\exists!"] & P \end{tikzcd}$$

(encodes all G-invariant morphisms)

$\Longrightarrow$ unique up to unique isomorphisms

It is denoted $M/\!/G$ (if it exists).

two slashes because it tends to identify orbits. GIT: $x \sim y \Leftrightarrow \overline{G\cdot x} \cap \overline{G\cdot y} \neq \emptyset$.

Geometric Invariant Theory

G complex algebraic group $\circlearrowright$ M complex algebraic variety

Goal: Construct a quotient of M by G.

Problems:
- M/G is in general not algebraic
- $M/\!/G$ might not exist or be too small

Solution: Find a large open subset $M^{L\text{-}ss} \subseteq M$ on which $M^{L\text{-}ss}/\!/G$ exists.

Depends on a choice of line bundle $L \to M$ and lift $G \circlearrowright L$.

The affine case. Let:

- $M \subseteq \mathbb{C}^n$ smooth affine variety
- $G \subseteq GL(n,\mathbb{C})$ preserving M
- suppose G is **reductive**: all finite-dimensional representations decompose into irreducible summands

$(\Longleftrightarrow G = K_{\mathbb{C}})$ (e.g. $GL(n,\mathbb{C})$, all complex semisimple groups)

$\boxed{\text{**Nagata's Theorem.** The algebra } \mathbb{C}[M]^G \text{ of } G\text{-invariant polynomials on } M \text{ is finitely generated.}}$

$\Longrightarrow M/\!/G := \mathrm{Spec}(\mathbb{C}[M]^G)$ is a categorical quotient

where $M \xrightarrow{\pi} M/\!/G$ is induced by $\mathbb{C}[M]^G \subseteq \mathbb{C}[M]$.

concretely: $M \subseteq \mathbb{C}^n$, $\mathbb{C}[M] = \mathbb{C}[x_1, \dots, x_n]/I$

Nagata $\Rightarrow$ $\mathbb{C}[M]^G \simeq \langle f_1, \dots, f_k \rangle$, $f_i \in \mathbb{C}[M]^G \subseteq \mathbb{C}[M]$

$$0 \longrightarrow \underset{\langle g_1, \dots, g_m \rangle}{\overset{\displaystyle J}{\shortparallel}} \longrightarrow \mathbb{C}[y_1, \dots, y_k] \longrightarrow \mathbb{C}[M]^G \longrightarrow 0$$

$$y_i \longmapsto f_i$$

$$\mathbb{C}[M]^G \cong \mathbb{C}[y_1, \dots, y_k]/\langle g_1, \dots, g_m \rangle$$

so $M/\!/G = \{ y \in \mathbb{C}^k : g_i(y) = 0, \forall i \}$

$$M \subseteq \mathbb{C}^n \longrightarrow M/\!/G \subseteq \mathbb{C}^k$$

$$x \longmapsto (f_1(x), \dots, f_k(x)).$$

Mumford's Theorem.

- M complex algebraic variety
- G complex reductive group $\circlearrowright M$ algebraically
- $L \to M$ line bundle with lift $G \circlearrowright L$ (linearization)

Define the set of semistable points

$$M^{L\text{-}ss} := \bigcup_{\substack{\sigma \in H^0(L^{\otimes r})^G \\ r \geq 1,\ M_\sigma \text{ affine}}} M_\sigma \quad , \quad M_\sigma := \{ x \in M : \sigma(x) \neq 0 \}$$

Then, there is a categorical quotient for $G \circlearrowright M^{L\text{-}ss}$, denoted $M /\!\!/_L G$.

Proof. Glue $M_\sigma /\!\!/ G := \operatorname{Spec} \mathbb{C}[M_\sigma]^G$ together. $\square$

$M/\!/_L G$ is also a good quotient:

- $(\pi_* \mathcal{O}_M)^G = \mathcal{O}_{M/\!/_L G}$
- $\pi^{-1}(\text{affine})$ is affine

$\Rightarrow M/\!/_L G = M^{L\text{-ss}}/\sim, \quad x \sim y \Leftrightarrow \overline{G\cdot x} \cap \overline{G\cdot y} \cap M^{L\text{-ss}} \neq \phi.$

Polystable points: $M^{L\text{-ps}} = \{p \in M^{L\text{-ss}} : G\cdot p \text{ is closed in } M^{L\text{-ss}}\}$

Every $\overline{G\cdot p} \cap M^{L\text{-ss}}$ contains a unique closed orbit, so

$$M/\!/_L G = M^{L\text{-ss}}/\sim \;=\; M^{L\text{-ps}}/G$$

Stable points: $M^{L\text{-s}} = \{p \in M^{L\text{-s}} : \dim G_p = 0\} \underset{\text{open}}{\subseteq} M^{L\text{-ss}}$

$$\underbrace{M^{L\text{-s}}/G}_{\text{algebraic variety}} \underset{\text{open}}{\subseteq} M/\!/_L G$$

GIT for projective-over-affine varieties

Definition. A **projective-over-affine variety** is a closed subvariety $M \subseteq \mathbb{C}^n \times \mathbb{P}^m$ for some $n, m \geq 0$.

Equivalently, $M = \mathrm{Proj}(R)$, where $R = \bigoplus_{k=0}^{\infty} R_k$ is finitely generated in R_0 and R_1.

Concretely: $R = \langle u_1, \dots, u_n, v_0, \dots, v_m \rangle$, $\quad u_i \in R_0$, $v_i \in R_1$

$$0 \longrightarrow I \longrightarrow \mathbb{C}[x_1, \dots, x_n, y_0, \dots, y_m] \longrightarrow R \longrightarrow 0$$

$$x_i \longmapsto u_i \qquad y_i \longmapsto v_i$$

$I = \langle f_1, \dots, f_k \rangle$, f_i homogeneous

$$R \cong \mathbb{C}[x_1, \dots, x_n, y_0, \dots, y_m] / \langle f_1, \dots, f_k \rangle$$

$$M = \{ (x, [y]) \in \mathbb{C}^n \times \mathbb{P}^m : f_i(x, y) = 0, \ \forall i \}$$

$$
\begin{tikzcd}
\operatorname{Proj}\left(\bigoplus_{k=0}^{\infty} R_k\right) \arrow[d] & = & M \arrow[d, "\pi"] & \subseteq & \mathbb{C}^n \times \mathbb{P}^m \arrow[d, "pr_1"] \\
\operatorname{Spec}(R_0) & = & N & \subseteq & \mathbb{C}^n
\end{tikzcd}
$$

projective morphism over an affine variety

Definition. A line bundle L on a p.o.a. M is very ample if $M \cong \operatorname{Proj} \bigoplus_{r=0}^{\infty} H^0(L^r)$, i.e.

$$\mathbb{C}[M] = H^0(M, L^0) = \langle u_1, \ldots, u_n \rangle$$

$$H^0(M, L) = \langle v_0, \ldots, v_m \rangle$$

and
$$
\begin{aligned}
M &\longrightarrow \mathbb{C}^n \times \mathbb{P}^m \\
p &\longmapsto (u(p), [v(p)])
\end{aligned}
$$
is a closed embedding.

A line bundle L is ample if $L^{\otimes r}$ is very ample for some $r \geq 1$.

If $M \subseteq \mathbb{C}^n \times \mathbb{P}^m$, $L = pr_2^* \mathcal{O}(1)|_M$ is ample, $pr_2 : \mathbb{C}^n \times \mathbb{P}^m \to \mathbb{P}^m$

Let $G \circlearrowright M$ by $G \to GL(n) \times GL(m+1)$. Then $G \circlearrowright L$.

Theorem. Let M be a projective-over-affine variety with very ample line bundle L and G acting linearly on L. Then

$$R^G := \bigoplus_{r=0}^{\infty} H^0(L^r)^G$$

is finitely generated in degree 0 and 1, $M/\!/_L G = \operatorname{Proj}(R^G)$, $M^{L\text{-}ss} = \{p \in M : \exists \sigma \in H^0(L)^G, \sigma(p) \neq 0\}$, and $M^{L\text{-}ss} \to M/\!/_L G$ is induced by $R^G \subseteq R$.

Concretely: $R^G = \langle u_1, \dots, u_r, v_0, \dots, v_s \rangle$, $u_i \in \mathbb{C}[M]^G$, $v_i \in H^0(M,L)^G$

$$\begin{aligned} M^{L\text{-}ss} \subseteq \mathbb{C}^n \times \mathbb{P}^m &\longrightarrow M/\!/_L G \subseteq \mathbb{C}^r \times \mathbb{P}^s \\ (x, [y]) &\longmapsto (u(x), [v(y)]) \end{aligned}$$

$$M^{L\text{-}ss} = \{(x,[y]) \in M : \exists i, v_i(y) \neq 0\}$$

Hilbert-Mumford Criterion

"semistable w.r.t. G $\iff$ semistable w.r.t. all $\mathbb{C}^* \leq G$"

Let:
- M projective-over-affine
- $L \to M$ very ample
- $G \circlearrowright L$ linearly

Let $\lambda : \mathbb{C}^* \to G$ be a group homomorphism (1-parameter subgroup)

Let $p \in M$, and let $p_0 = \lim_{t \to 0} \lambda(t) \cdot p$, if it exists.

Then, $\mathbb{C}^* \circlearrowright L_{p_0} \cong \mathbb{C}$ so $\lambda(t) \cdot v = t^{-k} v$, $k \in \mathbb{Z}$, $\forall v \in L_{p_0}$.

$\mu^L(p, \lambda) := k$ or $\mu^L(p, \lambda) = \infty$ if p_0 does not exist.

$$\boxed{\begin{array}{l} \textbf{Theorem} \cdot\ p \in M^{L\text{-}ss} \iff \mu^L(p,\lambda) \geq 0,\ \forall \lambda \\ \qquad\qquad\quad \cdot\ p \in M^{L\text{-}s} \iff \mu^L(p,\lambda) > 0,\ \forall \lambda \end{array}}$$

OVERVIEW

- M complex algebraic variety (non-singular) $\Rightarrow (M, I)$ complex manifold
- g Kähler metric on M $\Rightarrow (M, \omega)$ symplectic manifold
- K compact Lie group s.t. $G := K_{\mathbb{C}} \circlearrowright (M, I)$, $K \circlearrowright (M, \omega)$

3 types of quotients:

symplectic

algebraic (GIT)

choice of linearization L (line bundle)

$$\mu^{-1}(0)/K \cong M/\!\!/_L G = M^{L\text{-ss}}/G$$

categorical quotient in AG

choice of moment map μ

· homeomorphisms

· stratified

· Kähler

$\cong$ (left) $\qquad \cong$ (right)

true $\Leftrightarrow M^{\mu\text{-ss}} = M^{L\text{-ss}}$

$\Leftrightarrow$ • compatibility between μ and L

• good asymptotic properties at ∞ of M (always true if M is compact)

always true

$$M/\!\!/_\mu G = M^{\mu\text{-ss}}/G$$

complex analytic

categorical quotient in complex analytic spaces

Symplectic Reduction

K compact Lie group $\circlearrowright (M,\omega)$ symplectic manifold.

Goal: Construct a quotient of M by K.

Problems:
- M/K is not symplectic (unless $\dim K = 0$)
- $\nexists$ categorical quotient $M \to N$

Reason: subsets and symplectomorphisms are not the "right" K-invariant objects

Solution: (M,ω) should be thougth of as a "phase space" in classical mechanics. The "right" K-invariant objects are then particle motions under K-invariant hamiltonians.
This leads to the Marsden-Weinstein reduction.

Definition. A **moment map** for $K \circlearrowright (M,\omega)$ is a map $\mu : M \longrightarrow \mathfrak{k}^*$ such that

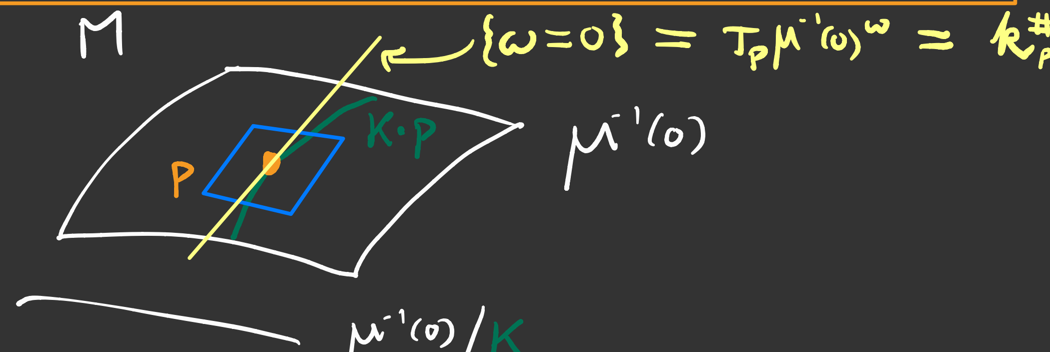
(1) $d\mu^x = i_{x^\#}\omega$ for all $x \in \mathfrak{k}$, where

- $\mu^x : M \to \mathbb{R}$, $\mu^x(p) = \mu(p)(x)$
- $x^\# \in \mathfrak{X}(M)$, $x^\#_p = \frac{d}{dt}\Big|_{t=0} e^{tx}\cdot p$

(2) $\mu(k\cdot p) = \mathrm{Ad}^*_k\, \mu(p)$.

If a moment map exists, the action is called **Hamiltonian.**

Note. Moment maps are not unique. If μ is a moment map and $\xi \in \mathfrak{k}^*$ is fixed by K, $\mu - \xi$ is a new moment map



Theorem. (Marsden–Weinstein 1974, Sjamaar–Lerman 1991)

Let:
- (M,ω) symplectic manifold
- K compact Lie group $\circlearrowright (M,\omega)$
- $\mu : M \longrightarrow \mathfrak{k}^*$ moment map

If K acts freely on $\mu^{-1}(0)$, then 0 is a regular value of μ, so $\mu^{-1}(0)$ and $\mu^{-1}(0)/K$ are smooth. Moreover there is a unique symplectic form $\bar{\omega}$ on $\mu^{-1}(0)/K$ such that $\pi^*\bar{\omega} = i^*\omega$ where

$$\begin{array}{ccc} \mu^{-1}(0) & \xhookrightarrow{i} & M \\ \downarrow \pi & & \\ \mu^{-1}(0)/K & & \end{array}$$

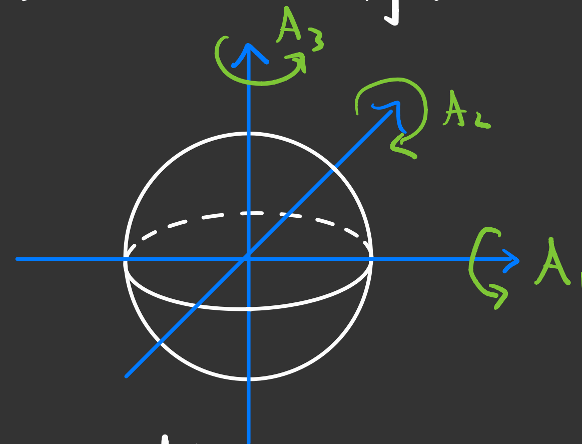
More generally, $\mu^{-1}(0)/K$ is stratified into symplectic manifolds

$$\mu^{-1}(0)/K = \bigcup_{(H)} \underbrace{\mu^{-1}(0)_{(H)}/K}_{\text{symplectic}}, \quad \mu^{-1}(0)_{(H)} = \{p \in \mu^{-1}(0) : K_p \text{ conjugate to } H \text{ in } K\}$$

Example. $S^2 \subseteq \mathbb{R}^3$ with standard area form ω

$SO(3) \circlearrowright S^2$

$\mathfrak{so}(3) = \mathrm{span}\{A_1, A_2, A_3\}$



$$\mu_i : S^2 \to \mathbb{R}, \quad d\mu_i = i_{A_i^\#}\omega$$
$$x \longmapsto x_i$$

$$\omega = d\theta \wedge dx_3, \quad A_3^\# = \frac{\partial}{\partial\theta}, \quad i_{A_3^\#}\omega = dx_3$$

$$\boxed{\mu : S^2 \hookrightarrow \mathbb{R}^3 \text{ after identifying } \mathfrak{so}(3)^* \cong \mathbb{R}^3 \text{ by } A_1, A_2, A_3}$$

More generally: $\mathfrak{k}$ Lie algebra, $\mathcal{O} \subseteq \mathfrak{k}^*$ coadjoint orbit,

$\Rightarrow \mathcal{O}$ symplectic (Kirillov-Kostant-Souriau), $K \circlearrowright \mathcal{O}$

$\mu : \mathcal{O} \hookrightarrow \mathfrak{k}^*$.

Kähler quotients.

Riemannian metric

symplectic form

Kähler manifold: (M, I, g, ω) $\qquad \omega(\cdot,\cdot) = g(I\cdot,\cdot)$

complex structure

$K \circlearrowright (M, I, g, \omega)$, $\quad \mu : M \longrightarrow \mathfrak{k}^*$

compact Lie group $\qquad$ moment map

$\Longrightarrow$ $\mu^{-1}(0)/K$ is Kähler (or stratified into Kähler mfs)

$$\begin{tikzcd} \mu^{-1}(0) \arrow[r, "i"] \arrow[d, "\pi"] & M \\ \mu^{-1}(0)/K & \end{tikzcd}$$

- $\pi^*\bar\omega = i^*\omega$, $\bar\omega$ symplectic
- $\bar g(u,v) = g(u^*, v^*)$ $\qquad$ horizontal lift wrt. $g|_{\mu^{-1}(0)}$
- $\bar I u = d\pi(I u^*)$

Example.

$S^1 \curvearrowright \mathbb{C}^n$, $\lambda \cdot z = \lambda z$, $\mathrm{Lie}(S^1) = i\mathbb{R}$

$\mu(z) = \frac{i}{2}(|z_1|^2 + \cdots + |z_n|^2)$, $i \in \mathrm{Lie}(S^1)$ is fixed

$(\mu - i)^{-1}(0)/S^1 = \mu^{-1}(i)/S^1 \cong \mathbb{P}^{n-1}$ with Fubini-Study metric

Examples: • $U(n) \circlearrowright \mathbb{C}^n$

$\mu : \mathbb{C}^n \to \mathfrak{u}(n) \cong \mathfrak{u}(n)^*$, $\mu(z) = \frac{i}{2} z z^* = \frac{i}{2}\begin{pmatrix} z_1 \\ \vdots \\ z_n \end{pmatrix}(\bar{z}_1 \cdots \bar{z}_n)$

• $U(n+1) \circlearrowright \mathbb{P}^n$, $\mu(z) = \frac{i}{2\pi} \frac{z z^*}{|z|^2}$

Operations: $K \circlearrowright (M, \omega)$, $\mu : M \to \mathfrak{k}^*$

• **(Subgroup)** $H \subseteq K$, $M \xrightarrow{\mu} \mathfrak{k}^* \to \mathfrak{h}^*$ moment map for $H \circlearrowright M$

• **(subspace)** $N \subseteq M$, K-invariant symplectic submanifold $\rightsquigarrow$ $\mu|_N : N \to \mathfrak{k}^*$

• **(Product)** $K_i \circlearrowright (M_i, \omega_i)$, $\mu_i : M_i \to \mathfrak{k}_i^*$ $\rightsquigarrow$ $\mu_1 \times \cdots \times \mu_n : M_1 \times \cdots \times M_n \to \mathfrak{k}_1^* \times \cdots \times \mathfrak{k}_n^*$

• $M \subseteq \mathbb{C}^n \times \mathbb{P}^m$ projective-over-affine ($\Rightarrow$ Kähler)

$K \subseteq U(n) \times U(m+1)$ preserving M

$\Rightarrow$ $\exists$ canonical moment map $\mu : M \to \mathfrak{k}^*$

OVERVIEW

- M complex algebraic variety (non-singular) $\Rightarrow (M, I)$ complex manifold
- g Kähler metric on M $\Rightarrow (M, \omega)$ symplectic manifold
- K compact Lie group s.t. $G := K_{\mathbb{C}} \circlearrowright (M, I)$, $K \circlearrowright (M, \omega)$

3 types of quotients:

symplectic

$$\mu^{-1}(0)/K \cong M/\!\!/_L G = M^{L\text{-ss}}/G$$

algebraic (GIT)

choice of linearization L (line bundle)

categorical quotient in AG

choice of moment map μ

- homeomorphisms
- stratified
- Kähler

$\mu^{-1}(0)/K \cong M/\!\!/_\mu G$

always true

$M/\!\!/_\mu G \cong M/\!\!/_L G$: true $\Leftrightarrow M^{\mu\text{-ss}} = M^{L\text{-ss}}$

$\Leftrightarrow$
- compatibility between μ and L
- good asymptotic properties at ∞ of M (always true if M is compact)

$$M/\!\!/_\mu G = M^{\mu\text{-ss}}/G$$

complex analytic

categorical quotient in complex analytic spaces

Analytic Kempf-Ness Theorem

Definition. A complex analytic space is a topological space X together with a sheaf of $\mathbb{C}$-algebras $\mathcal{O}_X$ which is locally isomorphic to

$$Z = \{x \in U \subseteq \mathbb{C}^n : f_1(x) = \cdots = f_r(x) = 0\}, \quad f_1, \ldots, f_r : U \subseteq \mathbb{C}^n \to \mathbb{C} \text{ holomorphic.}$$

Examples.

- complex manifold
- complex algebraic variety X. The underlying complex analytic space is called the analytification, denoted X^{an}.

Definition. G complex reductive group $\circlearrowright (X, \mathcal{O}_X)$

An analytic good quotient for $G \circlearrowright X$ is a G-invariant surjective holomorphic map $\pi : X \to Y$ such that

- $\mathcal{O}_Y = (\pi_* \mathcal{O}_X)^G$
- $\pi^{-1}(\text{Stein})$ is Stein

$\Rightarrow$ categorical quotient (hence unique) denoted $X /\!/ G$

Example. $M^{L\text{-}ss} \longrightarrow M /\!/_L G$ GIT quotient

$\Rightarrow (M^{L\text{-}ss})_{an} \longrightarrow (M /\!/_L G)_{an}$ analytic good quotient

Idea. We can construct an analytic good quotient on a large open set $M^{\mu\text{-}ss} \subseteq M$ by choosing a moment map.

Theorem. (Atiyah-Bott 1982, Guillemin-Sternberg 1982, Kirwan 1984, Sjamaar 1994, Heinzner-Loose 1994)

- (M, I, ω) Kähler manifold
- K compact Lie group, $G = K_{\mathbb{C}} \circlearrowright (M, I)$
- $K \circlearrowright (M, \omega)$, $\mu : M \longrightarrow \mathfrak{k}^*$ moment map

Define the set of analytically semistable points

$$M^{\mu\text{-}ss} = \{ p \in M \; : \; \overline{G \cdot p} \cap \mu^{-1}(0) \neq \phi \} \underset{\text{open}}{\subseteq} M$$

Then, there is an analytic good quotient for $G \circlearrowright M^{\mu\text{-}ss}$ denoted $M /\!\!/_{\mu} G := M^{\mu\text{-}ss} /\!\!/ G$. Moreover

$$\begin{tikzcd}
\mu^{-1}(0) \arrow[r, hook] \arrow[d] & M^{\mu\text{-}ss} \arrow[d] \\
\mu^{-1}(0)/K \arrow[r, "\cong"] & M /\!\!/_{\mu} G
\end{tikzcd}$$

- homeomorphism
- stratified spaces
- biholomorphisms

$$
\begin{tikzcd}
\mu^{-1}(0) \arrow[r, hook] \arrow[d] & M^{\mu\text{-ss}} \arrow[r, phantom, "\overset{?}{=}"] \arrow[d] & M^{L\text{-ss}} \arrow[d] \\
\mu^{-1}(0)/K \arrow[r, "\cong"] & M/\!\!/_{\mu} G \arrow[r, phantom, "="] & M/\!\!/_{L} G
\end{tikzcd}
$$

Kempf-Ness type theorem

By uniqueness of categorical quotients, $M/\!\!/_{\mu} G \cong M_L/\!\!/G$ iff

$$\boxed{M^{\mu\text{-ss}} \overset{?}{=} M^{L\text{-ss}}}$$

analytic semistability $\overset{?}{=}$ algebraic semistability

Theorem. (Kempf-Ness 1979, Ness 1984, Kirwan 1984)

Let $M \subseteq \mathbb{C}^n \times \mathbb{P}^m$ be projective-over-affine with the standard Kähler structure $K \subseteq U(n) \times U(m+1)$, $\mu : M \to \mathfrak{k}^*$ the standard moment map and $L = \mathcal{O}(1)|_M$.
Then, $M^{\mu\text{-ss}} = M^{L\text{-ss}}$.

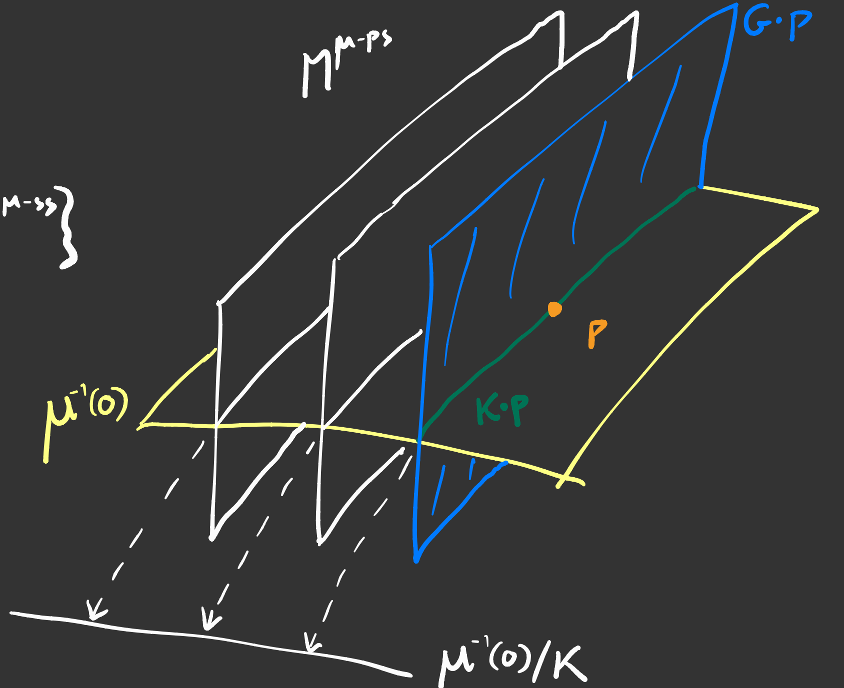
Idea of proof of the analytic KN theorem:

$$M/\!\!/_\mu G = M^{\mu\text{-ss}}/\sim, \quad x\sim y \Leftrightarrow \overline{G\cdot x} \cap \overline{G\cdot y} \cap M^{\mu\text{-ss}} \neq \emptyset$$

every equivalence class contains a unique closed orbit, so

$$M/\!\!/_\mu G \cong M^{\mu\text{-ps}}/G$$

$$M^{\mu\text{-ps}} := \{p \in M^{\mu\text{-ss}} : G\cdot p \text{ is closed in } M^{\mu\text{-ss}}\}$$



Take an invariant inner product on $\mathfrak{k}$

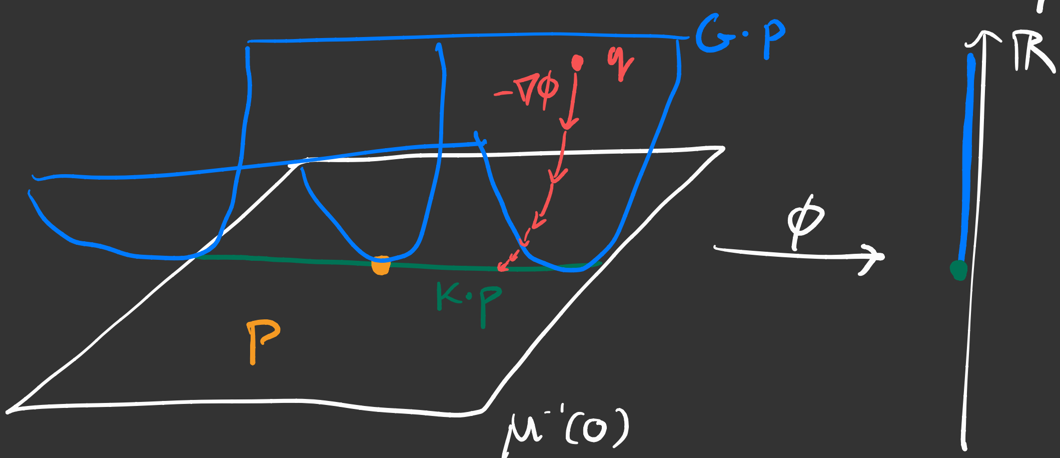
$\mu : M \longrightarrow \mathfrak{k}^* \cong \mathfrak{k}$, $\quad \phi := \|\mu\|^2 : M \longrightarrow \mathbb{R}$

$\nabla\phi = \mathrm{grad}(\phi) \in \mathfrak{X}(M)$ (using the Kähler metric)

Lemma. $(\nabla\phi)_p = 2 I \mu(p)^{\#}_p$ $\Rightarrow$ flow of $\nabla\phi$ contained in G-orbits

If the action is free: Lemma $\Rightarrow$ $\mathrm{crit}(\phi) = \mu^{-1}(0)$

F_t = flow of $-\nabla\phi$, $\quad F_\infty : M^{\mu\text{-ss}} \to \mu^{-1}(0)$ continuous retraction



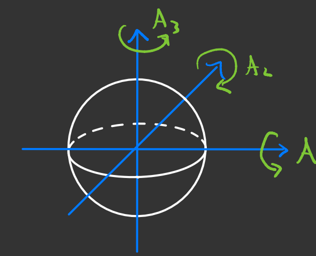
$$\begin{tikzcd} M^{\mu\text{-ss}} \arrow[r, "F_\infty"] \arrow[d] & \mu^{-1}(0) \arrow[d] \\ M/\!\!/_\mu G \arrow[r] & \mu^{-1}(0)/K \end{tikzcd}$$

inverse map

$\square$

Polygon Spaces

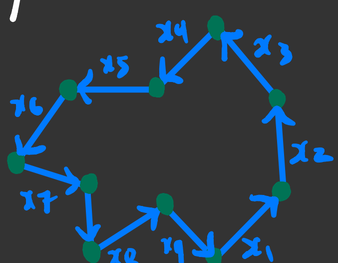
$SO(3) \circlearrowright (S^2, \omega)$, $\quad \mu : S^2 \hookrightarrow \mathbb{R}^3 \cong \mathfrak{so}(3)^*$



$$SO(3) \underset{\text{diagonally}}{\circlearrowright} \underbrace{S^2 \times \cdots \times S^2}_{n \text{ times}} =: M, \qquad \begin{aligned} \mu : M &\longrightarrow \mathbb{R}^3 \\ (x_1, \dots, x_n) &\longmapsto x_1 + \cdots + x_n \end{aligned}$$

$$\mu^{-1}(0)/SO(3) = \{(x_1, \dots, x_n) \in \mathbb{R}^3 : |x_i| = 1,\ x_1 + \cdots + x_n = 0\} / SO(3)$$

= space of n-gons in $\mathbb{R}^3$ with side lengths 1 up to rigid motions



$S^2 = \mathbb{P}^1$, $\omega = 4\pi\,\omega_{FS}$, $M = (\mathbb{P}^1)^n$

$L := pr_1^*\mathcal{O}(1) \otimes \cdots \otimes pr_n^*\mathcal{O}(1)$ very ample (Segre $(\mathbb{P}^1)^n \hookrightarrow \mathbb{P}^{(n+1)^2-1}$)

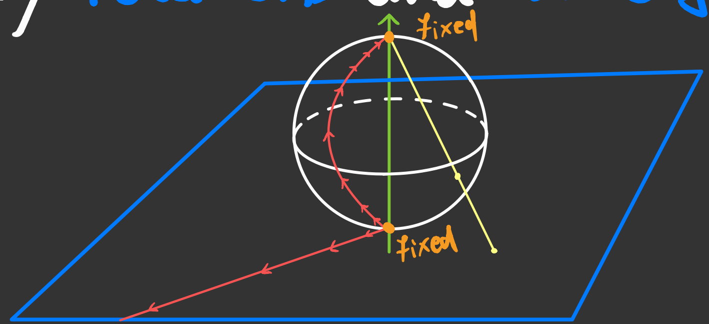
$SU(2) \longrightarrow SO(3)$ double cover, Kernel $\mathbb{Z}_2$

$SU(2) \circlearrowright M$ (via standard $SU(2) \circlearrowright \mathbb{C}^2$)

Extends holomorphically to $SL(2,\mathbb{C}) \circlearrowright M$ (Mobius transformations)

$SL(2,\mathbb{C}) \circlearrowright \mathbb{P}^1$, $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \zeta = \dfrac{c+d\zeta}{a+b\zeta}$, $[1:\zeta] \in \mathbb{P}^1$.

Generated by rotations and stereographic scalings $\zeta \mapsto r\zeta$, $r>0$



$SL(2,\mathbb{C}) \curvearrowright L = pr_1^* \mathcal{O}(1) \otimes \cdots \otimes pr_n^* \mathcal{O}(1)$

GIT quotient: $M /\!/_L SL(2,\mathbb{C})$ projective variety

= moduli space of ordered n-tuples of points on $\mathbb{P}^1$ up to Mobius transformations.

$$\mu^{-1}(0) = \left\{ (x_1, \dots, x_n) \in (\mathbb{P}^1)^n : \text{center of mass} = \frac{x_1 + \cdots + x_n}{n} = 0 \right\}$$

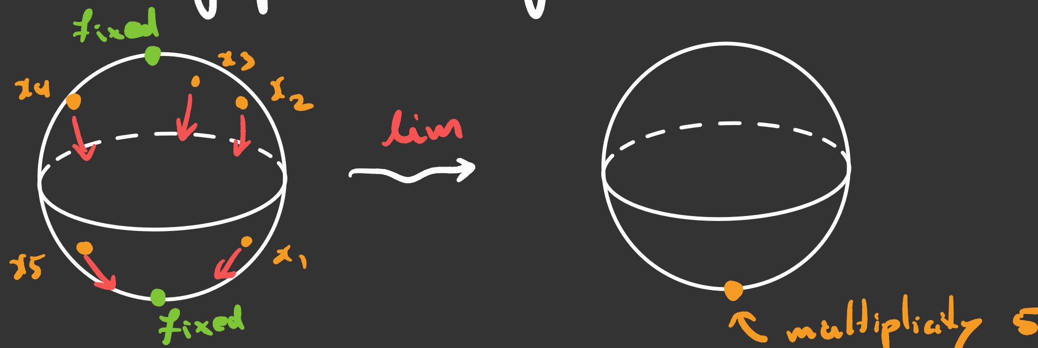
Kempf-Ness Theorem: $\mu^{-1}(0) \subseteq M^{L-ss}$ descends to

$$\mu^{-1}(0)/SO(3) \cong M /\!/_L SL(2,\mathbb{C})$$

Corollary. Every stable configuration of points on $\mathbb{P}^1$ can be moved by a Mobius transformation to have its center of mass in the centre.

Hilbert-Mumford Criterion. The 1-parameter subgroups $\lambda : \mathbb{C}^* \to SL(2,\mathbb{C})$ are conjugate to stereographic scallings $\zeta \mapsto t\zeta$

$$\lim_{t\to 0} \lambda(t)\cdot x$$

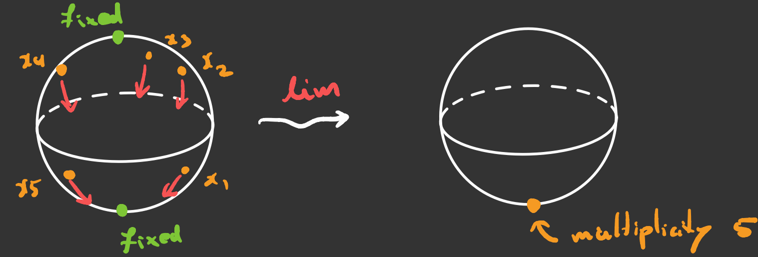


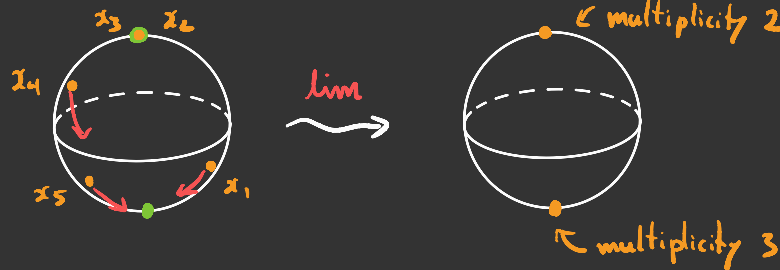
$$\lim_{t\to 0} \lambda(t)\cdot x$$

$\mathbb{C}^* \underset{\lambda}{\circlearrowright} L$ 

, $\lambda(t)\cdot\xi = t^{-2(r-s)}\xi$, $\mu^L(\lambda, x) = 2(r-s) \geq 0$

$\iff r \geq s$





$$\mu^L(x,\lambda) = \#\{x_i \text{ dragged to } 0 \text{ by } \lambda\} - \#\{x_i \text{ at } \infty\}$$

$x \in M^{L\text{-}ss} \Leftrightarrow \mu^L(x,\lambda) \geq 0, \forall \lambda \Leftrightarrow x$ has no point of multiplicity $> \frac{n}{2}$

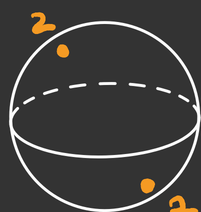
$x \in M^{L\text{-}s} \Leftrightarrow \mu^L(x,\lambda) \geq 0, \forall \lambda \Leftrightarrow x$ has no point of multiplicity $\geq \frac{n}{2}$

Corollary. A configuration of n points on S^2 with no point of multiplicity $\geq n/2$ can be moved by a Mobius transformation to have its centre of mass at the centre of the sphere.

$n=4$

stable

semistable but not stable

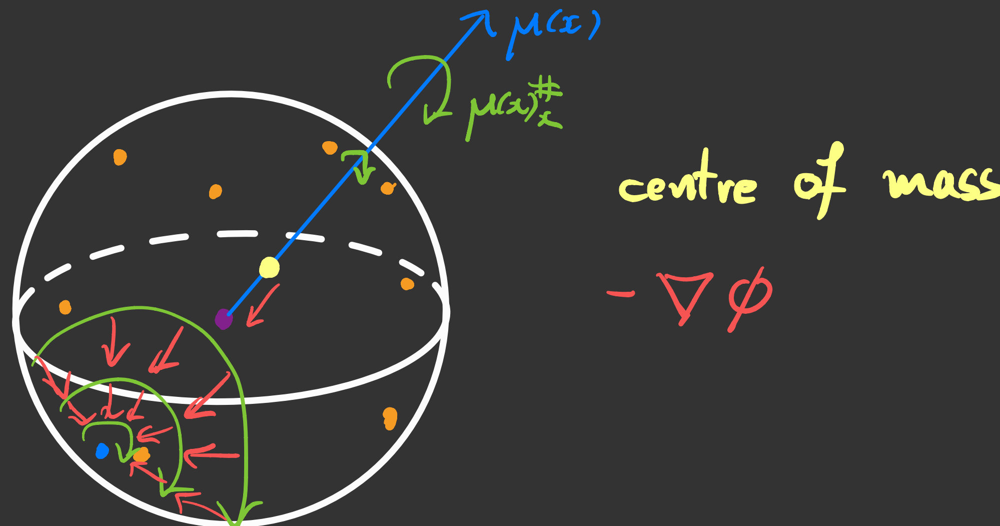


polystable but not stable

unstable

$$\phi = \|\mu\|^2, \qquad F_\infty : \text{flow by } -(\nabla\phi)_x = -2I\mu(x)^\#_x$$

$$F_\infty : M^{\mu\text{-}ss} \longrightarrow \mu^{-1}(0)$$



OVERVIEW

- M complex algebraic variety (non-singular) $\Rightarrow (M, I)$ complex manifold
- g Kähler metric on M $\Rightarrow (M, \omega)$ symplectic manifold
- K compact Lie group s.t. $G := K_{\mathbb{C}} \circlearrowright (M, I)$, $K \circlearrowright (M, \omega)$

3 types of quotients:

symplectic — algebraic (GIT)

$$\mu^{-1}(0)/K \;\cong\; M/\!\!/_L G = M^{L\text{-ss}}/G$$

categorical quotient in AG

choice of linearization L (line bundle)

choice of moment map μ

- homeomorphisms
- stratified
- Kähler

$\mu^{-1}(0)/K \cong M/\!\!/_\mu G$: always true

$M/\!\!/_L G \cong M/\!\!/_\mu G$: true $\Leftrightarrow M^{\mu\text{-ss}} = M^{L\text{-ss}}$

$\Leftrightarrow$
- compatibility between μ and L
- good asymptotic properties at ∞ of M (always true if M is compact)

$$M/\!\!/_\mu G = M^{\mu\text{-ss}}/G$$

complex analytic

categorical quotient in complex analytic spaces

The General Kempf-Ness Theorem

Definition. A Kähler manifold (M, I, ω) is *integral* if $\omega = \frac{i}{2\pi} F_\nabla$, where F_∇ is the curvature of a holomorphic line bundle $L \to M$ with hermitian metric $\|\cdot\|^2$.
($\Leftrightarrow$ $[\omega] \in H^2(M, \mathbb{Z})$).

Kodaira embedding theorem. A compact Kähler manifold is projective if and only if it is integral.

Warning: The embedding $M \subseteq \mathbb{P}^n$ is not necessarily an isometry $\omega \neq \omega_{FS}|M$.

Examples. • $M \subseteq \mathbb{C}^n \times \mathbb{P}^m$ projective-over-affine with standard Kähler str

• (M, I, ω), $\omega = 2i \partial\bar\partial f$, $L = M \times \mathbb{C}$

Moment map vs linearization

Let (M, I, ω, L) be integral

Let $K \circlearrowright (M, I, \omega)$. Then

$$\boxed{\begin{array}{ccc} \text{moment maps} & & \text{infinitesimal lifts of } K \circlearrowright M \\ \text{for } K \circlearrowright (M,\omega) & \overset{1:1}{\longleftrightarrow} & \text{to } L \text{ preserving } \|\cdot\|^2. \end{array}}$$

Suppose $K \circlearrowright L$ preserving $\|\cdot\|^2$ and lifing $K \circlearrowright M$. For all $x \in \mathfrak{k}$

$$x^\#_L = (x^\#_M)^* + 2\pi i \mu^x \xi$$

horizontal lift

$\mu^x : M \to \mathbb{R}$

Euler vector field on L

Lemma. $x \mapsto x^\#_L$ is a Lie algebra action $\Leftrightarrow$ $\mu : M \to \mathfrak{k}^*$ is a moment map

Concretely. Let (M, I, ω, L) be integral and $K \circlearrowright L$ preserving $\|\cdot\|^2$.

$$\begin{tikzcd} L^* \setminus \{0\} \arrow[d] \arrow[dr, "\hat{\mu}"] & \\ M \arrow[r, "\mu"] & \mathfrak{k}^* \end{tikzcd}$$

moment map for $K \circlearrowright (M, \omega)$

$$\hat{\mu}(v)(x) = \frac{d}{dt}\Big|_{t=0} \frac{1}{4\pi} \log \| e^{itx} \cdot v \|^2$$

Setup for Kempf-Ness theorem

- (M, I, ω, L) integral and algebraic
- K compact Lie group, $G = K_{\mathbb{C}}$.
- $G \circlearrowright L$ such that K preserves $\|\cdot\|^2$.

$\rightsquigarrow \mu : M \to \mathfrak{k}^*$

Question. $M^{\mu\text{-ss}} \stackrel{?}{=} M^{L\text{-ss}}$

Definition. A (M, L, G) (data for GIT) satisfies the **geometric condition** if

$$M^{L-ss} = \{ p \in M : \exists\, \hat{p} \in L^* \setminus \{0\} \mapsto p \,,\ \overline{G \cdot \hat{p}} \cap \{\text{zero section}\} = \phi \}$$

Note. This is an **algebraic condition** (does not involve the Kähler str)

Example. Projective-over-affine variety with ample line bundle.

The General Kempf-Ness Theorem. Let

- (M, I, ω, L) integral and algebraic, $G = K_{\mathbb{C}}$
- $G \circlearrowright L$ satisfying the geometric criterion (e.g. projective-over-affine) and such that K preserves $\|\cdot\|^2$ $\leadsto$ $\mu : M \to \mathfrak{k}^*$

Suppose that $\|\cdot\|^2 : L^* \to \mathbb{R}$ is proper on closed G-orbits disjoint from the zero section. Then

$$M^{\mu\text{-ss}} = M^{L\text{-ss}}$$

so

$$\boxed{\mu^{-1}(0)/K \cong M /\!\!/_L G}$$

- homeomorphism
- preserves stratifications
- biholomorphisms on the strata

The General Kempf-Ness Theorem. Let

- (M, I, ω, L) integral and algebraic, $G = K_{\mathbb{C}}$
- $G \circlearrowright L$ satisfying the geometric criterion (e.g. projective-over-affine)

  and such that K preserves $\|\cdot\|^2 \rightsquigarrow \mu : M \to \mathfrak{k}^*$

Suppose that $\|\cdot\|^2 : L^* \to \mathbb{R}$ is proper on closed G-orbits disjoint from the zero section. Then $M^{\mu\text{-ss}} = M^{L\text{-ss}}$.

Example. (Transcendental Projective – Sjamaar 1994)

(M, I, ω, L) integral and compact, $G \circlearrowright L$

Kodaira $\Rightarrow$ M projective, so the geometric condition holds

$\|\cdot\|^2 : L^* \to \mathbb{R}$ is proper so $M^{\mu\text{-ss}} = M^{L\text{-ss}}$.

Note. $\omega \neq \omega_{FS}|M$ so $\mu(z) \neq \frac{i}{2\pi} \frac{zz^*}{|z|^2}$. μ might be transcendental.

Kähler potentials. $M \subseteq \mathbb{C}^n$ affine variety, $G \subseteq GL(n,\mathbb{C})$

$\omega = 2i\partial\bar\partial f$, $f: M \to \mathbb{R}$, K-invariant Kähler potential

(e.g. $f = \|\cdot\|^2 \rightsquigarrow$ standard Kähler structure induced from $\mathbb{C}^n$)

$L = M \times \mathbb{C}$, $\|(p,z)\|^2 = e^{-4\pi f(p)}|z|^2 \rightsquigarrow \omega = \frac{i}{2\pi}F_\nabla$

so M is integral

$G \circlearrowright L$, $g\cdot(p,z) = (g\cdot p, z)$

$\rightsquigarrow \mu(p)(x) = df(Ix^\#_p)$ for $p \in M$, $x \in \mathfrak{k}$

Kempf-Ness theorem holds if f is proper and bounded below.

L trivial $\Rightarrow M^{L\text{-}ss} = M$

$$\boxed{\mu^{-1}(0)/K \cong M/\!\!/G}$$

(even if $\mu(z) \neq \frac{i}{2}zz^*$)

- $K \circlearrowright (M, I, \omega)$, $\omega = 2i\partial\bar{\partial} f$, $f: M \to \mathbb{R}$ K-invariant proper
- $\mu(p)(x) = df(I x_p^\#)$

Suppose that the action is free.

Then $M/\!/G = M/G$ since all orbits are closed

<u>Goal:</u> $\mu^{-1}(0)/K \cong M/G$, i.e. $\forall p \in M, \exists g \in G, \mu(g \cdot p) = 0$ and g is unique up to K. It suffices to show that

$$\begin{aligned} \mathcal{F}_p : K\backslash G &\longrightarrow \mathbb{R} \\ Kg &\longmapsto f(g \cdot p) \end{aligned}$$

(Kempf-Ness function)

has a unique critical point. In fact, it has a unique global minimum.

$\mathcal{F}_p : H = K\backslash G \longrightarrow \mathbb{R}, \quad Kg \longmapsto f(g\cdot p)$

Existence of minimum.

$G\cdot p$ closed, f proper and bounded below

$\Rightarrow f(G\cdot p)$ has a minimum, so $\mathcal{F}_p$ has a minimum g.

$\mu(g\cdot p) = 0$.

Uniqueness. $\cdot\, \mathcal{H}$ is a Riemannian symmetric space.

- It is geodesically convex: every 2 points are joined by a unique minimizing geodesic $(h(t) = e^{itx} g)$.
- $\frac{d^2}{dt^2} \mathcal{F}_p(e^{itx} g) = \| x^{\#}_{e^{itx}g} \|^2_{\text{Kähler metric}} > 0$

$\Rightarrow \mathcal{F}_p$ geodesically convex

Twisting by characters

<u>Recall:</u> (M, I, ω, L) integral, $G \circlearrowright L$, K preserves $\|\cdot\|^2$ on L

$$\rightsquigarrow \mu : M \to \mathfrak{k}^*.$$

Given a character $\chi : G \to \mathbb{C}^*$, we can change the linearization to $L_\chi = L$

$$G \times L \longrightarrow L, \quad g \cdot_\chi v = \chi(g)\, g \cdot v.$$

$$\rightsquigarrow \mu_\chi = \mu - \xi, \qquad \xi = \frac{i}{2\pi} d\chi_1 : \mathfrak{k} \to \mathbb{R}$$

Hence, if the conditions of the KN theorem hold,

$$\mu^{-1}(\xi)/K \cong M /\!\!/_{L_\chi} G$$

Can be used to resolve the singularities.

Twisted affine (King 1994)

- $M \subseteq \mathbb{C}^n$ smooth complex affine variety
- $G \subseteq GL(n,\mathbb{C})$ reductive $\curvearrowright M$
- $L = M \times \mathbb{C}$, $\quad g\cdot(p,z) = (g\cdot p, z)$

$$\Rightarrow \quad M/\!/_L G = M/\!/G = \operatorname{Spec}(\mathbb{C}[M]^G)$$

Problem: $M/\!/G$ might be too small or highly singular

Solution: Twist by $\chi : G \to \mathbb{C}^*$, $\quad g\cdot(p,z) = (g\cdot p, \chi(g)z)$

$$M/\!/_\chi G := M/\!/_{L_\chi} G = M^{L_\chi\text{-}ss}/\!/G$$

We can view M as a projective-over-affine $M \subseteq \mathbb{C}^n \times \mathbb{P}^0$

$$G \to GL(n,\mathbb{C}) \times GL(1,\mathbb{C}), \quad g \mapsto (g, \chi(g^{-1}))$$

$$L_\chi^{\otimes r} = M \times \mathbb{C}, \quad g \cdot (p, z) = (g \cdot p, \chi(g)^r z)$$

$$\Rightarrow M /\!/_\chi G = \operatorname{Proj} \bigoplus_{r=0}^{\infty} H^0(L^{\otimes r})^G = \operatorname{Proj} \bigoplus_{r=0}^{\infty} \mathbb{C}[M]^{\chi^r}$$

$$\mathbb{C}[M]^{\chi^r} = \{ f \in \mathbb{C}[M] : f(g \cdot p) = \chi(g)^r f(p) \}$$

$$\mathbb{C}[M]^{\chi^0} = \mathbb{C}[M]^G$$

$M /\!/_\chi G \longrightarrow M /\!/ G$ projective morphism

$M \subseteq \mathbb{C}^n$, $\omega = 2i\partial\bar\partial f$, $f = \|\cdot\|^2|_M : M \to \mathbb{R}$

(M, I, ω, L) integral, $\|(p,z)\|^2 = e^{-4\pi\|p\|^2}|z|^2$

To apply the general Kempf-Ness theorem, we must show that for all $(p,z) \in L^* \setminus \{0\}$ such that

$$S := G\cdot(p,z) = \{(g\cdot p, \chi(g)^{-1}z) : g \in G\} \subseteq L^* = M \times \mathbb{C}$$

is closed,

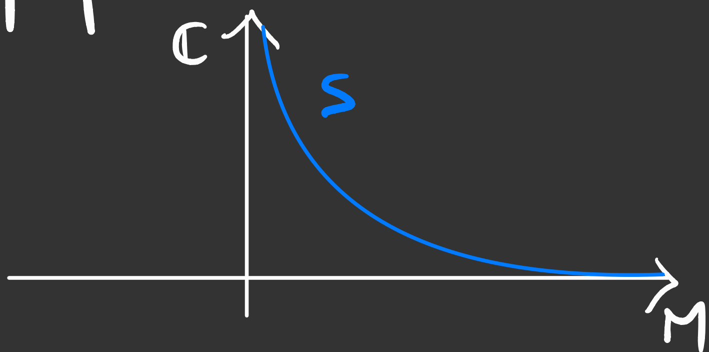
$$\|\cdot\|^2 : S \longrightarrow \mathbb{R}, \quad (q,w) \mapsto e^{4\pi\|q\|^2}|w|^2$$

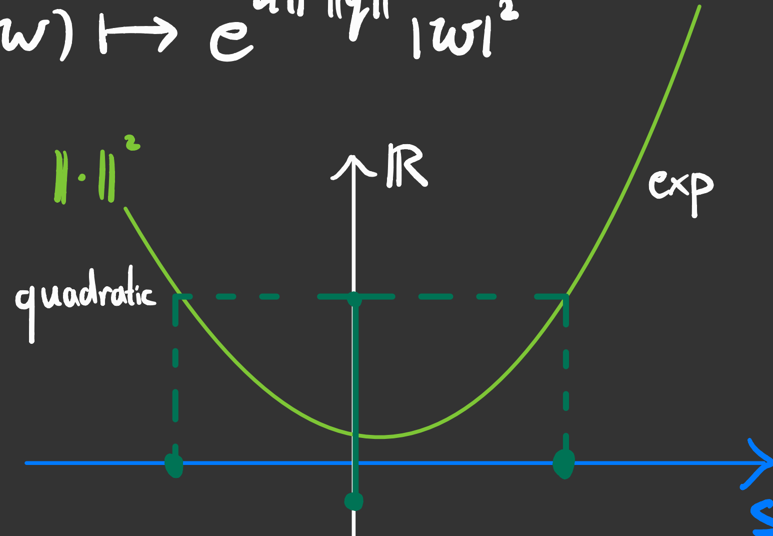
is proper.

Lemma. If $S \subseteq M \times \mathbb{C}^*$ is closed in $M \times \mathbb{C}$, then

$$\|\cdot\|^2 : S \longrightarrow \mathbb{R}, \quad (q, w) \mapsto e^{4\pi \|q\|^2} |w|^2$$

is proper.





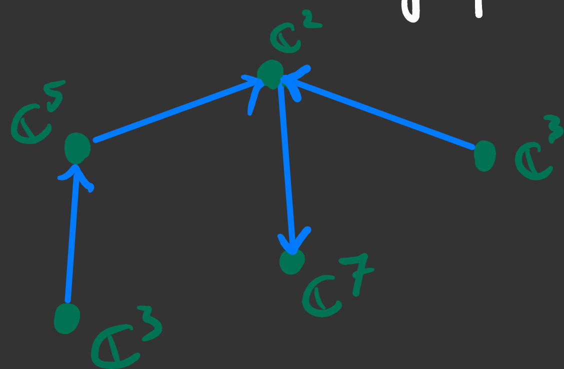
Hence, the Kempf-Ness theorem holds, so

$$\mu^{-1}\left(\tfrac{i}{2\pi} dx\right)/K \cong \operatorname{Proj} \bigoplus_{r=0}^{\infty} \mathbb{C}[M]^{\chi^r}$$

Quiver Varieties

A **quiver** is a directed graph with an integer on each node.

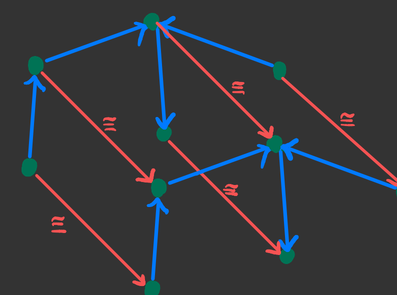
Q :



A **representation** of Q is a linear map on each edge

$$R = \bigoplus_{\text{edge } ij} \mathrm{Hom}(\mathbb{C}^{n_i}, \mathbb{C}^{n_j})$$

$$G = \prod_{\text{node } i} GL(n_i, \mathbb{C}) \circlearrowright R$$



isomorphism of representations

Problem. If Q has no cycle, $R/\!/G = pt.$

$$\chi_\theta : G \longrightarrow \mathbb{C}^*$$
$$g \longmapsto \prod_i \det(g_i)^{\theta_i}, \quad \theta = (\theta_1, \dots, \theta_n) \in \mathbb{Z}^{\#\text{ of nodes}}$$

$$R/\!/_{\chi_\theta} G \longrightarrow R/\!/G = pt \Rightarrow R/\!/_{\chi_\theta} G \text{ projective.}$$

If $\sum \theta_i n_i = 0$,

$$\phi \in R^{L_{\chi_\theta}\text{-}s} \iff \forall \text{ subrepresentation } \phi' \le \phi$$
$$\sum \theta_i n_i' < 0$$

$$\phi \in R^{L_{\chi_\theta}\text{-}ps} \iff \phi = \text{sum of stable representations}$$

$$U = \prod_i U(n_i) \subseteq G \subset R$$

$$\mu(\phi) = \sum_{ij} \phi_{ij}\phi_{ij}^* - \phi_{ij}^*\phi_{ij}$$

$$i\,dx = \sum_i \theta_i \, \mathrm{Id}_{\mathbb{C}^{n_i}}$$

$$\mu(\phi) = i\,dx \iff \sum_j \phi_{ji}\phi_{ji}^* - \phi_{ij}^*\phi_{ij} = \theta_i \qquad (*)$$

Corollary. A representation ϕ is polystable if and only if there is a metric on each $\mathbb{C}^{n_i}$ such that $(*)$ holds. This metric is unique up to the action of U.

Summary.

- We have an algebraic moduli problem
  - Configuration of points on $\mathbb{P}^1$ up to Möbius transformations
  - Representations of quivers up to isomorphisms
- GIT gives a natural "stability" condition and gives a moduli space of stable objects.
  - multiplicity $< n/2$
  - every subrepresentation $\phi' \leq \phi$ has $\sum \theta_i n_i' < 0$.
- There is a "real" or "unitary" condition invariant under a compact subgroup
  - centre of mass at zero
  - special hermitian metric
- The moduli space of stable objects is isomorphic to the space of unitary objects, which is a symplectic reduction.

Hyperkähler Reduction

Definition (Calabi 1979) A **hyperkähler manifold** is a Riemannian manifold (M,g) with 3 complex structures I, J, K s.t.

$$I^2 = J^2 = K^2 = IJK = -1$$

and which are Kähler with respect to g.

$$\Rightarrow \left.\begin{aligned} \omega_I &= g(I\cdot,\cdot) \\ \omega_J &= g(J\cdot,\cdot) \\ \omega_K &= g(K\cdot,\cdot) \end{aligned}\right\} \text{ 3 symplectic forms}$$

$\Omega_I := \omega_J + i\omega_K$ holomorphic symplectic w.r.t. I

Yau's solution to the Calabi conjecture $\Rightarrow$ compact holomorphic symplectic manifolds are hyperkähler

Example. $\mathbb{H}^n$

The only hyperkähler submanifolds of $\mathbb{H}^n$ are copies of $\mathbb{H}^k$ embedded linearly.

We need other construction methods:

- Hyperkähler reduction
- Twistor theory

K compact Lie group $\circlearrowright (M, g, I, J, K)$ hyperkähler

A **hyperkähler moment map** is a map

$$\mu = (\mu_I, \mu_J, \mu_K) : M \longrightarrow \mathfrak{k}^* \times \mathfrak{k}^* \times \mathfrak{k}^*$$

such that μ_I, μ_J, μ_K are moment maps w.r.t. $\omega_I, \omega_J, \omega_K$ respectively

<u>Theorem (HKLR 1987)</u> If K acts freely on $\mu^{-1}(0)$ then $\mu^{-1}(0)/K$ is a smooth hyperkähler manifold.

<u>Proof</u> $\mu_{\mathbb{C}} := \mu_J + i\mu_K : M \to \mathfrak{g}^*$ is holomorphic w.r.t. I.

$X = \mu_{\mathbb{C}}^{-1}(0) \subseteq M$ complex submanifold w.r.t. $I \Rightarrow$ Kähler

$\mu_I|_X : X \to \mathfrak{k}^*$ moment map $\Longrightarrow$ $(\mu_I|_X)^{-1}(0)/K$ Kähler $(\bar{I}, \bar{\omega}_I)$

$(\mu_I|_X)^{-1}(0)/K = \mu_I^{-1}(0) \cap \mu_J^{-1}(0) \cap \mu_K^{-1}(0) / K$

Repeat with J and K to get $(\bar{J}, \overline{\omega_J}), (\bar{K}, \overline{\omega_K})$ $\square$

- $K \circlearrowright (M, g, I, J, K)$
- suppose it extends to $G = K_{\mathbb{C}} \circlearrowright (M, I)$

Then, G preserves $\Omega_I = \omega_J + i\omega_K$ and $\mu_{\mathbb{C}} = \mu_J + i\mu_K$ is a holomorphic moment map

Then,
$$\boxed{\mu^{-1}(0)/K \cong \mu_{\mathbb{C}}^{-1}(0) /\!\!/_{\mu_I} G}$$

categerical quotient in complex analytic spaces

where $\mu_{\mathbb{C}}^{-1}(0) /\!\!/_{\mu_I} G = \mu_{\mathbb{C}}^{-1}(0) \cap M^{\mu_I - ss} /\!\!/ G$

Hence if there is a linearization $G \circlearrowright L$ such that $M^{\mu_{\mathbb{R}}-ss} = M^{L-ss}$ then

GIT quotient

$$\boxed{\mu^{-1}(0)/K \cong \mu_{\mathbb{C}}^{-1}(0) /\!\!/_L G}$$

An infinite-dimensional example

K compact Lie group, $G = K_{\mathbb{C}}$

Polar decomposition: $K \times \mathfrak{k} \xrightarrow{\cong} G$ diffeomorphism
$(k, x) \mapsto k e^{ix}$

$$G \cong K \times \mathfrak{k} \cong K \times \mathfrak{k}^* = T^*K$$

complex structure

choice of inner product on $\mathfrak{k}$

symplectic structure

Kähler

Can be proved by an infinite-dimensional version of the Kempf-Ness Theorem.

$\mathcal{A} = \{ A : [0,1] \longrightarrow \mathfrak{g} \mid A \in C^\infty \}$ (∞-dim analogue of M)

∞-dimensional hermitian vector space : $\langle A, B \rangle = \int_0^1 \langle A(t), B(t) \rangle \, dt$

$\Rightarrow \mathcal{A}$ ∞-dim Kähler manifold

$\mathcal{K}_0 = \{ k : [0,1] \to K \mid k(0) = k(1) = 1 \}$ (∞-dim analogue of K)

$\mathcal{G}_0 = \{ g : [0,1] \to G \mid g(0) = g(1) = 1 \}$ (∞-dim analogue of G)

$\mathcal{G}_0 \circlearrowright \mathcal{A}$, $\quad g \cdot A = gAg^{-1} - \dot{g}g^{-1}$ (gauge transformations)

$\mathcal{K}$ preserves $\langle \cdot , \cdot \rangle$ and hence the symplectic form ω

<u>moment map:</u>

$$\mu : \mathcal{A} \longrightarrow C^\infty([0,1], \mathfrak{k})$$
$$A = A_0 + iA_1 \longmapsto \dot{A}_1 + [A_0, A_1]$$

moment map: $\mu : \mathcal{A} \longrightarrow C^\infty([0,1], \mathfrak{k})$

$A = A_0 + iA_1 \longmapsto \dot{A}_1 + [A_0, A_1]$

$$\mu^{-1}(0)/\mathcal{K}_0 = \{(A_0, A_1) : [0,1] \to \mathfrak{k}^2 : \dot{A}_1 + [A_0, A_1] = 0\}/\mathcal{K}$$

$\forall A \in \mu^{-1}(0), \exists !\, k : [0,1] \to K,\ kA_0k^{-1} - \dot{k}k^{-1} = 0,\ k(0) = 1 \quad (\dot{k} = kA_0)$

$\Rightarrow \ k \cdot A = A_0' + iA_1' \in \mu^{-1}(0),\quad A_0' = 0,\quad \dot{A}_1' + [\cancel{A_0'}, A_1'] = 0 \Rightarrow A_1' \equiv x$

$$\mu^{-1}(0)/\mathcal{K} \xrightarrow{\cong} K \times \mathfrak{k} \cong T^*K$$

$$A \longmapsto (k(1), x)$$

symplectomorphism

Infinite-dimensional Kempf-Ness: $\mu^{-1}(0)/\mathcal{K}_0 \cong \mathcal{A}/\mathcal{G}_0$

Goal: • For all $A : [0,1] \to \mathfrak{g}$, there exists $g : [0,1] \to G$, $g(0) = g(1) = 1$ such that $A' := g \cdot A = gAg^{-1} - \dot{g}g^{-1}$ solves $\dot{A}'_1 + [A'_0, A'_1] = 0$
• g is unique up to $\mathcal{K}$

Kähler potential: $f(A) = \frac{1}{2}\int_0^1 \|A_1(t)\|^2 \, dt$

Fix $A \in \mathcal{A}$ and let

$\mathcal{F} : \mathcal{G} \longrightarrow \mathbb{R}$, $\mathcal{F}(g) = f(g \cdot A)$ (Kempf-Ness function)

<u>As in the finite-dim case:</u>

we need to minimize the functional $\mathcal{F}$

Wlog, $A_0 = 0$.

$\mathcal{F} : K\backslash\mathcal{G} \to \mathbb{R}$, $\mathcal{F}(g) = f(g\cdot A)$, $f(A) = \frac{1}{2}\int_0^1 \|A_1(t)\|^2 dt$

$K\backslash\mathcal{G} = \{[0,1] \to K\backslash G\} = \left\{\begin{array}{l}\text{paths in the Riemannian symmetric} \\ \text{space } \mathcal{H} = K\backslash G\end{array}\right\}$

$\mathcal{F}(h) = \frac{1}{2}\int_0^1 \|\dot{h}(t)\|^2_{\mathcal{H}}\, dt$ = path length of h in $\mathcal{H}$

There is a unique minimizing geodesic between any two points in $\mathcal{H}$ (since $\mathcal{H}$ is simply connected of sectional curvature ≤ 0) so $\mathcal{F} : \mathcal{G}_0 \to \mathbb{R}$ has a unique minimum up to K_0. This proves the Kempf-Ness theorem.

$$T^*K \cong \mu^{-1}(0)/\mathcal{K}_0 \cong \mathcal{A}/\mathcal{G}_0 \cong G$$

For all $A \in \mathcal{A}$, $\exists!\, g \in \mathcal{G}$, $\dot{g} = gA$, $g(0) = 1$

$A \longmapsto g(1)$

Hyperkähler version

$\mathcal{A} = \{ A : [0,1] \to \mathfrak{k} \otimes \mathbb{H} \mid A \in C^\infty \}$

infinite-dimensional hyperkähler manifold

$\mathcal{K}_0 \curvearrowright \mathcal{A}$, $k \cdot A = kAk^{-1} - \dot{k}k^{-1}$

hyperkähler moment map:

$$\mu(A) = \begin{pmatrix} \dot{A}_1 + [A_0, A_1] + [A_2, A_3] \\ \dot{A}_2 + [A_0, A_2] + [A_3, A_1] \\ \dot{A}_3 + [A_0, A_3] + [A_1, A_2] \end{pmatrix}$$

$\mu^{-1}(0)/\mathcal{K}_0$ = moduli space of solutions to Nahm's equations (finite-dimensional hyperkähler manifold)

∞-dim KN: (Donaldson 1984) $\mu_{\mathbb{C}} = \mu_J + i\mu_K$,

$$\mu^{-1}(0)/\mathcal{K}_0 \cong \mu_{\mathbb{C}}^{-1}(0)/\mathcal{G}_0.$$

$\mathcal{A} = \{(A_0, A_1) : [0,1] \to \mathfrak{g}^2\}$, $\mu_{\mathbb{C}}(A_0, A_1) = \dot{A}_1 + [A_0, A_1]$

$\Rightarrow \mu_{\mathbb{C}}^{-1}(0)/\mathcal{G}_0 \cong T^*G$

Theorem (Kronheimer 1988) T^*G is hyperkähler

Proof of Donaldson's Theorem. Same as above, but now

$$\mathcal{F} : C^\infty([0,1], \mathcal{H}) \to \mathbb{R}, \quad \mathcal{F}(h) = \frac{1}{2}\int_0^1 \left(\|\dot{h}\|_{\mathcal{H}}^2 + V(h)\right)$$

$V(h) = \|\mathrm{Ad}_h x\|^2$ "particle moving under a potential"

By Calculus of variation: For each $h_1, h_2 \in \mathcal{H}$, $\exists!$ path from h_1 to h_2 minimizing h. $\square$

References

1 Introductions

Symplectic reduction and geometric invariant theory

[1] Donaldson, S.K., Kronheimer, P.B.: The geometry of four-manifolds. Oxford Mathematical Monographs. The Clarendon Press, Oxford University Press, New York (1990). Oxford Science Publications

[2] Hoskins, V.: Geometric invariant theory and symplectic quotients. `https://userpage.fu-berlin.de/hoskins/GITnotes.pdf`

[3] Kirillov Jr., A.: Quiver representations and quiver varieties, *Graduate Studies in Mathematics*, vol. 174. American Mathematical Society, Providence, RI (2016)

[4] Kirwan, F.: Momentum maps and reduction in algebraic geometry. Differential Geom. Appl., pp. 135–171 (1998).

[5] Mumford, D., Fogarty, J., Kirwan, F.: Geometric invariant theory, *Ergebnisse der Mathematik und ihrer Grenzgebiete (2) [Results in Mathematics and Related Areas (2)]*, vol. 34, third edn. Springer-Verlag, Berlin (1994)

[6] Proudfoot, N.: Geometric invariant theory and projective toric varieties. In: Snowbird lectures in algebraic geometry, *Contemp. Math.*, vol. 388, pp. 161–167. Amer. Math. Soc., Providence, RI (2005)

[7] Proudfoot, P.: Lectures on toric varieties. `https://pages.uoregon.edu/njp/tv.pdf`

[8] Schwarz, G.W.: The topology of algebraic quotients. In: Topological methods in algebraic transformation groups (New Brunswick, NJ, 1988), *Progr. Math.*, vol. 80, pp. 135–151. Birkhäuser Boston, Boston, MA (1989)

[9] Thomas, R.P.: Notes on GIT and symplectic reduction for bundles and varieties. In: Surveys in differential geometry. Vol. X, *Surv. Differ. Geom.*, vol. 10, pp. 221–273. Int. Press, Somerville, MA (2006)

[10] Woodward, C.T.: Moment maps and geometric invariant theory. arXiv preprint arXiv:0912.1132 (2009)

Symplectic reduction

[1] Abraham, R., Marsden, J.E.: Foundations of mechanics. Benjamin/Cummings Publishing Co., Inc., Advanced Book Program, Reading, Mass. (1978). Second edition, revised and enlarged, With the assistance of Tudor Raţiu and Richard Cushman

[2] Cannas da Silva, A.: Lectures on symplectic geometry, *Lecture Notes in Mathematics*, vol. 1764. Springer-Verlag, Berlin (2001)

[3] Guillemin, V., Sternberg, S.: Symplectic techniques in physics, second edn. Cambridge University Press, Cambridge (1990)

Geometric invariant theory

[1] Dolgachev, I.: Lectures on invariant theory, *London Mathematical Society Lecture Note Series*, vol. 296. Cambridge University Press, Cambridge (2003)

[2] Schmitt, A.H.W.: Geometric invariant theory and decorated principal bundles. Zurich Lectures in Advanced Mathematics. European Mathematical Society (EMS), Zürich (2008)

[3] Mumford, D., Fogarty, J., Kirwan, F.: Geometric invariant theory, *Ergebnisse der Mathematik und ihrer Grenzgebiete (2) [Results in Mathematics and Related Areas (2)]*, vol. 34, third edn. Springer-Verlag, Berlin (1994)

Complex-analytic quotients

[1] Heinzner, P., Huckleberry, A.: Analytic Hilbert quotients. In: Several complex variables (Berkeley, CA, 1995–1996), *Math. Sci. Res. Inst. Publ.*, vol. 37, pp. 309–349. Cambridge Univ. Press, Cambridge (1999)

[2] Heinzner, P., Huckleberry, A.: Kählerian structures on symplectic reductions. In: Complex analysis and algebraic geometry, pp. 225–253. de Gruyter, Berlin (2000)

Hyperkähler quotients

[1] Dancer, A.S.: Hyper-Kähler manifolds. In: Surveys in differential geometry: essays on Einstein manifolds, *Surv. Differ. Geom.*, vol. 6, pp. 15–38. Int. Press, Boston, MA (1999)

[2] Hitchin, N.: Hyper-Kähler manifolds. 206, pp. Exp. No. 748, 3, 137–166 (1992). Séminaire Bourbaki, Vol. 1991/92

[3] Hitchin, N.J., Karlhede, A., Lindström, U., Roček, M.: Hyper-Kähler metrics and supersymmetry. Comm. Math. Phys. **108**(4), 535–589 (1987)

2 Research papers

[1] Calabi, E.: Métriques kählériennes et fibrés holomorphes. Ann. Sci. École Norm. Sup. (4) **12**(2), 269–294 (1979)

[2] Atiyah, M.F., Bott, R.: The Yang-Mills equations over Riemann surfaces. Philos. Trans. Roy. Soc. London Ser. A **308**(1505), 523–615 (1983)

[3] Azad, H., Loeb, J.J.: Plurisubharmonic functions and the Kempf-Ness theorem. Bull. London Math. Soc. **25**(2), 162–168 (1993)

[4] Dancer, A., Swann, A.: Hyper-Kähler metrics associated to compact Lie groups. Math. Proc. Cambridge Philos. Soc. **120**(1), 61–69 (1996)

[5] Dancer, A., Swann, A.: The geometry of singular quaternionic Kähler quotients. Internat. J. Math. **8**(5), 595–610 (1997)

[6] Donaldson, S.K.: Nahm's equations and the classification of monopoles. Comm. Math. Phys. **96**(3), 387–407 (1984)

[7] Guillemin, V., Sternberg, S.: Geometric quantization and multiplicities of group representations. Invent. Math. **67**(3), 515–538 (1982)

[8] Gulbrandsen, M.G., Halle, L.H., Hulek, K.: A relative Hilbert-Mumford criterion. Manuscripta Math. **148**(3-4), 283–301 (2015)

[9] Heinzner, P., Loose, F.: Reduction of complex Hamiltonian G-spaces. Geom. Funct. Anal. **4**(3), 288–297 (1994)

[10] Hitchin, N.J., Karlhede, A., Lindström, U., Roček, M.: Hyper-Kähler metrics and supersymmetry. Comm. Math. Phys. **108**(4), 535–589 (1987)

[11] Hitchin, N.J.: The self-duality equations on a Riemann surface. Proc. London Math. Soc. (3) **55**(1), 59–126 (1987)

[12] Kempf, G., Ness, L.: The length of vectors in representation spaces. In: Algebraic geometry (Proc. Summer Meeting, Univ. Copenhagen, Copenhagen, 1978), *Lecture Notes in Math.*, vol. 732, pp. 233–243. Springer, Berlin (1979)

[13] King, A.D.: Moduli of representations of finite-dimensional algebras. Quart. J. Math. Oxford Ser. (2) **45**(180), 515–530 (1994)

[14] Kirwan, F.C.: Cohomology of quotients in symplectic and algebraic geometry, *Mathematical Notes*, vol. 31. Princeton University Press, Princeton, NJ (1984)

[15] Klyachko, A.A.: Spatial polygons and stable configurations of points in the projective line. In: Algebraic geometry and its applications (Yaroslavl, 1992), Aspects Math., E25, pp. 67–84. Friedr. Vieweg, Braunschweig (1994)

[16] Kronheimer, P.: A hyperkahler structure on the cotangent bundle of a complex lie group. MSRI preprint (1988)

[17] Marsden, J., Weinstein, A.: Reduction of symplectic manifolds with symmetry. Rep. Mathematical Phys. **5**(1), 121–130 (1974)

[18] Mayrand, M.: Kempf–Ness type theorems and Nahm equations. J. Geom. Phys. **136**, 138–155 (2019)

[19] Ness, L.: A stratification of the null cone via the moment map. Amer. J. Math. **106**(6), 1281–1329 (1984). With an appendix by David Mumford

[20] Schwarz, G.W.: The topology of algebraic quotients. In: Topological methods in algebraic transformation groups (New Brunswick, NJ, 1988), *Progr. Math.*, vol. 80, pp. 135–151. Birkhäuser Boston, Boston, MA (1989)

[21] Sjamaar, R.: Holomorphic slices, symplectic reduction and multiplicities of representations. Ann. of Math. (2) **141**(1), 87–129 (1995)

[22] Sjamaar, R., Lerman, E.: Stratified symplectic spaces and reduction. Ann. of Math. (2) **134**(2), 375–422 (1991)